Strategy-Proof voting on Compact Ranges*

by

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Abstract: Strategy-proof social choice functions are characterized for societies where the space of alternatives is any full dimensional compact subset of a Euclidean space and all voters have generalized single-peaked preferences. Our results build upon and extend those obtained for cartesian product ranges by Border and Jordan (1983). By admitting a large set of non-cartesian ranges, we give a partial answer to the major open question left unresolved in this pioneering article. We prove that our class is composed by generalized median voter schemes which satisfy an additional condition, called the intersection property (Barberà, Massó, and Neme (1997)). *Journal of Economic Literature* Classification Number: D71.

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1. Introduction

For societies with n agents facing a set Z of alternatives, a social choice function determines what alternative to choose for each possible profile of preferences. The Gibbard-Satterthwaite Theorem (see Gibbard (1973) and Satterthwaite (1975)) establishes that all social choice functions whose range contains more than two alternatives are either dictatorial or manipulable. This clear-cut conclusion is obtained at some costs: one of them is the assumption of universal domain, according to which all possible preferences over alternatives are admissible for all agents.

In many cases, the nature of the social decision problem induces a specific structure on the set of alternatives, and this structure suggests, in turn, some restrictions on the set of admissible individual preferences. It is then natural to investigate how changed is the conclusion of the Gibbard-Satterthwaite Theorem when social choice functions are only required to operate on a restricted preference domain.

Different authors have investigated the possibility of designing nondictatorial, strategy-proof social choice functions for specific environments. Some domain restrictions like continuity, are not sufficient to avoid the incompatibility between these two desirable properties (see Barberà and Peleg (1990)). Others allow for more positive results. Moulin (1980), Sprumont (1991), and Alcalde and Barberà (1994) identify situations admitting efficient, nondictatorial and strategy-proof social choice functions. Their results apply, respectively, to the choice of level for one public good, the distribution of a fixed amount of private good, and the stable solution of matching problems, under appropriate domain restrictions. Many other environments allowing for nondictatorial and strategy-proof but not necessarily efficient social choice functions have been studied: see for example, Serizawa (1996) on economies with one public and one private good, Barberà and Jackson (1995) on exchange economies, Gibbard (1977), and Barberà, Bogomolnaia, and van der Stel (1997) on the choice of lotteries as social outcomes.[1]

Another family of interesting environments arises when alternatives can be described as points in the Euclidean space. In this paper we reconsider the possibility of designing strategy-proof social choice functions when the set of feasible alternatives is a full dimensional compact set in $\Re^m$ and preferences satisfy an

[1] We make no attempt to be exhaustive. For accounts of recent research on strategy-proofness in restricted domains, the reader is referred to Sprumont (1995), or Barberà (1997).

appropriate version of single-peakedness.[2] This question was addressed in a pioneering paper by Border and Jordan (1983), following Moulin's (1980) initial analysis of the one-dimensional case. Border and Jordan's results refer to the case where any element of $\Re^m$ can be a possible outcome, and all star-shaped and separable preferences are admissible. Their results are important, as they show the existence of a large class of nondictatorial and strategy-proof social choice functions for meaningful, yet restricted domains of preferences. Specifically, this class of functions, which we call generalized median voter schemes, can be informally described as follows:[3] (1) each agent declares her preferred alternative, each of these ideal points is projected onto each of the coordinate axes, one point is chosen in each coordinate, and then these coordinate values form the social outcome vector; (2) moreover, the choice within each coordinate axis is based on some variant of the median voter rule, which may vary from one dimension to another. The first part of this description reveals that strategy-proof rules in these environments must be decomposable, a fact that extends to more general cases, as shown by Le Breton and Sen (1995). The need to use generalized median voter rules on each dimension is implied by single-peakedness.

The results of Border and Jordan (1983) were marred by the assumption that the range of the functions was the whole Euclidean space or, equivalently, by a unanimity requirement. In the words of Border and Jordan, "the most obvious (open) question (in their paper) is: what happens if the unanimity assumption is dropped? Dropping the unanimity assumption is equivalent to restricting the range of the mechanism, and in economic environments such restrictions arise as feasibility constraints".

Before addressing this open question, let us qualify the statement. Since Border and Jordan (1983) admit preferences with ideals on any point in $\Re^m$, it is certainly the case that the range of any function defined on their admissible profiles and respecting unanimity must coincide with $\Re^m$, and that any function whose range is a proper subset of $\Re^m$ must violate unanimity, as long as all the above preferences are admissible. Remark, however, that some restrictions on the range (and thus, some violations of unanimity), are easy to deal with. Take, in

[2] Our full dimensionality condition excludes cases where the set of alternatives is finite. For analysis of the m-dimensional finite alternatives case see Barberà, Sonneschein, and Zhou (1991), Barberà, Gul, and Stacchetti (1993), and Barberà, Massó, and Neme (1997).

[3] See Border and Jordan (1983), Barberà, Gul, and Stacchetti (1993), Serizawa (1994), and Barberà, Massó, and Neme (1997) for alternative descriptions. Section 2 contains a formal definition.

particular, any cartesian product of one interval in each dimension, and let each agent vote for her preferred element in this set. The same class of procedures which were strategy-proof when any point in $\Re^m$ were admissible will still be strategy-proof when the range is a cartesian product and agents are no longer asked to vote for their preferred alternative, but for their best among those which are feasible (i.e., in the "a priori" fixed range). This first remark shows that introducing restrictions, or equivalently, dropping unanimity, need not always make our analysis of strategy-proof rules any harder. It also leads us to a second remark: when the set of admissible alternatives is not a cartesian product, then new difficulties can arise, even if we maintain the unanimity assumption by reducing our set of admissible preferences to those whose ideals are always feasible.

The purpose of our paper is to carefully analyze situations where the set of feasible alternatives is not necessarily cartesian, to show the nature of the new difficulties that arise in this case, and to characterize the strategy-proof social choice functions which can be defined given a set of feasible alternatives. We provide a full characterization for the case where the domain of admissible preferences is also restricted accordingly, thus allowing the candidate social choice functions to respect unanimity even if their range does not cover all of $\Re^m$. This characterization provides a satisfactory analysis of one of the implicit issues raised by Border and Jordan (1983). Namely, the type of difficulties added to the analysis of strategy-proof rules in the presence of exogenous constraints. Our results show that new interesting issues arise *even* if we reduce the domains of admissible preferences in order to respect unanimity. But we certainly do not give a full answer to Border and Jordan (1983), since we do not address the compounded difficulties that arise when the range is restricted *and* the unanimity requirement is dropped (or, equivalently, when the set of admissible alternatives is reduced but the set of admissible preferences is not).

The starting point of our analysis is thus the distinction between conceivable and feasible alternatives. Some conceivable alternatives may never be chosen because they are unfeasible, and this may change our conclusions regarding the possibility of designing nondictatorial, strategy-proof social choice functions. In order to address this question in a noncircular way,[4] we concentrate on functions

[4]Given a set Z of alternatives, it is always possible to define social choice functions whose range is a subset A of Z, even if all elements of Z were feasible. By identifying the range of our social choice functions with the set of feasible alternatives we avoid discussions over why a feasible alternative might not be chosen even in the case of unanimity, or what is the difference between objective and de facto unfeasibility.

whose range coincides with the set of feasible alternatives, and we simply restrict these ranges to be compact and to satisfy a full dimensionality condition.

