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MEMORY STRATEGIES IN NONATOMIC REPEATED GAMES

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Resumen Con el objeto de considerar comportamientos con memoria acotada en juegos repetidos no atómicos, se restringen los conjuntos de estrategias de los jugadores de tal manera que éstos sólo pueden tener en cuenta los promedios de las proporciones de las jugadas agregadas anteriores. Se impone un requerimiento mínimo de racionalidad, se define una solución para el juego y se demuestra la existencia de soluciones estables. El resultado más importante obtenido es que bajo hipótesis de continuidad de las funciones de pagos, una solución estable constituye un equilibrio de Nash del juego no restringido.

Abstract In order to account for bounded memory in a nonatomic repeated game, the strategy sets of the players are restricted in such a way that the players can take into account only the averages of the proportions of the previous aggregate plays. A minimal rationality requirement is imposed, a solution for the game is defined, and the existence of stable solutions is established. The main result obtained is that under continuous payoffs a stable solution constitutes a Nash equilibrium of the unrestricted game.

1. INTRODUCTION

Repeated games have been used in economics to analyze conflict situations lasting over time. Examples of their applicability include things as diverse as a firm and a union bargaining over the wage rate on the one hand, and institutional arrangements for the buying and selling of commodities such as auctions and competitive markets on the other (1). Here, we focus on a particular type of repeated game, the main characteristic of which is the presence of a large number of players. A large class of economic phenomena possess such characteristic.

The most well-known model of a repeated game is the one that assumes perfect monitoring: at any point in time a player is able to remember the past actions taken by everybody else. In such a setting, a player's current action may be contingent to different histories of play. Smale (1980) argued that for reasons of bounded memory this way of modeling players' behavior is not a very sensible one. He suggested a continuous stationary behavior based on a finite-dimensional summary of the history of the game. He showed that if the aggregate behavior is stable and players maximize payoffs at the stationary point, then players don't have incentives to become more sophisticated (using all the information generated by the play of the game) if one restricts the strategies of the

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(1) See, for example, Perry and Solon (1985), Feinstein, Block and Nold (1985), Rotemberg and Saloner (1986) and Green (1980).

players (the set of possible deviations from the stationary behavior) to satisfy a very strong requirement of continuity.

Here, we would like to strengthen Smale result in a context in which to assume players with bounded rationality makes even more sense: the game that is repeated over time has a large set of players. There are at least two reasons for considering models of bounded rationality with a large set of individuals. First, a wide class of economic situations is characterized by a large set of decision makers. Second, the larger is the number of agents, the greater is the complexity of the fully-rational individual decision process if it has to take into account the behavior of all other agents.

The main motivation of the paper is to prove in a formal model the intuition that bounded rational behavior (in particular, bounded memory behavior) is less restrictive when the nature of the conflict that agents face has a competitive flavor: individual actions have no influence in the aggregate. I do this by mimicing Smale's set up and using Schmeidler (1973) existence results.

We are therefore interested in the equilibrium implications of boundedly rational behavior in a dynamic context where the number of agents is large. In particular, we analyze a model (a nonatomic repeated game) in which agents can only remember an aggregate summary of previous outcomes. Thus, at any point in time, players are restricted to choose their actions according to a memory strategy which takes into account only some average of the history of the system. Following Smale (1980), we define a solution of a repeated game to be a stationary point of the dynamical system generated by an aggregate memory strategy. A solution is said to satisfy the Nash property if at the stationary point the aggregate memory strategy prescribes a Nash equilibrium of the one-shot game (i.e.; payoff maximization behavior is required only at the stationary point); such solutions are shown to exist. Our main result is that, roughly, under continuous payoffs any stable solution with the Nash property constitutes a Nash equilibrium in the original unrestricted, repeated game (with long-run average payoff criterion) and for every $\epsilon > 0$ there is a discount factor such that it constitutes an ϵ-best reply of the discounted repeated game.

It is easy to think of general economic models that without fitting this set up could reproduce the results of this paper; in particular models where: (i) there is a continuum of firms selling goods over time in a competitive market, that using some statistic of the past (not necessarily a statistic of the past quantities) decide the present amount to sell; (ii) there is a large number of buyers and/or sellers in a stock market deciding their current behavior taking into account only past statistics of the market (average Dow Jones indexes, etc.); and (iii) there is a continuum of buyers in a given set of markets deciding the purchases over time of different goods and using past information on price indexes, quality indexes, and so on.

The next section of the paper describes the mixed extensions of nonatomic repeated games. In section 3, the core of the paper, we define memory strategies and present the main results. At the end of the paper the reader will find an appendix with some proofs.

