Game Theory

Nash Equilibrium

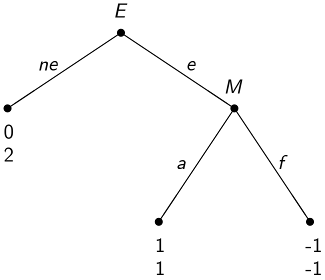
Jordi Massó

International Doctorate in Economic Analysis (IDEA)
Universitat Autònoma de Barcelona (UAB)

4.1.- Introduction

- We will first consider Nash equilibrium as a *necessary* condition of rationality.
  - We will ask: are there other stronger tests or properties that an strategy has to pass to be considered rational?
  - This approach (of imposing additional requirements to the Nash conditions) is known as the program of *perfecting* or *refining* the set of Nash equilibria.
  - There are two ways of doing so (they are related, but it is useful to look at them separately):
    - refinements in the extensive form and
    - refinements in the normal form.
  - For example, in extensive form games with perfect information, we may select those Nash equilibria that are obtained by backwards induction.

4.1.- Introduction



E/M	a	f
ne	$0, 2$	$0, 2$
e	$1, 1$	$-1, -1$

4.1.- Introduction

- In Sections 4.10 and 4.11 we will change the point of view and we will consider Nash equilibrium as a *sufficient* condition for rational behavior instead of a necessary one.
    - We will enlarge the set of Nash equilibria.

4.2.- Dominant strategies

- Let $G = (I, (S_i)_{i \in I}, (h_i)_{i \in I})$ be a game in normal form.
- **Definitions**
  - s_i *strictly dominates* s'_i if
    $$h_i(s_i, s_{-i}) > h_i(s'_i, s_{-i})$$
    for all $s_{-i} \in S_{-i}$.
  - s_i *(weakly) dominates* s'_i if
    $$h_i(s_i, s_{-i}) \geq h_i(s'_i, s_{-i})$$
    for all $s_{-i} \in S_{-i}$, and there exists $s'_{-i} \in S_{-i}$ such that $h_i(s_i, s'_{-i}) > h_i(s'_i, s'_{-i})$.

4.2.- Dominant strategies: examples

(1)

1/2	C	D
C	3, 3	0, 4
D	4, 0	1, 1

D strictly dominates C for both players.

(2)

1/2	L	R
T	10, 0	5, 2
B	10, 1	2, 0

T dominates B for player 1.

4.2.- Dominant strategies: mixed strategies

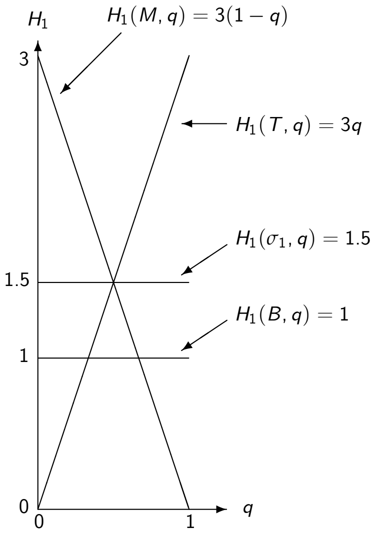
- Importance of using mixed strategies.
- **Example:**

	q	$1-q$
$1/2$	L	R
T	$3, 0$	$0, 1$
M	$0, 0$	$3, 1$
B	$1, 1$	$1, 0$

- No (pure) strategy dominates any other (pure) strategy for both players.
- However, the mixed strategy $\sigma_1(T) = \sigma_1(M) = \frac{1}{2}$ and $\sigma_1(B) = 0$ strictly dominates B since for all $q \in [0, 1]$,

$$\begin{aligned} H_1(\sigma_1, q) &= 3q\tfrac{1}{2} + 3(1-q)\tfrac{1}{2} = \tfrac{3}{2} \\ &> 1 = H_1(B, q). \end{aligned}$$

4.2.- Dominant strategies: mixed strategies



4.2.- Dominant strategies: mixed strategies

- **Definition** σ_i *strictly dominates* σ'_i if

$$H_i(\sigma_i, \sigma_{-i}) > H_i(\sigma'_i, \sigma_{-i})$$

  for all $\sigma_{-i} \in \Sigma_{-i}$.

- **Definition'** σ_i *strictly dominates* σ'_i if

$$H_i(\sigma_i, s_{-i}) > H_i(\sigma'_i, s_{-i})$$

  for all $s_{-i} \in S_{-i}$.

- **Fact** Definition and Definition' are equivalent.
  - The use of Definition' simplifies the test.

4.2.- Dominant strategies: proof of Fact

$$
\begin{aligned}
H_i(\sigma_i, \sigma_{-i}) &= \sum_{s \in S} \prod_{j \in I} \sigma_j(s_j) h_i(s) \\
&= \sum_{s_{-i} \in S_{-i}} \prod_{j \in I \setminus \{i\}} \sigma_j(s_j) \underbrace{\left[\sum_{s_i \in S_i} \sigma_i(s_i) h_i(s_i, s_{-i}) \right]}_{= H_i(\sigma_i, s_{-i})}.
\end{aligned}
$$

- Consider $\sigma_{-i} \in \Sigma_{-i}$ and $\sigma_i, \sigma_i' \in \Sigma_i$. Then

$$
H_i(\sigma_i, \sigma_{-i}) - H_i(\sigma_i', \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \prod_{j \in I \setminus \{i\}} \sigma_j(s_j) \left[H_i(\sigma_i, s_{-i}) - H_i(\sigma_i', s_{-i}) \right] \quad (1)
$$

4.2.- Dominant strategies: proof of fact

- $\Longrightarrow$) Assume that for all $\sigma_{-i} \in \Sigma_{-i}$,

$$H_i(\sigma_i, \sigma_{-i}) - H_i(\sigma'_i, \sigma_{-i}) > 0.$$

Then, and since pure strategy profiles s_{-i} belong Σ_{-i},

$$H_i(\sigma_i, s_{-i}) - H_i(\sigma'_i, s_{-i}) > 0$$

for all $s_{-i} \in S_{-i}$.

- $\Longleftarrow$) Assume that for all $s_{-i} \in S_{-i}$,

$$H_i(\sigma_i, s_{-i}) - H_i(\sigma'_i, s_{-i}) > 0.$$

Then, by the previous expression in (1),

$$H_i(\sigma_i, \sigma_{-i}) - H_i(\sigma'_i, \sigma_{-i}) > 0$$

for all $\sigma_{-i} \in \Sigma_{-i}$. $\blacksquare$

4.2.- Dominant strategies

- **Definitions**
  - s_i is an *strictly dominant* strategy if for all $s'_i \in S_i \backslash \{s_i\}$,

    $$h_i(s_i, s_{-i}) > h_i(s'_i, s_{-i})$$

    for all $s_{-i} \in S_{-i}$. Namely, s_i strictly dominates all other strategies.
  - s_i is a *(weakly) dominant* strategy if for all $s'_i \in S_i$,

    $$h_i(s_i, s_{-i}) \geq h_i(s'_i, s_{-i})$$

    for all $s_{-i} \in S_{-i}$ and [condition (*)] there exists $s'_{-i} \in S_{-i}$ such that

    $$h_i(s_i, s'_{-i}) > h_i(s'_i, s'_{-i}).$$

    Namely, [with condition (*)] s_i (weakly) dominates all other strategies.
- In Example (1), D is an strictly dominant strategy.
- In Example (2), T is a dominant strategy.

4.2.- Dominant strategies

- **Definition (normal form)** Let G be a game in normal form. We say that $s^* \in S^*$ is a (strictly) *dominant strategy equilibrium* of G if for all $i \in I$, s_i^* is a (strictly) dominant strategy.
- In Example (1), (D, D) is an strictly dominant strategy equilibrium.
- In Example (2), there is no equilibrium in dominant strategies: (T, R) is the unique Nash equilibrium, but R is not a dominant strategy for player 2.
- This concept is the strongest and less controversial one.
    - It transforms the game (a multi-agent problem) into several one-agent problems.
    - It does not require that the game be common knowledge; in particular, to compute a dominant strategy a player does not need to know the other players' payoffs.
    - However, often the set of dominant strategy equilibria is empty.

4.2.- Dominant strategies

- **Mechanism design (or implementation theory):** To select mechanisms to obtain a social goal. Namely, the game is not given, but rather it has to be designed with the objective that the set of equilibria has some properties; for instance, the set of dominant strategies is non-empty and "implements" the social goal.
- **Examples:**
    - Auctions.
    - Voting.
    - Decision on a public good.
    - Etc.

4.3- Elimination of dominated strategies

- **Principle:** Never play a dominated strategy.
- **Example:**

(2)

1/2	L	R
T	$10, 0$	$5, 2$
B	$10, 1$	$2, 0$

  - Two Nash equilibria: $S^* = \{(B, L), ((T, R)\}$.
  - T dominates B for player 1. Player 1 should not play B.
  - We have refined the set S^*. We have a unique prediction: (T, R).

- However,

(2')

1/2	L	R
T	$10, 0$	$5, 2$
B	$10, 11$	$2, 0$

- Is it also (B, L) a non sensible prediction?

4.3- Elimination of dominated strategies: Iterated

$1/2$	L	M	R
t	$4,3^*$	$5,1$	$6,2$
m	$2,1$	$8,4$	$3,6$
b	$3,0$	$9,6^*$	$2,6$

1/2	L	M	R
t	4, 3	5, 1	6, 2
m	2, 1	8, 4	3, 6
b	3, 0	9, 6	2, 6

R dominates $M \longrightarrow$

1/2	L	R
t	4, 3	6, 2
m	2, 1	3, 6
b	3, 0	2, 6

4.3- Elimination of dominated strategies: Iterated

1/2	L	M	R
t	4, 3	5, 1	6, 2
m	2, 1	8, 4	3, 6
b	3, 0	9, 6	2, 6

R dominates M $\longrightarrow$

1/2	L	R
t	4, 3	6, 2
m	2, 1	3, 6
b	3, 0	2, 6

t dominates m (or t dominates b) $\longrightarrow$

1/2	L	R
t	4, 3	6, 2

1/2	L	M	R
t	4, 3	5, 1	6, 2
m	2, 1	8, 4	3, 6
b	3, 0	9, 6	2, 6

R dominates M $\longrightarrow$

1/2	L	R
t	4, 3	6, 2
m	2, 1	3, 6
b	3, 0	2, 6

t dominates m (or t dominates b) $\longrightarrow$

1/2	L	R
t	4, 3	6, 2

L dominates R $\longrightarrow$

1/2	L
t	4, 3

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- Only one Nash equilibrium has survived.
- Is it important the order of elimination? (Homework).

4.3- Dominance solvability and Sophisticated equilibrium

- Let $G = (I, (S_i)_{i \in I}, (h_i)_{i \in I})$ be a game in normal form.
  - It could also be defined for the mixed extension of G.
- Let $A = \prod_{i \in I} A_i \subseteq S$ be a Cartesian product subset of S. For every $i \in I$ define

$$UD_i(A) = \left\{ s_i \in A_i \mid \nexists s_i' \in A_i \text{ s.t. } s_i' \text{ dominates } s_i \right\}.$$

- Given G, the successive elimination of dominated strategies is made up of the sequences: for every $i \in I$,

$$S_i = S_i^0 \supseteq S_i^1 \supseteq ... \supseteq S_i^t \supseteq S_i^{t+1} \supseteq ...,$$

where for all $t \geq 0$,

$$S_i^{t+1} = UD_i(S^t).$$

4.3- Dominance solvability and Sophisticated equilibrium

- Denote $S_i^\infty = \bigcap_{t=0}^{\infty} S_i^t$.
- **Definition** We say that G is *dominant solvable* if
  1. $S^\infty \neq \varnothing$ and
  2. for all $i \in I$, $h_i(s) = h_i(s')$ for all $s, s' \in S^\infty$.
- The non-empty set S^∞ is called the set of *sophisticated equilibria.*
- Moulin, H. "Dominance Solvable Voting Schemes," *Econometrica* 47, 1979.

