Game Theory

Nash Equilibrium

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4.1.- Introduction

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  - There are two ways of doing so (they are related, but it is useful to look at them separately):
    - refinements in the extensive form and
    - refinements in the normal form.
  - For example, in extensive form games with perfect information, we may select those Nash equilibria that are obtained by backwards induction.

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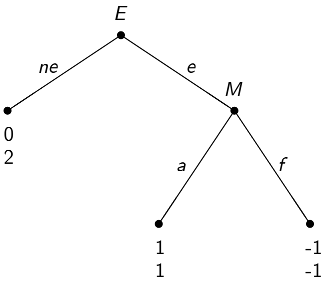

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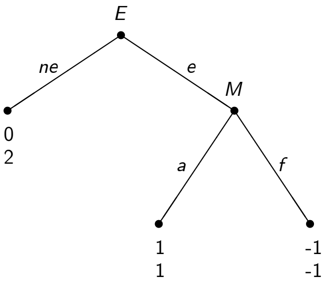

E/M	a	f
ne	$0, 2$	$0, 2$
e	$1, 1$	$-1, -1$

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    - We will enlarge the set of Nash equilibria.

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  - s_i *(weakly) dominates* s_i' if
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    for all $s_{-i} \in S_{-i}$, and there exists $s'_{-i} \in S_{-i}$ such that $h_i(s_i, s'_{-i}) > h_i(s_i', s'_{-i})$.

4.2.- Dominant strategies: examples

(1)

1/2	C	D
C	3, 3	0, 4
D	4, 0	1, 1

D strictly dominates C for both players.

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1/2	C	D
C	3, 3	0, 4
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(2)

1/2	L	R
T	10, 0	5, 2
B	10, 1	2, 0

T dominates B for player 1.

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- **Example:**

	q	$1-q$
$1/2$	L	R
T	$3,0$	$0,1$
M	$0,0$	$3,1$
B	$1,1$	$1,0$

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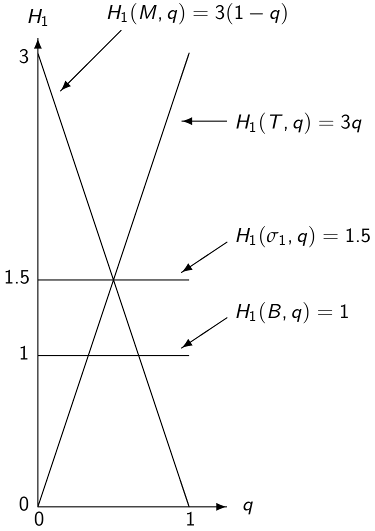
B	$1,1$	$1,0$

- No (pure) strategy dominates any other (pure) strategy for both players.
- However, the mixed strategy $\sigma_1(T) = \sigma_1(M) = \frac{1}{2}$ and $\sigma_1(B) = 0$ strictly dominates B since for all $q \in [0,1]$,

$$\begin{aligned} H_1(\sigma_1, q) &= 3q\tfrac{1}{2} + 3(1-q)\tfrac{1}{2} = \tfrac{3}{2} \\ &> 1 = H_1(B, q). \end{aligned}$$


H_1
$H_1(M,q) = 3(1-q)$
3
$H_1(T,q) = 3q$
$H_1(\sigma_1,q) = 1.5$
1.5
$H_1(B,q) = 1$
1
0
0
1
q

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- **Definition** σ_i *strictly dominates* σ_i' if

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- **Fact** Definition and Definition' are equivalent.
  - The use of Definition' simplifies the test.

$$
\begin{aligned}
H_i(\sigma_i, \sigma_{-i}) &= \sum_{s \in S} \prod_{j \in I} \sigma_j(s_j) h_i(s) \\
&= \sum_{s_{-i} \in S_{-i}} \prod_{j \in I \setminus \{i\}} \sigma_j(s_j) \underbrace{\left[\sum_{s_i \in S_i} \sigma_i(s_i) h_i(s_i, s_{-i}) \right]}_{= H_i(\sigma_i, s_{-i})}.
\end{aligned}
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\end{aligned}
$$

- Consider $\sigma_{-i} \in \Sigma_{-i}$ and $\sigma_i, \sigma_i' \in \Sigma_i$. Then

$$
H_i(\sigma_i, \sigma_{-i}) - H_i(\sigma_i', \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \prod_{j \in I \setminus \{i\}} \sigma_j(s_j) \left[H_i(\sigma_i, s_{-i}) - H_i(\sigma_i', s_{-i}) \right] \tag{1}
$$

4.2.- Dominant strategies: proof of fact

- $\Longrightarrow$) Assume that for all $\sigma_{-i} \in \Sigma_{-i}$,

$$H_i(\sigma_i, \sigma_{-i}) - H_i(\sigma_i', \sigma_{-i}) > 0.$$

Then, and since pure strategy profiles s_{-i} belong Σ_{-i},

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Then, by the previous expression in (1),

$$H_i(\sigma_i, \sigma_{-i}) - H_i(\sigma_i', \sigma_{-i}) > 0$$

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- In Example (2), T is a dominant strategy.

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  - It transforms the game (a multi-agent problem) into several one-agent problems.
  - It does not require that the game be common knowledge; in particular, to compute a dominant strategy a player does not need to know the other players' payoffs.
  - However, often the set of dominant strategy equilibria is empty.

- **Mechanism design (or implementation theory):** To select mechanisms to obtain a social goal. Namely, the game is not given, but rather it has to be designed with the objective that the set of equilibria has some properties; for instance, the set of dominant strategies is non-empty and "implements" the social goal.

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- **Examples:**

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  - Decision on a public good.
  - Etc.

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- **Example:**

(2)

1/2	L	R
T	$10, 0$	$5, 2$
B	$10, 1$	$2, 0$

4.3- Elimination of dominated strategies

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  - Two Nash equilibria: $S^* = \{(B, L), ((T, R)\}$.

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  - Two Nash equilibria: $S^* = \{(B, L), ((T, R)\}$.
  - T dominates B for player 1. Player 1 should not play B.
  - We have refined the set S^*. We have a unique prediction: (T, R).

- However,

(2')	1/2	L	R
	T	10, 0	5, 2
	B	10, 11	2, 0

- However,

(2')

1/2	L	R
T	$10, 0$	$5, 2$
B	$10, 11$	$2, 0$

- Is it also (B, L) a non sensible prediction?

4.3- Elimination of dominated strategies: Iterated

1/2	L	M	R
t	$4, 3^*$	$5, 1$	$6, 2$
m	$2, 1$	$8, 4$	$3, 6$
b	$3, 0$	$9, 6^*$	$2, 6$

4.3- Elimination of dominated strategies: Iterated

1/2	L	M	R
t	4, 3	5, 1	6, 2
m	2, 1	8, 4	3, 6
b	3, 0	9, 6	2, 6

R dominates M $\longrightarrow$

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t dominates m (or t dominates b) $\longrightarrow$

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L dominates R $\longrightarrow$

1/2	L
t	4, 3

.

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t dominates m (or t dominates b) $\longrightarrow$

1/2	L	R
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L dominates R $\longrightarrow$

1/2	L
t	4, 3

.

- Only one Nash equilibrium has survived.

4.3- Elimination of dominated strategies: Iterated

1/2	L	M	R
t	4, 3	5, 1	6, 2
m	2, 1	8, 4	3, 6
b	3, 0	9, 6	2, 6

R dominates M $\longrightarrow$

1/2	L	R
t	4, 3	6, 2
m	2, 1	3, 6
b	3, 0	2, 6

t dominates m (or t dominates b) $\longrightarrow$

1/2	L	R
t	4, 3	6, 2

L dominates R $\longrightarrow$

1/2	L
t	4, 3

.

- Only one Nash equilibrium has survived.
- Is it important the order of elimination? (Homework).

4.3- Dominance solvability and Sophisticated equilibrium

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- Given G, the successive elimination of dominated strategies is made up of the sequences: for every $i \in I$,

$$S_i = S_i^0 \supseteq S_i^1 \supseteq ... \supseteq S_i^t \supseteq S_i^{t+1} \supseteq ...,$$

where for all $t \geq 0$,

$$S_i^{t+1} = UD_i(S^t).$$

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- Moulin, H. "Dominance Solvable Voting Schemes," *Econometrica* 47, 1979.