2. THE UNRESTRICTED GAME

Let (I, M, λ) be a measure space where $I = [0, 1]$ is the set of players, M is the σ-algebra of Lebesgue measurable subsets of I, and λ is the Lebesgue measure on M. Each player $i \in I$ has available a finite, uniformly bounded number of actions; and, without loss of generality, we may assume that this number is n for each $i \in I$. In order to consider the mixed extension of the one-shot game, define

$$\Delta^n = \left\{ x = (x^1, x^2, \ldots, x^n) \in \boldsymbol{R}^n \mid x^k \geq 0 \text{ for } 1 \leq k \leq n \text{ and } \sum_{k=1}^{n} x^k = 1 \right\}$$

as the set of mixed actions for each player. A joint action is a measurable function a from I to Δ^n. Thus, $a = (a^1, a^2, \ldots, a^n)$, where a^k is a measurable real-valued function from I to $[0, 1]$. In this case a^k is Lebesgue-integrable, and we write $\int_I a$ for the vector $(\int_I a^1(i)\, d\lambda(i), \ldots, \int_I a^n(i)\, d\lambda(i))$. Let A be the set of equivalence classes of such functions where a and b are equivalent iff $\lambda(\{i \in I \mid a(i) \neq b(i)\}) = 0$. In the sequel we follow the usual convention of neglecting the distinction between integrable functions and their equivalence classes.

Let $\mathscr{A}$ be the (larger) set of equivalence classes of the set

$$\{a: I \to \boldsymbol{R}^n \mid a \text{ is Lebesgue-integrable}\}$$

and let $\| \ \|_E$ be the maximum norm in $\boldsymbol{R}^n$. We can define a norm $\| \ \|_A$ on $\mathscr{A}$ as follows: given $a \in \mathscr{A}$

$$\|a\|_A = \left\| \int_I |a| \right\|_E = \max \{\int_I |a^k| : 1 \leq k \leq n\}.$$

It is easy to show that $\mathscr{A}$ is a complete normed vector space and A is a closed, bounded, convex subset of it. Following Schmeidler (1973), we may look at $\mathscr{A}$ as a locally convex linear topological space if we endow it with the L_1-weak topology (2). Then A is a compact, convex subset of it. Also, by the Eberlein-Smulian Theorem (Dunford and Schwartz (1966)) it is also sequentially compact.

For each player $i \in I$, we define n bounded functions $u_i^k: A \to \boldsymbol{R}$, $k = 1, 2, \ldots, n$, where $u_i^k(a)$ is the payoff to player i if he chooses action k and the joint action is a. Therefore, the payoff function for the mixed extension of the game for player i is the function $h_i: \Delta^n \times A \to \boldsymbol{R}$ defined by

$$h_i(x, a) = \sum_{k=1}^{n} x^k u_i^k(a) \text{ for every } (x, a) \in \Delta^n \times A.$$

Let y_i be a bound of the payoff function h_i. In addition, we also assume that for every $a \in A$ the function $h_i(a(i), a)$ is measurable as a function of i.

Before proceeding, a brief comment related to the assumptions already made may be useful. What is really needed here is that the number of actions be uniformly bounded (i.e., there is an upper bound common to all players) and the payoff function for every player be bounded below. If this is the case, then there is no restriction in assuming that the number

(2) The weak topology coincides with the natural topology generated by the family of seminorms $\{\rho_\ell \mid \ell \in \mathscr{A}^*\}$ where $\rho_\ell(a) = |\ell(a)|$ and $\mathscr{A}^*$ is the dual space of $\mathscr{A}$.

of actions is the same for all players, since we can modify the individual action set and the corresponding payoff function by introducing artificial actions with payoffs smaller than $-y_i$. The solution concept defined later in the paper will not be modified since continuity of memory strategies together with the Nash property will guarantee that players assign zero probability, in the long run, to any of the actions introduced artificially.

Now, we can define a *Nash equilibrium* of G (the mixed extension of the initial finite action game) as an $a \in A$ such that for *a.e.* $i \in I$,

$$h_i(a(i), a) \geq h_i(x, a) \quad \forall x \in \Delta^n.$$

Let A^* denote the set of Nash equilibria of G.

For our main goal of analyzing bounded memory behavior in a repeated game, we will have to deal with games having equilibrium points. Following Schmeidler (1973), the following assumption guarantees that A^* is nonempty (3):

A.1: (i) For *a.e.* $i \in I$ and $\forall k = 1, 2, ..., n$ the functions $u_i^k(\cdot)$ are weak continuous.

(ii) For every $a \in A$ and $k, l = 1, 2, ..., n$ the sets $\{i \in I \mid u_i^k(a) > u_i^l(a)\}$ are measurable.

In order to play the infinitely-repeated game in a fully rational way, at each period, say $t+1$, a player's action might depend on the history of the game up to period t. Therefore his action at period $t+1$ should be a function from the t-fold Cartesian product of A with itself (denoted by A^t) to Δ^n, i.e. a strategy for player i in the unrestricted, infinitely-repeated game G^∞ is a sequence of functions $f_i = \{f_i^1, f_i^2, ...\}$ satisfying:

(i) $f_i^1 \in \Delta^n$; and $\forall t \geq 1$

(ii) $f_i^{t+1}: A^t \rightarrow \Delta^n$.