Since the starting point of our analysis is a given set of feasible alternatives Z, we concentrate on preferences defined on this set. Yet, our definitions of admissible preferences will still appeal to the underlying distinction between the set of conceivable alternatives (all points in $\Re^m$) and the set $Z \subset \Re^m$ of feasible ones. Specifically, we work on the domain of preferences which are restrictions to the set Z of multidimensional single-peaked preferences on $\Re^m$, with the added requirement that the unconstrained maximal element of these preferences belongs to Z.[5] For similar reasons, we are only interested in social choice functions which assign a feasible outcome to any profile of admissible preferences over feasible alternatives.

Within this setting, we obtain two major results. The first one is that, regardless of the exact shape of the set of feasible alternatives, any strategy-proof social choice function must be a generalized median voter scheme. This part of the result is similar and builds upon the one obtained by Border and Jordan (1983), but it requires additional work and new techniques of proof.

Notice that, when the feasible set is non-cartesian, not all generalized median voter schemes are proper social choice functions, because some of these schemes can recommend the choice of unfeasible alternatives even when all agents vote for feasible alternatives. Our second result (in the line of Barberà, Massó, and Neme (1997) for a finite framework) characterizes the set of all generalized median voter schemes which are indeed social choice functions, for each set of feasible alternatives (we call them feasibility preserving). Border and Jordan (1983) had stated that "it seems unlikely that a transparent characterization can be developed to cover all range restrictions". Our characterization is based on what we call the intersection property. This is a condition which guarantees the needed coordination across decision rules which are used to select the components of the social outcome in different dimensions. Indeed, it is not a simple condition, but it can be sharply stated for any kind of range; its implications, however, must be carefully analyzed for each special case. For some shapes of the range, the intersection property can only be satisfied by mechanisms which give some agents a lot of decision power. For other shapes, however, it leaves room for the use of much nicer voting schemes. At any rate, we can show that dictatorship is hardly a consequence of strategy-proofness. For essentially all the ranges under consid-

[5] This domain is strongly related to the one considered by Border and Jordan (1983). For precise comparisons, see Definitions 1 to 4 and Remark 1 in Section 2.

eration, it is possible to construct social choice functions which are nondictatorial and strategy-proof: the sharp conclusion of the Gibbard- Satterthwaite Theorem is not easily recovered under restricted domains.

The paper is organized as follows. Section 2 contains the notation, definitions and some preliminary results. In Section 3 we characterize strategy-proof social choice functions as generalized median voter schemes. Finally, in Section 4 we show the existence of nondictatorial rules in this class for general ranges.

2. Notation, Definitions, and Preliminary Results

Let $N = \{1, ..., n\}$ be a set of *agents* and $M = \{1, ..., m\}$ be a set of coordinates. We assume that $n, m \geq 2$.[6] Let the set Z of *alternatives* be a compact subset of the $m-$dimensional Euclidean space $\Re^m$ endowed with the L_1-norm. That is, for $x \in \Re^m$

$$\parallel x \parallel = \sum_{k \in M} \mid x_k \mid .$$

We interpret $\Re^m$ as the set of conceivable alternatives and the set Z as the set of *feasible* alternatives. We assume that Z satisfies the following full-dimensionality requirement: $Z = cl\,(int\,(Z))$, where cl and int denotes closure and interior, respectively.[7]

Given $A \subseteq \Re^m$ and $k \in M$, denote the projection of A on the $k-th$ coordinate by $Proj_k(A)$. Given $k \in M$, we simply write $Proj_k(Z) = Z_k$. To stress the role of coordinate k we often write the vector z as (z_k, z_{-k}). Given $x \in \Re^m$ and $x' \in \Re^m$, denote the closed segment connecting x and x' by $[x, x']$. Given $A \subseteq \Re^m$, the minimal box containing A is the smallest cartesian product set $\hat{B}(A)$ containing the set A. That is,

$$\hat{B}\,(A) = \prod_{k \in M} \left[\min Proj_k(A), \max Proj_k(A)\right].$$

Therefore, $\hat{B}\,(Z)$ is the smallest box containing the set of alternatives Z.

Preferences are continuous and complete preorders on alternatives. We shall often abuse language and identify preferences with their continuous numerical representations. We first recall some conditions for preferences defined on $\Re^m$ that

[6] For the case $m = 1$, see Moulin (1980).

[7] This assumption is required by our techniques of proof. We believe that the essence of the results would be kept for lower dimensional ranges, but this would require separate arguments.

will be used as reference points. Let $\widehat{\mathcal{U}}$ be the set of all continuous preferences on $\Re^m$.

Definition 1. *A preference $u^i \in \widehat{\mathcal{U}}$ is multidimensional single-peaked on $\Re^m$ if:*
(1) It has a unique maximal element $\tau(u^i) \in \Re^m$ (the "top" of u^i).
(2) For any $z, z' \in \Re^m$,

$$\left[z' \in \hat{B}\left(\left\{z, \tau\left(u^i\right)\right\}\right) \text{ and } z' \neq z\right] \Rightarrow \left[u^i(z') > u^i(z)\right].$$

This is the adaptation to continuous settings of a condition used in Barberà, Gul, and Stacchetti (1993), Serizawa (1995), and Barberà, Massó, and Neme (1997). Border and Jordan (1983) used the following proper subclass of preferences on $\Re^m$.

Definition 2. *A preference $u^i \in \widehat{\mathcal{U}}$ is star-shaped and separable on $\Re^m$ if:*
(1) It has a unique maximal element $\tau(u^i) \in \Re^m$ (the "top" of u^i).
(2) For any $z \in \Re^m$, $z \neq \tau(u^i)$, and $\lambda \in (0,1)$,

$$u^i\left(\tau\left(u^i\right)\right) > u^i\left(\lambda\tau\left(u^i\right) + (1-\lambda)z\right) > u^i(z).$$

(3) For all $k \in M$, and all z_k, z'_k, $\bar{z}_{-k}$, and $\tilde{z}_{-k}$ we have that

$$\left[u^i(z_k, \bar{z}_{-k}) \geq u^i(z'_k, \bar{z}_{-k})\right] \Leftrightarrow \left[u^i(z_k, \tilde{z}_{-k}) \geq u^i(z'_k, \tilde{z}_{-k})\right].$$

Given the set Z of *feasible* alternatives we denote by $\mathcal{U}_Z$ the set of all continuous preferences on Z.

Definition 3. *A preference $u^i \in \mathcal{U}_Z$ is multidimensional single-peaked if there exists $\hat{u}^i \in \widehat{\mathcal{U}}$ such that:*
(1) $\hat{u}^i$ is multidimensional single-peaked on $\Re^m$ and $\hat{u}^i(z) = u^i(z)$ for all $z \in Z$.
(2) $\tau(\hat{u}^i) \in Z$.

Definition 4. *A preference $u^i \in \mathcal{U}_Z$ is star-shaped and separable if there exists $\hat{u}^i \in \widehat{\mathcal{U}}$ such that:*
(1) $\hat{u}^i$ is star-shaped and separable on $\Re^m$ and $\hat{u}^i(z) = u^i(z)$ for all $z \in Z$.
(2) $\tau(\hat{u}^i) \in Z$.

Let $\mathcal{P} \subset \mathcal{U}_Z$ be the set of all multidimensional single-peaked preferences and let $\mathcal{P}_S^* \subset \mathcal{U}_Z$ be the set of all star-shaped and separable preferences.

Notice that Definitions 3 and 4 include two parts each: (1) requires that a preference satisfying property α on Z should come from restricting a preference satisfying the same condition α on $\Re^m$. Condition (2) is less natural, and requires that the original preference on $\Re^m$ should be saturated at a point in Z. This is a limitation of our analysis, but it allows us to concentrate on the consequences of defining social choice functions on any kind of range, while retaining the unanimity assumption, as already discussed in the Introduction.