4.3- Dominance solvability and Sophisticated equilibrium: Example

1/2	L	M	R
t	4, 3	5, 1	6, 2
m	2, 1	8, 4	3, 6
b	3, 0	9, 6	2, 6

$S_1^0 = \{t, m, b\}$	$S_2^0 = \{L, M, R\}$
$S_1^1 = \{t, m, b\}$	$S_2^1 = \{L, R\}$
$S_1^2 = \{t\}$	$S_2^2 = \{L, R\}$
$S_1^3 = \{t\}$	$S_2^3 = \{L\}$
...	...
$S_1^\infty = \{t\}$	$S_2^\infty = \{L\}$

- $S^\infty = \{(t, L)\}$ is the set of sophisticated equilibrium.
- Observe that neither t nor L are dominant strategies (they do not dominate any strategy).

Example (Guess the average).

- Each player $i \in I$ picks simultaneously an integer x_i between 1 and 999. Hence, $S_i = \{1, ..., 999\}$.
- Given $x = (x_1, ..., x_n) \in \{1, ..., 999\}^n$, let
$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$
- The winners are those players whose ballots are closest to $\frac{2}{3}\bar{x}$.
- Every strategy $x_i > 666$ is dominated by 666.
- Hence, for every $i \in I$, $S_i^1 = \{1, ..., 666\}$.
- Now, for every $i \in I$, $S_i^2 = \{1, ..., 444\}$.
- Proceeding this way, for every $i \in I$, $S_i^\infty = \{1\}$.

- **Remark 1** We have a severe problem of existence.

M/W	F	B	$Home$
F	$3,1$	$0,0$	$-1,-1$
B	$0,0$	$1,3$	$-1,-1$
$Home$	$-1,-1$	$-1,-1$	$-1,-1$

  Home is dominated, and the (original) Battle of Sexes is not dominant solvable.

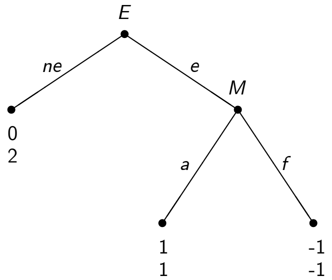
- **Remark 2** We could eliminate only strictly dominated strategies but then, the existence problem would be even worse.

- We will come back to this notion relating it to Subgame Perfect Equilibrium and razionalizable strategies.

- Selten, R. "Spieltheoretische Benhandlung eines Oligopolmodels mit Nachfragetrgheit," *Zeitschrift für die gesamte Saatswissenschaft* 12, 1965.

- Extensive Form refinement.

- **Idea:**



- Two Nash equilibria: $(ne, f), (e, a)$.

4.4- Subgame Perfect Equilibrium

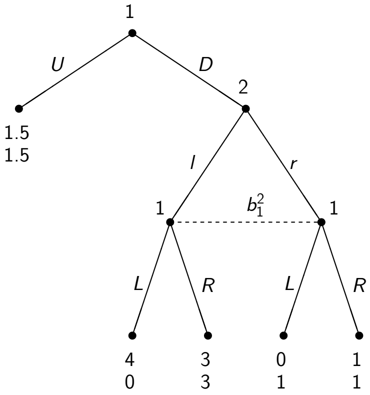
- Consider the Nash equilibrium (ne, f).
- ... and the telephone rings: Monopolist, it is your turn!!!
- M's information set is out of equilibrium path.
  - To play f was part of the optimal behavior because this information set was not reached.
  - The threat of playing f is what makes optimal for the entrant to play ne.
  - But, f is a non-credible threat, so E should not believe that M will play f (it is not rational for him) if he plays e.
  - Subgame Perfect Equilibrium requires rational behavior even in information sets that are not reached in equilibrium (equilibrium should not be based on incredible threats).
- Obtain (e, a) as the unique Subgame Perfect Equilibrium (which coincides with the one obtained by backwards induction).

4.4- Subgame Perfect Equilibrium

E/M	a	f
ne	$0, 2$	$0, 2$
e	$1, 1$	$-1, -1$

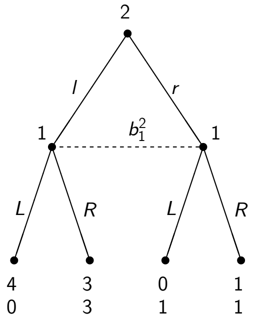
- (e, a) could be obtained also in the normal form as applying the principal "never a dominated strategy" since f is dominated by a.
- But this is not always true.

Example



4.4- Subgame Perfect Equilibrium

- $((U, R), r)$ is a Nash equilibrium.
- Backwards induction can not be used to eliminate it, but if player 2 believes that 1 will play R, then 2 should play l instead of r.
- What it is important here is that the subgame below looks like a game, and (r, R) is not a Nash equilibrium of the subgame.



4.4- Subgame Perfect Equilibrium

- Let $\Gamma = ((I, N), K, P, B, C, p, u)$ be a game in extensive form.
- Given $x \in X$ let $F(x)$ be the set of nodes that follow x; *i.e.*,

$$F(x) = \{y \in X \cup Z \mid x \in [x_1, y]\}.$$

- We say that $F(x)$ is a subtree of K, and it is denoted by K_x, if:
    - $\{x\} \in B_i$ for some $i \in I \cup \{N\}$; *i.e.*, the information set containing x is a singleton.
    - For every $x' \in F(x)$

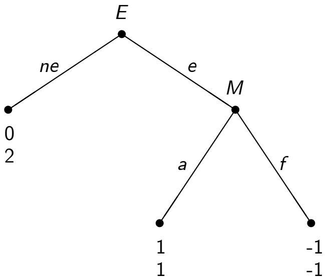
      $$x' \in b_j \Longrightarrow b_j \subseteq F(x);$$

      i.e., every node that follows x belongs to an information set that contains only nodes in $F(x)$.
    - The two requirements above make sure that all information sets are either contained in K_x or are disjoint with K_x.
- Given $x \in X$, the subgame Γ_x is the restriction of Γ in the subtree K_x.

4.4- Subgame Perfect Equilibrium

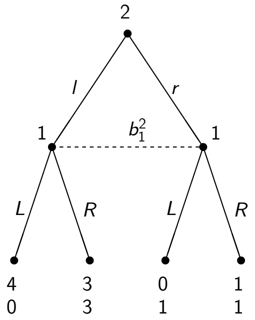
- Let $\sigma \in \hat{\Sigma}$ be a behavioral strategy in Γ, let $x \in X$ and consider Γ_x. Then σ can be decomposed as (σ^x, σ^{-x}) where σ^x describes behavior in Γ_x and σ^{-x} in $\Gamma_{-x} = \Gamma \backslash \Gamma_x$.
- **Definition** Let Γ be a game in extensive form. The behavioral strategy $\hat{\sigma} \in \Sigma$ is a *Subgame Perfect Equilibrium* (SPE) of Γ if for every possible subgame Γ_x, the restriction of σ in Γ_x, σ^x, is a Nash equilibrium of Γ_x.
- Note that Γ is also a subgame of itself ($\Gamma = \Gamma_{x_1}$ since $x_1 \in X$). Thus, for all Γ, SPE$\subseteq$NE.

4.4- Subgame Perfect Equilibrium



- Two Nash equilibria: $(ne, f), (e, a)$.
- (e, a) is the unique SPE of Γ and (ne, f) is not a SPE since f is not a Nash equilibrium of the subgame starting at the unique node that belongs to M.

- $((U, R), r)$ is not a SPE of the second example since (R, r) is not a Nash equilibrium of one subgame.



- **Fact:** Let Γ be a finite game in extensive form with perfect information. Then,

$$\text{SPE (in pure strategies)} = \bar{S}^* \neq \varnothing.$$

Theorem

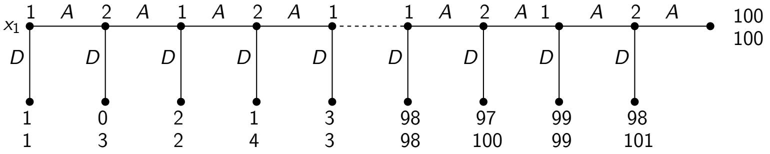
Selten (1965) Every finite game in extensive form Γ has at least one Subgame Perfect Equilibrium (not necessarily in pure strategies).

- **Proof (idea):** Apply the backwards induction argument in all possible subgames and apply the Nash-Kuhn Theorem to obtain a Nash equilibrium. The behavioral strategy obtained is a SPE of Γ.

4.4- Subgame Perfect Equilibrium

- **Remark 1** There are games Γ such that SPE$\subsetneq$NE.
- **Remark 2** There are games with SPE$=$NE.
- **Remark 3** If σ is the unique NE of Γ then σ is also a SPE of Γ.
- It is the most unchallenged refinement (and the most commonly used in Economics). However, ...

- Rosenthal's Centipede Game (*JET*, 1981).



- Unique SPE: "Always *D*."
- Bad prediction.
- Problem: full rationality (it is common knowledge).

- Subgame perfection requires 200 steps of backwards induction and therefore, 200 iterations of "Player 1 knows that Player 2 knows that Player 1 ... knows that Player 2 is rational."
- Rationality $\Longrightarrow$ SPE. But suppose that I (Player 2) am certain that 1 is rational and he knows that I am rational.
  - And suppose that Player 1 chooses A.
  - Then, my initial hypothesis (Player 1 is rational) is wrong but to justify D I have to believe ex-post that my (wrong) hypothesis still holds.
  - After all, Player 1 is not rational (he played D): try A.
  - Now, I can put myself in the position of Player 1 and realize that Player 2 (me) can do the above argument, and therefore try A and wait what happens.

- In general, SPE requires two different things:

  1. SPE gives a solution everywhere (in all subgames), even in subgames where the solution says that they will not be reached (information sets with zero probability).
  2. SPE imposes rational behavior everywhere, even in the subgames of the game that SPE says that cannot be reached. In out-of-equilibrium subgames, the "solution" is disapproved, yet players evaluate their actions taking as given the behavior of the other players, that have been demonstrated incorrect since we are in an out-of-equilibrium path.

4.5- Perfect Equilibrium

or Trembling-hand Perfect Equilibrium in the Extensive Form

- We are going to see this refinement in the extensive form.
- In 1965 Selten defined what we have named Subgame Perfect Equilibrium as Perfect Equilibrium.
- In 1975 he proposed another concept, and called it Perfect Equilibrium, and suggested to name the former concept Subgame Perfect Equilibrium.
- This new concept is also known as Trembling-hand Perfect Equilibrium.
- But there is still an additional problem (another source of terminological confusion): it is possible to define "Perfect Equilibrium" in the normal form which seems the natural extension, but it is not the same (it is if we consider the agent-normal form).

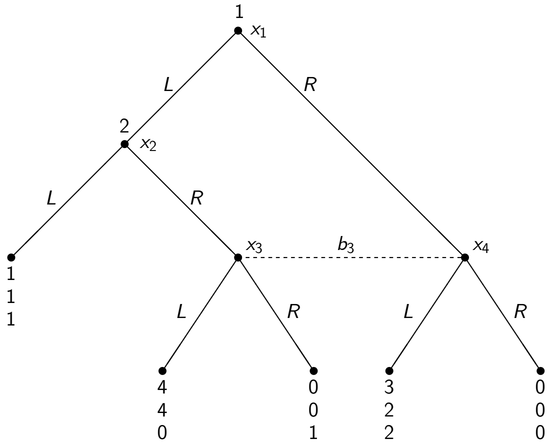
4.5- Perfect Equilibrium

- Selten, R. "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory* 4, 1975.

Main Idea-example

- Players are not absolutely perfect, there exists a possibility (that it may be very small) that they make *mistakes*
  - when computing the optimal behavior,
  - when implementing their strategies,
  - etc.
- Rationality should be understood as the limit of a process where mistakes tend to disappear.
- The Horse example illustrates the idea.

