Strategy-proofness and Equal-cost Sharing for Binary and Excludable Public Goods with Fixed Cost*

Jordi Massó† Antonio Nicolò‡ and Arunava Sen§

June 2010

Abstract

For the provision of a binary public good we characterize the set of all strategy-proof social choice functions and show that, if the binary public good is excludable and has a fixed cost, the equal-cost sharing rule minimizes the maximal welfare loss among the class of all strategy-proof and individually rational rules.

*We want to thank María Angeles de Frutos, Rajat Deb and Hervé Moulin for their comments. We are especially thankful to Shinji Ohseto for his comments and a very helpful suggestion that helped us to write a better paper. Support for the research of J. Massó was received through the prize "ICREA Acadèmia" for excellence in research, funded by the Generalitat de Catalunya. He also acknowledges the support of MOVE, where he is an affiliated researcher, and of the Barcelona Graduate School of Economics (through its Research Recognition Programme), where he is an affiliated professor. His work is also supported by the Spanish Ministry of Science and Innovation, through grants ECO2008-04756 (Grupo Consolidado-C) and CONSOLIDER-INGENIO 2010 (CDS2006-00016), and by the Generalitat de Catalunya, through grant SGR2009-419. The work of A. Nicolò is partially supported by the Italian Ministry of University and Research, through grant 2005137858.

†Departament d'Economia i d'Història Econòmica and CODE. Universitat Autonòma de Barcelona. 08193 Bellaterra (Barcelona), Spain. Email: jordi.masso@uab.es

‡Dipartimento di Scienze Economiche. Università degli Studi di Padova. Via del Santo 33. 35123, Padova, Italy. Email: antonio.nicolo@unipd.it

§Indian Statistical Institute. 7, S.J.S. Sansanwal Marg, New Delhi - 110016, India. Email: asen@isid.ac.in

Keywords: Excludable Public Good; Equal-cost Sharing; Strategy-proofness; Individual Rationality.
JEL Classification: D71.

1 Introduction

In many collective decision problems a group of agents has to make a joint decision about the costly provision of a public good that can be beneficial to more than one agent because there is no rivalry in consumption. Often, the public good is binary (either the public good is provided or it is not) and each agent assigns a (monetary) valuation to its use: her maximal willingness to pay. The public good is pure if no agent can be excluded from its use whenever the public good is provided. However, there are many circumstances under which the public good is not pure, because, even if all its users consume the same amount of the public good (or equivalently, it is technologically unfeasible or too costly to monitor the amount each agent consumes), agents can be excluded from its use; for example, when a key or a code, needed to have access to the consumption of the public good, is given only to the subset of users. A simple example is the decision of installing an elevator in a building: its use can be limited (using a key or a code) to those who contribute to pay the cost of installation. Other examples are the television and radio licence fees in UK or Italy and, more recently, the project of providing (through a satellite-based broadband platform) broadband internet connection at a flat service fee to households, to small and medium enterprises as well as to schools of ten countries in East and Central Africa.[1] In all these cases each agent contribution only depends on whether or not the agent is a user. Thus, an allocation associated to a binary public good, pure or excludable, is a triple consisting of: (i) a decision on whether or not the public good is provided, (ii) the set of its users (the empty set if the public good is not provided and either the empty set or the set of all agents if the public good is pure), and (iii) a list of agents' contributions (or prices). Society would like to choose an allocation according to agents' valuations. But since agents' valuations are private information they have to be elicited through a social choice function that maps profiles of valuations into allocations.

[1] See www.intersatafrica.com

The first desirable property we require a social choice function to satisfy, is *strategy-proofness.* A social choice function should induce truthful revelation of individual preferences in dominant strategies; that is, no agent can gain by misrepresenting her true valuation irrespective of her beliefs about the types of other agents.

Our first result (Theorem 1) characterizes, for any binary public good, the class of all strategy-proof social choice functions. The result is independent of whether the public good is pure or excludable and of the cost of providing it. Its main consequence (Corollary 1) is that a strategy-proof social choice function can be roughly described by a family of pairs of functions, one pair for each agent. The two functions associated with agent i depend on the vector of all valuations *except* agent i's own valuation (this is a well-known requirement of strategy-proofness). The first function selects a cut-off value with the property that if agent i's valuation is below this cut-off she is not a user while if it is above, she is a user. The second function selects a bonus-like contribution for agent i used to determine agent i's price: if i is a user, she has to pay a price that is equal to the cut-off value (the one selected by the first function) minus the bonus, and if i is not a user, she receives the bonus (*i.e.*, she has to pay a price that is equal to the negative value of the bonus). The proof of Theorem 1 uses the Fundamental Theorem of Calculus and follows arguments already used elsewhere to prove results on incentive compatibility mechanisms and auctions.[2]

Equipped with this general characterization of the full class of strategy-proof social choice functions for binary public goods, we then focus on binary and excludable public goods with a fixed cost of provision (*i.e.*, the cost of provision is independent of the set of its users) and restrict our attention to *feasible* social choice functions (we call them *rules*) which satisfy the additional requirement that if the public good is provided then, the sum of the prices paid by all agents (users and non-users) has to be larger or equal to the fixed cost of provision. We are interested in rules satisfying, in addition to strategy-proofness another basic and desirable property related to the voluntary participation of agents. A rule is *individually rational* if agents receive always a utility larger or equal to zero (the utility of not consuming the public good and not paying any price).