Remark 1. $\mathcal{P}_S^* \subsetneq \mathcal{P}$.[8]

This remark is relevant for comparison with Border and Jordan (1983), since we work with multidimensional single-peaked preferences, while they consider star-shaped and separable preferences.

When we want to emphasize the role of coalition $S \subset N$ we write $\left(u^S, u^{-S}\right)$ to represent the utility profile $u = (u^1, ..., u^n) \in \mathcal{U}_Z^n$, where $u^S = (u^i)_{i \in S} \in \mathcal{U}_Z^s$ and $u^{-S} = (u^i)_{i \in N \backslash S} \in \mathcal{U}_Z^{n-s}$.[9] Given $u^i \in \mathcal{U}_Z$ and $z \in Z$, define the *upper contour set* $UC\,(u^i, z)$ by

$$UC\left(u^i, z\right) = \left\{z' \in Z \mid u^i\left(z'\right) \geq u^i\left(z\right)\right\},$$

the *strict upper contour set* $SUC\,(u^i, z)$ by

$$SUC\left(u^i, z\right) = \left\{z' \in Z \mid u^i\left(z'\right) > u^i\left(z\right)\right\},$$

and the *lower contour set* $LC\,(u^i, z)$ by

$$LC\left(u^i, z\right) = \left\{z' \in Z \mid u^i\left(z'\right) \leq u^i\left(z\right)\right\}.$$

Given the set Z, let $\mathcal{V} \subset \mathcal{U}_Z$ denote any arbitrary subset of preferences having a unique maximal element on Z. Obviously, $\mathcal{P}$ or any subset of $\mathcal{P}$, are examples of such subsets.

A *social choice function* F on $\mathcal{V} \subseteq \mathcal{U}_Z$ is a function from $\mathcal{V}^n$ to Z. Since our primitives are preorders we only restrict attention to social choice functions which are invariant to the choice of their utility representation.

Definition 5. *A social choice function $F : \mathcal{V}^n \to Z$ respects unanimity if for any $u \in \mathcal{V}^n$ and for any $z \in Z$,*

$$\left[\forall i \in N, \tau\left(u^i\right) = z\right] \Rightarrow \left[F(u) = z\right].$$

[8] Notice that this inclusion holds when preferences are defined on the full $\Re^m$ and ideals can be any point in $\Re^m$, and thus also when preferences are defined on a set Z and their ideals are restricted to belong to this set Z.

[9] The notation $n - s$ stands for the cardinality of the set $N \backslash S$. In general, we denote sets by capital letters and their cardinality by the corresponding small letters.

Throughout the paper we assume that F respects unanimity and therefore it is onto Z.

A social choice function is strategy-proof if it is always in the interest of agents to report their preferences truthfully. Formally,

Definition 6. *A social choice function $F : \mathcal{V}^n \to Z$ is manipulable on $\mathcal{V}^n$ if there exists $u = (u^1, ..., u^n) \in \mathcal{V}^n$, $i \in N$ and $\hat{u}^i \in \mathcal{V}$ such that $u^i \left(F \left(\hat{u}^i, u^{-i} \right) \right) > u^i \left(F \left(u \right) \right)$. A social choice function $F : \mathcal{V}^n \to Z$ is strategy-proof on $\mathcal{V}^n$ if it is not manipulable on $\mathcal{V}^n$.*

Definition 7. *A social choice function $F : \mathcal{V}^n \to Z$ is tops-only if for any $u \in \mathcal{V}^n$ and for any $\hat{u} \in \mathcal{V}^n$,*

$$\left[\forall i \in N, \tau \left(u^i \right) = \tau \left(\hat{u}^i \right) \right] \Rightarrow \left[F \left(u \right) = F \left(\hat{u} \right) \right].$$

In what follows, we define generalized median voter schemes. These are based on a natural extension of the basic idea of the median voter rule, and their definition uses the auxiliary concept of right (or left) coalition systems.

Definition 8. *A right (left)-coalition system on $Z_k \equiv [a_k, b_k]$ is a correspondence $\mathcal{W}_k$ that assigns to every $z_k \in Z_k$ a collection $\mathcal{W}_k \left(z_k \right)$ of coalitions satisfying the following conditions:*
(1) *Voter sovereignty: For all $z_k \in (a_k, b_k]$ ($[a_k, b_k)$), $\mathcal{W}_k(z_k) \neq \emptyset$, $\emptyset \notin \mathcal{W}_k(z_k)$, and $\mathcal{W}_k \left(a_k \right) = 2^N \backslash \emptyset$ ($\mathcal{W}_k \left(b_k \right) = 2^N \backslash \emptyset$).*
(2) *Coalition monotonicity: If $W \in \mathcal{W}_k(z_k)$ and $W \subset W'$, then $W' \in \mathcal{W}_k(z_k)$.*
(3) *Outcome monotonicity: If $z'_k < (>) z_k$ and $W \in \mathcal{W}_k(z_k)$, then $W \in \mathcal{W}_k(z'_k)$.*
(4) *Upper semicontinuity: For any $W \subseteq N$, any $z_k \in Z_k$ and any sequence $\{z_k^t\} \subseteq Z_k$ such that $\lim_{t \to \infty} z_k^t = z_k$,*

$$\left[\forall t, W \in \mathcal{W}_k(z_k^t) \right] \Rightarrow \left[W \in \mathcal{W}_k(z_k) \right].$$

A family $\mathcal{R}$ of right-coalition systems on $\hat{B}\left(Z \right) = \prod_{k=1}^{m} Z_k$ is a collection $\{\mathcal{R}_k\}_{k=1}^m$ where each $\mathcal{R}_k$ is a right-coalition system on Z_k. Similarly, *a family $\mathcal{L}$ of left-coalition systems on $\hat{B}\left(Z \right) = \prod_{k=1}^{m} Z_k$* is a collection $\{\mathcal{L}_k\}_{k=1}^m$ where each $\mathcal{L}_k$ is a left-coalition system on Z_k. Now given a family of right coalition systems, let us describe how to construct its associated generalized median voter scheme (a parallel description holds for left coalition systems). Take the family $\mathcal{R}$ and a profile u. For each dimension k, let $(\tau_k(u^1), ..., \tau_k(u^n))$ be the vector of tops projected into dimension k. Now, choose the k-th component $F_k(u)$ of the social outcome to be the largest value such that the set of agents voting for values above it belong to the right coalition at $F_k(u)$. In this way, given a preference profile $u = (u^1, ..., u^n)$ and a right-coalition system $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^m$, one selects the social outcome $F(u) = (F_1(u), ..., F_m(u))$. Formally,

Definition 9. *Let Z be the set of alternatives and $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^m$ ($\mathcal{L} = \{\mathcal{L}_k\}_{k=1}^m$) a family of right (left)-coalition systems on $\hat{B}(Z)$. The generalized median voter scheme induced by $(Z, \mathcal{R})$ is the function $F : \mathcal{V}^n \to \hat{B}(Z)$ defined as follows: for every $u \in \mathcal{V}^n$ and every $k \in M$*

$$F_k(u) = \max\left\{z_k \in Z_k \mid \left\{i \in N \mid \tau_k\left(u^i\right) \geq z_k\right\} \in \mathcal{R}_k(z_k)\right\}$$
$$(F_k(u) = \min\left\{z_k \in Z_k \mid \left\{i \in N \mid \tau_k\left(u^i\right) \leq z_k\right\} \in \mathcal{L}_k(z_k)\right\}).$$

Remark 2. By (4) in the definition of right-coalition systems, the outcome $F_k(u)$ is well determined since $\max\{z_k \in Z_k \mid \{i \in N \mid \tau_k(u^i) \geq z_k\} \in \mathcal{R}_k(z_k)\}$ exists. Notice that by definition generalized median voter schemes are tops-only. When a coalition $W \subseteq N$ is in $\mathcal{R}_k(z_k)$ ($\mathcal{L}_k(z_k)$) for $k \in M$ and $z_k \in Z_k$, the coalition is said to be *right (left) winning* for z_k.

