Weighted Approval Voting*

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Abstract

To allow society to treat unequal alternatives distinctly we propose a natural extension of Approval Voting [4] by relaxing the assumption of neutrality. According to this extension, every alternative receives *ex-ante* a strictly positive and finite weight. These weights may differ across alternatives. Given the voting decisions of every individual (individuals are allowed to vote for, or approve of, as many alternatives as they wish to), society elects the alternative for which the product of total number of votes times exogenous weight is maximal. If the product is maximal for more than one alternative, a pre-specified tie-breaking rule is applied. Our main result is an axiomatic characterization of this family of voting procedures.

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1 Introduction

Approval Voting [4] is perhaps the most well known voting procedure that has been proposed as an alternative to the Plurality Rule. For example, in a recent survey Brams and Fishburn [5] lay out that the United Nations General Assembly, several scientific institutions (among others the Mathematical Association of America, the American Mathematical Society, and the American Statistical Association), and some political parties (*i.e.*, in Pennsylvania) adopted Approval Voting or had at least some experience with it. According to this rule, every voter can vote for, or approve of, as many alternatives as s/he wishes to and given the response profile of individual approvals, society elects the set of alternatives with the maximal number of votes.

Very recently, it has been looked at Approval Voting from different points of view. First, several case studies have been carried out. For instance, Laslier [12] studies the 1999 elections of the President and the Council of the Society for Social Choice and Welfare, where Approval Voting was the method being used but voters were also asked to submit their rankings under the Borda count. Laslier [13] analyzes Approval Voting by means of an experiment carried out in six places in France during the first round of the presidential election of 2002, in which Jean-Marie Le Pen came in second, defeating the socialist candidate Lionel Jospin, and thus obtained the right to compete in the second round against Jacques Chirac. Second, wide theoretical research has also been under way. Regenwetter and Tsetlin [15] compare Approval Voting with positional voting methods and identify conditions under which they tend to agree. Vorsatz [19] shows that, on the domain of dichotomous preferences, Approval Voting coincides with the Borda count. De Sinopoli *et al.* [7] analyze strategic behavior in Approval Voting games. Brams and Sanver [6] study Approval Voting under the assumption that voters do not only have preferences on the set of alternatives but also judgements about their acceptability. Nitzan and Baharad [14] study the consequence of modifying Approval Voting by restricting the minimal and maximal number of alternatives that can be voted for, and, finally, Dellis and Oak [8] compare Approval Voting with the Plurality Rule in a political competition model with endogenous candidacy entry.

It is inherent in the definition of Approval Voting that every vote counts the same, independently which alternative receives it. We believe that this neutrality assumption is relevant in democratic processes (*i.e.*, presidential elections) where all alternatives should be treated equally, but may not be as natural for group decision making problems in which the characteristics of the alternatives are objectively different and society agrees on the desirability to treat unequal alternatives distinctly (*i.e.*, in the case when alternatives are candidates characteristics such as seniority, age, education, race, and gender may matter). In some circumstances society may also wish to give a slight preference to the *status quo* alternative by using the following modification of Approval Voting: the *status quo* is maintained if it is one of the alternatives with the maximal number of votes; then, alternatives are not treated symmetrically any more and, consequently, the neutrality axiom is violated. In general, society might be willing to break ties according to some tie-breaking rule that is applied to the subset of alternatives with the maximal number of votes. Finally, consider the situation when alternatives can be identified as points in a multi-dimensional space in such a way that all preferences are unambiguously monotonic on one of the dimensions (for instance, the cost of producing each alternative). Then, weights of alternatives may be chosen according to the corresponding level on this dimension. It is our objective to propose a generalization of Approval Voting that can be applied to these kind of examples and present an axiomatic characterization of this new voting procedure.

The generalization we consider, *Weighted Approval Voting*, is simple and intuitive at the same time: Assign *ex-ante* a strictly positive and finite weight to every alternative. Observe that the weights are potentially different for distinct alternatives. Given the approvals of every voter (again voters can vote for as many alternatives as they wish to), society elects the alternative for which the product of total number of votes times weight is maximal. If the product is maximal for more than one alternative, a pre-specified tie-breaking is applied. It is defined as follows. Given a complete preorder (*i.e.*, a complete, reflexive, and transitive binary relation) on the set of alternatives, the tie-breaking rule selects, for each non-empty set of alternatives, the subset of alternatives that is maximal according to the complete preorder. This voting rule reduces to Approval Voting when the weights are identical for all alternatives and the complete preorder has a unique indifference class (*i.e.*, ties are not broken).

We are interested in general voting procedures that could operate in different voting situations in which the set of voters as well as the set of alternatives might vary (for instance, different choices have to be made over time). In particular, and given a universal set of potential voters and a set of conceivable alternatives, a voting procedure (a family of voting rules) should specify an outcome for every electorate (the subset of voters that indeed vote) and every set of feasible alternatives (the subset of alternatives that are indeed at stake). Our main result states that the family of all Weighted Approval Voting is characterized in this setting by means of the following five properties. *Consistency in alternatives*, which is the analogue of Arrow's Choice Axiom and implies that a general election from a set of feasible alternatives can be reduced to choices among pairs of alternatives only; *Consistency in voters*, which requires that if two disjoint electorates elect a common set out of two feasible alternatives, then exactly this set has to be elected when the two electorates are assembled; *Anonymity*, which is symmetry among voters; *No-Support*, which states that one alternative without any vote is elected, when confronted with another alternative, if and only if the second alternative does not receive any vote either; and *Coherence*, which asks that for every alternative, when confronted with another alternative, there must exist a situation (with strictly positive votes for both alternatives) at which the considered alternative is elected (perhaps together with the other one).

Several authors have analyzed Approval Voting axiomatically. Fishburn [10] shows that if the set of alternatives is fixed and the electorate is allowed to vary, then Approval Voting is characterized by means of consistency in voters, neutrality, anonymity, and disjoint equality (if two voters approve two nonempty and disjoint subsets of alternatives, then the union of these two sets has to be elected whenever there is no other voter). Fishburn [9] also characterizes Approval Voting by means of consistency in voters, neutrality, faithfulness (if there is only one voter and this voter approves at least one alternative, the voting procedure selects all alternatives this voter supports), and cancellation (if all alternatives receive the same number of votes, all alternatives are elected). Alós-Ferrer [1] shows that the latter characterization is not tight since neutrality follows from consistency in voters, faithfulness, and cancellation. Sertel [16] presents an alternative definition of Approval Voting that differs from the original one only in the situation when no alternative receives any vote. He assumes in this case that no alternative is elected. This, slightly different, voting rule is then characterized by anonymity, weak unanimity (if the society consists of only one voter, the voting procedure selects the set of alternatives this voter supports), weak consistency (this property weakens consistency in voters slightly without changing its main idea), and strong disjoint equality (disjoint equality is also defined for the case when some voter does not approve any alternative). A further characterization is due to Baigent and Xu [3]. They apply the properties of neutrality, strict monotonicity (if x is elected at a certain response profile and a second response profile is identical to the first one apart from the fact that x receives now an additional vote, then only x is elected at the second response profile) and independence of symmetric substitutions. The latter condition requires that the set of elected alternatives should be the same in the following two situations: In the first situation, some voter approves, among other alternatives, x but not y, whereas another voter approves, among other alternatives, y but not x. The second situation is identical to the first one with the only difference that the first voter approves now y but not x and the second voter approves now x but not y. Goodin and List [11] relate Approval Voting axiomatically to May's Theorem by showing that Approval Voting is characterized by anonymity (they define anonymity in a different way than we do, yet, the two properties turn out to be equivalent), neutrality, and strict monotonicity. Note that any Weighted Approval Voting satisfies strict monotonicity and independence of symmetric substitutions while it fails to satisfy axioms that have neutrality inherent in its definition such as disjoint equality, faithfulness, cancellation, and weak unanimity.

The remainder of the paper is organized as follows. In the next section, we introduce our notation and main definitions. In Section 3, we present the five axioms that characterize all Weighted Approval Voting. Afterwards, we prove our theorem. Finally, we establish the independence of the axioms and conclude with some remarks.