For pure public goods a rule is *efficient* if it prescribes the provision of the public good

[2] See, for instance, Mussa and Rosen (1978), Myerson (1981), Myerson and Satterthwaite (1983), and Krishna (2002).

if and only if the sum of agents' valuations exceeds its cost. A rule is *budget balanced* if the sum of the prices is always equal to the cost (after a normalization, 1 if the public good is provided and 0 otherwise). For pure public goods, Clarke-Groves rules are efficient and strategy-proof but typically violate individual rationality (as well as budget balancedness).[3] It may occur that, at some profiles of valuations, an agent has to pay a price that is strictly higher than her valuation when the good is provided and at other profiles of valuations, an agent has to pay a price that is strictly positive when the public good is not provided (and at other profiles of valuations the total amount paid by agents is strictly larger than the cost of provision). Hence, when the public good is excludable and no authority can force agents to participate, we have to restrict ourselves to use individually rational rules. But then, any rule selecting an allocation in which an agent with a strictly positive valuation is excluded is not efficient. Thus, we are left with rules which either cannot always guarantee an efficient decision, or else deal with the fact that agents may benefit by miss-reporting their valuations. We insist here on the incentives issue and adopt a second-best type of approach. Our aim is to identify−among the class of all strategy-proof and individually rational−a rule that *minimizes the maximal welfare loss*. The welfare loss of a rule at a profile of valuations is the difference between the aggregate welfare of the first best and the aggregate welfare of the rule, evaluated both at the given profile. The maximal welfare loss of a rule is the supremum, taken over all profiles of valuations, of its welfare loss. Then, each rule is evaluated according to its maximal welfare loss and the goal is to select a rule that minimizes it. Since we are interested in rules satisfying individual rationality it turns out that inefficiencies arise from the exclusion as users of some (or all) agents who have strictly positive valuations. The maximal welfare loss of a rule is then the sum of the valuations of all non-users of the good at the preference profile which maximizes this sum. In Theorem 2 we show that the equal-cost sharing rule minimizes the maximum welfare loss among all strategy-proof and individually rational rules. Given a profile of valuations, the *equal-cost sharing rule* chooses the allocation with the (set-inclusion) maximal group of users for which their contributions are smaller than their valuations, non-users pay a price equal to zero, and each user pays the cost of provision divided by the number of users (*i.e.*, the equal-cost share). Observe that the equal-cost sharing rule satisfies many other

[3]See Clarke (1971), Groves (1973), and Green and Laffont (1977 and 1979).

desirable properties; in particular, it is budget balanced.

The worst-case welfare criterion has been extensively used and brings to social choice theory a very well-established principle in other areas which states that the performance of a system should be evaluated according to the worst case scenario. The papers by Moulin and Shenker (2001), Moulin (2008), and Juarez (2008a and 2008b) are applications of this principle to the theory of public goods. In computer science the worst-case criterion has been adopted in the recent literature of the price of anarchy, introduced to measure the effects of selfish routing in a congested network (see Koutsoupias and Papadimitriou (1999), Roughgarden (2002), and Roughgarden and Tardos (2002)).

In the last years there has been an extensive literature on the characterization of cost sharing rules for excludable public goods. Most of this literature stems from the seminal paper by Moulin (1994) in which she proves that when the cost of production is convex the serial mechanism satisfies group strategy-proofness, equal treatment of equals, the stand alone test[4] and Pareto dominates the equal-costs mechanism. Deb and Razzolini (1999a and 1999b) focus on the case of a binary and excludable public good with fixed cost of provision and describe the equal-cost sharing rule as an auction like mechanism. Consider an excludable public good with a cost normalized to 1 and let n be the number of agents. At each step $k = 0, 1, 2, ..., n-1$ of the mechanism an auctioneer asks to agents who does want to be a user at price $\frac{1}{n-k}$. The mechanism stops at the smallest k such that there are $n-k$ agents who accept to be users at price $\frac{1}{n-k}$ (and in case there is no such k the public good is not provided). They characterize this mechanism as the unique one satisfying strategy-proofness, individual rationality, equal treatment of equals, and two additional conditions (directional nonbossiness and free entry in Deb and Razzolini (1999a), and upper semicontinuity and non-imposition in Deb and Razzolini (1999b)). Their characterization results show that equal treatment of equals substantially narrows down the class of strategy-proof mechanisms. Again, for the case of a binary and excludable public good with fixed cost, Ohseto (2000) characterizes the class of largest unanimous mechanisms as the set of strategy-proof, individually rational, demand-monotonic, and access-independent mechanisms. Access-independency requires that each agent should have access to either

[4] A stronger property than individual rationality: each agent should be guaranteed the utility level given by the amount of the good that maximizes her utility when she pays the full cost of provision.

level of the public good regardless of other agents' valuations; demand monotonicity imposes restrictions on the set of users when agents' valuations vary (namely, when all agents' valuations weakly increase then the set of users cannot shrink and when all users' valuations increase and all non-users' valuations decrease then the set of users does not vary). Largest unanimous mechanisms are such that, given a cost sharing method π,[5] the public good is provided for the largest coalition of agents whose members approve the provision of the public good coupled with the cost share specified by π at this coalition, and the public good is not provided if no such coalition exists. Clearly the equal-cost sharing rule belongs to this class and corresponds to the case in which π equally shares the cost of provision among the set of users. Moreover, Ohseto (2005) characterizes the class of augmented serial rules as the set of all strategy-proof, access-independent, envy-free, and non-bossy rules.[6] While there are augmented serial rules that are not individually rational all of them are non-subsidizing. Ohseto (2005) also discusses the trade-offs between individual rationality and the maximal welfare loss in the class of augmented serial rules. Observe that the two Ohseto's characterizations, as well as ours, do not use any property related to anonymity, symmetry or equal treatment of equals. In contrast, and for the case of allocating an indivisible unit of a private good, Moulin (2010) shows, among other results, the trade-offs between a notion of fairness and efficiency loss on the class of anonymous and strategy-proof rules.

The closest paper to our contribution is Moulin and Shenker (2001). They consider the provision of a binary and excludable public good when the cost function is a submodular function of the set of users. They show that the rule associated with the Shapley value cost sharing formula (which corresponds to the equal-cost sharing method for the case of a binary public good with fixed cost of provision) is the unique rule that satisfies the property that its maximum welfare loss is minimal among the class of rules that are defined from a cross monotonic cost sharing method and are group strategy-proof, individually rational, non-subsidizing (the cost shares are non negative), budget balanced, and satisfy

[5] A cost sharing method specifies for each set of users the price paid by each user with the property that the sum of the prices is equal to the cost of provision.