Notice that, in general, the image of a generalized median voter scheme belongs to $\hat{B}(Z)$ but not necessarily to Z (see Example 1 below). Thus, generalized median voter schemes are not always social choice functions. An important particular case arises when Z is a cartesian product. Then, $\hat{B}(Z) = Z$, and any generalized median voter scheme is a social choice function. Border and Jordan (1983) analyzed the particular case within this class, when Z equals $\Re^m$. What is important for their result is the fact that Z is a cartesian product, and we formulate their result in this slightly generalized form.

Theorem 2.1. *(Border and Jordan): Let the set Z of alternatives be box-shaped. A social choice function is strategy-proof on $(\mathcal{P}_S^*)^n$, the set of star-shaped and separable preferences, if and only if it is a generalized median voter scheme.*

Since we choose to work with multidimensional single-peaked preferences, rather than star-shaped and separable, it is worth checking that a result parallel to Border and Jordan's also holds for our domain. This is the contents of Theorem 2.2 below. Notice that, even if $\mathcal{P}_S^* \subsetneq \mathcal{P}$, none of the two results is implied by the other.

Theorem 2.2. *Let the set Z of alternatives be box-shaped. A social choice function is strategy-proof on $\mathcal{P}^n$, the set of multidimensional single-peaked preferences, if and only if it is a generalized median voter scheme.*

Proof. To show that a generalized median voter scheme is strategy-proof on $\mathcal{P}^n$ is straightforward, and therefore it is omitted. Let a social choice function F be strategy-proof on $\mathcal{P}^n$. We will establish that F is a generalized median voter scheme. Let $\hat{F}$ be the restriction of F to $(\mathcal{P}_S^*)^n$. Since $\hat{F}$ is strategy-proof on $(\mathcal{P}_S^*)^n$, it follows from Theorem 2.1 that $\hat{F}$ is a generalized median voter scheme. Now, we have only to show that for any $u \in \mathcal{P}^n$ and $\hat{u} \in (\mathcal{P}_S^*)^n$, if $\tau(u^i) = \tau(\hat{u}^i)$ for each $i \in N$, $F(u) = F(\hat{u})$.

Let $u \in \mathcal{P}^n$ and $\hat{u} \in (\mathcal{P}_S^*)^n$ be such that for each $i \in N$, $\tau(u^i) = \tau(\hat{u}^i)$. To get a contradiction, suppose that $F(u^1, \hat{u}^{-1}) \neq F(\hat{u})$. If $F(\hat{u}) \notin \hat{B}(\{\tau(\hat{u}^1), F(u^1, \hat{u}^{-1})\})$, there is $\tilde{u}^1 \in \mathcal{P}_S^*$ such that $\tau(\tilde{u}^1) = \tau(\hat{u}^1)$ and $\tilde{u}^1(F(u^1, \hat{u}^{-1})) > \tilde{u}^1(F(\hat{u}))$. Since $F(\tilde{u}^1, \hat{u}^{-1}) = F(\hat{u})$, this contradicts strategy-proofness for agent 1. Thus, $F(\hat{u}) \in \hat{B}(\{\tau(\hat{u}^1), F(u^1, \hat{u}^{-1})\})$. Since $u^1 \in \mathcal{P}$, $u^1(F(\hat{u})) > u^1(F(u^1, \hat{u}^{-1}))$, contradicting strategy-proofness. Therefore, $F(u^1, \hat{u}^{-1}) = F(\hat{u})$. By repeating the same argument for $i = 2, ..., n$, we have that $F(\hat{u}) = F(u)$. ■

Let us now return to our main concern: the extension of the above results to non-cartesian ranges. As already noted, when $Z \subsetneq \hat{B}(Z)$ nothing guarantees that the vector $(F_1(u), ..., F_m(u))$ selected throughout an arbitrary family of right-coalition systems $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^m$ will be an element of Z.[10] The following example shows that not any generalized median voter scheme, as defined previously, will preserve feasibility if Z is not box-shaped.

Example 1. Consider the case where the set of coordinates is $M = \{1, 2\}$, the set of alternatives is $Z = \left\{z \in \Re_+^2 \mid z_1 + z_2 \leq 1\right\}$, and the set of agents is $N = \{1, 2, 3\}$. Notice that $Z \subsetneq \hat{B}(Z) = \left\{z \in \Re_+^2 \mid z_1 \leq 1, z_2 \leq 1\right\}$. Let $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2\}$ be the family of right-coalition systems on $\hat{B}(Z)$ where $[W \in \mathcal{R}_1(z_1)] \Leftrightarrow [\#W \geq 2]$ for all $z_1 \in (0, 1]$, $[W \in \mathcal{R}_2(z_2)] \Leftrightarrow [\#W \geq 1]$ for all $z_2 \in (0, 1]$, and $\mathcal{R}_1(0) = \mathcal{R}_2(0) = 2^N \backslash \emptyset$.[11] Consider now any profile $u = (u_1, u_2, u_3) \in \mathcal{P}^3$ such that $\tau(u^1) = (5/8, 2/8)$, $\tau(u^2) = (4/8, 2/8)$, and $\tau(u^3) = (2/8, 5/8)$. Obviously,

[10] See Barberà, Massó, and Neme (1997) for a detailed discussion of the feasibility problem in the context of finite sets of alternatives.

[11] This is a particular case, called voting by quota, where winning coalitions are constant and defined just by its cardinality. In this example we would say that the generalized median voter scheme F defined by those right-coalition systems is an scheme of voting by quota 2 in the first coordinate and voting by quota 1 in the second one.

$\tau(u^i) \in Z$ for all $i = 1, 2, 3$. Now, since $(\tau_1(u^1), \tau_1(u^2), \tau_1(u^3)) = (5/8, 4/8, 2/8)$ and $(\tau_2(u^1), \tau_2(u^2), \tau_2(u^3)) = (2/8, 2/8, 5/8)$ we have that $F(u) = (4/8, 5/8) \notin Z$.

Therefore, we need some additional property to guarantee that a generalized median voter scheme always selects vectors in Z. In order to state this property, it is useful to understand the relationship between right and left coalition systems, $\mathcal{R}_k$ and $\mathcal{L}_k$, that select the same outcome for all $(\tau_k(u^1), ..., \tau_k(u^n))$.

Given $\mathcal{R}_k$, define $\mathcal{L}_k^*$ as follows:

$$\mathcal{L}_k^*(z_k) = \left\{W \in 2^N \mid \forall z_k' > z_k, \forall W' \in \mathcal{R}_k(z_k'), W \cap W' \neq \emptyset\right\}.$$

Is is easy to see that $\mathcal{R}_k$ and $\mathcal{L}_k$ will select the same outcome for all $(\tau_k(u^i))_{i \in N}$ if and only if $\mathcal{L}_k = \mathcal{L}_k^*$.