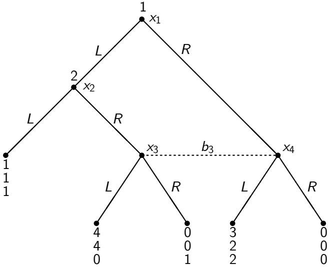
4.5- Perfect Equilibrium



4.5- Perfect Equilibrium

- **Remark:** Since there is only one subgame (the game itself), all Nash equilibria are SPE.
- Critical point: behavior out-of-equilibrium path.
- Let p_i be the probability that player i chooses L.
- There are two types of equilibria:
    - Type 1: $p_1 = 1$, $p_2 = 1$ and $p_3 \in [0, \frac{1}{4}]$.
    - Type 2: $p_1 = 0$, $p_2 \in [\frac{1}{3}, 1]$ and $p_3 = 1$.
- Consider first the particular equilibrium of type 2: $(p_1, p_2, p_3) = (0, 1, 1)$.
    - The same argument will work for all other equilibria of type 2, but it will be less transparent.

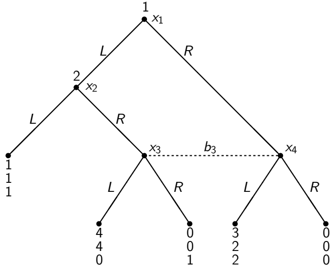
- Given $p_1 = 0$ and $p_3 = 1$, is it reasonable to think that player 2 will play $p_2 = 1$?



- NO. Or is $(0, 1, 1)$ an stable agreement? Suppose they agree on playing $(0, 1, 1)$.
  - Player 2 arrives home (he does not have to play) but suddenly, the telephone rings and says: "It is your turn, decide between L and R".
  - He knows that he is at x_2 (player 1 did a mistake), but given $p_3 = 1$, player 2 cannot play $p_2 = 1$ but rather he has to play R.

4.5- Perfect Equilibrium

- Type 2 equilibria are not sensible since they disappear as soon as there is a probability that players make mistakes when implementing their strategies.
- Consider now the type 1 equilibrium $(p_1, p_2, p_3) = (1, 1, 0)$.
- Now, suppose player 3 is called to play (an out-of-equilibrium play).



- $p_3 = 0$ is still rational since he can be either at x_3 or at x_4 (the mistake may come from either player 1 or player 2). Even with a probability of mistakes, $(1, 1, 0)$ is still rational.

4.5- Perfect Equilibrium

- A Perfect Equilibrium is a Nash equilibrium that is the limit of behavior where mistakes (hand trembles) are possible (Trembling-hand Perfect Equilibrium).
- Let Γ be a finite game in Extensive Form and let ε be a function that assigns to each choice $c \in C$ of Γ a positive number $\varepsilon_c > 0$ with the property that, for all $b \in B$,

$$\sum_{c \in C_b} \varepsilon_c < 1.$$

- For every $i \in I$, define $\hat{\Sigma}_i(\varepsilon)$ as the subset of behavioral strategies $\hat{\sigma}_i$ of player i with the property that for all $b_i \in B_i$, $\hat{\sigma}_i(c) \geq \varepsilon_c$ for all $c \in C_{b_i}$; namely, each action c at b_i has at least probability $\varepsilon_c > 0$.
- Define the perturbed game $\Gamma(\varepsilon)$ as the same game Γ when players can only choose strategies in $\hat{\Sigma}(\varepsilon)$.
- Crucial point: given $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$, all information sets have strictly positive probability to be reached.

Definition A behavioral strategy $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$ is an *equilibrium* of $\Gamma(\varepsilon)$ if for every $i \in I$,

$$H_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) \geq H_i(\hat{\sigma}'_i, \hat{\sigma}_{-i})$$

for all $\hat{\sigma}'_i \in \hat{\Sigma}_i(\varepsilon)$.

- **Remark** By a fixed-point argument, for every ε sufficiently small, the set of equilibria of $\Gamma(\varepsilon)$ is non-empty since $\hat{\Sigma}(\varepsilon)$ is a non-empty, compact and convex subset of a finitely-dimensional Euclidian space and the best-reply correspondence

$$B(\varepsilon) : \hat{\Sigma}(\varepsilon) \twoheadrightarrow \hat{\Sigma}(\varepsilon)$$

is upper-hemi continuous; moreover, for every $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$, $B(\varepsilon)(\hat{\sigma})$ is non-empty and convex.

Definition A behavioral strategy $\hat{\sigma} \in \hat{\Sigma}$ is a *perfect equilibrium* of Γ (or a *trembling-hand perfect equilibrium*) if there exist two sequences $\{\varepsilon^k\}_{k=1}^{\infty} \to 0$ and $\{\hat{\sigma}^k\}_{k=1}^{\infty}$ such that for every $k \geq 1$, $\hat{\sigma}^k$ is an equilibrium of $\Gamma(\varepsilon^k)$ and $\{\hat{\sigma}^k\}_{k=1}^{\infty} \to \hat{\sigma}$.

Theorem (Selten, 1975)

Every finite game in extensive form has at least a perfect equilibrium.

- **Proof (idea):**
  - Let $\{\varepsilon^k\}$ be any arbitrary sequence such that $\{\varepsilon^k\} \to 0$.
  - For every $k \geq 1$, consider the game $\Gamma(\varepsilon^k)$.
  - By the Remark above, $\Gamma(\varepsilon^k)$ has at least one equilibrium: $\hat{\sigma}^k$.
  - Since $\{\hat{\sigma}^k\}$ is a sequence in a compact set $\hat{\Sigma}$, it has a convergent subsequence: $\{\hat{\sigma}^{k_n}\}_{n=1}^{\infty} \to \hat{\sigma} \in \hat{\Sigma}$.
  - Hence, $\hat{\sigma}$ is a perfect equilibrium of Γ. ■

4.5- Perfect Equilibrium

- **Remark** There are games for which the set of perfect equilibria is an strict subset of the set of subgame perfect equilibria.
  - Type 2 equilibria of the horse game are subgame perfect but not perfect.
  - Type 1 equilibria of the horse game are perfect equilibria.

Theorem (Selten, 1975)

Let $\hat{\sigma}$ be a perfect equilibrium of Γ. Then, $\hat{\sigma}$ is a subgame perfect equilibrium of Γ.

- **Proof (idea):**
  - Along the sequence, players behave rationally except for the fact that all choices have to receive strictly positive probability.
  - Since payoff functions are continuous, in the limit also rational behavior is required, even in information sets that are out-of-equilibrium play.

4.5- Perfect Equilibrium

- **Lemma 1** *Every equilibrium* $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$ *of* $\Gamma(\varepsilon)$ *is a subgame perfect equilibrium of* $\Gamma(\varepsilon)$.
- **Proof of Lemma 1** Assume $\hat{\sigma}$ is not a subgame perfect equilibrium of $\Gamma(\varepsilon)$.
  - Then, there exists a subgame $\Gamma(\varepsilon)_x$ such that $\hat{\sigma}^x$ is not an equilibrium of $\Gamma(\varepsilon)_x$.
  - Namely, there exists $i \in I$ and $\tilde{\sigma}_i^x$ such that

$$H_i^x(\tilde{\sigma}_i^x, \hat{\sigma}_{-i}^x) > H_i^x(\hat{\sigma}_i^x, \hat{\sigma}_{-i}^x).$$

  - Let $\tilde{\sigma} \in \hat{\Sigma}(\varepsilon)$ coincide with $\hat{\sigma}$ except that player i follows $\tilde{\sigma}_i^x$ at $\Gamma(\varepsilon)_x$.
  - Since all information sets have strictly positive probability (in particular, x) and i gets strictly higher payoff at x,

$$H_i(\tilde{\sigma}_i, \hat{\sigma}_{-i}) > H_i(\hat{\sigma}_i, \hat{\sigma}_{-i}).$$

  - Thus, $\hat{\sigma}$ is not an equilibrium of $\Gamma(\varepsilon)$. ■

- **Lemma 2** *Let* $\hat{\sigma} \in \hat{\Sigma}$ *be a perfect equilibrium of* Γ. *Then, in each subgame* Γ_x, $\hat{\sigma}^x$ *is a perfect equilibrium of* Γ_x.

- **Proof of Lemma 2** Let $\hat{\sigma} \in \hat{\Sigma}$ be a perfect equilibrium of Γ.
  - Then, there exist two sequences $\{\varepsilon^k\} \to 0$ and $\{\hat{\sigma}^k\}$ such that for every $k \geq 1$, $\hat{\sigma}^k$ is an equilibrium of $\Gamma(\varepsilon^k)$ and $\{\hat{\sigma}^k\} \to \hat{\sigma}$.
  - Hence, for each subgame Γ_x, $\{(\hat{\sigma}^x)^k\} \to \hat{\sigma}^x$.
  - Since $\hat{\sigma}^k$ is an equilibrium of $\Gamma(\varepsilon^k)$, by Lemma 1, it is also a subgame perfect equilibrium of $\Gamma(\varepsilon^k)$.
  - Hence, $(\hat{\sigma}^x)^k$ is an equilibrium of $\Gamma(\varepsilon^k)_x$.
  - Thus, there exist two sequences $\{\varepsilon^k\} \to 0$ and $\{(\hat{\sigma}^x)^k\}$ such that for every $k \geq 1$, $(\hat{\sigma}^x)^k$ is an equilibrium of $\Gamma(\varepsilon^k)_x$ and $\{(\hat{\sigma}^x)^k\} \to \hat{\sigma}^x$.
  - Namely, $\hat{\sigma}^x$ is a perfect equilibrium of Γ_x. ∎

- **Lemma 3** *Let $\hat{\sigma}^* \in \hat{\Sigma}$ be a perfect equilibrium of Γ. Then, $\hat{\sigma}^*$ is an equilibrium of Γ.*
- **Proof of Lemma 3** Let $\hat{\sigma}^* \in \hat{\Sigma}$ be a perfect equilibrium of Γ.
  - Then, there exist two sequences $\{\varepsilon^k\} \to 0$ and $\{\hat{\sigma}^k\}$ such that for every $k \geq 1$, $\hat{\sigma}^k$ is an equilibrium of $\Gamma(\varepsilon^k)$ and $\{\hat{\sigma}^k\} \to \hat{\sigma}^*$.
  - Namely, for every $k \geq 1$ and every $i \in I$,
  $$H_i(\hat{\sigma}^k) \geq H_i(\tilde{\sigma}_i, \hat{\sigma}^k_{-i})$$
  for all $\tilde{\sigma}_i \in \hat{\Sigma}_i(\varepsilon^k)$.
  - Fix $m \geq 1$ and define $\hat{\Sigma}_i^m = \bigcap_{k \geq m} \Sigma_i(\varepsilon^k)$.
  - Now, for every $k \geq m$ and every $i \in I$,
  $$H_i(\hat{\sigma}^k) \geq H_i(\tilde{\sigma}_i, \hat{\sigma}^k_{-i})$$
  for all $\tilde{\sigma}_i \in \hat{\Sigma}_i^m \subseteq \hat{\Sigma}_i(\varepsilon^k)$.