[6] Envy-freeness requires that no agent prefers to be treated by the rule (in terms of her participation and contribution) as any other agent is treated. Non-bossiness requires that by changing her valuation, no agent is able to change the social decision without affecting the way the rule treats her.

consumer sovereignty. Cross monotonicity imposes that the price paid by each user weakly decreases when the set of users enlarges. Our result and Moulin and Shenker (2001) result are logically independent but complementary because we focus on the more restricted case in which the cost of provision is fixed, but we prove that the equal-cost sharing rule is worst-case minimizing among a much broader set of rules. Specifically, we first show that non-subsidizingness, budget balancedness, and consumer sovereignty are properties that are implied by the second-best efficiency criterion we adopt. Second, and more importantly, we do not impose that the rule be defined through a cross monotonic method. We show in the last section of the paper that there are rules that are strategy-proof, individually rational, and budget balanced which are neither group strategy-proof nor defined through a cross monotonic method. Our result shows that the worst-case minimizing criterion narrows down an extremely large class of rules to the equal-cost sharing rule. Notice that we reach this result without imposing any normative criterion to equalize agents' contributions like an equal treatment of equals property or like a cross monotonicity requirement, which also goes in the direction of imposing similar treatment (all users should in fact weakly benefit from an enlargement of the set of users): the (ex-post) equal treatment of users that the equal-cost sharing rule imposes, only derives from the (ex-ante) second-best efficiency criterion we adopt. Third, we only require strategy-proofness and not group strategy-proofness. We believe that strategy-proofness is a more compelling axiom than group strategy-proofness from a decision-theoretic perspective. If agents are ignorant of the types of other agents, assumptions about the ability of coalitions to coordinate their messages for mutual benefit require stronger justification. Group strategy-proofness can also be a demanding requirement in this setting as is shown by the result of Juarez (2008b) who proves that any group strategy-proof rule (remarkably, without imposing budget balancedness) is not efficient (except when the cost function is additive). Juarez (2008b) also shows that if the cost function has decreasing marginal cost then the average cost rule (each agent pays a unitary price equal to the average cost) is the worst-case minimizing rule among the set of group strategy-proof rules which satisfy equal treatment of equals.[7]

The paper is organized as follows. Section 2 introduces the basic model and definitions.

[7] Juarez (2008b) also proves that when the marginal cost is increasing then the sequential average cost is worst-case minimizing among the set of group strategy-proof rules.

Section 3 presents the main properties of social choice functions and gives a general characterization of all strategy-proof social choice functions (Theorem 1). Section 4 describes the efficiency criterion of minimizing the maximal welfare loss, defines the equal-cost sharing rule, and presents Theorem 2 stating that the equal-cost sharing rule minimizes the maximal welfare loss among the class of all strategy-proof and individually rational rules. In Section 5 we prove Theorem 2. Section 6 contains a final remark. An appendix at the end of the paper contains the proof of Theorem 1.

2 Preliminaries

Consider a finite set of agents $N = \{1, ..., n\}$ that has to decide on the provision of a binary public good. The public good is *binary* because it can either be provided (denoted by 1) or not provided (denoted by 0). Let $X = \{0, 1\}$ be the set of the two binary choices and let $x \in X$ be a generic choice. The public good is *excludable* whenever a subset of agents (called *non-users*) can be excluded from its use, even when $x = 1$. The set of agents that are not excluded are called *users*. A generic subset of users will be denoted by S. A public good is *pure* if no agent can be excluded from its consumption when the public good is produced; namely, when $x = 1$ the set of users S is the entire set of agents N. Since Theorem 1 will apply to any binary public good, independently of the cost of providing it, we do not make any assumption yet on its cost.

For each agent $i \in N$, let $\alpha_i \in \mathbb{R}_+$ be the (monetary) *valuation* that i assigns to the public good if it is produced and i is a user. By requiring that α_i is independent of the set of users we are implicitly assuming that there is no rivalry in the consumption of the public good. A *profile* $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}^N_+$ is a vector of valuations, one for each agent. For each subset of agents $S \subseteq N$, let $\mathbf{1}_S : N \to \{0, 1\}$ be the indicator function where for all $i \in N$,

$$\mathbf{1}_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

To simplify notation we write $\mathbf{1}^i_S$ instead of $\mathbf{1}_S(i)$. Let $p = (p_i)_{i \in N} \in \mathbb{R}^N$ be a vector of *prices* (or contributions).[8]

[8] We are admitting the possibility of negative prices. Later, we will impose that prices be positive.

The set of agents N has to decide whether or not to provide the public good ($x \in X$), its set of users $S \in 2^N$, and the vector of contributions $p \in \mathbb{R}^N$. An *allocation* is a triple $(x, S, p) \in X \times 2^N \times \mathbb{R}^N$ with the property that $x = 0$ implies $S = \emptyset$. Observe that we are not imposing yet any condition on the vector of prices p nor excluding the possibility that $x = 1$ and $S = \emptyset$. Denote by $A \equiv \{(x, S, p) \in X \times 2^N \times \mathbb{R}^N \mid x = 0 \text{ implies } S = \emptyset\}$ the set of all allocations. Agent i's preferences on the set of allocations A depend on i's valuation $\alpha_i \in \mathbb{R}_+$ and they are represented by the utility function $v_i : A \times \mathbb{R}_+ \to \mathbb{R}$, where for each $(x, S, p, \alpha_i) \in A \times \mathbb{R}_+$,

$$v_i(x, S, p, \alpha_i) = \mathbf{1}_S^i \cdot x \cdot \alpha_i - p_i.$$

Since the society N will remain fixed, a profile $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}_+^N$ completely describes a *problem*. We will write (α_i, α_{-i}) to emphasize the role of agent i in the profile α, and (α_S, α_{-S}) to emphasize the role of the subset of agents S.

A *social choice function* $f : \mathbb{R}_+^N \to A$ selects, for each profile $\alpha \in \mathbb{R}_+^N$, an allocation $f(\alpha) \in A$. Hence, a social choice function f can be identified with its three components $f = (x^f, S^f, p^f)$, where $x^f : \mathbb{R}_+^N \to \{0, 1\}$, $S^f : \mathbb{R}_+^N \to 2^N$, and $p^f : \mathbb{R}_+^N \to \mathbb{R}^N$. Namely, for each $\alpha \in \mathbb{R}_+^N$, $f(\alpha) = (x^f(\alpha), S^f(\alpha), p^f(\alpha))$; obviously, the triple must satisfy that for all $\alpha \in \mathbb{R}_+^N$, if $x^f(\alpha) = 0$ then $S^f(\alpha) = \emptyset$. When no confusion arises we omit the superscript f and write $f = (x, S, p)$.