Let F_i denote the set of f_i satisfying (i) and (ii) and let

$$F = \{(f_i)_{i \in I} \mid f_i \in F_i \ \forall i \in I; f_i^1 \text{ is measurable as a function of } i; \text{ and}$$
$$\forall t \geq 1 \text{ and } \forall (a^1, a^2, ..., a^t) \in A^t, f_i^{t+1}(a^1, a^2, ..., a^t) \text{ is measurable as a function of } i\}$$

denote the set of feasible joint strategies. Given $f \in F$, the play it produces is identified as follows: let $a^1(f) \in A$ be such that $a^1(f)(i) = f_i^1$ *a.e.* $i \in I$, and, recursively, $a^{t+1}(f) \in A$ be such that

$$a^{t+1}(f)(i) = f_i^{t+1}(a^1(f), ..., a^t(f)) \text{ a.e. } i \in I.$$

For player $i \in I$, let

$$h_i^T(f) = \frac{1}{T} \sum_{t=1}^{T} h_i(f_i^t(a^1(f), ..., a^{t-1}(f)), a^t(f)).$$

To define payoffs in G^∞, first we associate with each $f \in F$ the sequence $\{h_i^t(f)\}_{t=1}^\infty \in \ell_\infty$. Next, we take any Banach limit (4) on ℓ_∞ and we define the payoff function $H_i(f)$ to be

(3) For a more general existence result, in which aggregate actions are modeled via distributions, see Mas-Colell (1984).
(4) A Banach limit on ℓ_∞ is a linear functional LIM satisfying:

$$\liminf_{t \to \infty} y^t \leq LIM(y) \leq \limsup_{t \to \infty} y^t \quad \forall y \in \{y^1, y^2, ...\} \in \ell_\infty$$

The existence of such functionals is a well-known consequence of the Hahn-Banach Theorem.

the Banach limit evaluated at the sequence $\{h_i^t(f)\}_{t=1}^\infty$. Given $f \in F$ and $g_i \in F_i$, let $(f \mid g_i) \in F$ denote the joint strategy f with player i switching to g_i, defined by

$$(f \mid g_i)_j = \begin{cases} g_i & \text{if } j = i \\ f_j & \text{if } j \neq i \end{cases}$$

Now, $f \in F$ is a *Nash equilibrium* of G^∞ if for *a.e.* $i \in I$ $\ H_i(f) \geq H_i(f \mid g_i)\ \forall g_i \in F_i$.

Let F^* denote the set of Nash equilibria of G^∞.

3. MEMORY STRATEGIES

Since A is an infinite-dimensional space, each player might need an infinite memory in order to play his part of a Nash equilibrium in the repeated game. Considerations of bounded memory therefore suggest the possibility of modeling the behavior of players in a more restrictive way. In particular, we assume here that strategies are stationary and depend only on the average of proportions of players that have taken the different actions in the past.

Formally, define

$$\mathcal{S} = \{ s \mid s : \Delta^n \rightarrow \Delta^n, \text{ continuous}\}$$

as the set of *individual memory strategies* for the repeated game and define

$$S = \{s = (s_i)_{i \in I} \mid s_i \in \mathcal{S} \text{ for each } i \in I, \text{ and } \forall x \in \Delta^n\ s_i(x) \text{ is measurable as a function of } i\}$$

as the set of *aggregate memory strategies* (as before, identified as the set of its equivalence classes).

Given $s \in S$ and initial condition $z \in \Delta^n$, the repeated game is played as follows: at period $t = 1$ the aggregate play is $a^1 \in A$ defined by $a^1(i) = s_i(z)$ *a.e.* $i \in I$. Given a^1, define $\hat{a}^1 \in A$ by $\hat{a}^1 \in a^1$, and at the end of period 1 players only know and recall the vector $\int_I \hat{a}^1 \in \Delta^n$. In general, in period $t+1$, given $\int_I \hat{a}^t \in \Delta^n$, the aggregate play is $a^{t+1} \in A$ defined by $a^{t+1}(i) = s_i(\int_I \hat{a}^t)$ *a.e.* $i \in I$; and, at the end of period $t+1$, players only know and recall $\int_I \hat{a}^{t+1} \in \Delta^n$, where $\hat{a}^{t+1} \in A$ is defined by

$$\hat{a}^{t+1} = \frac{t}{t+1} \hat{a}^t + \frac{1}{t+1} a^{t+1}.$$

In the next period $(t+2)$ almost every $i \in I$ will play according to his s_i evaluated at $\int_I \hat{a}^{t+1}$. Thus, $s \in S$ and $z \in \Delta^n$ induce a sequence $\{a^t\}_{t=1}^\infty$ of functions in A which tells us how the repeated game is played in the aggregate.