- **Proof of Lemma 3** (continuation).
  - Fix $\tilde{\sigma}_i \in \hat{\Sigma}_i^m$, by continuity of $H_i(\cdot,\cdot)$ and $H_i(\tilde{\sigma}_i,\cdot)$,

$$H_i(\hat{\sigma}^*) \geq H_i(\tilde{\sigma}_i, \hat{\sigma}^*_{-i}).$$

  - But since $\hat{\Sigma}_i = cl(\bigcup_{m\geq 1} \hat{\Sigma}_i^m)$ and continuity of $H_i(\cdot)$,

$$H_i(\hat{\sigma}^*) \geq H_i(\sigma_i, \hat{\sigma}^*_{-i})$$

  for all $\sigma_i \in \hat{\Sigma}_i$.
  - Thus, $\hat{\sigma}^*$ is an equilibrium of Γ. ∎

4.5- Perfect Equilibrium

- **Proof of the Theorem**
  - Let $\hat{\sigma}$ be a perfect equilibrium of Γ.
  - By Lemma 2, for all subgames Γ_x, $\hat{\sigma}^x$ is a perfect equilibrium of Γ_x.
  - By Lemma 3, $\hat{\sigma}^x$ is an equilibrium of Γ_x.
  - By the definition of subgame perfection, $\hat{\sigma}$ is a subgame perfect equilibrium of Γ. $\blacksquare$

- **Summary**
  - PE$\subseteq$SPE.
  - PE$\neq \varnothing$.
  - There exists a game Γ (Selten's horse game) such that PE$\subsetneq$SPE.

4.6- Perfect Equilibrium in the normal form

- Suppose we proceed by extending the definition of perfect equilibrium from the extensive form to the (mixed extension) of the normal form as follows.
- Let $G = (I, (S_i)_{i \in I}, (h_i)_{i \in I})$ be a finite game in normal form and let ε be a function that assigns to every s_i an strictly positive number $\varepsilon_{s_i} > 0$ in such a way that for all $i \in I$,

$$\sum_{s_i \in S_i} \varepsilon_{s_i} < 1.$$

- Given the mixed extension $G^* = (I, (\Sigma_i)_{i \in I}, (H_i)_{i \in I})$ and the function ε, define the ε-perturbed game $G^*(\varepsilon) = (I, (\Sigma(\varepsilon)_i)_{i \in I}, (H_i)_{i \in I})$, where

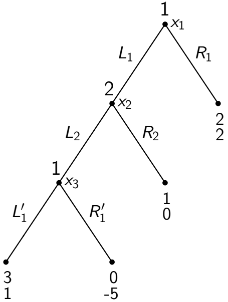
$$\Sigma(\varepsilon)_i = \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) \geq \varepsilon_{s_i} \text{ for all } s_i \in S_i\}.$$

4.6- Perfect Equilibrium in the normal form

Definition* A mixed strategy $\sigma \in \Sigma$ is a *normal-form perfect equilibrium* (or a *trembling-hand perfect equilibrium in the normal form*) of G^* if there exist two sequences $\{\varepsilon^k\} \to 0$ and $\{\sigma^k\}$ such that for every $k \geq 1$, σ^k is an equilibrium of $G^*(\varepsilon^k)$ and $\{\sigma^k\} \to \sigma$.

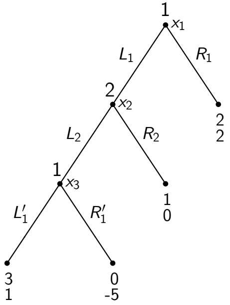
- Suppose we have a finite game in extensive form Γ and construct its associated normal form G.
- Question: Does every normal-form perfect equilibrium of G^* correspond to a perfect equilibrium of Γ?
- Answer: NO.

Example:



- There exists a unique subgame perfect equilibrium of Γ: $((L_1, L_1'), L_2)$. Hence, $((L_1, L_1'), L_2)$ is the unique perfect equilibrium of Γ.
- However, $(R_1, R_1'), R_2)$ is a perfect equilibrium (according to Definition*) in the normal form.

4.6- Perfect Equilibrium in the normal form



(L_1, L_1')
(L_1, R_1')
(R_1, L_1')
(R_1, R_1')

- The reason is that in the normal form trembles are correlated while in the extensive form trembles in different information sets are uncorrelated.
- In the example, trembles at x_1 and x_3 in the normal form are not independent (the same experiment is used for both), while in the extensive form we have to use uncorrelated trembles by performing two experiments, one at x_1 and the other at x_3.

4.6- Perfect Equilibrium in the normal form

$$
\begin{array}{ccc|c|c|}
 & & & \varepsilon_k & 1-\varepsilon_k \\
 & & 1/2 & L_2 & R_2 \\
\cline{4-5}
\frac{\varepsilon_k^2}{2} & & L_1 L_1' & 3,1 & 1,0 \\
\cline{4-5}
\frac{\varepsilon_k^2}{2} & & L_1 R_1' & 0,-5 & 1,0 \\
\cline{4-5}
\varepsilon_k & & R_1 L_1' & 2,2 & 2,2 \\
\cline{4-5}
1-\varepsilon_k-\varepsilon_k^2 & & R_1 R_1' & 2,2 & 2,2 \\
\cline{4-5}
\end{array}
$$

- Take any sequence $\{\varepsilon_k\} \to 0$ and define $\Sigma_1(\frac{\varepsilon_k^2}{2})$ and $\Sigma_2(\varepsilon_k)$, where $\varepsilon_1^k(s_1) = \frac{\varepsilon_k^2}{2}$ for all $s_1 \in S_1$ and $\varepsilon_2^k(s_2) = \varepsilon_k$ for all $s_2 \in S_2$.
- Given $\varepsilon_k > 0$ sufficiently small, consider the following strategy $(\sigma_1^k, \sigma_2^k) \in \Sigma_1(\frac{\varepsilon_k^2}{2}) \times \Sigma_2(\varepsilon_k)$:
  - $\sigma_1^k(L_1 L_1') = \frac{\varepsilon_k^2}{2}$, $\sigma_1^k(L_1 R_1') = \frac{\varepsilon_k^2}{2}$, $\sigma_1^k(R_1 L_1) = \varepsilon_k$, and $\sigma_1^k(R_1 R_1') = 1 - \varepsilon_k - \varepsilon_k^2$.
  - $\sigma_2^k(L_2) = \varepsilon_k$ and $\sigma_2^k(R_2) = 1 - \varepsilon_k$.

4.6- Perfect Equilibrium in the normal form

- Consider player 2 :
  - $H_2(\sigma_1^k, L_2) = \frac{\varepsilon_k^2}{2} - 5\frac{\varepsilon_k^2}{2} + 2\varepsilon_k + 2 - 2\varepsilon_k - 2\varepsilon_k^2 = 2 - 4\varepsilon_k^2$.
  - $H_2(\sigma_1^k, R_2) = 2\varepsilon_k + 2 - 2\varepsilon_k - 2\varepsilon_k^2 = 2 - 2\varepsilon_k^2$.
  - Hence, for all $\varepsilon_k > 0$ sufficiently small, $H_2(\sigma_1^k, R_2) > H_2(\sigma_1^k, L_2)$.
  - Thus, σ_2^k (to play R_2 with probability $1 - \varepsilon_k$) is the best-reply in $\Sigma_2(\varepsilon_k)$ against σ_1^k.
- Consider now player 1 :
  - $H_1(L_1 L_1', \sigma_2^k) = 3\varepsilon_k + (1 - \varepsilon_k)$.
  - $H_1(L_1 R_1', \sigma_2^k) = 1 - \varepsilon_k$.
  - $H_1(R_1 L_1', \sigma_2^k) = H_2(R_1 R_1', \sigma_2^k) = 2$.
  - Hence, for sufficiently small $\varepsilon_k > 0$, σ_1^k is a best-reply in $\Sigma_1(\frac{\varepsilon_k^2}{2})$ against σ_2^k.
- Thus, $(R_1 R_1', R_2)$ is a perfect equilibrium in the normal form (according to Definition*).

4.6- Perfect Equilibrium in the normal form

- Suppose that, given an extensive form game, we construct the agent-normal form, where every information set corresponds to an agent, and every player controls its agents. We will later give a formal definition of the agent-normal form.

- **Example** Agents 1 (at x_1) and 3 (at x_3) are agents of player 1.
  - New set of players: $1 = (1.1)$, $2 = (2.1)$ and $3 = (1.2)$.
  - L_1

2/3	L_1'	R_1'
L_2	3, 1, 3	0,-5, 0
R_2	1, 0, 1	1, 0, 1

    R_1

2/3	L_1'	R_1'
L_2	2, 2, 2	2, 2, 2
R_2	2, 2, 2	2, 2, 2

    .
  - We want to see that $(L_1 L_1', L_2)$ is the unique perfect equilibrium of this agent-normal form, and hence, it is the unique subgame perfect equilibrium and perfect equilibrium of the extensive form Γ (although we already knew that, since it is the unique subgame perfect equilibrium of Γ and all perfect equilibrium are subgame perfect).

4.6- Perfect Equilibrium in the normal form

- But before, we give an alternative and very useful formulation of perfect equilibrium of a game in normal form.

- Observe that its definition requires to check that every $\sigma^k \in \Sigma(\varepsilon^k)$, in the convergent sequence to σ, is a Nash equilibrium of $G^*(\varepsilon^k)$, which in general may be difficult.

Proposition 1 (Selten, 1975) *Let G be a finite game in normal form. Then, σ^* is a perfect equilibrium of G (according to Definition*) if and only if there exists a sequence $\{\sigma^k\} \to \sigma^*$ such that (a) σ^k is completely mixed (i.e., $\sigma^k \in int(\Sigma)$) and (b) for every $k \geq 1$, σ_i^* is a best reply to σ_{-i}^k for all $i \in I$.*

4.6- Perfect Equilibrium in the normal form

Proof $\Longleftarrow$) Assume there exists a sequence $\{\sigma^k\} \to \sigma^*$ such that (a) σ^k is completely mixed (i.e., $\sigma^k \in int(\Sigma)$) and (b) for every $k \geq 1$, σ_i^* is a best reply to σ_{-i}^k for all $i \in I$.

- Let $\{e_k\} \to 0$ be such that for all $k \geq 1$, $e_k > 0$ and for all $i \in I$ and all $s_i \in S_i$,
$$\sigma_i^k(s_i) > e_k. \tag{2}$$
  - Notice that such sequence $\{e_k\}$ does always exist since $\sigma^k \in int(\Sigma)$; for instance, we can always take
$$e_k = \frac{1}{k} \min_{i \in I} \min_{s_i \in S_i} \{\sigma_i^k(s_i)\} > 0.$$

- Define $\varepsilon^k(\cdot)$ as follows: for every $k \geq 1$ and every $s_i \in S_i$,
$$\varepsilon^k(s_i) = \begin{cases} \sigma_i^k(s_i) & \text{if } s_i \text{ is not a best reply to } \sigma_{-i}^k \text{ in } G \\ e_k & \text{otherwise.} \end{cases} \tag{3}$$

- Consider $G(\varepsilon^k)$.

4.6- Perfect Equilibrium in the normal form

- Objective: We want to show that $\{\varepsilon^k\} \to 0$ and σ^k is a Nash equilibrium of $G(\varepsilon^k)$. By assumption, $\{\sigma^k\} \to \sigma^*$.
- Assume $s_i \in S_i$ is not a best reply to σ^*_{-i}.
- This means that there exists $\bar{s}_i \in S_i$ such that
$$H_i(\bar{s}_i, \sigma^*_{-i}) > H_i(s_i, \sigma^*_{-i}). \tag{4}$$
- Since $\{\sigma^k\} \to \sigma^*$, (4) and continuity of H_i, we have that for all sufficiently large k,
$$H_i(\bar{s}_i, \sigma^k_{-i}) > H_i(s_i, \sigma^k_{-i}). \tag{5}$$
- Hence, and since σ^*_i is a best reply to σ^k_{-i}, we must have that $\sigma^*_i(s_i) = 0$.
- Thus, $\{\sigma^k_i(s_i)\} \to 0$.
- Therefore, if $s_i \in S_i$ is not a best reply to σ^*_{-i}, $\{\sigma^k_i(s_i)\} \to 0$.
- By definition in (3), $\{\varepsilon^k(s_i)\} \to 0$.