3 Basic Properties and Preliminary Results

A social choice function is strategy-proof if, at all profiles, to report truthfully is a dominant strategy for all agents. To state it formally, we need the notion of manipulation. Agent $i \in N$ *manipulates* $f : \mathbb{R}_+^N \to A$ *at profile* $\alpha \in \mathbb{R}_+^N$ if there exists $\alpha_i' \in \mathbb{R}_+$ such that

$$v_i(x^f(\alpha_i', \alpha_{-i}), S^f(\alpha_i', \alpha_{-i}), p^f(\alpha_i', \alpha_{-i}), \alpha_i) > v_i(x^f(\alpha_i, \alpha_{-i}), S^f(\alpha_i, \alpha_{-i}), p^f(\alpha_i, \alpha_{-i}), \alpha_i).$$

In this case we say that i manipulates f at α via α_i'.

Definition 1 A social choice function $f : \mathbb{R}_+^N \to A$ is *strategy-proof* if no agent manipulates f at any profile.

Theorem 1 below characterizes the class of all strategy-proof social choice functions. To state it, we need the following definition. A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is an *increasing function* *of* z if $z' > z'' \geq 0$ implies $g(z') \geq g(z'') \geq 0$.

Theorem 1 *A social choice function* $f : \mathbb{R}_+^N \to A$ *is strategy-proof if and only if for all* $i \in N$ *the following two conditions hold:*
(T1.a) *for all* $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$, $\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i})$ *is an increasing function of* α_i*;*
(T1.b) *for all* $\alpha \in \mathbb{R}_+^N$,

$$p_i^f(\alpha) = \mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) \cdot \alpha_i - \int_0^{\alpha_i} \mathbf{1}^i_{S^f(t,\alpha_{-i})} \cdot x^f(t, \alpha_{-i})dt - h_i^f(\alpha_{-i}), \tag{1}$$

where $h_i^f : \mathbb{R}_+^{N\setminus\{i\}} \to \mathbb{R}$ *is an arbitrary function.*

As we have already said in the Introduction the proof of Theorem 1 relies on the Fundamental Theorem of Calculus in a similar way as the proofs of other incentive compatibility results do in different settings like mechanism design and auction theory. For completeness, the interested reader will find the proof of Theorem 1 in an appendix at the end of the paper.

Theorem 1 is a very powerful result. Observe first that the characterization is general since it includes as particular cases pure and excludable public goods with fixed cost as well as any cost function (which may depend on the set of users). Moreover, it does not require any *a priori* relationship between the sum of the prices and the cost of providing the public good. Before proceeding, it is useful to describe, given α_{-i}, the increasing function $\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i})$ of α_i identified in Theorem 1 as follows. Given a social choice function $f : \mathbb{R}_+^N \to A$ and an agent $i \in N$, fix $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$ and let $\beta_i^f : \mathbb{R}_+ \to \{0, 1\}$ be the function (of α_i, given α_{-i}) such that for all $\alpha_i \in \mathbb{R}_+$, $\beta_i^f(\alpha_i) = \mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i})$. Hence, for each $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$ there exists a critical value $\phi_i^f(\alpha_{-i}) \in \mathbb{R}_+ \cup \{\infty\}$ (the point where β_i^f is discontinuous, if any) such that $\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) = 1$ for all $\alpha_i > \phi_i^f(\alpha_{-i})$, if any, and $\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) = 0$ for all $\alpha_i < \phi_i^f(\alpha_{-i})$, if any. That is, given α_{-i}, there exists $\phi_i^f(\alpha_{-i}) \in [0, +\infty]$ such that either

$$\beta_i^f(\alpha_i) = \begin{cases} 0 & \text{if } \alpha_i \leq \phi_i^f(\alpha_{-i}) \\ 1 & \text{if } \alpha_i > \phi_i^f(\alpha_{-i}) \end{cases}$$

or

$$\beta_i^f(\alpha_i) = \begin{cases} 0 & \text{if } \alpha_i < \phi_i^f(\alpha_{-i}) \\ 1 & \text{if } \alpha_i \geq \phi_i^f(\alpha_{-i}). \end{cases}$$

Figure 1 illustrates the two relevant cases of these β_i^f functions depending on whether β_i^f is right or left continuous at $\phi_i^f(\alpha_{-i})$.

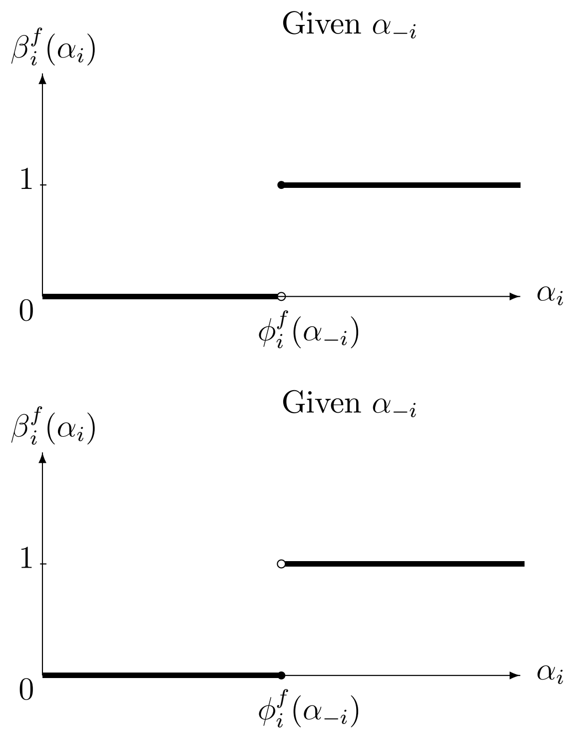

Figure 1

The two other cases are the constant functions where β_i^f is always equal to 0 (if $\phi_i^f(\alpha_{-i}) = 0$) or equal to 1 (if $\phi_i^f(\alpha_{-i}) = +\infty$).