For any pair of vectors $y, z \in \hat{B}(Z)$, let $M^+(y, z) = \{k \in M \mid z_k > y_k\}$ and $M^-(y, z) = \{k \in M \mid z_k < y_k\}$ be the set of dimensions in which the components of z are strictly greater or smaller than those of y, respectively.

Definition 10. *A family $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^m$ of right-coalition systems on $\hat{B}(Z)$ has the intersection property for Z if for any $y \in \hat{B}(Z) \backslash Z$, and any finite subset $\{z^1, ..., z^T\} \subset Z$*

$$\bigcap_{t=1}^{T} \left\{ [\bigcup_{k \in M^+(y, z^t)} l_k(y_k)] \bigcup [\bigcup_{k \in M^-(y, z^t)} r_k(y_k)] \right\} \neq \emptyset$$

for every $r_k(y_k) \in \mathcal{R}_k(y_k)$ with $k \in \bigcup_{t=1}^{T} M^-(y, z^t)$ and every $l_k(y_k) \in \mathcal{L}_k^*(y_k)$ with $k \in \bigcup_{t=1}^{T} M^+(y, z^t)$.

A generalized median voter scheme induced by $(Z, \mathcal{R})$ satisfies the *intersection property* if and only if $\mathcal{R}$ has the intersection property for Z.

The following Proposition gives a necessary and sufficient condition for a generalized median voter scheme induced by $(Z, \mathcal{R})$ to be a social choice function. We state it without proof since the proof of Theorem 1 in Barberà, Massó, and Neme (1997) for the finite case can be straigthtforwardly adapted to our setting.

Proposition 1. *A generalized median voter scheme induced by $(Z, \mathcal{R})$ is a social choice function if and only if it satisfies the intersection property.*

To illustrate the intersection property, consider again Example 1. We will check that the generalized median voter scheme induced by $(Z, \mathcal{R})$ defined there does not satisfy the intersection property. Consider the vectors $y = (3/4, 3/4) \in \hat{B}(Z) \backslash Z$, $z^1 = (3/4, 1/4) \in Z$, and $z^2 = (1/8, 3/4) \in Z$. In this case $M^+(z^1, y) = \{k \in M \mid z_k^1 > y_k\} = \emptyset$, $M^-(z^1, y) = \{k \in M \mid z_k^1 < y_k\} = \{2\}$, $M^+(z^2, y) = \{k \in M \mid z_k^2 > y_k\} = \emptyset$, and $M^-(z^2, y) = \{k \in M \mid z_k^2 < y_k\} = \{1\}$. Then, $M^+(z^1, y) \cup M^+(z^2, y) = \emptyset$ and $M^-(z^1, y) \cup M^-(z^2, y) = \{1, 2\}$. But $\{1, 2\} \in \mathcal{R}_1(y_1)$, $\{3\} \in \mathcal{R}_2(y_2)$, and $\{1, 2\} \cap \{3\} = \emptyset$, which is a violation of the intersection property. However, as we will see in Example 2 of Section 4, if we replace $\mathcal{R}_2$, the right coalition system of the second dimension, by $\mathcal{R}_2'$ where $[W \in \mathcal{R}_2'(z_2)] \Leftrightarrow [\#W \geq 2]$ for all $z_2 \in (0, 1]$, and $\mathcal{R}_2'(0) = 2^N \backslash \emptyset$, then $\mathcal{R}' = \{\mathcal{R}_1, \mathcal{R}_2'\}$ has the intersection property for Z.

3. The Characterization Result

We can now state and prove the main result of the paper.

Theorem 3.1. *A social choice function is strategy-proof on $\mathcal{P}^n$ if and only if it is a generalized median voter scheme satisfying the intersection property.*

The proof of Theorem 3.1 uses two interesting facts which are presented as Lemmata and the following concept of option sets.

Given a social choice function $F : \mathcal{V}^n \to Z$, a coalition $N' \subsetneq N$ and $u^{N'} = (u^i)_{i \in N'} \in \mathcal{V}^{n'}$, define the *option set left by* $u^{N'}$ as

$$\sigma^F(u^{N'}) = \left\{ z \in Z \mid \text{there exists } u^{-N'} \in \mathcal{V}^{n-n'} \text{ such that } F\left(u^{N'}, u^{-N'}\right) = z \right\}.$$

Lemma 1 below says that if F is strategy-proof on an arbitrary domain $\mathcal{V}^n \subseteq \mathcal{U}_Z^n$,[12] z is among the options left by $u^{N'}$, and z is maximal for all agents in $N \backslash N'$, then z must be chosen by F. Notice that it implies respect for unanimity when $N' = \emptyset$. We omit the proof, which is straightforward.

Lemma 1. *Let $F : \mathcal{V}^n \to Z$ be a strategy-proof social choice function. Let $N' \subseteq N$, $u^{N'} \in \mathcal{V}^{n'}$, $z \in \sigma^F\left(u^{N'}\right)$ and $u^{-N'} \in \mathcal{V}^{n-n'}$ be such that for each $j \in N \backslash N'$, $\tau(u^j) = z$. Then $F(u) = z$.*

[12] Notice that single-peakedness does not play any role here.

Lemma 2 states that if F is strategy-proof on the domain of multidimensional single-peaked preferences $\mathcal{P}^n$ and for some profile the maximal alternatives of all agents lie in a box within the range, then the outcome must belong to that box. We will say then that F satisfies the *Weak Minimal Box Property*.

Lemma 2. *(Weak Minimal Box Property) Let $F : \mathcal{P}^n \to Z$ be a strategy-proof social choice function, $u \in \mathcal{P}^n$, and $\hat{B}\left(\{\tau\left(u^i\right) \mid i \in N\}\right) \subseteq Z$. Then $F\left(u\right) \in \hat{B}\left(\{\tau\left(u^i\right) \mid i \in N\}\right)$.*

Proof. Let $u \in \mathcal{P}^n$, $z = F\left(u\right)$ and $k \in M$. We have to show that $\min_{i \in N}\{\tau_k\left(u^i\right)\} \leq z_k \leq \max_{i \in N}\{\tau_k\left(u^i\right)\}$. By contradiction, suppose not. Without loss of generality, we may assume that $k = 1$ and $z_1 < \min_{i \in N}\{\tau_1\left(u^i\right)\} = \tau_1\left(u^1\right)$. Let $z^* \in \Re^m$ be such that $z^*_1 = \min_{i \in N}\{\tau_1\left(u^i\right)\}$; and for each $k' \in M$, $z^*_{k'} = \min_{i \in N}\{\tau_{k'}\left(u^i\right)\}$ if $z_{k'} < \min_{i \in N}\{\tau_{k'}\left(u^i\right)\}$, $z^*_{k'} = z_{k'}$ if $\min_{i \in N}\{\tau_{k'}\left(u^i\right)\} \leq z_{k'} \leq \max_{i \in N}\{\tau_{k'}\left(u^i\right)\}$, $z^*_{k'} = \max_{i \in N}\{\tau_{k'}\left(u^i\right)\}$ if $z_{k'} > \max_{i \in N}\{\tau_{k'}\left(u^i\right)\}$. Since $z^* \in \hat{B}\left(\{\tau\left(u^i\right) \mid i \in N\}\right)$, it follows from $\hat{B}\left(\{\tau\left(u^i\right) \mid i \in N\}\right) \subseteq Z$ that $z^* \in Z$. Note that for any $i \in N$, $\| z - \tau\left(u^i\right) \| = \| z - z^* \| + \| \tau\left(u^i\right) - z^* \|$. Thus for any $i \in N$, there is $\hat{u}^i \in \mathcal{P}$ such that $\tau\left(\hat{u}^i\right) = z^*$ and $LC\left(u^i, z\right) \cap UC\left(\hat{u}^i, z\right) = \{z\}$. Then the fact that agent 1 can not manipulate F and $F\left(u\right) = z$ together imply $F\left(\hat{u}^1, u^{-1}\right) = z$. Repeating this argument for $i = 2, ..., n$, we get $F\left(\hat{u}\right) = z$. But since $z^* \in Z$, this contradicts respect for unanimity, and by Lemma 1, strategy-proofness of F. ■