2 Preliminaries

We consider elections in which the set of alternatives and the set of voters may vary. First, let $\mathcal{K}$ be the universal set of conceivable alternatives for election. Generic alternatives will be denoted by x, y, and z. The cardinality of $\mathcal{K}$, κ, is finite and greater or equal to 3 (if the set of conceivable alternatives contains only two alternatives, then the first axiom, consistency in alternatives, is superfluous in Theorem 1 as it will become clear from Lemma 3 later on). Since it may happen that not all conceivable alternatives are eligible, we restrict the set of *feasible alternatives* to be equal to $K \subseteq \mathcal{K}$. Alternatively, we will denote subsets of alternatives by the capital letters S and T. Second, we represent the universal set of voters by the set of natural numbers $\mathbb{N}$. We will consider situations in which the set of voters actually participating in the election, the *electorate* N, is a finite subset of the natural numbers. Often we will also use the capital letters A and B to denote electorates. The cardinalities of N and K are equal to $n \geq 1$ and $k \geq 2$, respectively.

For any voter $i \in \mathbb{N}$, let $M_i \in 2^{\mathcal{K}}$ be the set of alternatives i votes for. A *profile* $M = (M_i)_{i \in \mathbb{N}} \in (2^{\mathcal{K}})^{\mathbb{N}}$ is a list of all votes. Given a profile M and an electorate N, a *response profile* $M_N = (M_i)_{i \in N} \in (2^{\mathcal{K}})^N$ is the n-tuple of votes coming from the electorate N at profile M. We say that the response profiles M_A and M_B, corresponding to the electorates A and B of equal size, are *isomorphic* if there exists a one-to-one mapping $\pi : A \to B$ such that for all $i \in A$, $M_i = M_{\pi(i)}$. Given two disjoint electorates A and B and two response profiles M_A and M_B, denote by $M_A + M_B$ the response profile $(M_i)_{i \in A \cup B} \in (2^{\mathcal{K}})^{A \cup B}$. Finally, given the response profile M_N and alternative $x \in \mathcal{K}$, let $G_x(M_N) = |\{i \in N : x \in M_i\}|$ be the *support* of x at M_N.

Given a set of feasible alternatives K and an electorate N, a *voting rule* $v^{K,N} : (2^{\mathcal{K}})^{\mathbb{N}} \to 2^K \backslash \{\emptyset\}$ selects, for all profiles M, a nonempty set of feasible alternatives $v^{K,N}(M)$ with the property that for all $M, M' \in (2^{\mathcal{K}})^{\mathbb{N}}$ such that $M_N = M'_N$, $v^{K,N}(M) = v^{K,N}(M')$. This is the reason why, with a slight abuse of notation, we will write $v^K(M_N)$ instead of $v^{K,N}(M)$. Observe first that, although the empty set can be a component of response profiles, the images of a given voting rule are nonempty subsets of feasible alternatives.

We exclude the possibility to elect no alternative (even when all feasible alternatives get zero support), because we want to include the interpretation of the image as the set of pre-elected alternatives from which an ultimate winning alternative has still to be determined in a yet to be specified way (*i.e.*, a lottery). Additionally, we aim at generalizing Approval Voting which, for each response profile, elects the (always nonempty) subset of feasible alternatives with maximal support. Second, response profiles may include votes for unfeasible alternatives. These votes are redundant but this formulation simplifies later on the definition of consistency in alternatives.

A *family of voting rules* $\{v^{K,N} : (2^{\mathcal{K}})^{\mathbb{N}} \to 2^K \backslash \{\emptyset\}\}_{K,N}$ is a set of voting rules, one for every set of feasible alternatives K and electorate N. It is denoted by v. Given the family of voting rules v and a particular set of feasible alternatives K, we denote the subfamily of voting rules $\{v^{K,N} : (2^{\mathcal{K}})^{\mathbb{N}} \to 2^K \backslash \{\emptyset\}\}_N$ by v^K.

As we have already argued in the Introduction, there are meaningful situations in which not all alternatives are equally important. Thus, it is our objective to eliminate the neutrality assumption underlying Approval Voting by allowing for the possibility to discriminate among alternatives keeping the impact of a vote for a given alternative the same for all voters.[1] To define the natural non-neutral extension of Approval Voting, denote by $\mathbb{R}_{++}$ and $\mathbb{Q}_{++}$ the set of strictly positive real and rational numbers, respectively. Let $\succsim$ be a complete preorder on $\mathcal{K}$ (*i.e.*, a complete, reflexive, and transitive binary relation on $\mathcal{K}$). We refer to $\succsim$ as a tie-breaking rule. The asymmetric and symmetric parts of $\succsim$ are denoted by $\succ$ and $\sim$, respectively. Given a vector of strictly positive weights $p = (p_x)_{x \in \mathcal{K}} \in \mathbb{R}^{\kappa}_{++}$, the tie-breaking rule $\succsim$ on $\mathcal{K}$, and a set of feasible alternatives K, we denote by $p|_K$ and $\succsim|_K$ the restrictions of p and $\succsim$ on K, respectively; namely, $p|_K = (p_x)_{x \in K} \in \mathbb{R}^k_{++}$ and for all $x, y \in K$, $x \succsim|_K y$ if and only if $x \succsim y$. Obviously, $p|_{\mathcal{K}} = p$ and $\succsim|_{\mathcal{K}} = \succsim$.

[1] An alternative approach aims at allowing for different weights for distinct voters maintaining neutrality. It is also very prospective to analyze the normative foundations of this generalization of Approval Voting because one can identify a variety situations where this rule is applied. Examples include voting in the EU Member Council (the weight of a country is determined by its population size) and management boards (a vote from the CEO counts usually more than a vote from other board members). To our best knowledge, this rule has not been studied axiomatically so far.

Definition 1 The family of voting rules v is a *Weighted Approval Voting* if there exists a vector of weights $p = (p_x)_{x \in \mathcal{K}}$, with $p_x \in \mathbb{R}_{++}$ for all $x \in \mathcal{K}$, and a tie-breaking rule $\succsim$ such that for all sets of feasible alternatives K and all response profiles M_N,

$$x \in v^K(M_N) \Leftrightarrow \begin{array}{l} p_x \cdot G_x(M_N) \geq p_y \cdot G_y(M_N) \text{ for all } y \in K \text{ and} \\ x \succsim y \text{ for all } y \in K \text{ such that } p_x \cdot G_x(M_N) = p_y \cdot G_y(M_N) > 0. \end{array} \tag{1}$$

The family of Weighted Approval Voting with vector of weights p and tie-breaking rule $\succsim$ is denoted by $v_{(p,\succsim)}$. Approval Voting, denoted by v_A, is the special case of a Weighted Approval Voting when for all $x, y \in \mathcal{K}$, $p_x = p_y$ and $x \sim y$. Note that the family of all Weighted Approval Voting contains as a specially interesting subclass those lexicographic voting rules that always choose a unique alternative (except when no alternative receives any vote) by applying first Approval Voting (all weights are the same) and selecting afterwards, among the subset of alternatives with maximal support, the unique alternative that maximizes a given strict order $\succ$. Moreover, observe that the vector of weights $(p_x)_{x \in \mathcal{K}}$ of any Weighted Approval Voting has one degree of freedom because multiplying the weights by a strictly positive number does not have any effect on the result of the election.

Remark 1 *For all vectors of weights p and all $\lambda \in \mathbb{R}_{++}$, $v_{(\lambda \cdot p,\succsim)} = v_{(p,\succsim)}$.*

Finally, let $v = \{v^{K,N} : (2^{\mathcal{K}})^{\mathbb{N}} \to 2^K \backslash \{\emptyset\}\}_{K,N}$ be a family of voting rules and let K be a set of feasible alternatives. Given a vector of strictly positive weights $p^K = (p_x^K)_{x \in K}$ and a tie-breaking rule $\succsim^K$ on K, the subfamily of voting rules $v^K = \{v^{K,N} : (2^{\mathcal{K}})^{\mathbb{N}} \to 2^K \backslash \{\emptyset\}\}_N$ will be called the *Weighted Approval Voting relative to p^K and $\succsim^K$* if condition (1) holds when p is replaced by p^K and $\succsim$ by $\succsim^K$.