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$S_1^0 = \{t, m, b\}$	$S_2^0 = \{L, M, R\}$
$S_1^1 = \{t, m, b\}$	$S_2^1 = \{L, R\}$
$S_1^2 = \{t\}$	$S_2^2 = \{L, R\}$
$S_1^3 = \{t\}$	$S_2^3 = \{L\}$
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- $S^\infty = \{(t, L)\}$ is the set of sophisticated equilibrium.
- Observe that neither t nor L are dominant strategies (they do not dominate any strategy).

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- Proceeding this way, for every $i \in I$, $S_i^\infty = \{1\}$.

- **Remark 1** We have a severe problem of existence.

M/W	F	B	$Home$
F	$3,1$	$0,0$	$-1,-1$
B	$0,0$	$1,3$	$-1,-1$
$Home$	$-1,-1$	$-1,-1$	$-1,-1$

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- We will come back to this notion relating it to Subgame Perfect Equilibrium and razionalizable strategies.

4.4- Subgame Perfect Equilibrium

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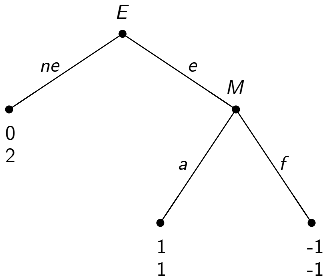

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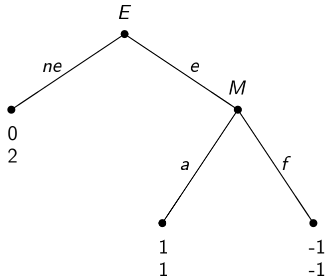

- Two Nash equilibria: $(ne, f), (e, a)$.

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  - Subgame Perfect Equilibrium requires rational behavior even in information sets that are not reached in equilibrium (equilibrium should not be based on incredible threats).
- Obtain (e, a) as the unique Subgame Perfect Equilibrium (which coincides with the one obtained by backwards induction).

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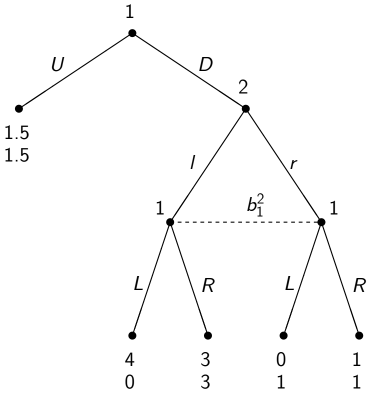
- But this is not always true.

Example



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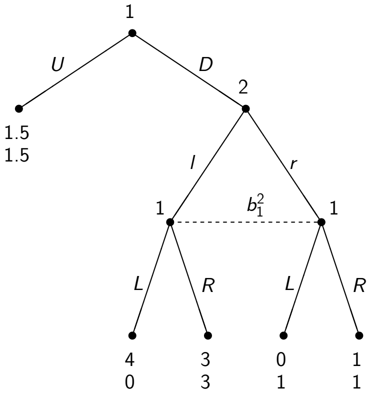
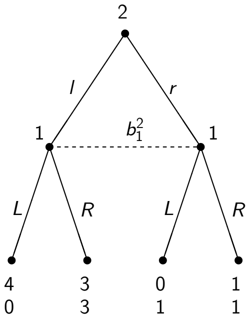


- $((U, R), r)$ is a Nash equilibrium.



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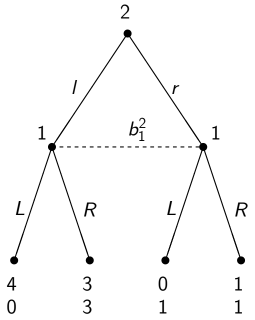
- $((U, R), r)$ is a Nash equilibrium.
- Backwards induction can not be used to eliminate it, but if player 2 believes that 1 will play R, then 2 should play l instead of r.



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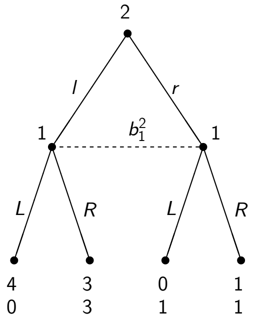
- What it is important here is that the subgame below looks like a game, and (r, R) is not a Nash equilibrium of the subgame.



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  - The two requirements above make sure that all information sets are either contained in K_x or are disjoint with K_x.
- Given $x \in X$, the subgame Γ_x is the restriction of Γ in the subtree K_x.

- Let $\sigma \in \hat{\Sigma}$ be a behavioral strategy in Γ, let $x \in X$ and consider Γ_x. Then σ can be decomposed as (σ^x, σ^{-x}) where σ^x describes behavior in Γ_x and σ^{-x} in $\Gamma_{-x} = \Gamma \backslash \Gamma_x$.

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- **Definition** Let Γ be a game in extensive form. The behavioral strategy $\hat{\sigma} \in \Sigma$ is a *Subgame Perfect Equilibrium* (SPE) of Γ if for every possible subgame Γ_x, the restriction of σ in Γ_x, σ^x, is a Nash equilibrium of Γ_x.

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- Note that Γ is also a subgame of itself ($\Gamma = \Gamma_{x_1}$ since $x_1 \in X$). Thus, for all Γ, SPE$\subseteq$NE.

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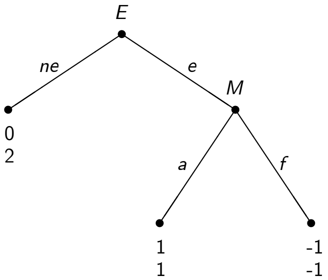

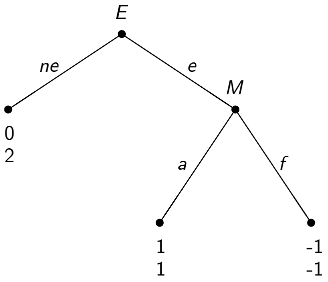

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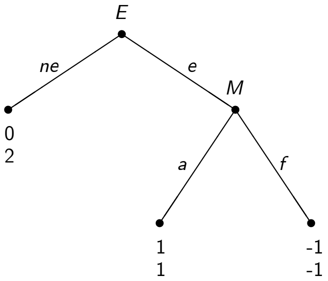

- Two Nash equilibria: $(ne, f), (e, a)$.
- (e, a) is the unique SPE of Γ and (ne, f) is not a SPE since f is not a Nash equilibrium of the subgame starting at the unique node that belongs to M.

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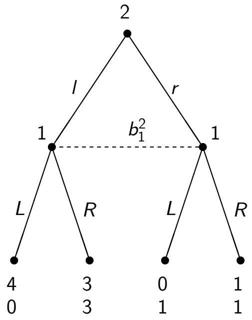
- $((U, R), r)$ is not a SPE of the second example since (R, r) is not a Nash equilibrium of one subgame.



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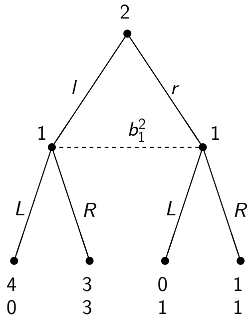

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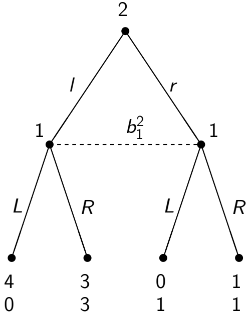

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- **Proof (idea):** Apply the backwards induction argument in all possible subgames and apply the Nash-Kuhn Theorem to obtain a Nash equilibrium. The behavioral strategy obtained is a SPE of Γ.

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- It is the most unchallenged refinement (and the most commonly used in Economics). However, ...