By (T1.b) in Theorem 1, if $\alpha_i > \phi_i^f(\alpha_{-i})$ then,

$$\begin{aligned} p_i(\alpha) &= \mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) \cdot \alpha_i - \int_{\phi_i^f(\alpha_{-i})}^{\alpha_i} \mathbf{1}^i_{S^f(t,\alpha_{-i})} \cdot x^f(t, \alpha_{-i}) dt - h_i^f(\alpha_{-i}) \\ &= \alpha_i - (\alpha_i - \phi_i^f(\alpha_{-i})) - h_i^f(\alpha_{-i}) \\ &= \phi_i^f(\alpha_{-i}) - h_i^f(\alpha_{-i}). \end{aligned}$$

Moreover, if either (i) $\alpha_i < \phi_i^f(\alpha_{-i})$ or (ii) $\alpha_i = \phi_i^f(\alpha_{-i})$ and $\mathbf{1}^i_{S^f(\phi_i^f(\alpha_i),\alpha_{-i})} \cdot x^f(\phi_i^f(\alpha_{-i}), \alpha_{-i}) = 0$ then, $p_i^f(\alpha) = -h_i^f(\alpha_{-i})$. Finally, if $\alpha_i = \phi_i^f(\alpha_{-i})$ and $\mathbf{1}^i_{S^f(\phi_i^f(\alpha_i),\alpha_{-i})} \cdot x^f(\phi_i^f(\alpha_{-i}), \alpha_{-i}) = 1$ then, $p_i^f(\alpha) = \phi_i^f(\alpha_{-i}) - h_i^f(\alpha_{-i})$. Hence, as a consequence of Theorem 1 we can state Corollary 1.

Corollary 1 *A social choice function $f : \mathbb{R}_+^N \to A$ is strategy-proof if and only if for each $i \in N$ there exist two functions $\phi_i^f : \mathbb{R}_+^{N\backslash\{i\}} \to \mathbb{R}_+ \cup \{\infty\}$ and $h_i^f : \mathbb{R}_+^{N\backslash\{i\}} \to \mathbb{R}$ such that*
(C1.a) *if $\alpha_i > \phi_i^f(\alpha_{-i})$ then $\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) = 1$ and $p_i^f(\alpha) = \phi_i^f(\alpha_{-i}) - h_i^f(\alpha_{-i})$;*
(C1.b) *if $\alpha_i < \phi_i^f(\alpha_{-i})$ then $\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) = 0$ and $p_i^f(\alpha) = -h_i^f(\alpha_{-i})$; and*
(C1.c) *if $\alpha_i = \phi_i^f(\alpha_{-i})$ then either $[\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) = 1$ and $p_i^f(\alpha) = \phi_i^f(\alpha_{-i}) - h_i^f(\alpha_{-i})]$* or *$[\mathbf{1}^i_{S^f(\alpha_i,\alpha_{-i})} \cdot x^f(\alpha_i, \alpha_{-i}) = 0$ and $p_i^f(\alpha) = -h_i^f(\alpha_{-i})]$.*

From now on we will only consider cases where the *cost* of providing the binary public good is constant, and independent on the set of users; we normalize the cost of providing the public good to be equal to 1 while the cost of not providing it to be equal to 0. Thus, an allocation $(x, S, p) \in A$ is *feasible* if $x = 1$ implies $\sum_{i\in N} p_i \geq 1$. Let FA be the set of feasible allocations.

Definition 2 A social choice function $f : \mathbb{R}_+^N \to A$ is *feasible* if for all $\alpha \in \mathbb{R}_+^N$, $f(\alpha) \in FA$; namely, for all $\alpha \in \mathbb{R}_+^N$, $x^f(\alpha) = 0$ implies $S^f(\alpha) = \emptyset$, and $x^f(\alpha) = 1$ implies $\sum_{i\in N} p_i^f(\alpha) \geq 1$.

A *rule* is a feasible social choice function $f : \mathbb{R}_+^N \to FA$. From now on we will only consider social choice functions that are rules.

A rule is individually rational at a profile if no agent obtains a lower utility than the utility she would have obtained by not participating.

Definition 3 A rule $f : \mathbb{R}_+^N \to FA$ is *individually rational at* $\alpha \in \mathbb{R}_+^N$ if for all $i \in N$, $v_i(x^f(\alpha), S^f(\alpha), p^f(\alpha), \alpha_i) \geq 0$. A rule $f : \mathbb{R}_+^N \to FA$ is *individually rational* if it is individually rational at all profiles.

Consider a strategy-proof and individually rational rule $f = (x, S, p)$. Fix $i \in N$ and $\alpha_{-i} \in \mathbb{R}_+^{N\backslash\{i\}}$. By individual rationality,

$$v_i(x(0, \alpha_{-i}), S(0, \alpha_{-i}), p(0, \alpha_{-i}), 0) \geq 0.$$

Hence, $\mathbf{1}^i_{S(0,\alpha_{-i})} \cdot x(0, \alpha_{-i}) \cdot 0 - p_i(0, \alpha_{-i}) \geq 0$; namely, $p_i(0, \alpha_{-i}) \leq 0$. Then, by (C1.b) if $\phi_i(\alpha_{-i}) > 0$ or by (C1.c) if $\phi_i(\alpha_{-i}) = 0$, $p_i(0, \alpha_{-i}) = -h_i(\alpha_{-i}) \leq 0$. Hence, $h_i(\alpha_{-i}) \geq 0$. We state this fact as Remark 1 below.

Remark 1 Let $f : \mathbb{R}_+^N \to FA$ be a strategy-proof and individually rational rule. Then, for all $i \in N$ and all $\alpha_{-i} \in \mathbb{R}_+^{N\backslash\{i\}}$, $h_i^f(\alpha_{-i}) \geq 0$.

Let Φ be the class of strategy-proof and individually rational rules. We are interested in selecting among all rules in Φ a second-best efficient rule.

4 Efficiency, Welfare Loss and Equal-cost Sharing

4.1 Purely Efficient Rules

There is a natural notion of (first-best) efficiency for pure public goods. Remember that a public good is *pure* if once the public good is produced no agent can be excluded from its consumption ($x = 1$ implies $S = N$). Assume the public good is pure. Then, the following notion of efficiency is natural.

Definition 4 A rule $f : \mathbb{R}^N_+ \to FA$ is *(purely) efficient* if for all $\alpha \in \mathbb{R}^N_+$:
(i) $\sum_{i \in N} \alpha_i \geq 1$ implies $x^f(\alpha) = 1$ and $S^f(\alpha) = N$.
(ii) $\sum_{i \in N} \alpha_i < 1$ implies $x^f(\alpha) = 0$ and $S^f(\alpha) = \emptyset$.