- Now observe that $\sigma_i^k \in \Sigma_i(\varepsilon^k)$ since
  - if s_i is not a best reply to σ_{-i}^k then, $\sigma_i^k(s_i) = \varepsilon^k(s_i)$,
  - if s_i is a best reply to σ_{-i}^k then, by (2) and (3), $\sigma_i^k(s_i) > \varepsilon^k(s_i) = e_k$.
- Hence, σ_i^k has the property that non-best replies receive the minimum probability; *i.e.*, σ^k is a Nash equilibrium of $G(\varepsilon^k)$.
- Thus, there exist $\{\varepsilon^k\} \to 0$ and $\{\sigma^k\} \to \sigma^*$ such that for all $k \geq 1$, σ^k is a Nash equilibrium of $G(\varepsilon^k)$, implying that σ^* is a perfect equilibrium of G.

4.6- Perfect Equilibrium in the normal form

Proof $\Longrightarrow$) Assume σ^* is a perfect equilibrium (according to Definition*) of G.

- Then, there exist $\{\varepsilon^k\} \to 0$ and $\{\sigma^k\} \to \sigma^*$ such that for all $k \geq 1$, σ^k is a Nash equilibrium of $G(\varepsilon^k)$.

- For every $k \geq 1$ and $i \in I$, define

$$T_i^k = \{s_i \in S_i \mid \underset{\sigma_i^*(s_i)}{\underset{\downarrow\ (*)}{\sigma_i^k(s_i)}} > \underset{0}{\underset{\downarrow}{\varepsilon^k(s_i)}}\}.$$

- Since σ^k is a Nash equilibrium of $G(\varepsilon^k)$, $s_i \in T_i^k$ implies that s_i is a best reply against σ_{-i}^k.

- However, T_i^k may not contain all of them. By $(*)$, there exists K such that for all $k \geq K$, if $\sigma_i^*(s_i) > 0$ then $s_i \in T_i^k$. Without loss of generality, assume $K = 1$.

4.6- Perfect Equilibrium in the normal form

- Summing up: every $s_i \in S_i$ with $\sigma_i^*(s_i) > 0$ is in T_i^k and every $s_i \in T_i^k$ is a best reply to σ_{-i}^k.
- Thus,
  - (a) σ^k is completely mixed (i.e., $\sigma^k \in int(\Sigma)$) and
  - (b) for every $k \geq 1$, σ_i^* is a best reply to σ_{-i}^k for all $i \in I$. ■

Corollary

If σ^* is a perfect equilibrium of the game in normal form G then, for every $i \in I$, σ_i^* is not a dominated strategy.

4.6- Perfect Equilibrium in the normal form

- Let Γ be a finite game in extensive form.
- For every $i \in I$, let $B_i = \{b_i^1, ..., b_i^{K_i}\}$ and define the set of agents of G as $I^a = \bigcup_{i \in I} \bigcup_{t=1}^{K_i} (i.t)$, and for every $(i.t) \in I^a$, define $S^a_{(i.t)} = C_{b_i^t}$ and $h^a_{(i.t)} = h_i$.
- Let $G^a = (I^a, (S^a_{(i.t)})_{(i.t) \in I^a}, (h^a_{(i.t)})_{(i.t) \in I^a})$ be the *agent-normal form* of Γ.

Proposition 2 *Let Γ be a finite game in extensive form and let G^a be its corresponding agent-normal form of Γ. Then, σ is a perfect equilibrium of Γ if and only if σ is a perfect equilibrium (according to Definition*) of G^a.*

- **Example** We want to show that $(L_1 L_1', L_2)$ is the unique perfect equilibrium of this agent-normal form, where agents 1 (at x_1) and 3 (at x_3) are agents of player 1.
  - New set of players: $1 \equiv (1.1)$, $2 \equiv (2.1)$ and $3 \equiv (1.2)$.

L_1

	2/3	$1-\varepsilon_3^k$ L_1'	ε_3^k R_1'
$1-\varepsilon_2^k$	L_2	3, 1, 3	0,-5, 0
ε_2^k	R_2	1, 0, 1	1, 0, 1

R_1

	2/3	$1-\varepsilon_3^k$ L_1'	ε_3^k R_1'
$1-\varepsilon_2^k$	L_2	2, 2, 2	2, 2, 2
ε_2^k	R_2	2, 2, 2	2, 2, 2

- Assume σ is a perfect equilibrium of G^a.
- Notice that L_1' dominates R_1'.
  - Hence, by Corollary above, $\sigma_3(L_1') = 1$.
  - This already shows that $(R_1 R_1', R_2)$ cannot be a perfect equilibrium of the agent-normal form.
  - Hence, by Proposition 2, $(R_1 R_1', R_2)$ is not a perfect equilibrium of Γ (we already knew that since it is not subgame perfect).

4.6- Perfect Equilibrium in the normal form

- Let $\{\varepsilon_3^k\} \to 0$ be arbitrary.
- Take any completely mixed sequence $\{\sigma^k\} \to \sigma$ with the property that for all $k \geq 1$, $\sigma_3^k(L_1') = 1 - \varepsilon_3^k$.
-
$$\begin{aligned} H_2(\sigma_1^k, L_2, \sigma_3^k) &= \sigma_1^k(L_1)[(1-\varepsilon_3^k) - 5\varepsilon_3^k] + 2(1-\sigma_1^k(L_1)) \\ &= \sigma_1^k(L_1) - 6\varepsilon_3^k\sigma_1^k(L_1) + 2 - 2\sigma_1^k(L_1) \\ &= 2 - \sigma_1^k(L_1)(1+6\varepsilon_3^k). \end{aligned}$$
-
$$\begin{aligned} H_2(\sigma_1^k, R_2, \sigma_3^k) &= \sigma_1^k(L_1)\cdot 0 + (1-\sigma_1^k(L_1))2 \\ &= 2 - 2\sigma_1^k(L_1) \\ &< 2 - \sigma_1^k(L_1)(1+6\varepsilon_3^k), \end{aligned}$$

  since for large k, $1 + 6\varepsilon_3^k < 2$.
- By Proposition 1, σ_2 is a best reply to σ_{-2}^k. Thus, $\sigma_2(L_2) = 1$.

- Let $\{\varepsilon_2^k\} \to 0$ be arbitrary.
- $$\begin{aligned} H_1(L_1, \sigma_2^k, \sigma_3^k) &= 3(1-\varepsilon_2^k)(1-\varepsilon_3^k) + \varepsilon_2^k \\ &= 3 - 3\varepsilon_2^k - 3\varepsilon_3^k + 3\varepsilon_2^k\varepsilon_3^k + \varepsilon_2^k \\ &= 3 - 2\varepsilon_2^k - 3\varepsilon_3^k + 3\varepsilon_2^k\varepsilon_3^k. \end{aligned}$$
- $$\begin{aligned} H_1(R_1, \sigma_2^k, \sigma_3^k) &= 2 \\ &< H_1(L_1, \sigma_2^k, \sigma_3^k), \end{aligned}$$

  for sufficiently large k.
- By Proposition 1, σ_1 is a best reply to σ_{-1}^k. Hence, $\sigma_1(L_1) = 1$.
- Thus, we have proved, using Proposition 1, that if σ is a perfect equilibrium of G then, $\sigma_1(L_1) = 1$, $\sigma_2(L_2) = 1$ and $\sigma_3(L_3) = 1$.

4.7- Perfect Equilibrium and undominated strategies

- Question: Is the principle "Nash equilibrium plus never a dominated strategy" a characterization of perfect equilibria?

- **Example**

1/2	L	R
T	$10, 0$	$5, 2$
B	$10, 1$	$2, 0$

  - T dominates B (as well as all completely mixed strategies).
  - (T, R) is the unique perfect equilibrium.

- Answer: Yes for $n = 2$, but not in general.

- Fact: *Let G be a finite game in normal form with $\#I = 2$. Then, σ is a perfect equilibrium of G if and only if* (a) *σ is a Nash equilibrium of G and* (b) *for all $i = 1, 2$, σ_i is an undominated strategy.*

4.7- Perfect Equilibrium and undominated strategies

Counter-example (with $\#I = 3$).

$l \quad 1-\varepsilon_3^k$

	$1-\varepsilon_2^k$	ε_2^k
$1/2$	L	R
T	1, 1, 1	1, 0, 1
B	1, 1, 1	0, 0, 1

$r \quad \varepsilon_3^k$

	$1-\varepsilon_2^k$	ε_2^k
$1/2$	L	R
T	1, 1, 0	0, 0, 0
B	0, 1, 0	1, 0, 0

.

- (B, L, l) is a Nash equilibrium and none of the three strategies is dominated.
- However, (B, L, l) is not a perfect equilibrium of G.

4.7- Perfect Equilibrium and undominated strategies

- Assume the contrary, (B, L, l) is a perfect equilibrium of G and let $\{\varepsilon_2^k\} \to 0$ and $\{\varepsilon_3^k\} \to 0$ be arbitrary. For every $k \geq 1$ define $\sigma_2^k(L) = 1 - \varepsilon_2^k$ and $\sigma_3^k(l) = 1 - \varepsilon_3^k$.
- Then,

$$H_1(T, \sigma_2^k, \sigma_3^k) = (1 - \varepsilon_3^k) + \varepsilon_3^k(1 - \varepsilon_2^k) \tag{6}$$

and

$$H_1(B, \sigma_2^k, \sigma_3^k) = (1 - \varepsilon_3^k)(1 - \varepsilon_2^k) + \varepsilon_2^k \varepsilon_3^k. \tag{7}$$

- For sufficiently large $k \geq 1$,

$$(1 - \varepsilon_3^k) > (1 - \varepsilon_3^k)(1 - \varepsilon_2^k) \tag{8}$$

and

$$\varepsilon_3^k(1 - \varepsilon_2^k) > \varepsilon_2^k \varepsilon_3^k. \tag{9}$$

4.7- Perfect Equilibrium and undominated strategies

- Hence, (8) and (9) imply that $H_1(T, \sigma_2^k, \sigma_3^k) > H_1(B, \sigma_2^k, \sigma_3^k)$.
- This means that B is not a best reply to (σ_2^k, σ_3^k) for sufficiently large k.
- Thus, by Proposition 1, (B, L, l) is not a perfect equilibrium of G.

4.7- Perfect Equilibrium and undominated strategies

Fact: *Let G be a finite game in normal form with $\#I = 2$. Then, σ is a perfect equilibrium of G if and only if* (a) *σ is a Nash equilibrium of G and* (b) *for all $i = 1, 2$, σ_i is an undominated strategy.*

Proof of Fact: $\Longrightarrow$) It follows from Lemma 3 and Corollary.

Claim Let G be a finite game in normal form. Assume σ_i^* is not dominated. Then, there exists $\hat{\sigma}_{-i} \in int(\Sigma_{-i})$ such that σ_i^* is a best reply to $\hat{\sigma}_{-i}$.

Proof of Fact: $\Longleftarrow$) Assume $\#I = 2$, σ^* is a Nash equilibrium of G and for $i = 1, 2$, σ_i^* is undominated.

- By Claim above, there exists $\hat{\sigma}_2 = \hat{\sigma}_{-1} \in int(\Sigma_2)$ such that σ_1^* is a best reply to $\hat{\sigma}_2$.
- By Claim above, there exists $\bar{\sigma}_1 = \bar{\sigma}_{-2} \in int(\Sigma_1)$ such that σ_2^* is a best reply to $\bar{\sigma}_1$.
- Remark: If $j \in I \setminus \{1, 2\}$, nothing guarantees that $\hat{\sigma}_j = \bar{\sigma}_j$. Hence, we could not proceed with the proof.