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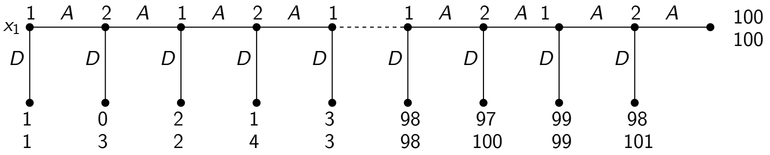

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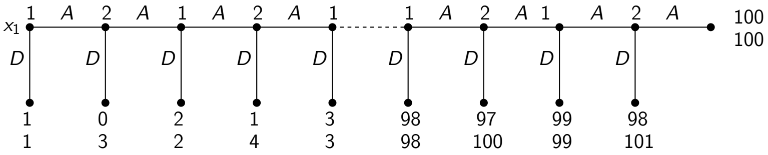

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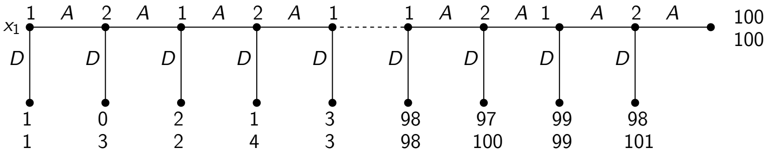

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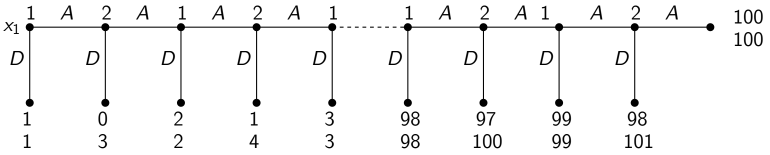

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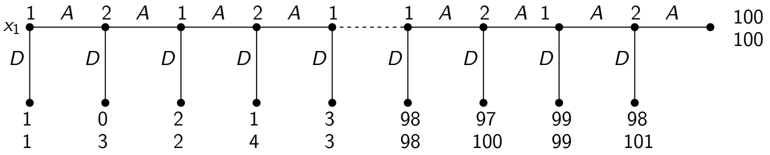

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    - Now, I can put myself in the position of Player 1 and realize that Player 2 (me) can do the above argument, and therefore try A and wait what happens.

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  1. SPE gives a solution everywhere (in all subgames), even in subgames where the solution says that they will not be reached (information sets with zero probability).
  2. SPE imposes rational behavior everywhere, even in the subgames of the game that SPE says that cannot be reached. In out-of-equilibrium subgames, the "solution" is disapproved, yet players evaluate their actions taking as given the behavior of the other players, that have been demonstrated incorrect since we are in an out-of-equilibrium path.

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- But there is still an additional problem (another source of terminological confusion): it is possible to define "Perfect Equilibrium" in the normal form which seems the natural extension, but it is not the same (it is if we consider the agent-normal form).

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- The Horse example illustrates the idea.

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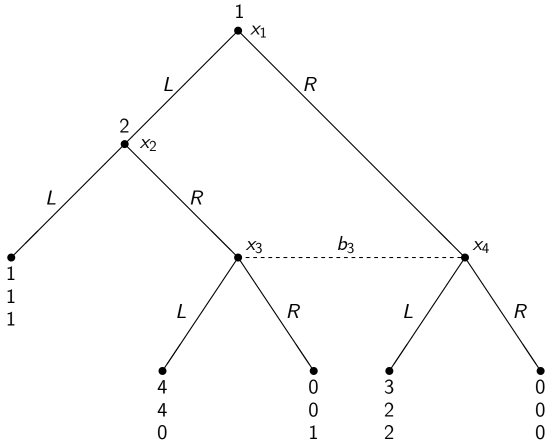

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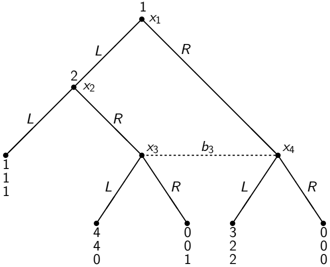
- Consider first the particular equilibrium of type 2: $(p_1, p_2, p_3) = (0, 1, 1)$.
  - The same argument will work for all other equilibria of type 2, but it will be less transparent.

- Given $p_1 = 0$ and $p_3 = 1$, is it reasonable to think that player 2 will play $p_2 = 1$?



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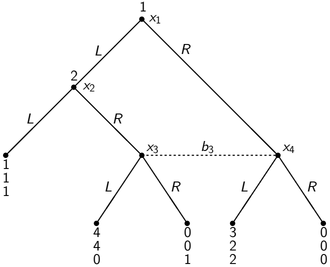

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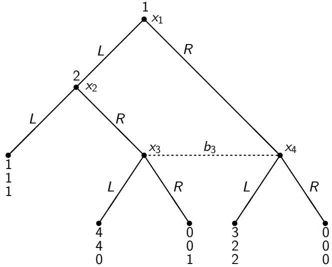

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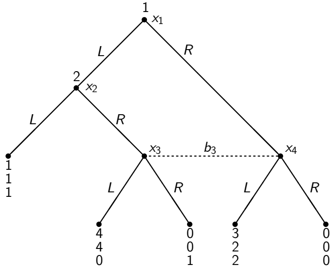

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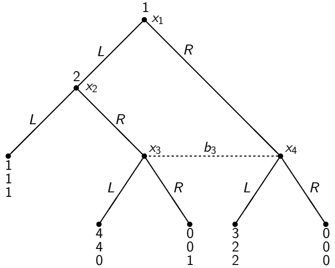

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  - Player 2 arrives home (he does not have to play) but suddenly, the telephone rings and says: "It is your turn, decide between L and R".
  - He knows that he is at x_2 (player 1 did a mistake), but given $p_3 = 1$, player 2 cannot play $p_2 = 1$ but rather he has to play R.

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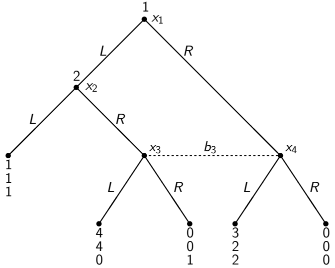

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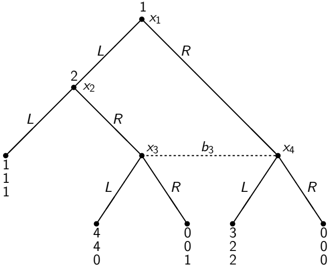

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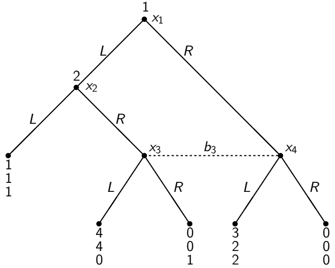

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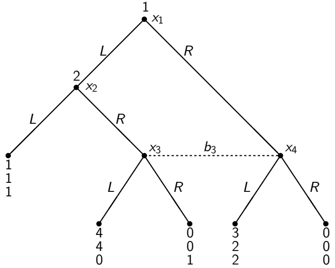

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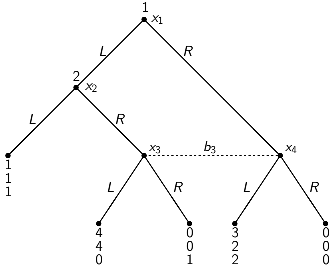

- $p_3 = 0$ is still rational since he can be either at x_3 or at x_4 (the mistake may come from either player 1 or player 2). Even with a probability of mistakes, $(1, 1, 0)$ is still rational.

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- For every $i \in I$, define $\hat{\Sigma}_i(\varepsilon)$ as the subset of behavioral strategies $\hat{\sigma}_i$ of player i with the property that for all $b_i \in B_i$, $\hat{\sigma}_i(c) \geq \varepsilon_c$ for all $c \in C_{b_i}$; namely, each action c at b_i has at least probability $\varepsilon_c > 0$.

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- Define the perturbed game $\Gamma(\varepsilon)$ as the same game Γ when players can only choose strategies in $\hat{\Sigma}(\varepsilon)$.
- Crucial point: given $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$, all information sets have strictly positive probability to be reached.

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Definition A behavioral strategy $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$ is an *equilibrium* of $\Gamma(\varepsilon)$ if for every $i \in I$,

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- **Remark** By a fixed-point argument, for every ε sufficiently small, the set of equilibria of $\Gamma(\varepsilon)$ is non-empty since $\hat{\Sigma}(\varepsilon)$ is a non-empty, compact and convex subset of a finitely-dimensional Euclidian space and the best-reply correspondence

$$B(\varepsilon) : \hat{\Sigma}(\varepsilon) \twoheadrightarrow \hat{\Sigma}(\varepsilon)$$

is upper-hemi continuous; moreover, for every $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$, $B(\varepsilon)(\hat{\sigma})$ is non-empty and convex.

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Theorem (Selten, 1975)

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- **Summary**
  - PE$\subseteq$SPE.
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  - There exists a game Γ (Selten's horse game) such that PE$\subsetneq$SPE.