Observe that (purely) efficient rules refer to pure public goods; as soon as there is exclusion (and a non-user has strictly positive valuation) efficiency is violated. In particular, to see that there is no strategy-proof, individually rational, and (purely) efficient rule consider the case where $N = \{1, 2, 3\}$ and $\alpha_1 = \alpha_2 = 4/9$ and $\alpha_3 = 2/9$. By Corollary 1 and efficiency, any such f has the property that $\phi^f_1(4/9, 2/9) = \phi^f_2(4/9, 2/9) = 3/9$ and $\phi^f_3(4/9, 4/9) = 1/9$. Hence, $x^f(\alpha) = 1$. By individually rationality, Corollary 1 and Remark 1, $p^f_i(\alpha) \leq \phi^f_i(\alpha_{-i})$ for all $i \in N$. Thus, $\sum_{i \in N} p^f_i(\alpha) \leq \sum_{i \in N} \phi^f_i(\alpha_{-i}) = 7/9 < 1$, which violates feasibility.

4.2 Minimizing the Maximal Welfare Loss

For any binary public good, pure or excludable, the first best at profile $\alpha \in \mathbb{R}^N_+$ requires provision of the public good if $\sum_{i \in N} \alpha_i \geq 1$ and non-provision if $\sum_{i \in N} \alpha_i < 1$. Given $\alpha \in \mathbb{R}^N_+$, let $W(FB, \alpha) \equiv \max\{\sum_{i \in N} \alpha_i - 1, 0\}$ be the welfare of the first best at profile α. We will show in Theorem 2 that, among the rules in Φ, the equal-cost sharing rule minimizes the maximal welfare loss from the first best.

Consider again the case of a public good with exclusion. Fix $f \in \Phi$ and consider $\alpha \in \mathbb{R}^N_+$.

The welfare of f at α is

$$W(f,\alpha) = \begin{cases} \sum\limits_{i \in S^f(\alpha)} \alpha_i - 1 & \text{if } x^f(\alpha) = 1 \\ 0 & \text{if } x^f(\alpha) = 0. \end{cases}$$

Hence, the welfare loss from the first best of f at α is

$$WL(f,\alpha) = W(FB,\alpha) - W(f,\alpha) \tag{2}$$

$$= \begin{cases} \max\{\sum\limits_{i \in N} \alpha_i - 1, 0\} - (\sum\limits_{i \in S^f(\alpha)} \alpha_i - 1) & \text{if } x^f(\alpha) = 1 \\ \max\{\sum\limits_{i \in N} \alpha_i - 1, 0\} & \text{if } x^f(\alpha) = 0 \end{cases}$$

$$= \begin{cases} \max\{\sum\limits_{i \notin S^f(\alpha)} \alpha_i, -\sum\limits_{i \in S^f(\alpha)} \alpha_i + 1\} & \text{if } x^f(\alpha) = 1 \\ \max\{\sum\limits_{i \in N} \alpha_i - 1, 0\} & \text{if } x^f(\alpha) = 0. \end{cases}$$

Thus, the maximal welfare loss from the first best of f is

$$MWL(f) = \sup_{\alpha \in \mathbb{R}^N_+} WL(f,\alpha).$$

We want to minimize the maximal welfare loss on Φ. That is, we want to find a rule $\hat{f} \in \Phi$ with the property that $MWL(\hat{f}) \leq MWL(f)$ for all $f \in \Phi$.

4.3 Equal-cost Sharing for Binary and Excludable Public Goods

The equal-cost sharing rule splits equally the cost of providing the binary and excludable public good among the maximal set of users for whom equal split is individually rational. In Theorem 2 below we will show that the equal-cost sharing rule minimizes the maximal welfare loss among the class of all strategy-proof and individually rational rules. Formally, for each $\alpha \in \mathbb{R}^N_+$, define the family of subsets of agents

$$U(\alpha) = \{S \in 2^N \mid \alpha_i \geq \frac{1}{\#S} \text{ for all } i \in S \text{ and } \alpha_j \leq \frac{1}{\#S+1} \text{ for all } j \notin S\}.$$

Namely, given $\alpha \in \mathbb{R}^N_+$, $U(\alpha)$ is the family of sets of users that (i) satisfy individual rationality at α when the cost of the public good is uniformly distributed among the set of users, and (ii) non-users do not strictly prefer to become a user by joining the group of users and pay the corresponding uniform contribution. Namely, a set in the family $U(\alpha)$ satisfies an internal and external stability property at α if the cost of providing the binary and excludable public good is equally shared among the set of users. Observe that for some profile $\alpha \in \mathbb{R}^N_+$, the family $U(\alpha)$ may contain only the empty set while for other profiles $\alpha' \in \mathbb{R}^N_+$ the family $U(\alpha')$ may contain more than one subset of agents. However, if $S, S' \in U(\alpha')$ then $S \cup S' \in U(\alpha')$. Therefore, for all $\alpha \in \mathbb{R}^N_+$, there exists a unique (set-inclusion) maximal set in $U(\alpha)$. Denote it by S_α.

Definition 5 The *equal-cost sharing rule* $f^{EC} : \mathbb{R}^N_+ \to FA$ is the rule that, for each $\alpha \in \mathbb{R}^N_+$,

$$(x^{f^{EC}}(\alpha), S^{f^{EC}}(\alpha)) = \begin{cases} (1, S_\alpha) & \text{if } U(\alpha) \neq \{\emptyset\} \\ (0, \emptyset) & \text{if } U(\alpha) = \{\emptyset\} \end{cases}$$

and, for all $i \in N$,

$$p_i^{f^{EC}}(\alpha) = \begin{cases} \frac{1}{\#S^{f^{EC}}(\alpha)} & \text{if } i \in S^{f^{EC}}(\alpha) \\ 0 & \text{if } i \notin S^{f^{EC}}(\alpha). \end{cases}$$

Theorem 2 is the main result of the paper.