3 Properties and Characterization

We present now formally the properties that characterize all Weighted Approval Voting. Two consistency properties prescribe how the elected set of alternatives varies as the set of feasible alternatives or the electorate changes.

Consistency in Alternatives: The family of voting rules v is *consistent in alternatives* if for all sets of feasible alternatives $S \subset T \subseteq \mathcal{K}$, all profiles $M \in (2^{\mathcal{K}})^{\mathbb{N}}$, and all electorates N such that $v^T(M_N) \cap S \neq \emptyset$,

$$v^S(M_N) = v^T(M_N) \cap S.$$

This property means the following. Assume first that a particular set of alternatives is feasible and society elects a subset of them. If it turns out afterwards that fewer alternatives are feasible, then the set of elected alternatives is restricted accordingly (see [2]). Consistency in alternatives plays a crucial role in the proof of our characterization because it establishes the transitivity of the weights and the tie-breaking rule. Additionally, it allows us to extend the two alternatives case to any set of alternatives. For the latter reason we only have to state the other four properties with respect to two alternatives.

The second consistency property requires that if two disjoint electorates elect some common alternatives, then exactly these alternatives are elected whenever all voters within and no voters outside these two electorates participate in the election (see [17]). This property insures the additivity of the votes.

Consistency in Voters: The family of voting rules v is *consistent in voters* if for all alternatives $x, y \in \mathcal{K}$, all profiles $M \in (2^{\mathcal{K}})^{\mathbb{N}}$, and all disjoint electorates $A, B \subseteq \mathbb{N}$ such that $v^{\{x,y\}}(M_A) \cap v^{\{x,y\}}(M_B) \neq \emptyset$,

$$v^{\{x,y\}}(M_A + M_B) = v^{\{x,y\}}(M_A) \cap v^{\{x,y\}}(M_B).$$

According to the third property the set of elected alternatives depends only on the support of the alternatives (see [9]). Hence, the weights will be independent of the identity of the voters.

Anonymity: The family of voting rules v is *anonymous* if for all alternatives $x, y \in \mathcal{K}$, all profiles $M, M' \in (2^{\mathcal{K}})^{\mathbb{N}}$, and all electorates A and B such that $G_x(M_A) = G_x(M'_B)$ and $G_y(M_A) = G_y(M'_B)$,

$$v^{\{x,y\}}(M_A) = v^{\{x,y\}}(M'_B).$$

The fourth axiom refers to response profiles with the property that the support of at least one of the two feasible alternatives is zero.

No-Support: The family of voting rules v satisfies the *no-support condition* if for all alternatives $x, y \in \mathcal{K}$, all profiles $M \in (2^{\mathcal{K}})^{\mathbb{N}}$, and all electorates N such that $G_x(M_N) = 0$,

$$x \in v^{\{x,y\}}(M_N) \text{ if and only if } G_y(M_N) = 0.$$

The last property, coherence, requires that every alternative x, when confronted with any alternative y, is elected (perhaps with y) for some profile and some electorate at which both alternatives have a strictly positive support.

Coherence: The family of voting rules v is *coherent* if for all $x \in \mathcal{K}$ and all $y \in \mathcal{K}\setminus\{x\}$, there exists a profile $M \in (2^{\mathcal{K}})^{\mathbb{N}}$ and an electorate N such that $G_x(M_N) > 0$, $G_y(M_N) > 0$, and

$$x \in v^{\{x,y\}}(M_N).$$

In Theorem 1 we state an axiomatic characterization of all Weighted Approval Voting based on these five properties.

Theorem 1 *The family of voting rules v is consistent in alternatives and voters, anonymous, coherent, and satisfies the no-support condition if and only if v is a Weighted Approval Voting.*

4 Proof of the Characterization

We start by proving that, in the case of two feasible alternatives, the relevant information is not the absolute support of the alternatives (as it follows directly from anonymity) but rather their relative support. Afterwards, we prove a monotonicity like property. First, let $F_{xy}(M_N) = \frac{G_x(M_N)}{G_y(M_N)}$ be the relative support of alternative x with respect to y at the response profile M_N, provided that $G_y(M_N) > 0$.

Lemma 1 *Assume that the family of voting rules v is consistent in voters and anonymous. Then, for all alternatives $x, y \in \mathcal{K}$, all profiles $M, M' \in (2^{\mathcal{K}})^{\mathbb{N}}$, and all electorates A and B such that $G_y(M_A) > 0$, $G_y(M'_B) > 0$, and $F_{xy}(M_A) = F_{xy}(M'_B)$,*

$$v^{\{x,y\}}(M_A) = v^{\{x,y\}}(M'_B).$$

Proof: Let $\{x, y\}$ be the set of feasible alternatives and take any two response profiles M_A and M'_B that satisfy the hypothesis of Lemma 1. Consider two electorates $\bar{A}$ and $\bar{B}$ of sizes $|A| \cdot G_y(M'_B)$ and $|B| \cdot G_y(M_A)$, respectively. Let $\bar{M}_{\bar{A}}$ and $\bar{M}'_{\bar{B}}$ be two response profiles obtained by replicating $G_y(M'_B)$-times the response profile M_A and $G_y(M_A)$-times the response profile M'_B, respectively. Namely, the response profile $\bar{M}_{\bar{A}}$ is the union of $G_y(M'_B)$-isomorphic copies of M_A (denoted by $M_{A_1}, ..., M_{A_{G_y(M'_B)}}$) and the response profile $\bar{M}'_{\bar{B}}$ is the union of $G_y(M_A)$-isomorphic copies of M'_B (denoted by $M'_{B_1}, ..., M'_{B_{G_y(M_A)}}$), where all electorates A_t, $t = 1, ..., G_y(M'_B)$ and all B_r, $r = 1, ..., G_x(M_A)$ are disjoint. Observe that $G_x(\bar{M}_{\bar{A}}) = G_x(M_A) \cdot G_y(M'_B)$ and $G_x(\bar{M}'_{\bar{B}}) = G_x(M'_B) \cdot G_y(M_A)$. By assumption,

$$G_x(\bar{M}_{\bar{A}}) = G_x(\bar{M}'_{\bar{B}}). \tag{2}$$

Moreover, $G_y(\bar{M}_{\bar{A}}) = G_y(M_A) \cdot G_y(M'_B)$ and $G_y(\bar{M}'_{\bar{B}}) = G_y(M'_B) \cdot G_y(M_A)$. Thus,

$$G_y(\bar{M}_{\bar{A}}) = G_y(\bar{M}'_{\bar{B}}). \tag{3}$$

By anonymity, (2) and (3) imply

$$v^{\{x,y\}}(\bar{M}_{\bar{A}}) = v^{\{x,y\}}(\bar{M}'_{\bar{B}}). \tag{4}$$

Also, by anonymity, for all $t = 1, ..., G_y(M'_B)$ and all $r = 1, ..., G_y(M_A)$, $v^{\{x,y\}}(M_{A_t}) = v^{\{x,y\}}(M_A)$ and $v^{\{x,y\}}(M'_{B_r}) = v^{\{x,y\}}(M'_B)$. Then, by iterating on the properties of consistency in voters and anonymity,

$$v^{\{x,y\}}(\bar{M}_{\bar{A}}) = v^{\{x,y\}}\left(\sum\nolimits_{t=1}^{G_y(M'_B)} M_{A_t}\right) = \bigcap_{t=1}^{G_y(M'_B)} v^{\{x,y\}}(M_{A_t}) = v^{\{x,y\}}(M_A)$$

and

$$v^{\{x,y\}}(\bar{M}'_{\bar{B}}) = v^{\{x,y\}}\left(\sum\nolimits_{r=1}^{G_y(M_A)} M'_{B_r}\right) = \bigcap_{r=1}^{G_y(M_A)} v^{\{x,y\}}(M'_{B_r}) = v^{\{x,y\}}(M'_B).$$