Proof of Theorem 3.1. Since the “if” part is straightforward, we will show the converse. Let F be a strategy-proof social choice function. To establish that $F : \mathcal{P}^n \to Z$ is a generalized median voter scheme on Z we proceed as follows. For each z in the interior of Z, we select a box containing z and strictly contained in Z (here, we use the full dimensionality of Z). Lemma 2 allows us to apply Theorem 2.2 ”locally” to each one of these boxes, whose union constitutes the interior of Z. From this we get a collection of generalized median voter schemes, one for each box, and we use an overlapping argument to show that, in fact, they are all restrictions of a common generalized median voter scheme defined on the interior of the minimal box containing Z. Finally, we use continuity to show that this generalized median voter scheme still applies when we consider the boundaries of Z, as well as the interior. Notice that although Theorem 3.1 extends Border and Jordan’s result, our proof uses their result initially, when we apply it locally to the boxes associated to each z. Therefore, our proof builds upon theirs, and is not an alternative to it.

To proceed formally with the proof, let I be the interior of Z and $\mathcal{P}^I = \{u^i \in \mathcal{P} \mid \tau(u^i) \in I\}$, and let F^I be the restriction of F to $\left(\mathcal{P}^I\right)^n$.

Claim: F^I is a generalized median voter scheme on I.

Proof of the Claim: Let B be a non-degenerate box contained in Z and $\mathcal{P}_B = \{u^i \in \mathcal{P} \mid \tau(u^i) \in B\}$. Then by the weak minimal box property (Lemma 2), for any $u \in \mathcal{P}_B^n$, $F(u) \in B$; and by respect for unanimity (Lemma 1), for any $z \in B$, there is $u \in \mathcal{P}_B^n$ such that $F(u) = z$. Thus the restriction of F to $\mathcal{P}_B^n$ is considered to be a social choice function with a box-shaped set of alternatives. Then, Theorem 2.2 is applied, which implies that the restriction of F to $\mathcal{P}_B^n$ is a generalized median voter scheme on B, so that there is a family of right-coalition systems associated with it, call it $\mathcal{R}^B$. For any non-degenerate box in Z, the associated family of right-coalition systems is derived similarly. For any $z \in I$, there is a non-degenerate box $B \subseteq Z$ such that z is in the interior of B, thus for each $k \in M$, a set of winning coalitions $\mathcal{R}_k^B(z_k)$ is derived. We establish (3) (outcome monotonicity) between the associated lists of winning coalitions.

Let $k \in M$, $z \in I$ and $z' \in I$ be such that $z_k \geq z'_k$. Let B and B' be non-degenerated boxes in Z such that z and z' are in the interiors of B and B' respectively. Let $\mathcal{R}_k^B(z_k)$ and $\mathcal{R}_k^{B'}(z'_k)$ be the lists of winning coalitions associated with B and B' respectively. Let $W \in \mathcal{R}_k^B(z_k)$. We want to show $W \in \mathcal{R}_k^{B'}(z'_k)$.

Let $x \in B$, $y \in B$, $x' \in B'$ and $y' \in B'$ be such that $y_k < z_k < x_k$, $y'_k < z'_k < x'_k$, $y_{-k} = x_{-k}$, and $y'_{-k} = x'_{-k}$. Let $u \in \mathcal{P}_B^n$ be such that $\tau(u^i) = x$ for any $i \in W$ and $\tau(u^i) = y$ for any $i \notin W$. Let $u' \in \mathcal{P}_{B'}^n$ be such that for any i and $j \in W$, $u'^i = u'^j$, $\tau(u'^i) = x'$, and $u'^i(z'') > u^i(z''')$ for any $z'' \in Z$ and $z''' \in Z$ with $z'''_k < z'_k$ and $x'_k \leq z''_k$, and $\tau(u'^i) = y'$ for any $i \notin W$. Since $W \in \mathcal{R}_k^B(z_k)$, $F_k(u) \geq z_k$. Without loss of generality, let $N \backslash W = \{1, ..., n-w\}$ and $W = \{n-w+1, ..., n\}$, where $w = \#W$.

We will show that $F_k(u'^1, u^{-1}) \geq z_k$. By contradiction, suppose not. Then there is $\hat{u}^1 \in \mathcal{P}_B$ such that $\tau(\hat{u}^1) = y$ and $\hat{u}^1(F(u'^1, u^{-1})) > \hat{u}^1(z'')$ for $z'' \in Z$ with $z''_k \geq z_k$. Since generalized median voter scheme are tops-only, $F(\hat{u}^1, u^{-1}) = F(u)$. Thus $\hat{u}^1(F(u'^1, u^{-1})) > \hat{u}^1(F(\hat{u}^1, u^{-1}))$, contradicting strategy-proofness. Therefore $F_k(u'^1, u^{-1}) \geq z_k$. Next we will show that if $2 \notin W$, $F_k\left(u'^1, u'^2, u^{-\{1,2\}}\right) \geq z_k$. Let $2 \notin W$. Suppose not. Then there is $\hat{u}^2 \in \mathcal{P}_B$ such that $\tau(\hat{u}^2) = y$ and $\hat{u}^2\left(F\left(u'^1, u'^2, u^{-\{1,2\}}\right)\right) > \hat{u}^2(z'')$ for $z'' \in Z$ with $z''_k \geq z_k$. Since generalized median voter schemes are tops-only, $F(\hat{u}^2, u^{-2}) = F(u)$. Thus we can show that $F_k\left(u'^1, \hat{u}^2, u^{-\{1,2\}}\right) \geq z_k$ in the same way we showed that $F_k(u'^1, u^{-1}) \geq z_k$.

Thus $\hat{u}^2\left(F\left(u'^1,u'^2,u^{-\{1,2\}}\right)\right) > \hat{u}^2\left(F\left(u'^1,\hat{u}^2,u^{-\{1,2\}}\right)\right)$, contradicting strategy-proofness. Therefore $F_k\left(u'^1,u'^2,u^{-\{1,2\}}\right) \geq z_k$. Repeating this argument for $i=3,...,n-w$, we can show that $F_k\left(u'^{N\backslash W},u^W\right) \geq z_k$.

Since for any $i \in W$, $u'^i(z'') > u^i(z''')$ for any $z'' \in Z$ and $z''' \in Z$ with $z'''_k < z'_k$ and $x'_k \leq z''_k$, if $F_k\left(u'^{N\backslash W},u'^{n-w+1},u^{W\backslash\{n-w+1\}}\right) < z'_k$, then it follows that $u'^{n-w+1}\left(F\left(u'^{N\backslash W},u'^{n-w+1},u^{W\backslash\{n-w+1\}}\right)\right) < u'^{n-w+1}\left(F\left(u'^{N\backslash W},u^W\right)\right)$. This contradicts strategy-proofness. Therefore $F_k\left(u'^{N\backslash W},u'^{n-w+1},u^{W\backslash\{n-w+1\}}\right) \geq z'_k$. By the same condition as above, $F_k\left(u'^{N\backslash W},u'^{n-w+1},u'^{n-w+2},u^{W\backslash\{n-w+1,n-w+2\}}\right) \geq z'_k$ because $u'^{n-w+2}=u'^{n-w+1}$. Repeating this argument for $i=n-w+3,...,n$, we can show that $F_k(u') \geq z'_k$. Accordingly $W \in \mathcal{R}_k^{B'}(z'_k)$. It is easy to check that the other conditions required for a coalition system hold. Hence F^I is a generalized median voter scheme on I, which proves the Claim.