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- Let $G = (I, (S_i)_{i \in I}, (h_i)_{i \in I})$ be a finite game in normal form and let ε be a function that assigns to every s_i an strictly positive number $\varepsilon_{s_i} > 0$ in such a way that for all $i \in I$,

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- Given the mixed extension $G^* = (I, (\Sigma_i)_{i \in I}, (H_i)_{i \in I})$ and the function ε, define the ε-perturbed game $G^*(\varepsilon) = (I, (\Sigma(\varepsilon)_i)_{i \in I}, (H_i)_{i \in I})$, where

$$\Sigma(\varepsilon)_i = \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) \geq \varepsilon_{s_i} \text{ for all } s_i \in S_i\}.$$

4.6- Perfect Equilibrium in the normal form

Definition* A mixed strategy $\sigma \in \Sigma$ is a *normal-form perfect equilibrium* (or a *trembling-hand perfect equilibrium in the normal form*) of G^* if there exist two sequences $\{\varepsilon^k\} \to 0$ and $\{\sigma^k\}$ such that for every $k \geq 1$, σ^k is an equilibrium of $G^*(\varepsilon^k)$ and $\{\sigma^k\} \to \sigma$.

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- Question: Does every normal-form perfect equilibrium of G^* correspond to a perfect equilibrium of Γ?

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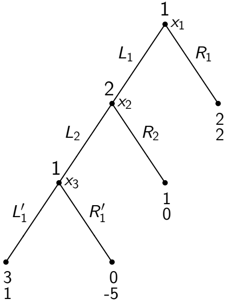
- Suppose we have a finite game in extensive form Γ and construct its associated normal form G.
- Question: Does every normal-form perfect equilibrium of G^* correspond to a perfect equilibrium of Γ?
- Answer: NO.

4.6- Perfect Equilibrium in the normal form

Example:



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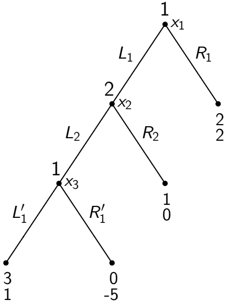

- There exists a unique subgame perfect equilibrium of Γ: $((L_1, L_1'), L_2)$. Hence, $((L_1, L_1'), L_2)$ is the unique perfect equilibrium of Γ.

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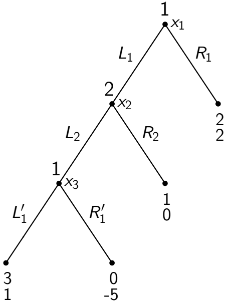
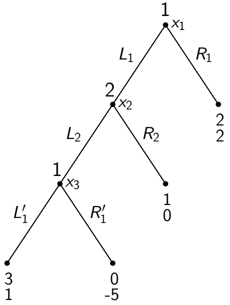
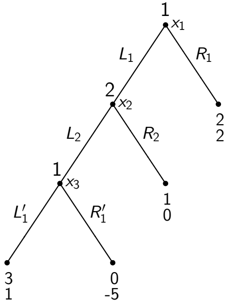


- There exists a unique subgame perfect equilibrium of Γ: $((L_1, L'_1), L_2)$. Hence, $((L_1, L'_1), L_2)$ is the unique perfect equilibrium of Γ.
- However, $(R_1, R'_1), R_2)$ is a perfect equilibrium (according to Definition*) in the normal form.

4.6- Perfect Equilibrium in the normal form



(L_1, L_1')
(L_1, R_1')
(R_1, L_1')
(R_1, R_1')



(L_1, L_1')
(L_1, R_1')
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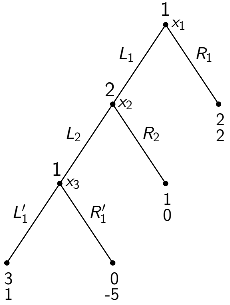
- The reason is that in the normal form trembles are correlated while in the extensive form trembles in different information sets are uncorrelated.

4.6- Perfect Equilibrium in the normal form



(L_1, L_1')
(L_1, R_1')
(R_1, L_1')
(R_1, R_1')

- The reason is that in the normal form trembles are correlated while in the extensive form trembles in different information sets are uncorrelated.
- In the example, trembles at x_1 and x_3 in the normal form are not independent (the same experiment is used for both), while in the extensive form we have to use uncorrelated trembles by performing two experiments, one at x_1 and the other at x_3.

4.6- Perfect Equilibrium in the normal form

$$
\begin{array}{rc|c|c|}
 & & \varepsilon_k & 1-\varepsilon_k \\
 & 1/2 & L_2 & R_2 \\
\hline
\frac{\varepsilon_k^2}{2} & L_1 L_1' & 3,1 & 1,0 \\
\hline
\frac{\varepsilon_k^2}{2} & L_1 R_1' & 0,-5 & 1,0 \\
\hline
\varepsilon_k & R_1 L_1' & 2,2 & 2,2 \\
\hline
1-\varepsilon_k-\varepsilon_k^2 & R_1 R_1' & 2,2 & 2,2 \\
\hline
\end{array}
$$

		ε_k	$1-\varepsilon_k$
	$1/2$	L_2	R_2
$\frac{\varepsilon_k^2}{2}$	$L_1 L_1'$	$3, 1$	$1, 0$
$\frac{\varepsilon_k^2}{2}$	$L_1 R_1'$	$0, -5$	$1, 0$
ε_k	$R_1 L_1'$	$2, 2$	$2, 2$
$1-\varepsilon_k-\varepsilon_k^2$	$R_1 R_1'$	$2, 2$	$2, 2$

- Take any sequence $\{\varepsilon_k\} \to 0$ and define $\Sigma_1(\frac{\varepsilon_k^2}{2})$ and $\Sigma_2(\varepsilon_k)$, where $\varepsilon_1^k(s_1) = \frac{\varepsilon_k^2}{2}$ for all $s_1 \in S_1$ and $\varepsilon_2^k(s_2) = \varepsilon_k$ for all $s_2 \in S_2$.

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- Given $\varepsilon_k > 0$ sufficiently small, consider the following strategy $(\sigma_1^k, \sigma_2^k) \in \Sigma_1(\frac{\varepsilon_k^2}{2}) \times \Sigma_2(\varepsilon_k)$:

		ε_k	$1-\varepsilon_k$
	$1/2$	L_2	R_2
$\frac{\varepsilon_k^2}{2}$	L_1L_1'	$3,1$	$1,0$
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4.6- Perfect Equilibrium in the normal form

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  - $\sigma_2^k(L_2) = \varepsilon_k$ and $\sigma_2^k(R_2) = 1-\varepsilon_k$.

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- Consider now player 1 :

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  - $H_1(L_1 L_1', \sigma_2^k) = 3\varepsilon_k + (1 - \varepsilon_k)$.

- Consider player 2 :
  - $H_2(\sigma_1^k, L_2) = \frac{\varepsilon_k^2}{2} - 5\frac{\varepsilon_k^2}{2} + 2\varepsilon_k + 2 - 2\varepsilon_k - 2\varepsilon_k^2 = 2 - 4\varepsilon_k^2$.
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4.6- Perfect Equilibrium in the normal form

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  - $H_1(R_1 L_1', \sigma_2^k) = H_2(R_1 R_1', \sigma_2^k) = 2$.

- Consider player 2 :
  - $H_2(\sigma_1^k, L_2) = \frac{\varepsilon_k^2}{2} - 5\frac{\varepsilon_k^2}{2} + 2\varepsilon_k + 2 - 2\varepsilon_k - 2\varepsilon_k^2 = 2 - 4\varepsilon_k^2$.
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  - Hence, for sufficiently small $\varepsilon_k > 0$, σ_1^k is a best-reply in $\Sigma_1(\frac{\varepsilon_k^2}{2})$ against σ_2^k.
- Thus, $(R_1 R_1', R_2)$ is a perfect equilibrium in the normal form (according to Definition*).

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- Suppose that, given an extensive form game, we construct the agent-normal form, where every information set corresponds to an agent, and every player controls its agents. We will later give a formal definition of the agent-normal form.

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  - L_1

2/3	L_1'	R_1'
L_2	3, 1, 3	0,-5, 0
R_2	1, 0, 1	1, 0, 1

R_1

2/3	L_1'	R_1'
L_2	2, 2, 2	2, 2, 2
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    .

  - We want to see that $(L_1 L_1', L_2)$ is the unique perfect equilibrium of this agent-normal form, and hence, it is the unique subgame perfect equilibrium and perfect equilibrium of the extensive form Γ (although we already knew that, since it is the unique subgame perfect equilibrium of Γ and all perfect equilibrium are subgame perfect).