By (4), $v^{\{x,y\}}(M_A) = v^{\{x,y\}}(M'_B)$. $\square$

Lemma 2 *Assume that the family of voting rules v is consistent in voters, anonymous, and satisfies the no-support condition. Then, for all alternatives $x, y \in \mathcal{K}$, all profiles $M, M' \in (2^{\mathcal{K}})^{\mathbb{N}}$, and all electorates A and B such that $G_y(M_A) > 0$, $G_y(M'_B) > 0$, $F_{xy}(M_A) > F_{xy}(M'_B)$, and $x \in v^{\{x,y\}}(M'_B)$,*

$$v^{\{x,y\}}(M_A) = \{x\}.$$

Proof: Let $\{x, y\}$ be the set of feasible alternatives and take any two response profiles M_A and M'_B that satisfy the hypothesis of Lemma 2. Consider two electorates $\bar{A}$ and $\bar{B}$ of sizes $|A| \cdot G_y(M'_B)$ and $|B| \cdot G_y(M_A)$, respectively. Let $\bar{M}_{\bar{A}}$ and $\bar{M}'_{\bar{B}}$ be the two response profiles obtained by replicating $G_y(M'_B)$-times the response profile M_A and $G_y(M_A)$-times the response profile M'_B, respectively. Namely, the response profile $\bar{M}_{\bar{A}}$ is the union of $G_y(M'_B)$-isomorphic copies of M_A and the response profile $\bar{M}'_{\bar{B}}$ is the union of $G_y(M_A)$ -isomorphic copies of M'_B. By consistency in voters and anonymity,

$$v^{\{x,y\}}(\bar{M}_{\bar{A}}) = v^{\{x,y\}}(M_A) \text{ and } v^{\{x,y\}}(\bar{M}'_{\bar{B}}) = v^{\{x,y\}}(M'_B). \tag{5}$$

Observe that $G_y(\bar{M}_{\bar{A}}) = G_y(M_A) \cdot G_y(M'_B) = G_y(\bar{M}'_{\bar{B}})$. Moreover, by hypothesis, $G_x(\bar{M}_{\bar{A}}) = G_x(M_A) \cdot G_y(M'_B) > G_y(M_A) \cdot G_x(M'_B) = G_x(\bar{M}'_{\bar{B}})$.

Now, take two response profiles $\hat{M}_C$ and $\hat{M}_D$ corresponding to the disjoint electorates C and D, with the properties that $G_y(\hat{M}_D) = G_y(\bar{M}'_{\bar{B}})$, $G_x(\hat{M}_D) = G_x(\bar{M}'_{\bar{B}})$, $G_y(\hat{M}_C) = 0$, and $G_x(\hat{M}_C) = G_x(\bar{M}_{\bar{A}}) - G_x(\bar{M}'_{\bar{B}}) > 0$. By anonymity,

$$v^{\{x,y\}}(\hat{M}_D) = v^{\{x,y\}}(\bar{M}'_{\bar{B}}). \tag{6}$$

Since $G_x(\bar{M}_{\bar{A}}) = G_x(\hat{M}_C) + G_x(\bar{M}'_{\bar{B}}) = G_x(\hat{M}_C) + G_x(\hat{M}_D)$, $G_y(\bar{M}_{\bar{A}}) = G_y(\hat{M}_C) + G_y(\hat{M}_D)$, and the electorates C and D are disjoint, $G_x(\bar{M}_{\bar{A}}) = G_x(\hat{M}_C + \hat{M}_D)$ and $G_y(\bar{M}_{\bar{A}}) = G_y(\hat{M}_C + \hat{M}_D)$. By anonymity,

$$v^{\{x,y\}}(\bar{M}_{\bar{A}}) = v^{\{x,y\}}(\hat{M}_C + \hat{M}_D). \tag{7}$$

By the no-support condition,

$$v^{\{x,y\}}(\hat{M}_C) = \{x\}.$$

Since, by hypothesis, $x \in v^{\{x,y\}}(M'_B)$, conditions (5) and (6) imply $x \in v^{\{x,y\}}(\hat{M}_D)$.Thus,

$$v^{\{x,y\}}(\hat{M}_C) \cap v^{\{x,y\}}(\hat{M}_D) = \{x\}. \tag{8}$$

By consistency in voters,

$$v^{\{x,y\}}(\hat{M}_C + \hat{M}_D) = v^{\{x,y\}}(\hat{M}_C) \cap v^{\{x,y\}}(\hat{M}_D). \tag{9}$$

Conditions (7), (8), and (9) imply that $v^{\{x,y\}}(\bar{M}_{\bar{A}}) = \{x\}$. Finally, it follows from (5) that $v^{\{x,y\}}(M_A) = \{x\}$. □

Lemma 3 *Assume that the family of voting rules v is consistent in voters, anonymous, coherent, and satisfies the no-support condition. Then, for all alternatives $x, y \in \mathcal{K}$, there exist two weights $p_x^{\{x,y\}}, p_y^{\{x,y\}} \in \mathbb{R}_{++}$ and a tie-breaking rule $\succsim^{\{x,y\}}$ on $\{x, y\}$ such that $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x^{\{x,y\}}, p_y^{\{x,y\}})$ and $\succsim^{\{x,y\}}$.*

Proof: Let $\{x, y\}$ be the set of feasible alternatives and take any v that satisfies the hypothesis of Lemma 3. For all profiles M and all electorates N, if M_N is such that $G_x(M_N) = G_y(M_N) = 0$, then $v^{\{x,y\}}(M_N) = \{x, y\}$ by the no-support condition. Thus, it remains to be shown that there exist two weights $p_x^{\{x,y\}}, p_y^{\{x,y\}} \in \mathbb{R}_{++}$ and a tie-breaking rule $\succsim^{\{x,y\}}$ on $\{x, y\}$ with the property that for all response profiles M_N satisfying $G_x(M_N) + G_y(M_N) > 0$,

$$\begin{aligned} x \in v^{\{x,y\}}(M_N) \Leftrightarrow \quad & \text{either } p_x^{\{x,y\}} \cdot G_x(M_N) > p_y^{\{x,y\}} \cdot G_y(M_N) \text{ or} \\ & p_x^{\{x,y\}} \cdot G_x(M_N) = p_y^{\{x,y\}} \cdot G_y(M_N) \text{ and } x \succsim^{\{x,y\}} y. \end{aligned} \tag{10}$$

To insure that condition (10) holds, we investigate the restrictions that response profiles impose on the weights and the tie-breaking rule. Consider any electorate N and let the response profiles M_N and M'_N be such that $G_x(M_N) = G_y(M'_N) = 1$ and $G_y(M_N) = G_x(M'_N) = 0$. By the no-support condition, $v^{\{x,y\}}(M_N) = \{x\}$ and $v^{\{x,y\}}(M'_N) = \{y\}$. Then, condition (10) holds for any $p_x^{\{x,y\}} > 0$ and $p_y^{\{x,y\}} > 0$, and any tie-breaking rule $\succsim^{\{x,y\}}$ on $\{x, y\}$. To further restrict the weights and the tie-breaking rule, we have to consider the response profiles in which both alternatives get at least one vote. Formally, for any electorate N, define $\mathcal{M}_N = \{\tilde{M}_N \in (2^{\mathcal{K}})^N : G_x(\tilde{M}_N) > 0 \text{ and } G_y(\tilde{M}_N) > 0\}$. We divide the analysis into four cases.

1. Assume that for all electorates B and all $\tilde{M}_B \in \mathcal{M}_B$, $v^{\{x,y\}}(\tilde{M}_B) = \{x\}$. This contradicts coherence, and therefore, this case cannot be.

2. Assume that for all electorates B and all $\tilde{M}_B \in \mathcal{M}_B$, $v^{\{x,y\}}(\tilde{M}_B) = \{y\}$. This contradicts coherence, and therefore, this case cannot be.