To finish the proof of the Theorem, let G be a generalized median voter scheme on Z such that for any $u \in \left(\mathcal{P}^I\right)^n$, $G(u)=F^I(u)$. Since the family of right coalition systems are upper semicontinuous and satisfy outcome monotonicity, G is unique. We need to establish that $F=G$. Let $u \in \mathcal{P}^n$. We want to show that $F(u)=G(u)$. Suppose not. We derive a contradiction by induction. When $\tau(u^i) \in I$ for all $i \in N$, $F(u)=G(u)$. As induction hypothesis, assume that when the number of agents whose top elements are in $Z\backslash I$ is less than $n'(\leq n)$, $F(u)=G(u)$. Let the number of agents whose top elements are in $Z\backslash I$ be equal to n'. Without loss of generality, we may let that $\tau(u^1) \in Z\backslash I$. Since Z is compact and has full dimension, there is a sequence $\{\hat{u}_t^1\}_{t=1}^{\infty}$ such that $\hat{u}_t^1 \in \mathcal{P}^I$ for all $t \geq 1$ and $\tau(\hat{u}_t^1)$ goes to $\tau(u^1)$ as t goes infinity. It follows from the induction hypothesis that $F(\hat{u}_t^1,u^{-1})=G(\hat{u}_t^1,u^{-1})$ for all t. Since G is continuous, $\lim_{t\to\infty}F(\hat{u}_t^1,u^{-1})=G(u)$. Note that $G(u) \in \hat{B}(\{\tau(u^1),F(u)\})$ or $G(u) \notin \hat{B}(\{\tau(u^1),F(u)\})$. First consider the case that $G(u) \in \hat{B}(\{\tau(u^1),F(u)\})$. Since $F(u) \neq G(u)$ implies $u^1(G(u)) > u^1(F(u))$, it follows from $\lim_{t\to\infty}F(\hat{u}_t^1,u^{-1})=G(u)$ that $u^1(F(\hat{u}_t^1,u^{-1})) > u^1(F(u))$ for t sufficiently large. This is a contradiction to strategy-proofness. Next consider the case where $G(u) \notin \hat{B}(\{\tau(u^1),F(u)\})$. Since $\lim_{t\to\infty}F(\hat{u}_t^1,u^{-1})=G(u)$, $F(\hat{u}_t^1,u^{-1}) \notin \hat{B}(\{\tau(u^1),F(u)\})$ for t sufficiently large. Then there is $\tilde{u}^1 \in \mathcal{P}^n$ such that $\tau(\tilde{u}^1)=\tau(\hat{u}_t^1)$ and $\tilde{u}_t^1(F(u)) > \tilde{u}_t^1(F(\hat{u}_t^1,u^{-1}))$. Since G is tops-only, $F(\tilde{u}_t^1,u^{-1})=F(\hat{u}_t^1,u^{-1})$. Thus $\tilde{u}_t^1(F(u)) > \tilde{u}_t^1(F(\tilde{u}_t^1,u^{-1}))$. This is a contradiction to strategy-proofness. ■

4. The Existence of Non Dictatorial Social Choice Functions for General Ranges

The shape of the set of alternatives Z will determine the subclass of generalized median voter schemes which can actually be a social choice functions onto this set. Specifically, this will only hold for schemes which satisfy the intersection property, a condition whose bite depends on the shape of Z. In this section we present a set of examples to illustrate the fact that, although restrictive, the intersection property may allow for some reasonable schemes. Moreover, we show by example that it will *not* in general precipitate the existence of full dictators.

We start by showing that, when Z is a "triangular" set, then there exist anonymous voting schemes which are social choice functions onto Z. We are intentionally avoiding the "budget set" vocabulary because it may suggest a setting where preferences over the underlying universal set of alternatives are among other things monotonic. Remember that here we are always dealing with preferences saturated on the set Z

Example 2. Consider the family of problems where the set of feasible alternatives Z can be described as a "triangular" set; that is, given an strictly positive vector $x = (x_1, ..., x_m) \in \Re^m_{++}$, and an strictly positive number $X > 0$, the set of feasible alternatives is defined by

$$Z = \left\{ z \in \Re^m_+ \mid \sum_{k=1}^{m} x_k z_k \leq X \right\}.$$

Notice again that $Z \subsetneq \hat{B}(Z) = \left\{ z \in \Re^m_+ \mid z_k \leq X/x_k \text{ for all } k = 1, ..., m \right\}$.[13] The scheme of voting by quota Q, where $1 \leq Q \leq n$, can be defined by the family of right-coalition systems $\mathcal{R} = \{\mathcal{R}_k\}_{k=1}^m$ on $\hat{B}(Z)$, where for all $k = 1, ..., m$,

$$\begin{array}{rcl} [W \in \mathcal{R}_k(z_k)] & \Leftrightarrow & [\#W \geq Q] \text{ for } z_k \in (0, X/x_k], \text{ and} \\ \mathcal{R}_k(0) & = & 2^N \backslash \emptyset. \end{array}$$

Now, to determine the values of Q for which $\mathcal{R}$ has the intersection property for Z, notice that it is sufficient to look at the property only for any $y \in \hat{B}(Z) \backslash Z$ such that $y \gg 0$ and for the subset of feasible alternatives $\{z^1, ..., z^m\} \subset Z$, where for all $t = 1, ..., m$, the vector z^t is defined as follows: $z^t_k = y_k$ if $t = k$, and

[13] The case considered in Example 1 is a "triangular" set for $m = 2$ and $x_1 = x_2 = X = 1$.

$z_k^t = 0$ if $t \neq k$.[14] In this case, and since $y_k > 0$ for all $k = 1,..,m$, we have that $M^+(y, z^t) = \emptyset$ and $M^-(y, z^t) = M \setminus \{t\}$ for all $t = 1, ..., m$. Therefore, the intersection property says that

$$\bigcap_{t=1}^{m} \left\{ \bigcup_{k \in M \setminus \{t\}} r_k(y_k) \right\} \neq \emptyset$$

for every $r_k(y_k) \in \mathcal{R}_k(y_k)$, which implies, by an induction argument on m, that $mQ > (m-1)n$. For instance, if $m = 2$ and $n = 3$, as in Example 1, $Q = 2$ and $Q = 3$ are the two quotas satisfying the intersection property for Z. In the other hand, for large m and n smaller than m, the unique quota satisfying the intersection property for Z is $Q = n$; that is, unanimity on the right (or vetoer on the left).

The preceding example is quite positive, since it gives equal power to all agents and it is thus very far from dictatorship. Yet, it may require almost unanimity to change decisions at all levels and for all dimensions. Our next example will exhibit another family of rules, still defined on "triangular" sets. Here, we have freedom to choose any structure for the coalition systems, on some cartesian subset $\bar{Z}$ of the range Z, while requiring unanimous agreement to make decisions which would lead to outcomes outside $\bar{Z}$.