Given $s \in S$ we may describe the dynamics of the system by the function $\phi_s: \Delta^n \rightarrow \Delta^n$ and the sequence of functions $\{\varphi_s^t: \Delta^n \rightarrow \Delta^n\}_{t=1}^\infty$ defined by:

$$\phi_s(x) = \int_I s_i(x)\, d\lambda(i), \text{ and}$$

$$\varphi_s^1 = \phi_s \text{ for } t = 1, \text{ and, } \forall t \geq 1,\ \varphi_s^{t+1}(x) = \frac{t}{t+1}\, x + \frac{1}{t+1}\, \phi_s(x).$$

Now, we can define a solution of the game as well as prove its existence.

Definition: A pair (s, x), where $s \in S$ and $x \in \Delta^n$, is a *solution* if x is stationary for the dynamics defined by ϕ_s and $\{\varphi_s^t\}_{t=1}^{\infty}$; i.e., if x is a fixed point of ϕ_s.

Proposition 1: For every $s \in S$ there exists $x \in \Delta^n$ such that (s, x) is a solution.

Proof: Δ^n is a compact and convex subset of $\boldsymbol{R}^n$, thus by Brouwer's Fixed Point Theorem we need only to show that ϕ_s is continuous. Let $\{x_m\}_{m=1}^{\infty}$ be an arbitrary sequence in Δ^n such that $\lim_{m\to\infty} x_m = x$. We have to show that $\lim_{m\to\infty} \phi_s(x_m) = \phi_s(x)$. By continuity of $s_i(\cdot)$, $\lim_{m\to\infty} s_i(x_m) = s_i(x)$, and for every $m \geq 1$ and for $a.e.\ i \in I$, $\| s_i(x_m) \|_E \leq 1$. Moreover, $\{s_i(x_m)\}_{m=1}^{\infty}$ is a sequence of measurable functions defined on I; therefore, by the Bounded Convergence Theorem,

$$\lim_{m\to\infty} \phi_s(x_m) = \lim_{m\to\infty} \int_I s_i(x_m)\, d\lambda(i) = \int_I \lim_{m\to\infty} s_i(x_m)\, d\lambda(i) = \int_I s_i(x)\, d\lambda(i) = \phi_s(x).$$

■

From our point of view a solution must also satisfy some stability property in order to be interesting. After defining global stability in our context, we will look for a set of sufficient conditions guaranteeing that a solution satisfies this requirement.

Definition: A solution (s, x) is *globally stable* if for every initial $z \in \Delta^n$,

$$\{\textstyle\int_I \hat{a}^t(i)\, d\lambda(i)\}_{t=1}^{\infty} \to x.$$

Proposition 2: Suppose $s \in S$ satisfies the following conditions: For $a.e.\ i \in I$ the function $s_i(\cdot)$ is differentiable. In addition, there exists $\eta \in [0, 1)$ such that $\forall x \in \Delta^n$ and $\forall k, l = 1, 2, \ldots, n$ and for $a.e.\ i \in I$

$$\left| \frac{\partial s_i^k(x)}{\partial x^l} \right| \leq \frac{\eta}{n}.$$

Then there exists a unique $x^* \in \Delta^n$ such that (s, x^*) is a solution. Moreover, (s, x^*) is globally stable.

Proof: See Appendix at the end of the paper.

The solution associated with any memory strategy so far has no relation to payoff maximization. We will now define a solution with a minimum requirement of rationality by imposing that at the (unique) stationary point the solution has to satisfy a one-shot Nash equilibrium property; i.e.,

Definition: A solution (s, x^*) has the *Nash property* if for $a.e.\ i \in I$,

$$h_i(s_i(x^*), a) \geq h_i(x, a) \ \ \forall x \in \Delta^n \ \text{ where } \ a(i) = s_i(x^*)\ a.e.\ i \in I.$$

The next two propositions give sufficient conditions for the existence of a globally stable solution with the Nash property (in the case of Proposition 4, the solution is in pure actions). Both are direct applications of results in Schmeidler (1973) and the proofs use only memoryless strategies.

Proposition 3: If A.1 is satisfied there exists a globally stable solution (s, x^*) with the Nash property.

Proof: By Theorem 1 in Schmeidler (1973), $\exists a^* \in A$ such that for $a.e.\ i \in I$,

$$h_i(a^*(i), a^*) \geq h_i(x, a^*)\ \forall x \in \Delta^n. \text{ Let } x^* = \textstyle\int_I a^*(i)\, d\lambda(i) \text{ and let } s_i(x) = a^*(i)\ \forall x \in \Delta^n.$$

The hypothesis of Proposition 2 as well as the Nash property are clearly satisfied.

■

Now, an additional assumption in order to guarantee that players use a pure action in the stationary point.

A.2: The payoff functions satisfy: For every $k = 1, 2, \ldots, n$ and for $a.e.\ i \in I$, if $\int_I a = \int_I b$ then $u_i^k(a) = u_i^k(b)$.