4.7- Perfect Equilibrium and undominated strategies

- Let $1 > \varepsilon > 0$ be arbitrary. Define
  - $\sigma_2^\varepsilon = (1-\varepsilon)\sigma_2^* + \varepsilon\hat{\sigma}_2 \in int(\Sigma_2)$.
  - $\sigma_1^\varepsilon = (1-\varepsilon)\sigma_1^* + \varepsilon\bar{\sigma}_1 \in int(\Sigma_1)$.
- Now,

$$\begin{aligned} H_1(\sigma_1^*, \sigma_2^\varepsilon) &= (1-\varepsilon)H_1(\sigma_1^*, \sigma_2^*) + \varepsilon H_1(\sigma_1^*, \hat{\sigma}_2) \\ &\geq (1-\varepsilon)H_1(\sigma_1', \sigma_2^*) + \varepsilon H_1(\sigma_1', \hat{\sigma}_2) \quad \text{for all } \sigma_1' \in \Sigma_1 \\ &= H_1(\sigma_1', \sigma_2^\varepsilon) \quad \text{for all } \sigma_1' \in \Sigma_1. \end{aligned}$$

- Hence, σ_1^* is a best reply to σ_2^ε.
- Similarly,

$$\begin{aligned} H_2(\sigma_1^\varepsilon, \sigma_2^*) &= (1-\varepsilon)H_2(\sigma_1^*, \sigma_2^*) + \varepsilon H_2(\bar{\sigma}_1, \sigma_2^*) \\ &\geq (1-\varepsilon)H_2(\sigma_1^*, \sigma_2') + \varepsilon H_2(\bar{\sigma}_1, \sigma_2') \quad \text{for all } \sigma_2' \in \Sigma_2 \\ &= H_2(\sigma_1^\varepsilon, \sigma_2') \quad \text{for all } \sigma_2' \in \Sigma_2. \end{aligned}$$

- Hence, σ_2^* is a best reply to σ_1^ε.

4.7- Perfect Equilibrium and undominated strategies

- Take an arbitrary sequence $\{\varepsilon^k\} \to 0$.
- For every $k \geq 1$ and $i = 1, 2$, define $\sigma_i^{\varepsilon^k}$ as above.
- Observe that $\{\sigma^{\varepsilon^k}\} \to \sigma^*$ and for $i = 1, 2$, σ_i^* is a best reply to $\sigma_{-i}^{\varepsilon^k}$ for all $k \geq 1$.
- Thus, by Proposition 1, σ^* is a perfect equilibrium of G. ■
- We will return to perfect equilibrium to study its relationship with sequential equilibrium in the context of incomplete information.

4.8- Proper Equilibrium

- Myerson, R. "Refinements of the Nash Equilibrium Concept," *International Journal of Game Theory* 7 (1978).

- Normal form refinement.

- **Example-idea:**

1/2	L_2	R_2
L_1	1, 1	0, 0
R_1	0, 0	0, 0

.

- G has a unique perfect equilibrium: (L_1, L_2).

- Consider now the game G' :

1/2	L_2	R_2	A_1
L_1	1, 1	0, 0	-1,-2
R_1	0, 0	0, 0	0,-2
A_1	-2,-1	-2, 0	-2,-2

.

- Notice that G' is obtained from G after adding an strictly dominated strategy for every player (A_i).

4.8- Proper Equilibrium

- Both L_i and R_i strictly dominate A_i.
- One could argue that the addition of this strategy should be irrelevant; in particular, it should not change the set of "stable" outcomes.
- Hence, (L_1, L_2) should be the unique "stable" outcome of G'.
- However, (R_1, R_2) is a perfect equilibrium of G'!!
- Why? If players agree to play (R_1, R_2) and the mistakes to play A_i are more likely than to play L_i then, (R_1, R_2) may be obtained as the limit of "rational" trembles.
- Thus, adding strictly dominated strategies may change the set of perfect equilibria.

4.8- Proper Equilibrium

		$\frac{1}{k^2}$	$1-\frac{1}{k}-\frac{1}{k^2}$	$\frac{1}{k}$
	1/2	L_2	R_2	A_1
$\frac{1}{k^2}$	L_1	1, 1	0, 0	-1,-2
$1-\frac{1}{k}-\frac{1}{k^2}$	R_1	0, 0	0, 0	0,-2
$\frac{1}{k}$	A_1	-2,-1	-2, 0	-2,-2

  .
- Define for $i=1,2$, σ_i^k as follows: $\sigma_i^k(L_i)=\frac{1}{k^2}$, $\sigma_i^k(R_i)=1-\frac{1}{k}-\frac{1}{k^2}$ and $\sigma_i^k(A_i)=\frac{1}{k}$.
- Observe that for $i=1,2$, and all $k\geq 1$,
  - $H_i(L_i,\sigma_{-i}^k)=\frac{1}{k^2}-\frac{1}{k}<0$.
  - $H_i(R_i,\sigma_{-i}^k)=0$.
  - $H_i(A_i,\sigma_{-i}^k)=-2$.
- Namely, for all $k\geq 1$, R_i is a best reply against σ_{-i}^k.
- Thus, by Proposition 1, (R_1,R_2) is a perfect equilibrium of G'.

4.8- Proper Equilibrium

- Proper equilibrium requires that players make mistakes in a more rational way: more costly mistakes are less likely since players put more effort to prevent them.
- In the example, players should not expect the mistake A_i with higher probability than the mistake L_i.
- Then, (R_1, R_2) cannot be obtained as the limit of this more rational mistakes.
- **Definition** Let G be a finite game in normal form and let $\varepsilon > 0$ be given. An ε-*proper equilibrium* of G is a totally mixed strategy $\sigma \in int(\Sigma)$ such that for all $i \in I$,

$$\text{if } H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i}) \text{ then } \sigma_i(s_i) \leq \varepsilon\sigma_i(s_i').$$

4.8- Proper Equilibrium

- **Definition** Let G be a finite game in normal form. A strategy $\sigma \in \Sigma$ is a *proper equilibrium* of G if there exist $\{\varepsilon^k\} \to 0$ and $\{\sigma^k\} \to \sigma$ such that for all $k \geq 1$, $\varepsilon^k > 0$ and σ^k is an ε^k-proper equilibrium of G.

Theorem

Myerson (1978) Every finite game in normal form has at least a proper equilibrium.

We will first prove the following Lemma that will be useful to prove the Theorem.

Lemma *Let G be a finite game in normal form and let $\varepsilon > 0$ be sufficiently small. Then, G has at least one ε-proper equilibrium.*

Proof of the Lemma (sketch) Let G be a finite game in normal form and let $\varepsilon > 0$ be sufficiently small.

- For each $i \in I$, construct

$$\Sigma_i^\varepsilon = \left\{ \sigma_i \in int(\Sigma_i) \mid \sigma_i(s_i) \geq \frac{\varepsilon^m}{m} \text{ for each } s_i \in S_i \right\},$$

where $m = \max_{i \in I} \# S_i$. Observe that if $\varepsilon > 0$ is sufficiently small, $\Sigma_i^\varepsilon \neq \varnothing$.

- For each $i \in I$, consider now the constrained best-reply correspondence $r_i^\varepsilon : \Sigma^\varepsilon \twoheadrightarrow \Sigma_i^\varepsilon$ defined as follows: for every $\sigma \in \Sigma^\varepsilon$,

$$r_i^\varepsilon(\sigma) = \left\{ \sigma_i' \in \Sigma_i^\varepsilon \mid \text{if } H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i}) \text{ then } \sigma_i'(s_i) \leq \varepsilon \sigma_i'(s_i') \right\}.$$

4.8- Proper Equilibrium

- It is possible to show that, since $\sigma_i'(s_i) \leq \varepsilon \sigma_i'(s_i')$ are linear weak inequalities, r_i^ε is convex and compact valued, and upper hemi-continuous.
- Fix $\sigma \in \Sigma^\varepsilon$. To prove that $r_i^\varepsilon(\sigma) \neq \varnothing$, consider any $s_i \in S_i$ and let

$$\rho(s_i) = \#\{s_i' \in S_i \mid H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i})\}.$$

- Then,
  - if $\rho(s_i) = 0$ for all $s_i \in S_i$, then $r_i^\varepsilon(\sigma) = \Sigma_i^\varepsilon \neq \varnothing$,
  - if there exists $\hat{s}_i \in S_i$ such that $\rho(\hat{s}_i) > 0$ then consider the strategy $\hat{\sigma}_i \in \Sigma_i$ where for every $s_i \in S_i$,

$$\hat{\sigma}_i(s_i) = \frac{\varepsilon^{\rho(s_i)}}{\sum_{\bar{s}_i \in S_i} \varepsilon^{\rho(\bar{s}_i)}}.$$

- Since $\varepsilon^{\rho(s_i)} \geq \varepsilon^m$ because $\rho(s_i) \leq m$, $\varepsilon \leq 1$ (it is sufficiently small), and

$$\sum_{\bar{s}_i \in S_i} \varepsilon^{\rho(\bar{s}_i)} \leq \sum_{\bar{s}_i \in S_i : \rho(\bar{s}_i) > 0} \varepsilon + \#\{\bar{s}_i \in S_i \mid \rho(\bar{s}_i) = 0\} \leq m$$

holds, we have that

$$\hat{\sigma}_i(s_i) = \frac{\varepsilon^{\rho(s_i)}}{\sum_{\bar{s}_i \in S_i} \varepsilon^{\rho(\bar{s}_i)}} \geq \frac{\varepsilon^m}{m}.$$

- Then, $\hat{\sigma}_i \in \Sigma_i^{\varepsilon}$.

- To show that $\hat{\sigma}_i \in r_i^{\varepsilon}(\sigma)$, assume that $s_i, s_i' \in S_i$ are such that $H_i(s_i, \sigma_{-i}) < H_i(s_i', \sigma_{-i})$.

4.8- Proper Equilibrium

- Then, $\rho(s_i) > 0$ and $\rho(s_i) \geq \rho(s_i') + 1$. Hence,

$$\hat{\sigma}_i(s_i) = \frac{\varepsilon^{\rho(s_i)}}{\sum\limits_{\bar{s}_i \in S_i} \varepsilon^{\rho(\bar{s}_i)}} \overset{?}{\leq} \varepsilon \frac{\varepsilon^{\rho(s_i')}}{\sum\limits_{\bar{s}_i \in S_i} \varepsilon^{\rho(\bar{s}_i)}} = \varepsilon \hat{\sigma}_i(s_i').$$

- But $\overset{?}{\leq}$ holds because $\varepsilon^{\rho(s_i)} \leq \varepsilon^{\rho(s_i')}$ since $\varepsilon < 1$ and $\rho(s_i) > \rho(s_i')$.
- Thus, $r_i^\varepsilon(\sigma) \neq \varnothing$ for every $\sigma \in \Sigma^\varepsilon$.
- Now, applying the Kakutani Fixed Point Theorem to the correspondence $r^\varepsilon : \Sigma^\varepsilon \twoheadrightarrow \Sigma^\varepsilon$, we obtain that there exists $\sigma \in \Sigma^\varepsilon$ such that $\sigma \in r^\varepsilon(\sigma)$.
- Thus, σ is an ε-proper equilibrium of G.

Proof of the Theorem Let G be a finite game in normal form. We want to show that G has a proper equilibrium.

- Let $\{\varepsilon^k\} \to 0$ be an arbitrary sequence.
- Let $\{\sigma^k\}$ be a corresponding sequence, where for every $k \geq 1$, σ^k is an ε^k-proper equilibria of G, which exists by the previous Lemma.
- Since $\{\sigma^k\}$ lies in the compact set Σ, it has a convergent subsequence $\{\sigma^{k_n}\} \to \sigma$.
- By definition, σ is a proper equilibrium of G. ∎

Remark Let σ be a proper equilibrium of G. Then, σ is a perfect equilibrium of G.

4.9- Stable Sets of Equilibria

- Kohlberg, E. and J.F. Mertens. "On the Strategic Stability of Equilibria," *Econometrica* 54 (1986).

4.10- Rationalizable Strategic Behavior

- Bernheim, B. "Rationalizable Strategic Behavior," *Econometrica* 53 (1984). [Normal Form].
- Pearce, D. "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica* 52 (1984). [Extensive Form].
- It is not a refinement.
- Strategic uncertainty: Bayesian approach to the problem of strategic selection.
- The idea is to find restrictions on the behavior of players just coming from the hypothesis of rationality (and the common knowledge of it).