Example 3. Suppose now that $m = 2$ and the set of feasible alternatives is $Z = \left\{(z_1, z_2) \in \Re_+^2 \mid x_1 z_1 + x_2 z_2 \leq X\right\}$ given $x_1, x_2, X > 0$. Now, choose any $(\bar{z}_1, \bar{z}_2) \in Z$, and consider any family of right-coalition systems $\{\mathcal{R}_1, \mathcal{R}_2\}$ on $[0, X/x_1] \times [0, X/x_2]$ such that $\mathcal{R}_1(z_1) = \{N\}$ for $z_1 \in (\bar{z}_1, X/x_1]$ and $\mathcal{R}_2(z_2) = \{N\}$ for $z_2 \in (\bar{z}_2, X/x_2]$. Then, the generalized median voter scheme induced by $(Z, \mathcal{R})$ is a social choice function F onto Z. This is clearly the case, since (1) the outcome will always belong to $\bar{Z} = \left\{(z_1, z_2) \in \Re_+^2 \mid z_1 \leq \bar{z}_1, z_2 \leq \bar{z}_2\right\}$ whenever $(\min_{i \in N} \tau_1(u^i), \min_{i \in N} \tau_2(u^i)) \in \bar{Z}$ and (2) the outcome will belong to $Z \setminus \bar{Z}$ otherwise. Hence, F is a social choice function onto Z.

Notice that, with the same "triangular" set, we can have different social choice functions depending on (a) our choice of $(\bar{z}_1, \bar{z}_2)$ within the "triangular" set and (b) the choice of the right-coalition systems below those critical values.

Notice also that the same construction could be generalized to m dimensions by taking any box-shaped set $\bar{Z}$ contained in the "triangular" set, requiring unanimity to get out of $\bar{Z}$ and letting any generalized median voter scheme within the bounds

[14] See Barberà, Massó, and Neme (1997) for a full discussion of this sufficiency.

of $\bar{Z}$. Moreover, the construction can easily be generalized to any convex set Z. Just take a box $\bar{Z}$ inside Z, defined by upper and lower bounds $(\underline{z}_k, \bar{z}_k)$ for each dimension k, and take any generalized median voter scheme defined by $\mathcal{R}$ and $\mathcal{L}^*$ such that for all $k = 1, ..., m$: $\mathcal{L}_k^*(z_k) = \{N\}$ for $z_k < \underline{z}_k$ and $\mathcal{R}_k(z_k) = \{N\}$ for $z_k > \bar{z}_k$. We leave it for the reader to check that this generalized median voter scheme induced by $(Z, \mathcal{R})$ will be a strategy-proof social choice function onto Z.

Finally, we present an example involving sets of quite arbitrary shape. The purpose of the example is to show that, even in very strange cases, full dictatorship is not a consequence of strategy proofness. Clearly, the functions we describe give a lot of power to one individual, and we do not claim that they are attractive. But the example certainly proves that sharp statements à la Gibbard-Satterthwaite are not to be expected in our context, and that each individual decision problem attached to a feasible set of alternatives Z will require its own careful examination.

Example 4. Let Z be any given compact set in $\Re^2$ satisfying our full dimensionality requirement. For our construction to work, we must be able to find a box $B = \{(z_1, z_2) \in \Re^2 \mid \underline{x}_1 \leq z_1 \leq \bar{x}_1, \underline{x}_2 \leq z_2 \leq \bar{x}_2\}$ such that for all elements $z \in Z \backslash B$ we have that all points in the segment between z and the projection of z on B (call it $Pr\ z$) belong to Z. Figure 1 shows that several such B can be found for a given, and rather complicated shape of Z.

Figure 1

Now, consider the generalized median voter scheme induced by $(Z, \mathcal{R})$ where for $k = 1, 2$:

$$\begin{aligned} \mathcal{R}_k(z_k) &= \{S \subseteq N \mid 1 \in S\} \text{ for } \underline{x}_k \leq z_k \leq \bar{x}_k, \\ \mathcal{R}_k(z_k) &= \{N\} \text{ for } z_k > \bar{x}_k, \text{ and} \\ \mathcal{L}_k^*(z_k) &= \{N\} \text{ for } z_k < \underline{x}_k. \end{aligned}$$

In words, agent 1 is a dictator on the set $Z \cap B$ and a unanimous decision is required to go away from this set. Again, this is a globally nondictatorial strategy-proof social choice function onto Z. As already stated, we do not present this rule as a wonderful one. Rather, it is to make the point that full dictatorship is hard to obtain.

References

Alcalde, J. and Barberà, S. (1994). "Top Dominance and the Possibility of Strategy-proof Stable Solutions to Matching Problems," *Economic Theory* **4**, 417-435.

Barberà, S. (1997). "Notes on Strategy-proof Social Choice Functions," forthcoming in *Social Choice Re-examinated*, (K. Arrow, A. Sen, and K. Suzumura, Eds.). French version: (1996). "Sur les functions de choix non manipulables," *Revue d'Économie Politique* **106**, 61-81.

Barberà, S., Bogomolnaia, A., and van der Stel, H. (1997). "Strategy-proof Probabilistic Rules for Expected Utility Maximizers," forthcoming in M*athematical Social Sciences*.

Barberà, S., Gul, F. and Stacchetti, E. (1993). "Generalized Median Voter Schemes and Committees," *Journal of Economic Theory* **61**, 262-289.

Barberà, S. and Jackson, M. (1995). "Strategy-proof Exchange," *Econometrica* **63**, 51-87.

Barberà, S., Massó, J. and Neme, A. (1997). "Voting under Constraints," *Journal of Economic Theory* **76**, 298-321.

Barberà, S. and Peleg, B. (1990). "Strategy-Proof Voting Schemes with Continuous Preferences," *Social Choice and Welfare* **7**, 31-38.

Barberà, S., Sonnenschein, H., and Zhou, L. (1991). "Voting by Committees," *Econometrica* **59**, 595,609.

Border, K. C. and Jordan, J. S. (1983). "Straightforward Elections, Unanimity and Phantom Voters," *Review of Economic Studies* **50**, 153-170.

Gibbard, A. (1973). "Manipulation of Voting Schemes: A General Result," *Econometrica* **41**, 587-601.

Gibbard, A. (1977). "Manipulation of Schemes that Mix Voting with Chance," *Econometrica* **45**, 665-681.

Le Breton, M. and Sen, A. (1995). "Strategyproofness and Decomposability: Weak Orderings," Indian Statistical Institute, Discussion Paper 95-04.

Moulin, H. (1980). “On Strategy-Proofness and Single-Peakedness,” *Public Choice* **35**, 437-455.

Satterthwaite, M.A. (1975). “Strategy-proofness and Arrow’s Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions,” *Journal of Economic Theory* **10**, 187-217.

Serizawa, S. (1994): “An Impossibility Theorem in Pure Public Goods Economies with Feasibility Constraints: Voting by Committees in Non-Rectangular Feasible Sets”, *Japanese Economic Review* **47**, 368-383.

Serizawa, S. (1995): “Power of Voters and Domain of Preferences where Voting by Committees is Strategy-Proof”, *Journal of Economic Theory* **67**, 599-608.

Serizawa, S. (1996). “Strategy-proof and Symmetric Social Choice Function for Public Good Economies,” mimeo.

Sprumont, Y. (1991). “The Division Problem with Single-peaked Preferences: A Characterization of the Uniform Allocation Rule,” *Econometrica* **59**, 509-519.

Sprumont, Y. (1995). “Strategyproof Collective Choice in Economic and Political Environments, ” *Canadian Journal of Economics* **28**, 68-107.