3. Assume that there exists an electorate B and a response profile $\tilde{M}_B \in \mathcal{M}_B$ such that $v^{\{x,y\}}(\tilde{M}_B) = \{x, y\}$. Consider any electorate $A \neq B$ and any response profile $\hat{M}_A \in \mathcal{M}_A$. Assume at first that $F_{xy}(\hat{M}_A) > F_{xy}(\tilde{M}_B)$. Since $x \in v^{\{x,y\}}(\tilde{M}_B)$ by assumption, Lemma 2 implies that $v^{\{x,y\}}(\hat{M}_A) = \{x\}$. Therefore,

$$\text{if } F_{xy}(\hat{M}_A) > F_{xy}(\tilde{M}_B), \text{ then } v^{\{x,y\}}(\hat{M}_A) = \{x\}. \tag{11}$$

Assume now that $v^{\{x,y\}}(\hat{M}_A) = \{x\}$. Since $v^{\{x,y\}}(\tilde{M}_B) = \{x, y\}$, Lemma 1 implies that $F_{xy}(\hat{M}_A) \neq F_{xy}(\tilde{M}_B)$. Assume at first that $F_{xy}(\hat{M}_A) < F_{xy}(\tilde{M}_B)$. Then, since $y \in v^{\{x,y\}}(\tilde{M}_B)$ by assumption, we obtain from Lemma 2 that $y \in v^{\{x,y\}}(\hat{M}_A) = \{x\}$. This is a contradiction, and therefore, $F_{xy}(\hat{M}_A) > F_{xy}(\tilde{M}_B)$. Hence,

$$\text{if } v^{\{x,y\}}(\hat{M}_A) = \{x\}, \text{ then } F_{xy}(\hat{M}_A) > F_{xy}(\tilde{M}_B). \tag{12}$$

We conclude from (11) and (12) that

$$v^{\{x,y\}}(\hat{M}_A) = \{x\} \text{ if and only if } F_{xy}(\hat{M}_A) > F_{xy}(\tilde{M}_B). \tag{13}$$

Symmetrically, we can obtain that

$$v^{\{x,y\}}(\hat{M}_A) = \{y\} \text{ if and only if } F_{xy}(\hat{M}_A) < F_{xy}(\tilde{M}_B). \tag{14}$$

It follows from (13) and (14) that condition (10) holds if and only if $p_x^{\{x,y\}} = G_y(\tilde{M}_B)$, $p_y^{\{x,y\}} = G_x(\tilde{M}_B)$, and $x \sim^{\{x,y\}} y$.

4. Assume that for all electorates B and all $\tilde{M}_B \in \mathcal{M}_B$, $v^{\{x,y\}}(\tilde{M}_B) \neq \{x, y\}$ and neither Case 1 nor Case 2 holds. Consider the electorate N and a response profile $\hat{M}_N$ with the property that $G_x(\hat{M}_N) = G_y(\hat{M}_N) = 1$. Suppose without loss of generality that $v^{\{x,y\}}(\hat{M}_N) = \{x\}$. Moreover, set $p_y^{\{x,y\}} \equiv 1$.

   Observe that by Lemma 1, $v^{\{x,y\}}$ is a function of the relative support and, by Lemma 2, this function is monotonic. Moreover, by the assumption defining Case 4, $v^{\{x,y\}}$

is a singleton set and, by coherence, there exists a response profile $\bar{M}_B \in \mathcal{M}_B$ with strictly positive support for both alternatives such that $v^{\{x,y\}}(\bar{M}_N) = \{y\}$. Hence, there exists a real number $r \geq 1$ such that for all electorates B and all response profiles $\tilde{M}_B$ with the property that $G_y(\tilde{M}_B) > G_x(\tilde{M}_B)$, at least one of the following two cases holds:

(a) if $F_{yx}(\tilde{M}_B) > r$, $v^{\{x,y\}}(\tilde{M}_B) = \{y\}$ and if $F_{yx}(\tilde{M}_B) \leq r$, $v^{\{x,y\}}(\tilde{M}_B) = \{x\}$ or

(b) if $F_{yx}(\tilde{M}_B) \geq r$, $v^{\{x,y\}}(\tilde{M}_B) = \{y\}$ and if $F_{yx}(\tilde{M}_B) < r$, $v^{\{x,y\}}(\tilde{M}_B) = \{x\}$.

Now, if $r \in \mathbb{Q}_{++}$, condition (10) holds for $p_x^{\{x,y\}} = r$, $p_y^{\{x,y\}} = 1$, and $x \succ^{\{x,y\}} y$ in the first case and $p_x^{\{x,y\}} = r$, $p_y^{\{x,y\}} = 1$, and $y \succ^{\{x,y\}} x$ in the second case. If $r \notin \mathbb{Q}_{++}$, condition (10) holds for $p_x^{\{x,y\}} = r$, $p_y^{\{x,y\}} = 1$, and any tie-breaking rule $\succsim^{\{x,y\}}$ on $\{x, y\}$. $\square$

In Lemma 3 we have shown that for any pair of alternatives $x, y \in \mathcal{K}$, there are two strictly positive and finite weights, $p_x^{\{x,y\}}$ and $p_y^{\{x,y\}}$, and a tie-breaking rule $\succsim^{\{x,y\}}$ on $\{x, y\}$ such that the subfamily of voting rules $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x^{\{x,y\}}, p_y^{\{x,y\}})$ and $\succsim^{\{x,y\}}$. Hence, so far we have constructed for every alternative $x \in \mathcal{K}$, $\kappa - 1$ weights and $\kappa - 1$ tie-breaking rules that can be applied when x is confronted with each alternative $y \neq x$. We show next that it is possible to construct a single weight for every alternative.

Lemma 4 *Assume that the family of voting rules v is consistent in alternatives and voters, anonymous, coherent, and satisfies the no-support condition. Then, there exists a κ-tuple of weights $(p_z)_{z\in\mathcal{K}} \in \mathbb{R}^{\kappa}_{++}$ and $\frac{\kappa(\kappa-1)}{2}$ tie-breaking rules $\succsim^{\{x,y\}}$, one for every pair of different alternatives $x, y \in \mathcal{K}$, such that for all alternatives $x, y \in \mathcal{K}$, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x, p_y)$ and $\succsim^{\{x,y\}}$.*

Proof: The proof is done by induction on the set of feasible alternatives. Take any $K \subset \mathcal{K}$ of cardinality two. By Lemma 3, there are two weights $(p_x^K)_{x\in K} \in \mathbb{R}^2_{++}$ and a tie-breaking rule $\succsim^K$ such that v^K is the Weighted Approval Voting relative to p^K and $\succsim^K$.

Induction Hypothesis: Suppose that given the set of feasible alternatives $K \subset \mathcal{K}$ of cardinality $k \geq 2$, there exists a k-tuple of weights $\left(p_x^K\right)_{x \in K} \in \mathbb{R}_{++}^k$ and $\frac{k(k-1)}{2}$ tie-breaking rules $\succsim^{\{x,y\}}$, one for every pair of different and feasible alternatives $x, y \in K$, such that for all $x, y \in K$, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x^K, p_y^K)$ and $\succsim^{\{x,y\}}$.

We have to prove that if the set of feasible alternatives is equal to $K \cup \{z\}$, $z \notin K$, then there exists a $k+1$-tuple of weights $(p_x^{K \cup \{z\}})_{x \in K \cup \{z\}} \in \mathbb{R}_{++}^{k+1}$ and $\frac{(k+1)k}{2}$ tie-breaking rules $\succsim^{\{x,y\}}$, one for every pair of different and feasible alternatives $x, y \in K \cup \{z\}$, such that for all $x, y \in K \cup \{z\}$, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x^{K \cup \{z\}}, p_y^{K \cup \{z\}})$ and $\succsim^{\{x,y\}}$.

For all alternatives $x \in K$, let $p_x^{K \cup \{z\}} = p_x^K$. Then, for all $x, y \in K$, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x^{K \cup \{z\}}, p_y^{K \cup \{z\}})$ and $\succsim^{\{x,y\}}$ by the induction hypothesis. By Lemma 3, we know that for all $x \in K$, there exist two strictly positive and finite weights, $p_x^{\{x,z\}}$ and $p_z^{\{x,z\}}$, and a tie-breaking rule $\succsim^{\{x,z\}}$ such that $v^{\{x,z\}}$ is the Weighted Approval Voting relative to $p^{\{x,z\}} = (p_x^{\{x,z\}}, p_z^{\{x,z\}})$ and $\succsim^{\{x,z\}}$. Consequently, it remains to determine the weight $p_z^{K \cup \{z\}}$.