4.10- Rationalizable Strategic Behavior

- The point of view in the refinement approach (for instance, the principle of "never a dominated strategy"): What strategies are not going to be played by a rational player?
- The starting point of rationalizability is the complementary question: What is the set of strategies that a rational player may play?
- Answer: A rational player will play *only* strategies that are best reply to some beliefs on the strategies of the other players. The contrapositive: A rational player will never play a strategy that is never a best reply for some belief on the strategies of the other players.
- Moreover, common knowledge of rationality implies that not all beliefs about other players' behavior are possible.
- This means that we have to face an infinite reasoning process. Let's model it.

4.10- Rationalizable Strategic Behavior

- Let $G^* = (I, (\Sigma_i)_{i \in I}, (H_i)_{i \in I})$ be a mixed extension of a finite game in normal form G.
- Notation: Given $X \subseteq \mathbb{R}^m$, the *convex hull* of X, denoted by $co(X)$, is the smallest convex set that contains X. Also

$$co(X) = \bigcap_{X \subseteq Y \subseteq \mathbb{R}^m : Y \text{ is convex}} Y.$$

- **Definition** Define for every $i \in I$, $\Sigma_i^0 = \Sigma_i$ and recursively, for all $t \geq 0$,

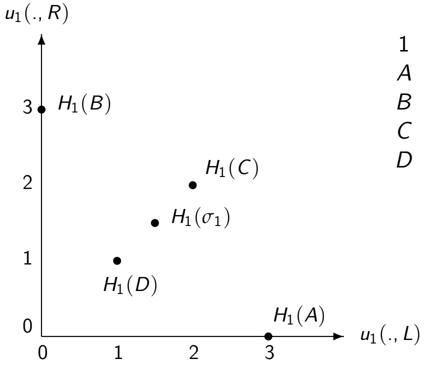
$$\Sigma_i^{t+1} = \{\sigma_i \in \Sigma_i^t \mid \exists \sigma_{-i} \in \prod_{j \neq i} co(\Sigma_j^t) \text{ s.t.}$$
$$H_i(\sigma_i, \sigma_{-i}) \geq H_i(\sigma_i', \sigma_{-i}) \ \forall \sigma_i' \in \Sigma_i^t\}.$$

- Namely, σ_i is a best reply against i's belief $\prod_{j \neq i} co(\Sigma_j^t)$.
- Why $co(\Sigma_j^t)$ instead of Σ_j^t? (later).

- The set of *rationalizable strategies* for player i is

$$R_i = \bigcap_{t=0}^{\infty} \Sigma_i^t.$$

- A strategy profile $\sigma \in \Sigma$ is *rationalizable* if $\sigma_i \in R_i$ for all $i \in I$.
- **Example** (Why $co(\Sigma_j^t)$ instead of Σ_j^t?)
  - It is possible that $\sigma_j', \sigma_j'' \in \Sigma_j^t$ but the mixture $\frac{1}{2}\sigma_j' + \frac{1}{2}\sigma_j'' \notin \Sigma_j^t$ (the belief that player j will play σ_j' with probability $\frac{1}{2}$ and σ_j'' with probability $\frac{1}{2}$).



1	L	R
A	3	0
B	0	3
C	2	2
D	1	1

- The strategy $\sigma_1(A) = \sigma_1(B) = \frac{1}{2}$ is dominated by C. Hence, $\sigma_1 \notin \Sigma_1^1$ but since $\sigma_1', \sigma_1'' \in \Sigma_1^1$, where $\sigma_1'(B) = \sigma_1''(A) = 1$, we want in Σ_1^1 the belief "with any probability, 1 will play A and the complementary probability, 1 will play B"; thus, we have $co(\Sigma_j^t)$ in the definition.

4.10- Rationalizable Strategic Behavior

- **Remark 1** If σ is a Nash equilibrium of G then, σ is rationalizable.
  - Observe that for any $t \geq 0$ and any $i \in I$, $\sigma_i \in \Sigma_i^t$ and hence, $\sigma_i \in R_i$.
  - Thus, $R \neq \varnothing$ for all finite G (since the set of Nash equilibria belongs to R).
  - Rationalizability is not a refinement. It is an enlargement of the set of solutions.
  - For instance, in the Battle of Sexes, $(F, B), (B, F) \in R$.
- **Remark 2** If $\#I = 2$ then, R coincides with the set of strategies that survive the iterate elimination of strictly dominated strategies (as in Moulin (*Econometrica*, 1979) to define sophisticated equilibrium, but now only *strictly* dominated strategies are eliminated).
  - Pearce (1984) proves this on the basis of the minimax Theorem.
  - In general, for $\#I > 2$, this statement does not hold.

4.11- Correlated Equilibrium

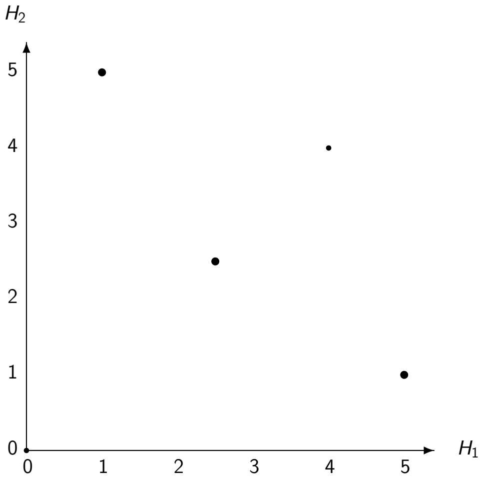
- R. Aumann, "Subjectivity and Correlation in Randomized Strategies," *Journal of Mathematical Economics* 1, (1974).
- Normal Form.
- The concept of Nash equilibrium assumes that players' mixed strategies are independent (uncorrelated).
- Question: What happens if players can correlate their strategies?
- Interpretation: Players, before playing the game, can communicate among them and reach agreements on playing mixed strategies coming from the same experiment (and hence, correlate their strategies). A correlated equilibrium is a profile of correlated mixed strategies that no player has incentives to change unilaterally.

Example-idea:

$$\begin{array}{cc|c|c|}
 & & p_2 & 1-p_2 \\
 & 1/2 & L & R \\
\hline
p_1 & U & 5,1 & 0,0 \\
\hline
1-p_1 & D & 4,4 & 1,5 \\
\hline
\end{array}$$

- $(1,1)$ and $(0,0)$ are two Nash equilibria in pure strategies.
- To find a completely mixed strategy equilibrium assume $(p_1, p_2) \in (0,1)^2$. Then,
  - $0 < p_2 < 1 \Longrightarrow$ $H_2(p_1, L) = p_1 + 4(1-p_1) = 5(1-p_1) = H_2(p_1, R)$. Hence, $p_1 = \frac{1}{2}$.
  - $0 < p_1 < 1 \Longrightarrow H_1(U, p_2) = 5p_2 = 4p_2 + (1-p_2) = H_2(D, p_2)$. Hence, $p_2 = \frac{1}{2}$.
- $(\frac{1}{2}, \frac{1}{2})$ is the mixed strategy equilibrium.
  - Payoffs: $H_1(\frac{1}{2}, \frac{1}{2}) = H_2(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}5 + \frac{1}{4}4 + \frac{1}{4}1 = 2.5$.
- Equilibrium payoffs: $(1,5), (5,1), (2.5, 2.5)$.

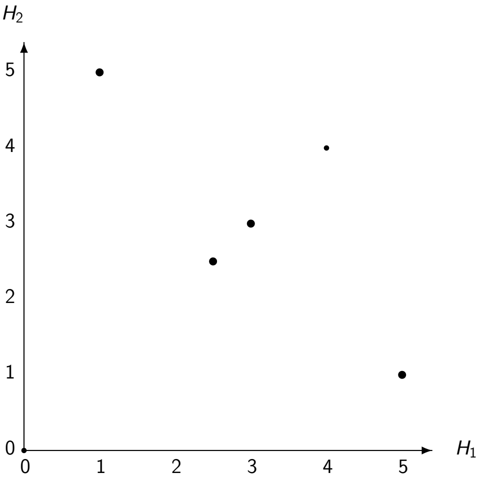
4.11- Correlated Equilibrium

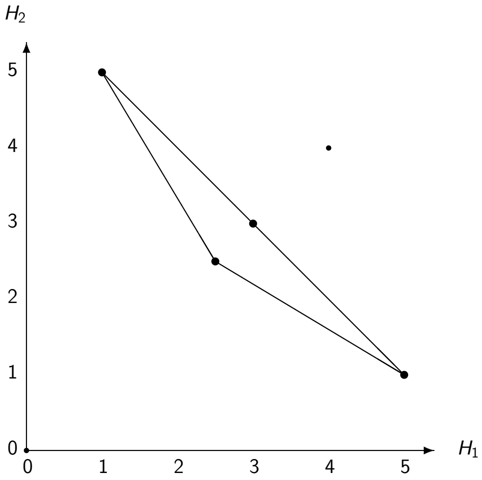


4.11- Correlated Equilibrium

- Assume a fair coin is flipped (there is an agreement among them to do so):
  - $\Omega = \{H, T\}$, $B_1 = \{\{H\}, \{T\}\}$, $B_2 = \{\{H\}, \{T\}\}$ and $p(H) = p(T) = \frac{1}{2}$.
  - If H then they play (U, L).
  - If T then they play (D, R).
  - Payoff$= \frac{1}{2}(1, 5) + \frac{1}{2}(5, 1) = (3, 3) > (2.5, 2, 5)$.
- They do not have incentives to violate the agreement.
- All convex combinations of Nash equilibrium are possible with this type of correlation (this was already known before Aumann's paper).

4.11- Correlated Equilibrium




H_2
5
4
3
2
1
0
0
1
2
3
4
5
H_1

4.11- Correlated Equilibrium

- However, they can even do better.

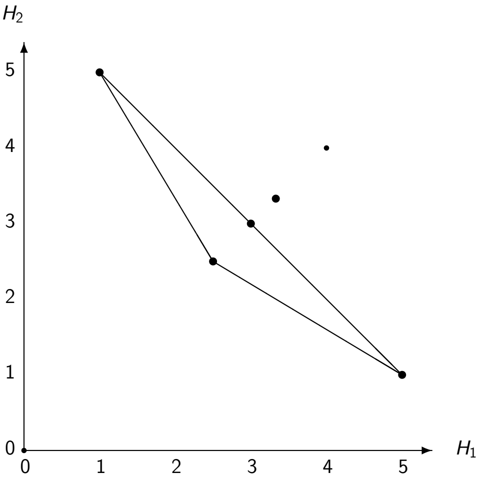
- Design the following experiment and information structure (signals that players will receive about the outcome of the experiment):

    - Sample space: $\Omega = \{w_1, w_2, w_3\}$.
    - Probability distribution p on Ω: $p(w_k) = \frac{1}{3}$ for all $k = 1, 2, 3$.
    - Information structure of player 1 (partition of Ω): $B_1 = \{\{w_1\}, \{w_2, w_3\}\}$.
    - Information structure of player 2 (partition of Ω): $B_2 = \{\{w_1, w_2\}, \{w_3\}\}$.

- A strategy now is a function from information to actions; *i.e.*, $\mathfrak{s}_i : B_i \longrightarrow S_i$. For instance:

    - $\mathfrak{s}_1(\{w_1\}) = U$, $\mathfrak{s}_1(\{w_2, w_3\}) = D$.
    - $\mathfrak{s}_2(\{w_3\}) = R$, $\mathfrak{s}_2(\{w_1, w_2\}) = L$.