Assumption A.2 means that the game is anonymous in the sense that the payoff for any player depends only on the action that he takes and the proportion of players taking the different actions.

Proposition 4: If A.1 and A.2 are satisfied, there exists a globally stable solution (s, x^*) with the Nash property such that for $a.e.\ i \in I$ and some $k_i \in \{1, 2, \ldots, n\}$, $s_i^{k_i}(x^*) = 1$ (i.e., at the stationary point almost all players use a pure action).

Proof: By Theorem 2 in Schmeidler (1973) there exists $a^* \in A$ such that for $a.e.\ i \in I$, $h_i(a^*(i), a^*) \geq h_i(x, a^*)$ $\forall x \in \Delta^n$ and for some $k_i \in \{1, 2, \ldots, n\}$, $a^{*k_i}(i) = 1$. The result follows as in the proof of Proposition 3.

■

A natural question to ask is under which conditions an aggregate memory strategy $s \in S$ is a Nash equilibrium in the restricted game (where the strategy set for all players is $\mathcal{S}$) or, even more generally, when s is a Nash equilibrium in the unrestricted game (when the strategy set for player $i \in I$ is F_i). The next theorem answers the more general question, but before stating it we need the following definition.

Definition: A solution (s, x^*) *generates a Nash equilibrium* of the unrestricted repeated game if $\forall z \in \Delta^n$, there exists $f \in F^*$ (possibly depending on z) such that the sequence $\{a^t\}_{t=1}^{\infty}$ induced by s and z is such that $\forall t \geq 1$, $a^t(f) = a^t$.

Theorem: If A.1 is satisfied and (s, x^*) is a globally stable solution with the Nash property, then (s, x^*) generates a Nash equilibrium of the unrestricted repeated game.

Proof: See Appendix at the end of the paper.

The above theorem answers in an affirmative way the question of whether or not a memory strategy is an equilibrium of the unrestricted repeated game if players are absolutely patient (i.e., they do not discount the future). We would like to answer the same question for the case players use the discounted sum criterion in evaluating an infinite sequence of payoffs. To do so we will define the payoff for the discounted repeated game $\boldsymbol{G}^{\beta}$ where $\beta \in [0, 1)$ is the discount factor.

Given a strategy $f \in F$ the payoff for player $i \in I$ in $\boldsymbol{G}^{\beta}$ is defined as

$$H_i^{\beta}(f) \equiv (1-\beta) \sum_{t=1}^{\infty} \beta^{t-1} h_i(a^t(f)(i), a^t(f)).$$

Now, $f \in F$ is an equilibrium of $\boldsymbol{G}^{\beta}$ if for *a.e.* $i \in I$ $H_i^{\beta}(f \mid g_i) \leq H_i^{\beta}(f)$ $\forall g_i \in F_i$. The «Anti-Folk» Theorem says that only sequences of one-shot equilibria are possible in an equilibrium of the repeated games $\boldsymbol{G}^{\beta}$ [see Massó (1993)]. The idea behind it is simple: if the payoff criterion of the repeated game has the property that increasing the payoff in any stage game raises the overall payoff, then, because of the fact that individual actions are not discernible by the others, the equilibrium set of the repeated game consists of sequences of one-shot equilibria. Therefore, the «Anti-Folk» Theorem tells us that in general memory strategies are not equilibria of $\boldsymbol{G}^{\beta}$. However, the next propositions give a partial affirmative answer in the sense that memory strategies constitute a best reply up to ϵ. More precisely,

Proposition 5: In addition to A.1 suppose that for *a.e.* $i \in I$ and for every $k = 1, 2, ..., n$ the functions $u_i^k(\cdot)$ are norm continuous, and let $\{a^t\}_{t=1}^{\infty}$ be any sequence of actions generated by a globally stable solution (s, x^*) with the Nash property. Then, $\forall \epsilon > 0$ and for *a.e.* $i \in I$, $\exists \beta_i \in (0, 1)$ such that $\forall \beta \in (\beta_i, 1)$

$$\left| (1-\beta) \sum_{t=1}^{\infty} \beta^{t-1} \left(h_i(a^t(i), a^t) - h_i(b_i^t, a^t) \right) \right| < \epsilon$$

for every sequence $\{b_i^1, b_i^2, ...\}$ of points in Δ^n such that

$$b_i^t \in \underset{x \in \Delta^n}{argmax}\ h_i(x, a^t) \text{ for every } t \geq 1.$$

Proof: See Appendix at the end of the paper.

Proposition 6: In addition to the assumptions of Proposition 5 suppose:

(i) For every $k = 1, 2, ..., n$ and *a.e.* $i \in I$ u_i^k's are uniformly norm continuous; i.e., if $\{a^t\}_{t=1}^{\infty} \to a$ in norm, then $\forall \epsilon > 0$ $\exists T \geq 1$ such that, $| h_i(a^t(i), a^t) - h_i(a(i), a) | < \epsilon$ $\forall t > T$.