By Remark 1, the weights $p_x^{\{x,z\}}$ and $p_z^{\{x,z\}}$ are determined up to proportional changes; that is, if we multiply both by $\lambda > 0$, then the result of the election when x is confronted with z does not change. Set λ equal to $p_x^{K \cup \{z\}} = \lambda \cdot p_x^{\{x,z\}}$, or, $\lambda = p_x^{K \cup \{z\}} / p_x^{\{x,z\}}$. Define $p_z^{K \cup \{z\}} = \lambda \cdot p_z^{\{x,z\}}$. Thus, $p_z^{K \cup \{z\}} = p_z^{\{x,z\}} \cdot p_x^{K \cup \{z\}} / p_x^{\{x,z\}}$. Without loss of generality we can also define $p_x^{K \cup \{z\}} \equiv 1$, by setting $\lambda' = 1/(\lambda \cdot p_x^{\{x,z\}})$. Then, $p_z^{K \cup \{z\}} = p_z^{\{x,z\}} / p_x^{\{x,z\}}$. Since, by Lemma 3, $v^{\{x,z\}}$ is the Weighted Approval Voting relative to $(p_x^{\{x,z\}}, p_z^{\{x,z\}})$ and $\succsim^{\{x,z\}}$, we conclude that this subfamily is also the Weighted Approval Voting relative to $p^{\{x,z\}} = (p_x^{K \cup \{z\}}, p_z^{K \cup \{z\}})$ and $\succsim^{\{x,z\}}$.

We still have to show that given any alternative $y \in K \backslash \{x\}$, $v^{\{y,z\}}$ is the Weighted Approval Voting relative to $p^{\{y,z\}} = (p_y^{K \cup \{z\}}, p_z^{K \cup \{z\}})$ and $\succsim^{\{y,z\}}$. To do so, we prove that there exists a $\mu > 0$ such that $p_y^{K \cup \{z\}} = \mu \cdot p_y^{\{y,z\}}$ and $p_z^{K \cup \{z\}} = \mu \cdot p_z^{\{y,z\}}$. Rewrite these two equations as $p_y^{K \cup \{z\}} \cdot p_z^{\{y,z\}} = p_z^{K \cup \{z\}} \cdot p_y^{\{y,z\}}$ and suppose otherwise. That is,

$$\delta \equiv p_z^{K\cup\{z\}} \cdot p_y^{\{y,z\}} - p_y^{K\cup\{z\}} \cdot p_z^{\{y,z\}} > 0.$$

Note that we can deal with the case $\delta < 0$ using a symmetric argument. Let $\bar{p}_z \equiv \frac{n_z}{m_z}$ and $\bar{p}_y \equiv \frac{n_y}{m_y}$ be two rational numbers such that $\bar{p}_z < p_z^{K\cup\{z\}}$, $\bar{p}_y > p_y^{K\cup\{z\}}$, and

$$\bar{p}_z \cdot p_y^{\{y,z\}} - \bar{p}_y \cdot p_z^{\{y,z\}} > 0. \tag{15}$$

Here, n_y, m_y, n_z and m_z are strictly positive integers. Observe that $\bar{p}_y$ and $\bar{p}_z$ must exist, because the set of rational numbers is dense in the set of real numbers. Rewrite equation (15) as

$$p_y^{\{y,z\}} \cdot (n_z \cdot m_y) > p_z^{\{y,z\}} \cdot (n_y \cdot m_z). \tag{16}$$

Consider now the electorate N of size $n \geq \min\{n_y \cdot n_z, n_y \cdot m_z, n_z \cdot m_y\}$ and let the response profile M_N be such that $G_x(M_N) = n_z \cdot n_y$, $G_y(M_N) = n_z \cdot m_y$, and $G_z(M_N) = n_y \cdot m_z$. Since, by Lemma 3, $v^{\{y,z\}}$ is the Weighted Approval Voting relative to $p^{\{y,z\}} = (p_y^{\{y,z\}}, p_z^{\{y,z\}})$ and $\succsim^{\{y,z\}}$, $v^{\{y,z\}}(M_N) = \{y\}$ by equation (16). This implies, by consistency in alternatives, that $z \notin v^{\{x,y,z\}}(M_N)$.

In addition, $G_x(M_N) = \bar{p}_y \cdot G_y(M_N) > p_y^{K\cup\{z\}} \cdot G_y(M_N)$. Since $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (1, p_y^{K\cup\{z\}})$ and $\succsim^{\{x,y\}}$ by construction, $v^{\{x,y\}}(M_N) = \{x\}$. This implies, by consistency in alternatives, that $y \notin v^{\{x,y,z\}}(M_N)$.

The two conditions $z \notin v^{\{x,y,z\}}(M_N)$ and $y \notin v^{\{x,y,z\}}(M_N)$ imply that $v^{\{x,y,z\}}(M_N) = \{x\}$. Hence, $v^{\{x,y,z\}}(M_N) \cap \{x, z\} = \{x\}$ and, by consistency in alternatives, $v^{\{x,z\}}(M_N) = \{x\}$. Finally, since $v^{\{x,z\}}$ is the Weighted Approval Voting relative to $p^{\{x,z\}} = (1, p_z^{K\cup\{z\}})$ and $\succsim^{\{x,z\}}$ by construction, $v^{\{x,z\}}(M_N) = \{x\}$ implies $p_z^{K\cup\{z\}} \cdot G_z(M_N) \leq 1 \cdot G_x(M_N)$. But $p_z^{K\cup\{z\}} \cdot G_z(M_N) > \bar{p}_z \cdot G_z(M_N) = \frac{n_z}{m_z} \cdot n_y \cdot m_z = G_x(M_N)$. This is a contradiction.
Hence, there is a $(k+1)$-tuple of strictly positive and finite weights $(p_x^{K\cup\{z\}})_{x\in K\cup\{z\}}$ and $\frac{(k+1)k}{2}$ binary relations $\succsim^{\{x,y\}}$ such that for all $x, y \in K\cup\{z\}$, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x^{K\cup\{z\}}, p_x^{K\cup\{z\}})$ and $\succsim^{\{x,y\}}$. The Lemma follows finally from the case $K \cup \{z\} = \mathcal{K}$ and $p_x \equiv p_x^{\mathcal{K}}$ for all $x \in \mathcal{K}$. □

So far, we have constructed the vector of weights p, but we still have not shown that the $\frac{\kappa(\kappa-1)}{2}$ tie-breaking rules (one for each pair of different alternatives $x, y \in \mathcal{K}$)obtained from the pairwise comparisons in Lemma 3 induce a unique tie-breaking rule $\succsim$ on $\mathcal{K}$ such that for every pair $x, y \in \mathcal{K}$, $\succsim|_{\{x,y\}} = \succsim^{\{x,y\}}$. This is done next.

Lemma 5 *Assume that the family of voting rules v is consistent in alternatives and voters, anonymous, coherent, and satisfies the no-support condition. Then, there exists a κ-tuple of weights $(p_z)_{z\in\mathcal{K}} \in \mathbb{R}^{\kappa}_{++}$ and a tie-breaking rule $\succsim$ on $\mathcal{K}$ such that for all $x, y \in \mathcal{K}$, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x, p_y)$ and $\succsim|_{\{x,y\}}$.*

Proof: Take any v that satisfies the hypothesis of the Lemma. By Lemma 4, there exist a κ-tuple of weights $(p_z)_{z\in\mathcal{K}} \in \mathbb{R}^{\kappa}_{++}$ and $\frac{\kappa(\kappa-1)}{2}$ tie-breaking rules $\succsim^{\{x,y\}}$, one for every pair of different alternatives $x, y \in \mathcal{K}$, such that $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x, p_y)$ and $\succsim^{\{x,y\}}$. Hence, we have to prove that it is possible to construct a complete, reflexive, and transitive binary relation $\succsim$ on $\mathcal{K}$ from $(\succsim^{\{x,y\}})_{\substack{x,y\in\mathcal{K}\\ x\neq y}}$ such that for every pair $x, y \in \mathcal{K}$, $\succsim|_{\{x,y\}} = \succsim^{\{x,y\}}$.