- Given $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$, we have generated a probability space $(S, 2^S, q_{\mathfrak{s}})$ where
  - $q_{\mathfrak{s}}(U, L) = p(\{w_1\}) = \frac{1}{3}$.
  - $q_{\mathfrak{s}}(U, R) = p(\{\varnothing\}) = 0$.
  - $q_{\mathfrak{s}}(D, R) = p(\{w_3\}) = \frac{1}{3}$.
  - $q_{\mathfrak{s}}(D, L) = p(\{w_2\}) = \frac{1}{3}$.
- $H_i(\mathfrak{s}_1, \mathfrak{s}_2) = \sum_{s \in S} q_{\mathfrak{s}}(s) h_i(s) = \frac{1}{3}5 + \frac{1}{3}4 + \frac{1}{3}1 = \frac{10}{3}$.
- $(\frac{10}{3}, \frac{10}{3})$ is outside the convex hull of the set of Nash equilibrium payoffs.



H_2
5
4
3
2
1
0
0
1
2
3
4
5
H_1

- We check now that no player wants to violate the agreement:
- Player 1:
  - When 1 knows that the true state of the world is w_1, he knows that 2 will play L and 1 wants to play U.
  - When 1 knows that the true state of the world belongs to $\{\{w_2\},\{w_3\}\}$, 1 can compute the posterior probabilities:
    - $p(\{w_2\} \mid \{w_2, w_3\}) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$.
    - $p(\{w_3\} \mid \{w_2, w_3\}) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$.
    - Hence, the expected payoffs are

$$
\begin{aligned}
H_1(U,\mathfrak{s}_2) &= p(w_2 | \, w_2 w_3) h_1(U,\mathfrak{s}_2(w_2)) + p(w_3 | \, w_2 w_3) h_1(U,\mathfrak{s}_2(w_3)) \\
&= \tfrac{1}{2} h_1(U,L) + \tfrac{1}{2} h_1(U,R) \\
&= \tfrac{1}{2} 5 + \tfrac{1}{2} 0 = 2.5.
\end{aligned}
$$

4.11- Correlated Equilibrium

- Player 1 (continuation):

$$
\begin{aligned}
H_1(D,\mathfrak{s}_2) &= p(w_2 |\ w_2 w_3) h_1(D,\mathfrak{s}_2(w_2)) + p(w_3 |\ w_2 w_3) h_1(D,\mathfrak{s}_2(w_3)) \\
&= \tfrac{1}{2} h_1(D,L) + \tfrac{1}{2} h_1(D,R) \\
&= \tfrac{1}{2} 4 + \tfrac{1}{2} 1 = 2.5,
\end{aligned}
$$

and he is playing D.

- Player 2:
  - When 2 knows that the true state of the world is w_3, he knows that 1 will play D and 2 wants to play R.
  - When 2 knows that the true state of the world belongs to $\{\{w_1\},\{w_2\}\}$, 2 can compute the posterior probabilities:
    - $p(\{w_1\} \mid \{w_1, w_2\}) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$.
    - $p(\{w_2\} \mid \{w_1, w_2\}) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$.

4.11- Correlated Equilibrium

- Player 2 (continuation):
    - His expected payoffs are

$$
\begin{aligned}
H_2(\mathfrak{s}_1, L) &= p(w_1 | \, w_1 w_2) h_2(\mathfrak{s}_1(w_1), L) + p(w_2 | \, w_1 w_2) h_2(\mathfrak{s}_1(w_2), L) \\
&= \tfrac{1}{2} h_2(U, L) + \tfrac{1}{2} h_2(D, L) \\
&= \tfrac{1}{2} 1 + \tfrac{1}{2} 4 = 2.5,
\end{aligned}
$$

and

$$
\begin{aligned}
H_2(\mathfrak{s}_1, R) &= p(w_1 | \, w_1 w_2) h_2(\mathfrak{s}_1(w_1), R) + p(w_2 | \, w_1 w_2) h_2(\mathfrak{s}_1(w_2), R) \\
&= \tfrac{1}{2} h_2(U, R) + \tfrac{1}{2} h_2(D, R) \\
&= \tfrac{1}{2} 0 + \tfrac{1}{2} 5 = 2.5,
\end{aligned}
$$

and he is playing L.

4.11- Correlated Equilibrium

- In fact, we have the following probability distribution on S :

1/2	L	R
U	$\frac{1}{3}$ w_1	0
D	$\frac{1}{3}$ w_2	$\frac{1}{3}$ w_3

.

- Observe that this probability distribution cannot be obtained with uncorrelated strategies.

- There are two alternative definitions of correlated equilibrium:
  - One makes explicit the information structure held by players about the (join) experiment.
  - The other formulates directly the equilibrium on the set of strategy profiles S, without explicitly modelling the experiment.

4.11- Correlated Equilibrium

- Let $G = (I, (S_i)_{i \in I}, (h_i)_{i \in I})$ be a finite game in normal form.
- Let $(\Omega, (B_i)_{i \in I}, p)$ be an information structure where
  - Ω is a finite set,
  - for each $i \in I$, $B_i = \{b_i^1, ..., b_i^{K_i}\}$ is a partition of Ω, and
  - p is a probability distribution on Ω (*i.e.*, for all $w \in \Omega$, $0 \leq p(w) \leq 1$ and $\sum_{w \in \Omega} p(w) = 1$).
- Nature selects $w \in \Omega$ according to p and each $i \in I$ is informed that the true state w is in $b_i(w) \equiv b_i^k$, where $w \in b_i^k$.
- Given $w \in \Omega$ and assuming that $p(b_i(w)) \equiv \sum_{w' \in b_i(w)} p(w') > 0$

  define the conditional probability on Ω, given $b_i(w)$, as follows: for each $\hat{w} \in \Omega$,

$$p(\hat{w} \mid b_i(w)) = \begin{cases} \frac{p(\hat{w})}{p(b_i(w))} & \text{if } \hat{w} \in b_i(w) \\ 0 & \text{otherwise.} \end{cases}$$

- Player i's strategy $\mathfrak{s}_i : B_i \longrightarrow S_i$.
- Equivalently, we can define a strategy of player i as a B_i-measurable function $\mathfrak{s}_i : \Omega \longrightarrow S_i$; *i.e.*, for every $w \in \Omega$,

$$\mathfrak{s}_i(w) = \mathfrak{s}_i(w')$$

for every $w' \in b_i(w)$.

- Let $\mathfrak{S}_i$ be the set of all B_i-measurable functions $\mathfrak{s}_i : \Omega \longrightarrow S_i$.
- Let $\mathfrak{S} = \prod_{i \in I} \mathfrak{S}_i$ be the set of strategy profiles.

4.11- Correlated Equilibrium

Definition 1 A *correlated equilibrium* $\mathfrak{s} = (\mathfrak{s}_1, ..., \mathfrak{s}_n) \in \mathfrak{S}$ of G relative to an information structure $(\Omega, (B_i)_{i \in I}, p)$ is a Nash equilibrium; namely, $\mathfrak{s}$ is a correlated equilibrium if

(Ex-ante) for all $i \in I$,

$$\sum_{w \in \Omega} p(w) h_i(\mathfrak{s}(w)) \geq \sum_{w \in \Omega} p(w) h_i(\mathfrak{s}'_i(w), \mathfrak{s}_{-i}(w))$$

for all $\mathfrak{s}'_i \in \mathfrak{S}_i$,

or equivalently,

(Interim) for all $i \in I$ and all $b_i \in B_i$ such that $p(b_i) > 0$,

$$\sum_{\{w \in \Omega | b_i(w) = b_i\}} p(w \mid b_i) h_i(\mathfrak{s}(w)) \geq \sum_{\{w \in \Omega | b_i(w) = b_i\}} p(w \mid b_i) h_i(\mathfrak{s}'_i(w), \mathfrak{s}_{-i}(w)$$

for all $\mathfrak{s}'_i \in \mathfrak{S}_i$.

4.11- Correlated Equilibrium

- The difficulty of Definition 1 is that it is relative to an information structure, and this is a very big space.
  - Given G, how do we find the set of all correlated equilibria?
  - We have to check all possible information structures: impossible.
- We will think now that the signal players receive from the experiment is directly a recommendation of an action s_i to play.
- Given G, $(\Omega, (B_i)_{i\in I}, p)$ and $\mathfrak{s} : \Omega \longrightarrow S$, for every $w \in \Omega$ we have

$$\begin{array}{ccc} b_1(w) & \longrightarrow & s_1 = \mathfrak{s}_1(w) \\ b_2(w) & \longrightarrow & s_2 = \mathfrak{s}_2(w) \\ \ldots & \ldots & \ldots \\ b_n(w) & \longrightarrow & s_n = \mathfrak{s}_n(w) \end{array}$$

- Namely, p and $\mathfrak{s}$ induce a probability distribution on S.
- Now, players will agree directly on a probability distribution on S.

4.11- Correlated Equilibrium

Definition 2 A *correlated equilibrium* of G is a probability distribution p on S such that for all $i \in I$ and all $d_i : S_i \longrightarrow S_i$,

$$\sum_{s \in S} p(s) h_i(s_i, s_{-i}) \geq \sum_{s \in S} p(s) h_i(d_i(s_i), s_{-i});$$

that is, every player wants to follow the recommendation s that is selected according to p.

Remarks

- If $\mathfrak{s}$ is a correlated equilibrium according to Definition 1 relative to the information structure $(\Omega, (B_i)_{i\in I}, \tilde{p})$, define for every $s \in S$,

$$p(s) = \sum_{\{w\in\Omega|\mathfrak{s}_i(w)=s_i \text{ for all } i\in I\}} \tilde{p}(w).$$

  - Then, p is a correlated equilibrium according to Definition 2.

- If p is a correlated equilibrium according to Definition 2, define
  - $\Omega = S$,
  - for every $i \in I$ and every $s \in S$,

$$b_i(s) = \{s' \in S \mid s'_i = s_i\},$$

  - for every $i \in I$, $\mathfrak{s}_i : \Omega \longrightarrow S_i$ by setting, for every $s \in S$, $\mathfrak{s}_i(s) = s_i$.
  - Then, $\mathfrak{s}$ is a correlated equilibrium relative to $(S, (S_i)_{i\in I}, p)$ according to Definition 1.

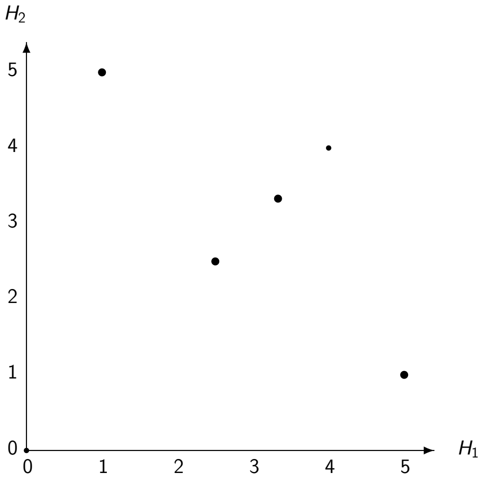
4.11- Correlated Equilibrium

Properties Let G be a finite game in normal form.

- (1) Let $s^* \in S^*$ be a pure strategy Nash equilibrium of G. Then, p with $p(s^*) = 1$ is a correlated equilibrium of G.
- (2) Let $\sigma^* \in \Sigma^*$ be a mixed strategy Nash equilibrium of G and let p be such that for every $s \in S$, $p(s) = \prod_{i \in S} \sigma_i^*(s_i)$. Then, p is a correlated equilibrium of G.
- (3) Therefore, every finite game in normal form G has at least a correlated equilibrium.
- (4) The set of correlated equilibria is convex [Homework].
- (5) By properties (2) and (4) above we have that for every finite game in normal form G,

$$co(\Sigma^*) \subseteq \text{Set of correlated equilibria of } G.$$

4.11- Correlated Equilibrium



4.11- Correlated Equilibrium

- (6) Unknown: How is the class of games for which the set of correlated equilibria is "very different" from $co(\Sigma^*)$?
  - T. Calvó-Armengol (2006) for 2×2 games.
- (7) Correlated equilibria appears as limits of several "adaptive" procedures.
  - D. Foster and R. Vohra. "Calibrated Learning and Correlated Equilibrium," *Games and Economic Behavior* 21 (1997).
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