(ii) $\exists y \in \boldsymbol{R}_+$ such that $\forall (x, a) \in \Delta^n \times A$ $| h_i(x, a) | < y$ for *a.e.* $i \in I$.

Then $\forall \epsilon > 0$, $\exists \hat{\beta} \in (0, 1)$ such that $\forall \beta \in (\hat{\beta}, 1)$ and for *a.e.* $i \in I$ the following is true:

$$\left| (1-\beta) \sum_{t=1}^{\infty} \beta^{t-1} \left(h_i(a^t(i), a^t) - h_i(b_i^t, a^t) \right) \right| < \epsilon$$

for every sequence $\{b_i^1, b_i^2, ...\}$ of points in Δ^n such that

$$b_i^t \in \underset{x \in \Delta^n}{argmax}\ h_i(x, a^t) \ \forall t \geq 1 \text{ (5)}.$$

Proof: The proof of Proposition 5 applies setting $T_i = T$, $y_i = y$ and $\beta_i = \hat{\beta}$ for *a.e.* $i \in I$. ■

Before finishing the paper a comment is in order. Following a remark in Schmeidler (1973), we think that a generalization of the contents of this paper is possible. Assume that Δ^n is a compact and convex subset of $\boldsymbol{R}^n$ (not necessarily the unit simplex) and that h_i's are continuous on $\Delta^n \times A$ and quasi-concave (not necessarily affine) on Δ^n. Then, existence of equilibrium follows, and all the analysis should go through (in the case of Proposition 4, we would have to replace the pure actions by extreme points of Δ^n). Another way of generalizing the paper, solving also the unpleasant feature of the mixed strategy behavior, and obtaining a model closer to economic applications would consist on adopting the more general framework of modeling aggregate behavior via the distributional approach of Mas-Colell (1984) and use his existence results.

APPENDIX

Proposition 2: Suppose $s \in S$ satisfies the following conditions: For *a.e.* $i \in I$ the function $s_i(\cdot)$ is differentiable. In addition, there exists $\eta \in [0, 1)$ such that $\forall x \in \Delta^n$ and $\forall k, l = 1, 2, ..., n$ and for *a.e.* $i \in I$

$$\left| \frac{\partial s_i^k(x)}{\partial x^l} \right| \leq \frac{\eta}{n}.$$

Then there exists a unique $x^* \in \Delta^n$ such that (s, x^*) is a solution. Moreover, (s, x^*) is globally stable.

Proof: The proof is in two steps. First, we will show that ϕ_s is a contraction, from which existence and uniqueness (but not global stability) (6) of such x^* follows easily. Second, we will show stability of (s, x^*).

Step 1: We show that $\phi_s: \Delta^n \to \Delta^n$ is a contraction with modulus η, i.e., for every x, $z \in \Delta^n$, $\| \phi_s(x) - \phi_s(z) \|_E \leq \eta \| x - z \|_E$. Let $x, z \in \Delta^n$ be arbitrary. Then

$$\| \phi_s(x) - \phi_s(z) \|_E = \| \int_I s_i(x)\, d\lambda(i) - \int_I s_i(z)\, d\lambda(i) \|_E =$$

$$\| \int_I [s_i(x) - s_i(z)]\, d\lambda(i) \|_E \leq max \{ \int_I | s_i^k(x) - s_i^k(z) |\, d\lambda(i) \mid 1 \leq k \leq n \}. \qquad [1]$$

(5) Notice that the difference between the conclusions of both propositions is that in Proposition 6, the lower bound for the discount factor β is uniform.

(6) In our situation, the contraction does not give us global stability immediately, since the average step disturbs the

Now, by the Mean-Value Theorem, $\forall j = 1, 2, ..., n$ and $a.e.\ i \in I$

$$\left| s_i^j(x) - s_i^j(z) \right| = \left| \left(\frac{\partial s_i^j(w_j)}{\partial x^1}, \ldots, \frac{\partial s_i^j(w_j)}{\partial x^n} \right)(x - z) \right| \qquad [2]$$

where $w_j = \alpha_j x + (1 - \alpha_j)z$ for some $\alpha_j \in (0, 1)$.
Therefore [2] is equal to

$$\left| \sum_{k=1}^{n} \frac{\partial s_i^j(w_j)}{\partial x^k}(x^k - z^k) \right| \leq \sum_{k=1}^{n} \left| \frac{\partial s_i^j(w_j)}{\partial x^k} \right| \left| x^k - z^k \right| \leq$$

$$\leq n \frac{\eta}{n} \left(\max_{1 \leq k \leq n} \left| x^k - z^k \right| \right) = \eta \| x - z \|_E.$$

Thus, [1] is less or equal to $\eta \| x - z \|_E$. Hence, by the Contraction Mapping Theorem there exists a unique $x^* \in \Delta^n$ such that $x^* = \phi_s(x^*)$, implying that (s, x^*) is the unique solution.

Step 2: (s, x^*) is globally stable.