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Proposition 1 (Selten, 1975) *Let G be a finite game in normal form. Then, σ^* is a perfect equilibrium of G (according to Definition*) if and only if there exists a sequence $\{\sigma^k\} \to \sigma^*$ such that (a) σ^k is completely mixed (i.e., $\sigma^k \in int(\Sigma)$) and (b) for every $k \geq 1$, σ_i^* is a best reply to σ_{-i}^k for all $i \in I$.*

4.6- Perfect Equilibrium in the normal form

Proof $\Longleftarrow$) Assume there exists a sequence $\{\sigma^k\} \rightarrow \sigma^*$ such that (a) σ^k is completely mixed (i.e., $\sigma^k \in int(\Sigma)$) and (b) for every $k \geq 1$, σ_i^* is a best reply to σ_{-i}^k for all $i \in I$.

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- Let $\{e_k\} \to 0$ be such that for all $k \geq 1$, $e_k > 0$ and for all $i \in I$ and all $s_i \in S_i$,

$$\sigma_i^k(s_i) > e_k. \tag{2}$$

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$$\sigma_i^k(s_i) > e_k. \tag{2}$$

  - Notice that such sequence $\{e_k\}$ does always exist since $\sigma^k \in int(\Sigma)$; for instance, we can always take

$$e_k = \frac{1}{k} \min_{i \in I} \min_{s_i \in S_i} \{\sigma_i^k(s_i)\} > 0.$$

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- Let $\{e_k\} \to 0$ be such that for all $k \geq 1$, $e_k > 0$ and for all $i \in I$ and all $s_i \in S_i$,
$$\sigma_i^k(s_i) > e_k. \quad (2)$$
  - Notice that such sequence $\{e_k\}$ does always exist since $\sigma^k \in int(\Sigma)$; for instance, we can always take
$$e_k = \frac{1}{k} \min_{i \in I} \min_{s_i \in S_i} \{\sigma_i^k(s_i)\} > 0.$$
- Define $\varepsilon^k(\cdot)$ as follows: for every $k \geq 1$ and every $s_i \in S_i$,
$$\varepsilon^k(s_i) = \begin{cases} \sigma_i^k(s_i) & \text{if } s_i \text{ is not a best reply to } \sigma_{-i}^k \text{ in } G \\ e_k & \text{otherwise.} \end{cases} \quad (3)$$

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- Consider $G(\varepsilon^k)$.

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4.6- Perfect Equilibrium in the normal form

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- Therefore, if $s_i \in S_i$ is not a best reply to σ^*_{-i}, $\{\sigma^k_i(s_i)\} \to 0$.
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- Hence, σ_i^k has the property that non-best replies receive the minimum probability; *i.e.*, σ^k is a Nash equilibrium of $G(\varepsilon^k)$.
- Thus, there exist $\{\varepsilon^k\} \to 0$ and $\{\sigma^k\} \to \sigma^*$ such that for all $k \geq 1$, σ^k is a Nash equilibrium of $G(\varepsilon^k)$, implying that σ^* is a perfect equilibrium of G.

Proof $\Longrightarrow$) Assume σ^* is a perfect equilibrium (according to Definition*) of G.

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- Then, there exist $\{\varepsilon^k\} \to 0$ and $\{\sigma^k\} \to \sigma^*$ such that for all $k \geq 1$, σ^k is a Nash equilibrium of $G(\varepsilon^k)$.
- For every $k \geq 1$ and $i \in I$, define

$$T_i^k = \{s_i \in S_i \mid \underset{\sigma_i^*(s_i)}{\overset{\sigma_i^k(s_i)}{\downarrow (*)}} > \underset{0}{\overset{\varepsilon^k(s_i)}{\downarrow}}\}.$$

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- However, T_i^k may not contain all of them. By $(*)$, there exists K such that for all $k \geq K$, if $\sigma_i^*(s_i) > 0$ then $s_i \in T_i^k$. Without loss of generality, assume $K = 1$.

- Summing up: every $s_i \in S_i$ with $\sigma_i^*(s_i) > 0$ is in T_i^k and every $s_i \in T_i^k$ is a best reply to σ_{-i}^k.

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  - (a) σ^k is completely mixed (i.e., $\sigma^k \in int(\Sigma)$) and
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  - (b) for every $k \geq 1$, σ_i^* is a best reply to σ_{-i}^k for all $i \in I$. ■

Corollary

If σ^* is a perfect equilibrium of the game in normal form G then, for every $i \in I$, σ_i^* is not a dominated strategy.

4.6- Perfect Equilibrium in the normal form

- Let Γ be a finite game in extensive form.

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- For every $i \in I$, let $B_i = \{b_i^1, ..., b_i^{K_i}\}$ and define the set of agents of G as $I^a = \bigcup_{i \in I} \bigcup_{t=1}^{K_i} (i.t)$, and for every $(i.t) \in I^a$, define $S^a_{(i.t)} = C_{b_i^t}$ and $h^a_{(i.t)} = h_i$.

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- Let $G^a = (I^a, (S^a_{(i.t)})_{(i.t) \in I^a}, (h^a_{(i.t)})_{(i.t) \in I^a})$ be the *agent-normal form* of Γ.

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Proposition 2 *Let Γ be a finite game in extensive form and let G^a be its corresponding agent-normal form of Γ. Then, σ is a perfect equilibrium of Γ if and only if σ is a perfect equilibrium (according to Definition*) of G^a.*

4.6- Perfect Equilibrium in the normal form

- **Example** We want to show that $(L_1 L_1', L_2)$ is the unique perfect equilibrium of this agent-normal form, where agents 1 (at x_1) and 3 (at x_3) are agents of player 1.

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L_1

		$1-\varepsilon_3^k$	ε_3^k
	2/3	L_1'	R_1'
$1-\varepsilon_2^k$	L_2	3, 1, 3	0,-5, 0
ε_2^k	R_2	1, 0, 1	1, 0, 1

R_1

		$1-\varepsilon_3^k$	ε_3^k
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$1-\varepsilon_2^k$	L_2	2, 2, 2	2, 2, 2
ε_2^k	R_2	2, 2, 2	2, 2, 2

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R_1

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$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{L_1} \\
 & & 1-\varepsilon_3^k & \varepsilon_3^k \\
 & 2/3 & L_1' & R_1' \\
\hline
1-\varepsilon_2^k & L_2 & 3,1,3 & 0,\text{-}5,0 \\
\hline
\varepsilon_2^k & R_2 & 1,0,1 & 1,0,1 \\
\hline
\end{array}
\qquad
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{R_1} \\
 & & 1-\varepsilon_3^k & \varepsilon_3^k \\
 & 2/3 & L_1' & R_1' \\
\hline
1-\varepsilon_2^k & L_2 & 2,2,2 & 2,2,2 \\
\hline
\varepsilon_2^k & R_2 & 2,2,2 & 2,2,2 \\
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\end{array}
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- Assume σ is a perfect equilibrium of G^a.
- Notice that $L_1^{'}$ dominates R_1'.

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- Notice that $L_1^{'}$ dominates R_1'.
  - Hence, by Corollary above, $\sigma_3(L_1') = 1$.

4.6- Perfect Equilibrium in the normal form

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$1-\varepsilon_2^k$	L_2	2, 2, 2	2, 2, 2
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- Assume σ is a perfect equilibrium of G^a.
- Notice that $L_1^{'}$ dominates R_1'.
  - Hence, by Corollary above, $\sigma_3(L_1') = 1$.
  - This already shows that $(R_1 R_1', R_2)$ cannot be a perfect equilibrium of the agent-normal form.
  - Hence, by Proposition 2, $(R_1 R_1', R_2)$ is not a perfect equilibrium of Γ (we already knew that since it is not subgame perfect).