According to Lemma 3, $\succsim^{\{x,y\}}$ is completely prescribed if and only if both alternatives have a rational weight. Otherwise, the binary relation can be chosen freely. So, let K be the set of all alternatives that have a rational weight; that is, $x \in K$ if and only if $p_x \in \mathbb{Q}_{++}$. We show that the $\frac{k(k-1)}{2}$ tie-breaking rules $(\succsim^{\{x,y\}})_{\substack{x,y\in K\\ x\neq y}}$ induce a unique complete, reflexive, and transitive binary relation $\succsim^K$ on K with the property that for all $x, y \in K$, $\succsim^K\big|_{\{x,y\}} = \succsim^{\{x,y\}}$.

Define the binary relation $\succsim^K$ on K as follows: for every pair of different alternatives $x, y \in K$, set $x \succsim^K y$ if and only if $x \succsim^{\{x,y\}} y$. Obviously, $\succsim^K$ is complete and reflexive. We have to show that $\succsim^K$ is transitive as well. Suppose otherwise; that is, there exists a triple $x, y, z \in K$ such that $x \succsim^{\{x,y\}} y$, $y \succsim^{\{y,z\}} z$, and $z \succ^{\{x,z\}} x$. For all $w \in \{x, y, z\}$, define $p_w = \frac{n_w}{m_w}$, where n_w and m_w are strictly positive integers. Now, consider any response profile M_N with the property that $G_x(M_N) = n_y \cdot m_x \cdot n_z$, $G_y(M_N) = n_x \cdot m_y \cdot n_z$, and $G_z(M_N) = n_x \cdot m_z \cdot n_y$. Then, $p_x \cdot G_x(M_N) = p_y \cdot G_y(M_N) = p_z \cdot G_z(M_N) = n_x \cdot n_y \cdot n_z$. Since $x \succsim^{\{x,y\}} y$ by assumption, $x \in v^{\{x,y\}}(M_N)$. It follows from consistency in alternatives that $x \in v^{\{x,y,z\}}(M_N)$ whenever $y \in v^{\{x,y,z\}}(M_N)$. Moreover, since $y \succsim^{\{y,z\}} z$ by assumption, $y \in v^{\{x,y\}}(M_N)$. It follows from consistency in alternatives that $y \in v^{\{x,y,z\}}(M_N)$ whenever

$z \in v^{\{x,y,z\}}(M_N)$. Both results together imply that $v^{\{x,y,z\}}(M_N) \in \{\{x\}, \{x,y\}, \{x,y,z\}\}$. Hence, $x \in v^{\{x,y,z\}}(M_N) \cap \{x,z\}$ and it follows from consistency in alternatives that $x \in v^{\{x,z\}}(M_N)$. But, $z \succ^{\{x,z\}} x$ and $p_x \cdot G_x(M_N) = p_z \cdot G_z(M_N)$ imply that $v^{\{x,z\}}(M_N) = \{z\}$, a contradiction. Hence, the $\frac{k(k-1)}{2}$ tie-breaking rules $(\succsim^{\{x,y\}})_{\substack{x,y \in \mathcal{K} \\ x \neq y}}$ induce a complete, reflexive, and transitive binary relation $\succsim^K$ on K with the property that for every pair $x, y \in \mathcal{K}$, $\succsim^K\big|_{\{x,y\}} = \succsim^{\{x,y\}}$.

Finally, observe that $\succsim^K$ induces a reflexive and transitive (but not complete) binary relation $\succsim^*$ on $\mathcal{K}$. However, according to Szpilrajn [18], any reflexive and transitive binary relation can be completed in a transitive way. $\square$

In the next and last step of the proof, we apply consistency in alternatives to generalize Lemma 5 to all sets of feasible alternatives.

Proof of Theorem 1: It is easy to check that any Weighted Approval Voting satisfies consistency in alternatives and voters, anonymity, the no-support condition, and coherence. To prove the other implication let v be a family of voting rules that satisfies consistency in alternatives and voters, anonymity, the no-support condition, and coherence. We show that the κ-tuple of strictly positive and finite weights $(p_x)_{x \in \mathcal{K}}$ constructed in Lemma 4 and the tie-breaking rule $\succsim$ on $\mathcal{K}$ identified in Lemma 5 are such that for all sets of feasible alternatives K and all response profiles M_N,

$$\begin{aligned} x \in v^K(M_N) \Leftrightarrow \quad & p_x \cdot G_x(M_N) \geq p_y \cdot G_y(M_N) \text{ for all } y \in K \text{ and} \\ & x \succsim y \text{ for all } y \in K \text{ satisfying } p_x \cdot G_x(M_N) = p_y \cdot G_y(M_N) > 0. \end{aligned}$$

Assume that $x \in v^K(M_N)$. Then, by consistency in alternatives, $x \in v^{\{x,y\}}(M_N)$ for all $y \in K \backslash \{x\}$. By Lemma 5, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x, p_y)$ and $\succsim|_{\{x,y\}}$. Hence, $p_x \cdot G_x(M_N) \geq p_y \cdot G_y(M_N)$ for all $y \in K$, and $x \succsim y$ for all $y \in K$ satisfying $p_x \cdot G_x(M_N) = p_y \cdot G_y(M_N) > 0$.

Assume that $p_x \cdot G_x(M_N) \geq p_y \cdot G_y(M_N)$ for all $y \in K$, and $x \succsim y$ for all $y \in K$ satisfying $p_x \cdot G_x(M_N) = p_y \cdot G_y(M_N) > 0$. Then, for all $y \in K \setminus \{x\}, x \in v^{\{x,y\}}(M_N)$ because, by Lemma 5, $v^{\{x,y\}}$ is the Weighted Approval Voting relative to $p^{\{x,y\}} = (p_x, p_y)$ and $\succsim|_{\{x,y\}}$.

If there is some $z \neq x$ such that $z \in v^K(M_N)$, then $v^K(M_N) \cap \{x, z\} \neq \emptyset$. Hence, $v^{\{x,z\}}(M_N) = v^K(M_N) \cap \{x, z\}$ by consistency in alternatives. Since, by Lemma 5, $v^{\{x,z\}}$ is the Weighted Approval Voting relative to $p^{\{x,z\}} = (p_x, p_z)$ and $\succsim|_{\{x,z\}}$, $x \in v^{\{x,z\}}(M_N)$. Hence, $x \in v^K(M_N)$. If there does not exist any alternative $z \neq x$ such that $z \in v^K(M_N)$, then $x \in v^K(M_N)$ as well because the set $v^K(M_N)$ cannot be empty. Finally, observe that if the response profile M_N is such that $G_x(M_N) = 0$ for all $x \in K$, then $v^K(M_N) = K$ by the no-support condition. This concludes the proof. □

5 Final Remarks

We show next, with the help of five examples, the independence of the properties used in Theorem 1. Finally, we argue that additional axioms are needed if the aim is to characterize, for a fixed electorate or a fixed set of feasible alternatives, the class of all Weighted Approval Voting without the corresponding consistency property.

5.1 Independence of the Axioms

Consistency in Alternatives: Fix $x \in \mathcal{K}$. Let the family of voting rules v be such that for all sets of feasible alternatives K of size two, all profiles M, and all electorates N, $v^K(M_N) = v_A^K(M_N)$. Otherwise, apply the Weighted Approval Voting with weights $p_x = 2$ and $p_y = 1$ for all $y \neq x$. Assume that ties are not broken. This family satisfies consistency in voters, anonymity, the no-support condition, and coherence. The following example shows that it is not consistent in alternatives. Let $\mathcal{K} = \{x, y, z\}$ and suppose that $N = \{i, j\}$. If $M_i = M_j = \mathcal{K}$, then $v^{\{x,y\}}(M_i + M_j) = \{x, y\}$ and $v^{\{x,y,z\}}(M_i + M_j) = \{x\}$. Consistency in alternatives would imply that $v^{\{x,y\}}(M_i + M_j) = v^{\{x,y,z\}}(M_i + M_j) \cap \{x, y\} = \{x\}$. Hence, v does not satisfy consistency in alternatives.