Let $z \in \Delta^n$ be arbitrary, we have to show that $\{\int \hat{a}^t\}_{t=1}^{\infty} \to x^*$. Let $t > 1$, then

$$\left\| \int \hat{a}^t - x^* \right\|_E = \left\| \frac{t-1}{t} \int \hat{a}^t + \frac{1}{t} \phi_s(\int \hat{a}^{t-1}) - \frac{t-1}{t} x^* - \frac{1}{t} \phi_s(x^*) \right\|_E \leq$$

$$\leq \frac{t-1}{t} \left\| \int \hat{a}^{t-1} - x^* \right\|_E + \frac{1}{t} \left\| \phi_s(\int \hat{a}^{t-1}) - \phi_s(x^*) \right\|_E \leq$$

$$\leq \frac{t-1}{t} \left\| \int \hat{a}^{t-1} - x^* \right\|_E + \frac{1}{t} \eta \left\| \int \hat{a}^{t-1} - x^* \right\|_E =$$

$$= \left(\frac{t-1}{t} + \frac{\eta}{t} \right) \left\| \int \hat{a}^{t-1} - x^* \right\|_E.$$

Therefore $\left\| \int \hat{a}^{t-1} - x^* \right\|_E \leq \prod_{r=1}^{t} \left(\frac{r-1}{r} + \frac{1}{r} \eta \right) \| z - x^* \|_E.$

Now, notice the following three facts:

(i) $\left(\frac{r-1}{r} + \frac{1}{r} \eta \right) = 1 - \left(1 - \frac{r-1}{r} \right)(1 - \eta);$

(ii) $\left(1 - \frac{r-1}{r} \right)(1 - \eta) \in [0, 1)\ \forall r \geq 1;$

(iii) $\sum_{r=1}^{\infty} \left(1 - \frac{r-1}{r} \right)(1 - \eta) = (1 - \eta) \sum_{r=1}^{\infty} \frac{1}{r}$ diverges.

Then, by Titchmarch's Lemma (7), $\lim_{t \to \infty} \prod_{r=1}^{t} \left(\frac{r-1}{r} + \frac{1}{r} \eta \right) = 0.$

Hence, $\lim_{t \to \infty} \left\| \int \hat{a}^t - x^* \right\|_E = 0$ implying that (s, x^*) is a globally stable solution.

■

Theorem: If A.1 is satisfied and (s, x^*) is a globally stable solution with the Nash property, then (s, x^*) generates a Nash equilibrium of the unrestricted repeated game.

Proof: Given (s, x^*) and $z \in \Delta^n$, construct $f \in F$ so that it agrees for every player i, every t, and every history with what s and z would generate. Since (s, x^*) is globally stable, $\{\int_I \hat{a}^t\}_{t=1}^{\infty} \to x^*$, and thus, by continuity of $s_i(\cdot)$, $\{s_i(\int_I \hat{a}^t)\}_{t=1}^{\infty} \to s_i(x^*)$ for $a.e.\ i \in I$. Define $a^*(i) = s_i(x^*)$ $a.e.\ i \in I$. Notice that for $a.e.\ i \in I$ $\{a^t(f)(i)\}_{t=1}^{\infty} \to a^*(i)$ because $a^t(f)(i) = s_i(\int_I \hat{a}^{t-1})$. Since (s, x^*) has the Nash property, for $a.e.\ i \in I$ $h_i(s_i(x^*), a^*) = h_i(a^*(i), a^*) \geq h_i(x, a^*)$ $\forall x \in \Delta^n$, implying that $a^* \in A^*$ (the set of one-shot Nash equilibria).

The next step is to show that $\{a^t(f)\}_{t=1}^{\infty} \to a^*$ in norm. Let $\epsilon > 0$ be given, we want to show that $\exists T$ such that $\forall t > T$, $\| a^t(f) - a^* \|_A < \epsilon$. As $s_i(\cdot)$ is continuous, $\forall k = 1, 2, ..., n$ there exists $T_k > 1$ such that $\forall t > T_k$, $\int_I \left| s_i^k(\int \hat{a}^{t-1}) - s_i^k(x^*) \right| d\lambda(i) < \epsilon$ by the Bounded Convergence Theorem. Take $T = \max\{T_k \mid 1 \leq k \leq n\}$; hence $\forall t > T$,

$$\| a^t(f) - a^* \|_A = \left\| \int_I \left| a^t(f)(i) - a^*(i) \right| d\lambda(i) \right\|_E =$$

$$= \max\{\int_I \left| s_i^k(\int \hat{a}^{t-1}) - s_i^k(x^*) \right| d\lambda(i) : 1 \leq k \leq n\} < \epsilon.$$

The last step is to show that the conditions of Theorem 2 in Massó and Rosenthal (1989) are satisfied and therefore $f \in F^*$. The set A is sequentially compact with the weak topology. Assumption A.1 means that for $a.e.\ i \in I$, h_i is continuous in its second argument. Since convergence in norm implies weak convergence and for $a.e.\ i \in I$ $\{a^t(f)(i)\}_{t=1}^{\infty} \to a^*(i)$, we have that $\{h_i(a^t(f)(i), a^t(f))\}_{t=1}^{\infty} \to h_i(a^*(i), a^*)$ for $a.e.\ i \in I$. Thus, by Theorem 2 in Massó and Rosenthal (1989), $f \in F^*$.