Theorem 2 *The equal-cost sharing rule minimizes the maximal welfare loss among the set of all strategy-proof and individually rational rules.*

5 Proof of Theorem 2

The structure of the proof of Theorem 2 is as follows. We will first restrict the rules in the class of strategy-proof and individually rational rules by successively requiring that they satisfy additional properties. Namely, we will further restrict our search of a rule that minimizes the maximal welfare loss on the class of rules that besides being strategy-proof and individually rational they are also non-subsidizing, budged balanced, demand monotonic, cross-monotonic, and satisfy consumer sovereignty. In Lemmata 1 to 4 we will show that to require that the additional conditions hold can be done without loss of generality because if a rule f does not satisfy one of these additional properties we can always find a rule $\tilde{f}$ satisfying them with an equal or smaller maximal welfare loss than the rule f. Then, we will show that the equal-cost sharing rule minimizes the maximal welfare loss among all strategy-proof, individually rational, non-subsidizing, budget balanced, demand monotonic, and cross monotonic rules that satisfy consumer sovereignty. Thus, by Lemmata 1 to 4 the equal-cost sharing rule minimizes the maximal welfare loss among all strategy-proof and individually rational rules.

5.1 Non-subsidizingness and budget balancedness

Some rules may require that at some profile an agent is subsidized (*i.e.*, pays a negative price). Rules that exclude this possibility are called non-subsidizing.

Definition 6 A rule $f : \mathbb{R}^N_+ \to FA$ is *non-subsidizing at* $\alpha \in \mathbb{R}^N_+$ if for all $i \in N$, $p^f_i(\alpha) \geq 0$. A rule $f : \mathbb{R}^N_+ \to FA$ is *non-subsidizing* if it is non-subsidizing at all profiles.

Another basic property of rules is that the sum of the prices be equal to the cost of providing the public good. For many applications we want to consider rules that balance the budget.

Definition 7 A rule $f : \mathbb{R}^N_+ \to FA$ is *budget balanced at* $\alpha \in \mathbb{R}^N_+$, if $x^f(\alpha) = 0$ implies $\sum_{i \in N} p^f_i(\alpha) = 0$, and $x^f(\alpha) = 1$ implies $\sum_{i \in N} p^f_i(\alpha) = 1$. A rule $f : \mathbb{R}^N_+ \to FA$ is *budget balanced* if it is budget balanced at all profiles.

To see that strategy-proofness, individual rationality and budget-balancedness do not imply non-subsidizingness consider the case where $N = \{1, 2\}$ and define the rule f by the quadruple $(\phi^f_1, \phi^f_2, h^f_1, h^f_2)$ of functions where for all $\alpha \in \mathbb{R}^N_+$, $\phi^f_1(\alpha_2) = 4/3$, $\phi^f_2(\alpha_1) = +\infty$, $h^f_1(\alpha_2) = 0$ and

$$h^f_2(\alpha_1) = \begin{cases} 1/3 & \text{if } \alpha_1 \geq 4/3 \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 1 and Remark 1, it is immediate to see that f is strategy-proof and individually rational. Since

$$\begin{aligned} p^f_1(\alpha_1, \alpha_2) &= \begin{cases} 4/3 & \text{if } \alpha_1 \geq 4/3 \\ 0 & \text{otherwise} \end{cases} \\ p^f_2(\alpha_1, \alpha_2) &= \begin{cases} -1/3 & \text{if } \alpha_1 \geq 4/3 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

f is budget balanced and subsidizing.

Lemma 1 below says that for our purpose we can restrict our search to the class of rules that in addition to be strategy-proof and individually rational they are also non-subsidizing and budget balanced.

Lemma 1 *Let $f : \mathbb{R}_+^N \to FA$ be a strategy-proof and individually rational rule. Then, there exists a strategy-proof, individually rational, non-subsidizing, and budget balanced rule $\tilde{f} : \mathbb{R}_+^N \to FA$ that has the same or a smaller maximal welfare loss than f.*

Proof To define $\tilde{f}$ from f, let $\alpha \in \mathbb{R}_+^N$ be arbitrary. Two cases are possible.

Case 1: $x^f(\alpha) = 0$. We now construct $\tilde{f}(\alpha) = (x^{\tilde{f}}(\alpha), S^{\tilde{f}}(\alpha), p^{\tilde{f}}(\alpha))$, individually rational, non-subsidizing and budget balanced at α, with the property that $WL(\tilde{f}, \alpha) = WL(f, \alpha)$. We do that by applying conditions (C1.a), (C1.b) and (C1.c) in Corollary 1 to the family of duples $(\phi_i^{\tilde{f}}(\alpha_{-i}), h_i^{\tilde{f}}(\alpha_{-i}))_{i\in N}$ defined as follows. Set $\phi_i^{\tilde{f}}(\alpha_{-i}) = \phi_i^f(\alpha_{-i})$ and $h_i^{\tilde{f}}(\alpha_{-i}) = 0$ for all $i \in N$. Note that $x^{\tilde{f}}(\alpha) = x^f(\alpha)$, $S^{\tilde{f}}(\alpha) = S^f(\alpha) = \varnothing$ and $p_i^{\tilde{f}}(\alpha) = 0$ for all $i \in N$. Moreover, $\tilde{f}$ is individually rational and non-subsidizing at α. Since $x^{\tilde{f}}(\alpha) = 0$ and $\sum_{i\in N} p_i^{\tilde{f}}(\alpha) = 0$, $\tilde{f}$ is budget balanced at α. Obviously, $WL(f, \alpha) = \sum_{i\in N} \alpha_i = WL(\tilde{f}, \alpha)$.

Case 2: $x^f(\alpha) = 1$. We now construct $\tilde{f}(\alpha) = (x^{\tilde{f}}(\alpha), S^{\tilde{f}}(\alpha), p^{\tilde{f}}(\alpha))$, individually rational, non-subsidizing and budget balanced at α, with the property that $WL(\tilde{f}, \alpha) \leq WL(f, \alpha)$. Again, we do that by applying conditions (C1.a), (C1.b) and (C1.c) in Corollary 1 to the family of duples $(\phi_i^{\tilde{f}}(\alpha_{-i}), h_i^{\tilde{f}}(\alpha_{-i}))_{i\in N}$ defined as follows. Set $h_i^{\tilde{f}}(\alpha_{-i}) = 0$ for all $i \in N$.