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$$\begin{aligned} H_2(\sigma_1^k, L_2, \sigma_3^k) &= \sigma_1^k(L_1)[(1-\varepsilon_3^k) - 5\varepsilon_3^k] + 2(1-\sigma_1^k(L_1)) \\ &= \sigma_1^k(L_1) - 6\varepsilon_3^k\sigma_1^k(L_1) + 2 - 2\sigma_1^k(L_1) \\ &= 2 - \sigma_1^k(L_1)(1+6\varepsilon_3^k). \end{aligned}$$

4.6- Perfect Equilibrium in the normal form

- Let $\{\varepsilon_3^k\} \to 0$ be arbitrary.
- Take any completely mixed sequence $\{\sigma^k\} \to \sigma$ with the property that for all $k \geq 1$, $\sigma_3^k(L_1') = 1 - \varepsilon_3^k$.
- $$\begin{aligned} H_2(\sigma_1^k, L_2, \sigma_3^k) &= \sigma_1^k(L_1)[(1-\varepsilon_3^k) - 5\varepsilon_3^k] + 2(1-\sigma_1^k(L_1)) \\ &= \sigma_1^k(L_1) - 6\varepsilon_3^k\sigma_1^k(L_1) + 2 - 2\sigma_1^k(L_1) \\ &= 2 - \sigma_1^k(L_1)(1+6\varepsilon_3^k). \end{aligned}$$
- $$\begin{aligned} H_2(\sigma_1^k, R_2, \sigma_3^k) &= \sigma_1^k(L_1)\cdot 0 + (1-\sigma_1^k(L_1))2 \\ &= 2 - 2\sigma_1^k(L_1) \\ &< 2 - \sigma_1^k(L_1)(1+6\varepsilon_3^k), \end{aligned}$$

  since for large k, $1 + 6\varepsilon_3^k < 2$.

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  since for large k, $1 + 6\varepsilon_3^k < 2$.
- By Proposition 1, σ_2 is a best reply to σ_{-2}^k. Thus, $\sigma_2(L_2) = 1$.

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  for sufficiently large k.
- By Proposition 1, σ_1 is a best reply to σ_{-1}^k. Hence, $\sigma_1(L_1) = 1$.
- Thus, we have proved, using Proposition 1, that if σ is a perfect equilibrium of G then, $\sigma_1(L_1) = 1$, $\sigma_2(L_2) = 1$ and $\sigma_3(L_3) = 1$.

- Question: Is the principle "Nash equilibrium plus never a dominated strategy" a characterization of perfect equilibria?

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- **Example**

1/2	L	R
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$l \quad 1-\varepsilon_3^k$

$1/2$	$1-\varepsilon_2^k$ L	ε_2^k R
T	$1,1,1$	$1,0,1$
B	$1,1,1$	$0,0,1$

$r \quad \varepsilon_3^k$

$1/2$	$1-\varepsilon_2^k$ L	ε_2^k R
T	$1,1,0$	$0,0,0$
B	$0,1,0$	$1,0,0$

.

Counter-example (with $\#I = 3$).

$$
\begin{array}{ccc}
 & & l \quad 1-\varepsilon_3^k \\
 & 1-\varepsilon_2^k & \varepsilon_2^k \\
1/2 & L & R \\
T & 1,1,1 & 1,0,1 \\
B & 1,1,1 & 0,0,1
\end{array}
\qquad
\begin{array}{ccc}
 & & r \quad \varepsilon_3^k \\
 & 1-\varepsilon_2^k & \varepsilon_2^k \\
1/2 & L & R \\
T & 1,1,0 & 0,0,0 \\
B & 0,1,0 & 1,0,0
\end{array}.
$$

- (B, L, l) is a Nash equilibrium and none of the three strategies is dominated.

4.7- Perfect Equilibrium and undominated strategies

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\begin{array}{cc}
l & 1-\varepsilon_3^k \\
\begin{array}{c|c|c|}
 & 1-\varepsilon_2^k & \varepsilon_2^k \\
1/2 & L & R \\
\hline
T & 1,1,1 & 1,0,1 \\
\hline
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\hline
\end{array}
&
\end{array}
\qquad
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\hline
T & 1,1,0 & 0,0,0 \\
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\hline
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$$

- (B, L, l) is a Nash equilibrium and none of the three strategies is dominated.
- However, (B, L, l) is not a perfect equilibrium of G.

4.7- Perfect Equilibrium and undominated strategies

- Assume the contrary, (B, L, l) is a perfect equilibrium of G and let $\{\varepsilon_2^k\} \to 0$ and $\{\varepsilon_3^k\} \to 0$ be arbitrary. For every $k \geq 1$ define $\sigma_2^k(L) = 1 - \varepsilon_2^k$ and $\sigma_3^k(l) = 1 - \varepsilon_3^k$.

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- Assume the contrary, (B, L, l) is a perfect equilibrium of G and let $\{\varepsilon_2^k\} \to 0$ and $\{\varepsilon_3^k\} \to 0$ be arbitrary. For every $k \geq 1$ define $\sigma_2^k(L) = 1 - \varepsilon_2^k$ and $\sigma_3^k(l) = 1 - \varepsilon_3^k$.
- Then,

$$H_1(T, \sigma_2^k, \sigma_3^k) = (1 - \varepsilon_3^k) + \varepsilon_3^k(1 - \varepsilon_2^k) \tag{6}$$

and

$$H_1(B, \sigma_2^k, \sigma_3^k) = (1 - \varepsilon_3^k)(1 - \varepsilon_2^k) + \varepsilon_2^k\varepsilon_3^k. \tag{7}$$

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- For sufficiently large $k \geq 1$,

$$(1 - \varepsilon_3^k) > (1 - \varepsilon_3^k)(1 - \varepsilon_2^k) \tag{8}$$

and

$$\varepsilon_3^k(1 - \varepsilon_2^k) > \varepsilon_2^k \varepsilon_3^k. \tag{9}$$

4.7- Perfect Equilibrium and undominated strategies

- Hence, (8) and (9) imply that $H_1(T, \sigma_2^k, \sigma_3^k) > H_1(B, \sigma_2^k, \sigma_3^k)$.

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- This means that B is not a best reply to (σ_2^k, σ_3^k) for sufficiently large k.
- Thus, by Proposition 1, (B, L, l) is not a perfect equilibrium of G.

4.7- Perfect Equilibrium and undominated strategies

Fact: *Let G be a finite game in normal form with $\#I = 2$. Then, σ is a perfect equilibrium of G if and only if* (a) *σ is a Nash equilibrium of G and* (b) *for all $i = 1, 2$, σ_i is an undominated strategy.*

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- Remark: If $j \in I \setminus \{1, 2\}$, nothing guarantees that $\hat{\sigma}_j = \bar{\sigma}_j$. Hence, we could not proceed with the proof.

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- We will return to perfect equilibrium to study its relationship with sequential equilibrium in the context of incomplete information.

4.8- Proper Equilibrium

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R_1	0, 0	0, 0	0,-2
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- Notice that G' is obtained from G after adding an strictly dominated strategy for every player (A_i).

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4.8- Proper Equilibrium

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- Thus, adding strictly dominated strategies may change the set of perfect equilibria.

4.8- Proper Equilibrium

-

		$\frac{1}{k^2}$	$1-\frac{1}{k}-\frac{1}{k^2}$	$\frac{1}{k}$
	1/2	L_2	R_2	A_1
$\frac{1}{k^2}$	L_1	1, 1	0, 0	-1,-2
$1-\frac{1}{k}-\frac{1}{k^2}$	R_1	0, 0	0, 0	0,-2
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4.8- Proper Equilibrium

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 & & \frac{1}{k^2} & 1-\frac{1}{k}-\frac{1}{k^2} & \frac{1}{k} \\
 & 1/2 & L_2 & R_2 & A_1 \\
\hline
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4.8- Proper Equilibrium

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4.8- Proper Equilibrium

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4.8- Proper Equilibrium

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- Namely, for all $k \geq 1$, R_i is a best reply against σ_{-i}^k.

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- Namely, for all $k \geq 1$, R_i is a best reply against σ_{-i}^k.
- Thus, by Proposition 1, (R_1, R_2) is a perfect equilibrium of G'.

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- In the example, players should not expect the mistake A_i with higher probability than the mistake L_i.
- Then, (R_1, R_2) cannot be obtained as the limit of this more rational mistakes.
- **Definition** Let G be a finite game in normal form and let $\varepsilon > 0$ be given. An ε-*proper equilibrium* of G is a totally mixed strategy $\sigma \in int(\Sigma)$ such that for all $i \in I$,

$$\text{if } H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i}) \text{ then } \sigma_i(s_i) \leq \varepsilon \sigma_i(s_i').$$

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We will first prove the following Lemma that will be useful to prove the Theorem.

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Theorem

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We will first prove the following Lemma that will be useful to prove the Theorem.

Lemma *Let G be a finite game in normal form and let $\varepsilon > 0$ be sufficiently small. Then, G has at least one ε-proper equilibrium.*

Proof of the Lemma (sketch) Let G be a finite game in normal form and let $\varepsilon > 0$ be sufficiently small.