■

Proposition 5: In addition to A.1 suppose that for $a.e.\ i \in I$ and for every $k = 1, 2, ..., n$ the functions $u_i^k(\cdot)$ are norm continuous, and let $\{a^t\}_{t=1}^{\infty}$ be any sequence of actions generated by a globally stable solution (s, x^*) with the Nash property. Then, $\forall \epsilon > 0$ and for $a.e.\ i \in I$, $\exists \beta_i \in (0, 1)$ such that $\forall \beta \in (\beta_i, 1)$

$$\left| (1 - \beta) \sum_{t=1}^{\infty} \beta^{t-1} \left(h_i(a^t(i), a^t) - h_i(b_i^t, a^t) \right) \right| < \epsilon$$

(7) Let $\{b_r\}_{r=1}^{\infty}$ be a sequence of real numbers such that $b_r \in [0, 1)$. Then $\prod_{r=1}^{t} (1 - b_r) \to 0$ provided that $\sum_{r=1}^{\infty} b_r$ di-

for every sequence $\{b_i^1, b_i^2, ...\}$ of points in Δ^n such that

$$b_i^t \in \underset{x \in \Delta^n}{argmax}\ h_i(x, a^t) \text{ for every } t \geq 1.$$

Proof: For *a.e.* $i \in I$ and $\forall t \geq 1$, let $b_i^t \in \Delta^n$ be s.t. $h_i(b_i^t, a^t) \geq h_i(a_i, a^t)$ $\forall a_i \in \Delta^n$ (the existence of such b_i^t follows from the Weierstrass Theorem). Let $\epsilon > 0$ be given; then for *a.e.* $i \in I$ and $\forall T_i \geq 1$,

$$\left| (1-\beta) \sum_{t=1}^{\infty} \beta^{t-1} [h_i(a^t(i), a^t) - h_i(b_i^t, a^t)] \right| \leq$$

$$\leq (1-\beta) \sum_{t=1}^{T_i} \beta^{t-1} 2y_i + (1-\beta) \sum_{t=T_i+1}^{\infty} \beta^{t-1} \left| h_i(a^t(i), a^t) - h_i(b_i^t, a^t) \right| =$$

$$= (1-\beta^{T_i}) 2y_i + (1-\beta) \sum_{t=T_i+1}^{\infty} \beta^{t-1} \Big| h_i(b_i^t, a^t) - h_i(s_i(x^*), a^*) +$$

$$+ h_i(s_i(x^*), a^*) - h_i(a^t(i), a^t) \Big|. \quad [3]$$

Since $\{(a^t(i), a^t)\}_{t=1}^{\infty} \to (s_i(x^*), a^*)$, by joint continuity of h_i we have that for *a.e.* $i \in I$: $\{h_i(a^t(i), a^t)\}_{t=1}^{\infty} \to h_i(s_i(x^*), a^*)$. Therefore for T_i sufficiently large, [3] is smaller than

$$(1-\beta^{T_i}) 2y_i + (1-\beta) \sum_{t=T_i+1}^{\infty} \beta^{t-1} \left(\frac{\epsilon}{3} + \left| h_i(b_i^t, a^t) - h_i(s_i(x^*), a^*) \right| \right). \quad [4]$$

Now, notice that for *a.e.* $i \in I$: $s_i(x^*) \in \underset{x \in \Delta^n}{argmax}\ h_i(x, a^*)$, for every $t \geq 1$ $b_i^t \in \underset{x \in \Delta^n}{argmax}\ h_i(x, a^t)$, and $h_i(\cdot, \cdot)$ is continuous. Therefore by the Maximum Theorem since $\{a^t\}_{t=1}^{\infty} \to a^*$, we have that

$$\{h_i(b_i^t, a^t)\}_{t=1}^{\infty} \to h_i(s_i(x^*), a^*).$$

Therefore, for T_i sufficiently large, [4] is smaller than

$$(1-\beta^{T_i}) 2y_i + \beta^{T_i} \left(\frac{\epsilon}{3} \right) + \beta^{T_i} \left(\frac{\epsilon}{3} \right). \quad [5]$$

Then, $\left(\text{assuming } y_i > \frac{\epsilon}{2} \right)$ letting $\hat{\beta}_i = \left(\frac{6y_i - 3\epsilon}{6y_i - 2\epsilon} \right)^{\frac{1}{T_i}}$, for every $\beta \in (\hat{\beta}_i, 1)$, [5] is smaller than ϵ. ∎

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