Consistency in Voters: Let the family of voting rules v be such that for all sets of feasible alternatives K, all profiles M, and all electorates N such that $G_x(M_N) > 1$ for some $x \in K$, $v^K(M_N) = v_A^K(M_N)$. Otherwise, apply the Weighted Approval Voting with weights

$p_x = 2$ and $p_y = 1$ for all $y \neq x$. Assume that ties are not broken. This family satisfies consistency in alternatives, anonymity, the no-support condition, and coherence. The following example shows that it is not consistent in voters. Let $\mathcal{K} = \{x, y, z\}$ and suppose that $N = \{i, j\}$. If $M_i = M_j = \{x, y\}$, then $v^{\{x,y\}}(M_i) = v^{\{x,y\}}(M_j) = \{x\}$ and $v^{\{x,y\}}(M_i + M_j) = \{x, y\}$. Consistency in voters would imply that $v^{\{x,y\}}(M_i + M_j) = v^{\{x,y\}}(M_i) \cap v^{\{x,y\}}(M_j) = \{x\}$. Hence, v does not satisfy consistency in voters.

Anonymity: Assign to each voter $i \in \mathbb{N}$ a strictly positive and finite number q_i in such a way that $q_i \neq q_j$ for some pair $i, j \in \mathbb{N}$. Now, let the family of voting rules v be such that for all sets of feasible alternatives K, all profiles M, and all electorates N, $x \in v^K(M_N)$ if and only if $\sum_{i \in N : x \in M_i} q_i \geq \sum_{i \in N : y \in M_i} q_i$ for all $y \in K$. This family satisfies consistency in alternatives and voters, the no-support condition, and coherence. The following example shows that it is not anonymous. Let $\mathcal{K} = \{x, y, z\}$ and suppose that $N = \{i, j\}$. Moreover, let $q_i = 2$ and $q_j = 1$. If $M_i = M'_j = \{x\}$ and $M'_i = M_j = \{y\}$, then $v^{\{x,y\}}(M_i + M_j) = \{x\}$ and $v^{\{x,y\}}(M'_i + M'_j) = \{y\}$. Hence, v does not satisfy anonymity.

No-Support: Let the family of voting rules v be such that for all sets of feasible alternatives K, all profiles M, and all electorates N, $x \in v^K(M_N)$ if and only if $G_x(M_N) \leq G_y(M_N)$ for all $y \in K$. This family satisfies consistency in alternatives and voters, anonymity, and coherence. The following example shows that it does not satisfy the no-support condition. Let $\mathcal{K} = \{x, y, z\}$ and suppose that $N = \{i, j\}$. If $M_i = M_j = \{y\}$, then $v^{\{x,y\}}(M_i + M_j) = \{x\}$. Hence, v does not satisfy the no-support condition.

Coherence: Let $\eta : \mathcal{K} \to \{1, ..., \kappa\}$ be any one-to-one mapping that assigns to every $x \in \mathcal{K}$ a positive integer between 1 and κ. Given η, let the family of voting rules v be such that for all sets of feasible alternatives K, all profiles M, and all electorates N, $v^K(M_N) = \{y \in K : G_y(M_N) > 0$ and $\eta(y) < \eta(z)$ for all $z \in K$ s.t. $G_z(M_N) > 0\}$. If no alternative gets any vote, then $v^K(M_N) = K$. This family satisfies consistency in alternatives and voters, anonymity, and the no-support condition. The following example shows that it does not satisfy coherence. Let $\mathcal{K} = \{x, y, z\}$ and define η to be such that $\eta(x) = 1$ and $\eta(y) = 2$.

Then, for all profiles M and all electorates N such that $G_x(M_N) > 0$ and $G_y(M_N) > 0$, $v^{\{x,y\}}(M_N) = \{x\}$. Hence, v does not satisfy coherence.

5.2 Consistency Properties

An additional point, also related to the independence of the properties, regards the question whether it is possible to obtain a similar characterization of all Weighted Approval Voting for a given electorate (or a given set of feasible alternatives); that is, if the electorate (or the set of feasible alternatives) is fixed at N (or at K) and the corresponding consistency property is dropped, is the class of all Weighted Approval Voting characterized by the remaining four properties? The following two examples show that this is not the case.

Example 1: Suppose that the electorate is equal to $N = \{1, 2\}$. Let the family of voting rules $\hat{v} = \{\hat{v}^{K,N} : (2^{\mathcal{K}})^N \to 2^K \backslash \{\emptyset\}\}_K$ be such that for all sets of feasible alternatives $K \subseteq \mathcal{K}$ and all response profiles $M_N \in (2^{\mathcal{K}})^N$ such that $G_r(M_N) > 0$ for some $r \in \mathcal{K}$, $\hat{v}^K(M_N) = \{r \in K : G_r(M_N) > 0\}$. If no alternatives receives any vote, the set K is elected. This family of voting rules satisfies consistency in alternatives, anonymity, coherence, and the no-support condition. Yet, $\hat{v}$ is not a Weighted Approval Voting because there does not exist a vector of weights $p = (p_x)_{x \in \mathcal{K}} \in \mathbb{R}^{\kappa}_{++}$ and a tie-breaking rule $\succsim$ on $\mathcal{K}$ such that for all sets of feasible alternatives K and all response profiles M_N,

$$r \in \hat{v}^K(M_N) \Leftrightarrow \begin{array}{l} p_r \cdot G_r(M_N) \geq p_s \cdot G_s(M_N) \text{ for all } s \in K \text{ and} \\ r \succsim s \text{ for all } s \in K \text{ such that } p_r \cdot G_r(M_N) = p_s \cdot G_s(M_N) > 0. \end{array} \tag{17}$$

To see it, let $\mathcal{K} = \{x, y, z\}$ and consider two response profiles M_N and M'_N with the property that $G_x(M_N) = G_y(M'_N) = 1$ and $G_x(M'_N) = G_y(M_N) = 2$. Observe that, by definition of $\hat{v}$, $\hat{v}^{\{x,y\}}(M_N) = \hat{v}^{\{x,y\}}(M'_N) = \{x, y\}$. Let $(p_x, p_y, p_z) \in \mathbb{R}^3_{++}$ be an arbitrary vector weights. Condition (17) implies simultaneously that $p_x = 2 \cdot p_y$ and $2 \cdot p_x = p_y$. Hence, $p_x = p_y = 0$. Observe that this argument works even if we admit zero weights. To see that, take any response profile M''_N which satisfies $G_x(M''_N) > 0$ and $G_y(M''_N) = 0$. Since $\hat{v}^K(M''_N) = \{x\}$ by definition of $\hat{v}^K$, condition (17) implies that $p_x > 0$. This contradicts $p_x = 0$. ■

Example 2: Suppose that the set of feasible alternatives is equal to $K = \{x, y, z\}$. Let the family of voting rules $\tilde{v} = \{\tilde{v}^{K,N} : (2^K)^{\mathbb{N}} \to 2^K \backslash \{\emptyset\}\}_N$ be such that for all profiles $M \in (2^K)^{\mathbb{N}}$ and all electorates N, (a) if $v_A^K(M_N) = \{x, y\}$, then $\tilde{v}^K(M_N) = \{x\}$, (b) if $v_A^K(M_N) = \{y, z\}$, then $\tilde{v}^K(M_N) = \{y\}$, (c) if $v_A^K(M_N) = \{x, z\}$, then $\tilde{v}^K(M_N) = \{z\}$, and (d) $\tilde{v}^K(M_N) = v_A^K(M_N)$ in all other situations. This family of voting rules is consistent in voters (if two disjoint electorates elect a common set of alternatives exactly those alternatives are elected when the two electorates are assembled), anonymous (the voting rule depends only on the amount of votes every alternative receives), coherent (given an alternative, there is a situation in which all alternatives have strictly positive support and the considered alternative belongs to the image), and satisfies the no-support condition (if an alternative does not get any vote it is selected if and only if all alternatives have zero support). Yet, $\tilde{v}$ is not a Weighted Approval Voting because the cycle induces a non-transitive tie-breaking rule. To see it, assume otherwise, and let $p = (p_x, p_y, p_z) \in \mathbb{R}^3_{++}$ and $\succsim$ be the vector of weights and the complete preoder associated to $\tilde{v}$. Then, (a) and (d) imply that $p_x = p_y$ and $x \succ y$, (b) and (d) imply that $p_y = p_z$ and $y \succ z$, and (c) and (d) imply that $p_z = p_x$ and $z \succ x$. Consequently, the tie-breaking rule is not transitive. ■

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