- For each $i \in I$, construct

$$\Sigma_i^{\varepsilon} = \left\{ \sigma_i \in int(\Sigma_i) \mid \sigma_i(s_i) \geq \frac{\varepsilon^m}{m} \text{ for each } s_i \in S_i \right\},$$

where $m = \max_{i \in I} \# S_i$. Observe that if $\varepsilon > 0$ is sufficiently small, $\Sigma_i^{\varepsilon} \neq \varnothing$.

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- For each $i \in I$, consider now the constrained best-reply correspondence $r_i^{\varepsilon} : \Sigma^{\varepsilon} \twoheadrightarrow \Sigma_i^{\varepsilon}$ defined as follows: for every $\sigma \in \Sigma^{\varepsilon}$,

$$r_i^{\varepsilon}(\sigma) = \left\{ \sigma_i' \in \Sigma_i^{\varepsilon} \mid \text{if } H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i}) \text{ then } \sigma_i'(s_i) \leq \varepsilon \sigma_i'(s_i') \right\}.$$

4.8- Proper Equilibrium

- It is possible to show that, since $\sigma'_i(s_i) \leq \varepsilon \sigma'_i(s'_i)$ are linear weak inequalities, r_i^{ε} is convex and compact valued, and upper hemi-continuous.

4.8- Proper Equilibrium

- It is possible to show that, since $\sigma_i'(s_i) \leq \varepsilon\sigma_i'(s_i')$ are linear weak inequalities, r_i^{ε} is convex and compact valued, and upper hemi-continuous.
- Fix $\sigma \in \Sigma^{\varepsilon}$. To prove that $r_i^{\varepsilon}(\sigma) \neq \varnothing$, consider any $s_i \in S_i$ and let
$$\rho(s_i) = \#\{s_i' \in S_i \mid H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i})\}.$$

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- Then,
    - if $\rho(s_i) = 0$ for all $s_i \in S_i$, then $r_i^\varepsilon(\sigma) = \Sigma_i^\varepsilon \neq \varnothing$,
    - if there exists $\hat{s}_i \in S_i$ such that $\rho(\hat{s}_i) > 0$ then consider the strategy $\hat{\sigma}_i \in \Sigma_i$ where for every $s_i \in S_i$,

$$\hat{\sigma}_i(s_i) = \frac{\varepsilon^{\rho(s_i)}}{\sum\limits_{\bar{s}_i \in S_i} \varepsilon^{\rho(\bar{s}_i)}}.$$

- Since $\varepsilon^{\rho(s_i)} \geq \varepsilon^m$ because $\rho(s_i) \leq m$, $\varepsilon \leq 1$ (it is sufficiently small), and

$$\sum_{\bar{s}_i \in S_i} \varepsilon^{\rho(\bar{s}_i)} \leq \sum_{\bar{s}_i \in S_i : \rho(\bar{s}_i) > 0} \varepsilon + \#\{\bar{s}_i \in S_i \mid \rho(\bar{s}_i) = 0\} \leq m$$

holds, we have that

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- To show that $\hat{\sigma}_i \in r_i^{\varepsilon}(\sigma)$, assume that $s_i, s_i' \in S_i$ are such that $H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i})$.

4.8- Proper Equilibrium

- Then, $\rho(s_i) > 0$ and $\rho(s_i) \geq \rho(s_i') + 1$. Hence,

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- Now, applying the Kakutani Fixed Point Theorem to the correspondence $r^{\varepsilon} : \Sigma^{\varepsilon} \twoheadrightarrow \Sigma^{\varepsilon}$, we obtain that there exists $\sigma \in \Sigma^{\varepsilon}$ such that $\sigma \in r^{\varepsilon}(\sigma)$.

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- Now, applying the Kakutani Fixed Point Theorem to the correspondence $r^\varepsilon : \Sigma^\varepsilon \twoheadrightarrow \Sigma^\varepsilon$, we obtain that there exists $\sigma \in \Sigma^\varepsilon$ such that $\sigma \in r^\varepsilon(\sigma)$.
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- By definition, σ is a proper equilibrium of G. $\blacksquare$

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Proof of the Theorem Let G be a finite game in normal form. We want to show that G has a proper equilibrium.

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Remark Let σ be a proper equilibrium of G. Then, σ is a perfect equilibrium of G.

- Kohlberg, E. and J.F. Mertens. "On the Strategic Stability of Equilibria," *Econometrica* 54 (1986).

4.10- Rationalizable Strategic Behavior

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- Strategic uncertainty: Bayesian approach to the problem of strategic selection.
- The idea is to find restrictions on the behavior of players just coming from the hypothesis of rationality (and the common knowledge of it).

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- Answer: A rational player will play *only* strategies that are best reply to some beliefs on the strategies of the other players. The contrapositive: A rational player will never play a strategy that is never a best reply for some belief on the strategies of the other players.

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- Moreover, common knowledge of rationality implies that not all beliefs about other players' behavior are possible.
- This means that we have to face an infinite reasoning process. Let's model it.

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- **Definition** Define for every $i \in I$, $\Sigma_i^0 = \Sigma_i$ and recursively, for all $t \geq 0$,

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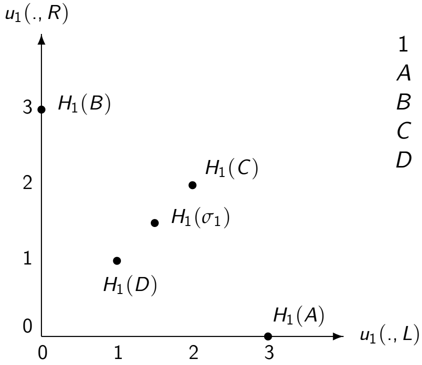
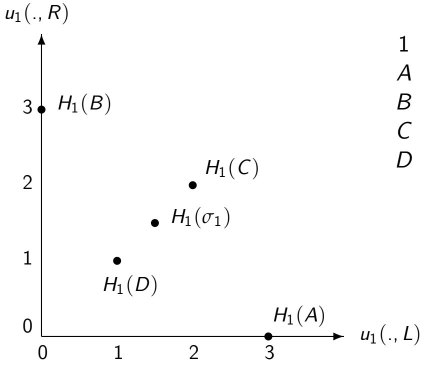
- **Example** (Why $co(\Sigma_j^t)$ instead of Σ_j^t?)
  - It is possible that $\sigma_j', \sigma_j'' \in \Sigma_j^t$ but the mixture $\frac{1}{2}\sigma_j' + \frac{1}{2}\sigma_j'' \notin \Sigma_j^t$ (the belief that player j will play σ_j' with probability $\frac{1}{2}$ and σ_j'' with probability $\frac{1}{2}$).



1	L	R
A	3	0
B	0	3
C	2	2
D	1	1



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- The strategy $\sigma_1(A) = \sigma_1(B) = \frac{1}{2}$ is dominated by C. Hence, $\sigma_1 \notin \Sigma_1^1$ but since $\sigma_1', \sigma_1'' \in \Sigma_1^1$, where $\sigma_1'(B) = \sigma_1''(A) = 1$, we want in Σ_1^1 the belief "with any probability, 1 will play A and the complementary probability, 1 will play B"; thus, we have $co(\Sigma_j^t)$ in the definition.

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  - In general, for $\#I > 2$, this statement does not hold.

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- The concept of Nash equilibrium assumes that players' mixed strategies are independent (uncorrelated).
- Question: What happens if players can correlate their strategies?
- Interpretation: Players, before playing the game, can communicate among them and reach agreements on playing mixed strategies coming from the same experiment (and hence, correlate their strategies). A correlated equilibrium is a profile of correlated mixed strategies that no player has incentives to change unilaterally.

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	$1/2$	L	R
p_1	U	$5,1$	$0,0$
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- $(\frac{1}{2}, \frac{1}{2})$ is the mixed strategy equilibrium.
  - Payoffs: $H_1(\frac{1}{2}, \frac{1}{2}) = H_2(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}5 + \frac{1}{4}4 + \frac{1}{4}1 = 2.5$.

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- Equilibrium payoffs: $(1,5), (5,1), (2.5, 2.5)$.

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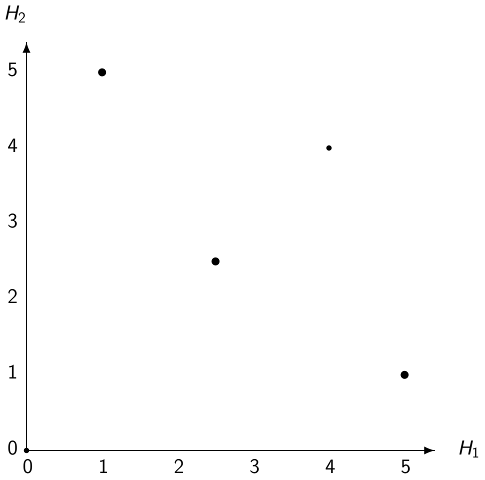

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- All convex combinations of Nash equilibrium are possible with this type of correlation (this was already known before Aumann's paper).

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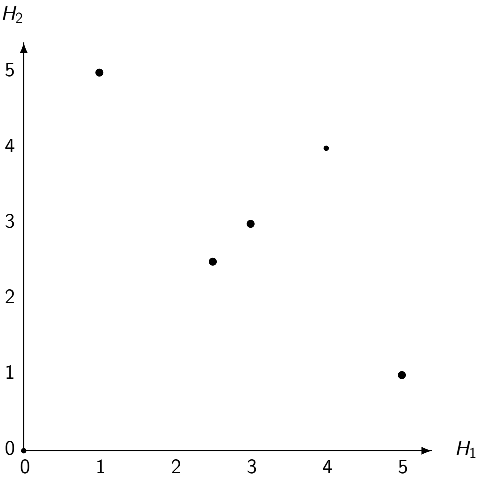

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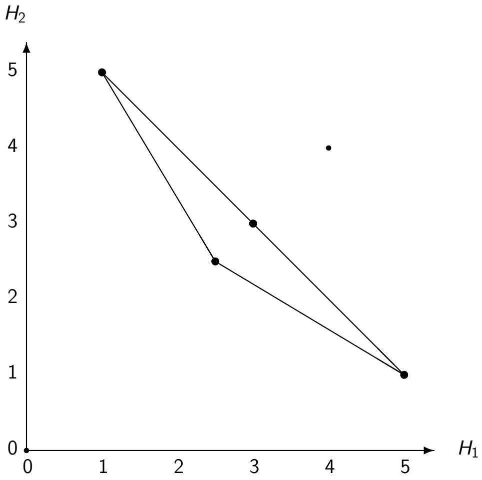

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4.11- Correlated Equilibrium

- However, they can even do better.
- Design the following experiment and information structure (signals that players will receive about the outcome of the experiment):
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  - $\mathfrak{s}_2(\{w_3\}) = R$, $\mathfrak{s}_2(\{w_1, w_2\}) = L$.

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- $H_i(\mathfrak{s}_1, \mathfrak{s}_2) = \sum_{s \in S} q_{\mathfrak{s}}(s) h_i(s) = \frac{1}{3}5 + \frac{1}{3}4 + \frac{1}{3}1 = \frac{10}{3}$.

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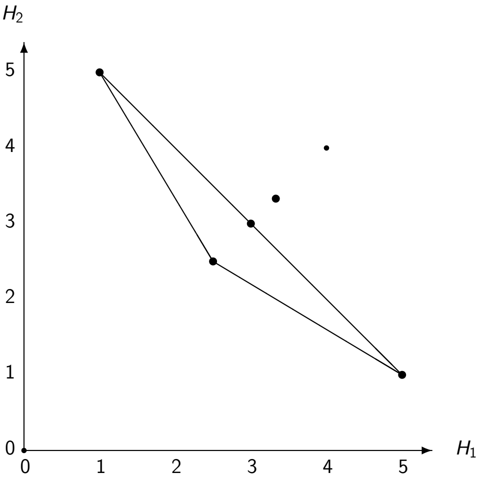
- $(\frac{10}{3}, \frac{10}{3})$ is outside the convex hull of the set of Nash equilibrium payoffs.

4.11- Correlated Equilibrium



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H_1(U,\mathfrak{s}_2) &= p(w_2 | \, w_2 w_3) h_1(U,\mathfrak{s}_2(w_2)) + p(w_3 | \, w_2 w_3) h_1(U,\mathfrak{s}_2(w_3)) \\
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&= \tfrac{1}{2} 5 + \tfrac{1}{2} 0 = 2.5.
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- Player 1 (continuation):

$$H_1(D,\mathfrak{s}_2) = p(w_2 | w_2w_3)h_1(D,\mathfrak{s}_2(w_2)) + p(w_3 | w_2w_3)h_1(D,\mathfrak{s}_2(w_3))$$

$$= \tfrac{1}{2}h_1(D,L) + \tfrac{1}{2}h_1(D,R)$$

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and he is playing D.

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and he is playing L.

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- In fact, we have the following probability distribution on S :

1/2	L	R
U	$\frac{1}{3}$ w_1	0
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- Observe that this probability distribution cannot be obtained with uncorrelated strategies.
- There are two alternative definitions of correlated equilibrium:
  - One makes explicit the information structure held by players about the (join) experiment.
  - The other formulates directly the equilibrium on the set of strategy profiles S, without explicitly modelling the experiment.

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- Nature selects $w \in \Omega$ according to p and each $i \in I$ is informed that the true state w is in $b_i(w) \equiv b_i^k$, where $w \in b_i^k$.
- Given $w \in \Omega$ and assuming that $p(b_i(w)) \equiv \sum_{w' \in b_i(w)} p(w') > 0$

  define the conditional probability on Ω, given $b_i(w)$, as follows: for each $\hat{w} \in \Omega$,

$$p(\hat{w} \mid b_i(w)) = \begin{cases} \frac{p(\hat{w})}{p(b_i(w))} & \text{if } \hat{w} \in b_i(w) \\ 0 & \text{otherwise.} \end{cases}$$

- Player i's strategy $\mathfrak{s}_i : B_i \longrightarrow S_i$.

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- Let $\mathfrak{S}_i$ be the set of all B_i-measurable functions $\mathfrak{s}_i : \Omega \longrightarrow S_i$.
- Let $\mathfrak{S} = \prod_{i \in I} \mathfrak{S}_i$ be the set of strategy profiles.

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Definition 1 A *correlated equilibrium* $\mathfrak{s} = (\mathfrak{s}_1, ..., \mathfrak{s}_n) \in \mathfrak{S}$ of G relative to an information structure $(\Omega, (B_i)_{i \in I}, p)$ is a Nash equilibrium; namely, $\mathfrak{s}$ is a correlated equilibrium if

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(Ex-ante) for all $i \in I$,

$$\sum_{w \in \Omega} p(w) h_i(\mathfrak{s}(w)) \geq \sum_{w \in \Omega} p(w) h_i(\mathfrak{s}'_i(w), \mathfrak{s}_{-i}(w))$$

for all $\mathfrak{s}'_i \in \mathfrak{S}_i$,

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- Namely, p and $\mathfrak{s}$ induce a probability distribution on S.
- Now, players will agree directly on a probability distribution on S.

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Definition 2 A *correlated equilibrium* of G is a probability distribution p on S such that for all $i \in I$ and all $d_i : S_i \longrightarrow S_i$,

$$\sum_{s \in S} p(s) h_i(s_i, s_{-i}) \geq \sum_{s \in S} p(s) h_i(d_i(s_i), s_{-i});$$

that is, every player wants to follow the recommendation s that is selected according to p.

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Properties Let G be a finite game in normal form.

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- (3) Therefore, every finite game in normal form G has at least a correlated equilibrium.
- (4) The set of correlated equilibria is convex [Homework].
- (5) By properties (2) and (4) above we have that for every finite game in normal form G,

$$co(\Sigma^*) \subseteq \text{Set of correlated equilibria of } G.$$

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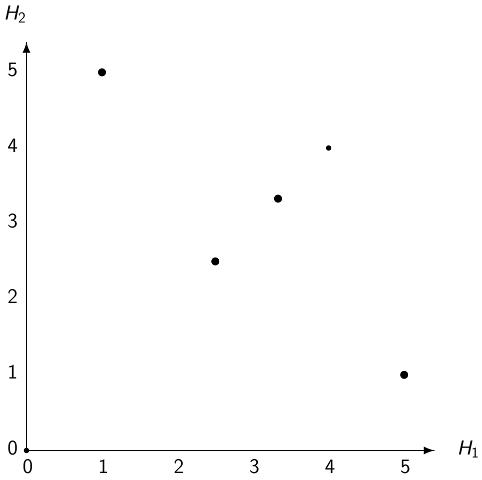

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