Assume first that $\sum_{i\in N} \alpha_i < 1$. Thus,

$$\textstyle\sum_{i\notin S^f(\alpha)} \alpha_i < 1 - \sum_{i\in S^f(\alpha)} \alpha_i.$$

Hence, by (2), $WL(f, \alpha) = 1 - \sum_{i\in S^f(\alpha)} \alpha_i$. For $i \notin S^f(\alpha)$, set $\phi_i^{\tilde{f}}(\alpha_{-i}) = \phi_i^f(\alpha_{-i})$ and for $i \in S^f(\alpha)$, set any $\phi_i^{\tilde{f}}(\alpha_{-i}) > \alpha_i$. Then, by Corollary 1, $x^{\tilde{f}}(\alpha) = 0$ and $S^{\tilde{f}}(\alpha) = \varnothing$. Since $p_i^{\tilde{f}}(\alpha) = 0$ for all $i \in N$, $\sum_{i\in N} p_i^{\tilde{f}}(\alpha) = 0$. Thus, $\tilde{f}$ is individually rational, non-subsidizing and budget balanced at α. Moreover, $WL(\tilde{f}, \alpha) = \max\{\sum_{i\in N} \alpha_i - 1, 0\} = 0 < 1 - \sum_{i\in S^f(\alpha)} \alpha_i = WL(f, \alpha)$. Hence, $\tilde{f}$ has a strictly smaller welfare loss than f at α.

Assume now that $\sum_{i\in N} \alpha_i \geq 1$. Thus,

$$\textstyle\sum_{i\notin S^f(\alpha)} \alpha_i \geq 1 - \sum_{i\in S^f(\alpha)} \alpha_i.$$

Hence, by (2), $WL(f, \alpha) = \sum_{i\notin S^f(\alpha)} \alpha_i$. We now distinguish between two subcases.

Subcase 2.1: $\sum_{i\in S^f(\alpha)} \alpha_i \geq 1$. Then, for all $i \notin S^f(\alpha)$ set $\phi_i^{\tilde{f}}(\alpha_{-i}) = \phi_i^f(\alpha)$ and for each $i \in S^f(\alpha)$ choose any $\phi_i^{\tilde{f}}(\alpha_{-i}) \leq \alpha_i$ with the property that $\sum_{i\in S^f(\alpha)} \phi_i^{\tilde{f}}(\alpha_{-i}) = 1$. Then, $x^{\tilde{f}}(\alpha) = 1$ and $S^{\tilde{f}}(\alpha) = S^f(\alpha)$. Since $p_i^{\tilde{f}}(\alpha) = \phi_i^{\tilde{f}}(\alpha_{-i}) \leq \alpha_i$ for all $i \in S^{\tilde{f}}(\alpha)$ and $p_i^{\tilde{f}}(\alpha) = 0$ for all $i \notin S^{\tilde{f}}(\alpha)$, $\tilde{f}$ is individually rational and non-subsidizing at α. Since $x^{\tilde{f}}(\alpha) = 1$ and $\sum_{i\in N} p_i^{\tilde{f}}(\alpha) = \sum_{i\in S^{\tilde{f}}(\alpha)} \phi_i^{\tilde{f}}(\alpha_{-i}) = 1$, $\tilde{f}$ is budget balanced at α. Moreover, $WL(\tilde{f}, \alpha) = \sum_{i\notin S^{\tilde{f}}(\alpha)} \alpha_i = \sum_{i\notin S^f(\alpha)} \alpha_i = WL(f, \alpha)$.

Subcase 2.2: $\sum_{i\in S^f(\alpha)} \alpha_i < 1$. Then, for all $i \notin S^f(\alpha)$ set $\phi_i^{\tilde{f}}(\alpha_{-i}) = \phi_i^f(\alpha)$ and for each $i \in S^f(\alpha)$ choose any $\phi_i^{\tilde{f}}(\alpha_{-i}) > \alpha_i$. Then, $x^{\tilde{f}}(\alpha) = 0$ and $S^{\tilde{f}}(\alpha) = \varnothing$. Since $p_i^{\tilde{f}}(\alpha) = 0$ for all $i \in N$, $\tilde{f}$ is individually rational, non-subsidizing and budget balanced at α. Finally, since $\sum_{i\in N} \alpha_i \geq 1$, by (2), $WL(\tilde{f}, \alpha) = \max\{\sum_{i\in N} \alpha_i - 1, 0\} = \sum_{i\in N} \alpha_i - 1 < \sum_{i\in N} \alpha_i - \sum_{i\in S^f(\alpha)} \alpha_i = \sum_{i\notin S^f(\alpha)} \alpha_i = WL(f, \alpha)$, where the strict inequality follows from the hypothesis of Subcase 2.2.

Thus, we have defined a rule $\tilde{f}$ that is budget balanced, non-subsidizing, individually rational and, by Corollary 1, strategy-proof. Moreover, since for all $\alpha \in \mathbb{R}_+^N$, $WL(\tilde{f}, \alpha) \leq WL(f, \alpha)$, we have that $MWL(\tilde{f}) \leq MWL(f)$. ■

Let $\Phi^{NS\cap BB} \subsetneq \Phi$ be the set of strategy-proof, individually rational, non-subsidizing, and budget balanced rules. Let $f \in \Phi^{NS\cap BB}$ and $\alpha \in \mathbb{R}_+^N$. Remark 2 below says that the welfare loss of f at α can be written in a very useful way.

Remark 2 Assume $x^f(\alpha) = 1$. Since f is individually rational, $\alpha_i - p_i^f(\alpha) \geq 0$ for all $i \in S^f(\alpha)$. Summing up, $\sum_{i\in S^f(\alpha)} \alpha_i - \sum_{i\in S^f(\alpha)} p_i^f(\alpha) \geq 0$. By individual rationality, non-subsidizingness, and budget balancedness, $\sum_{i\in S^f(\alpha)} \alpha_i - 1 \geq 0$. Thus, $WL(f, \alpha)$ can be written as

$$WL(f, \alpha) = \begin{cases} \sum\limits_{i\notin S^f(\alpha)} \alpha_i & \text{if } x^f(\alpha) = 1 \\ \max\{\sum\limits_{i\in N} \alpha_i - 1, 0\} & \text{if } x^f(\alpha) = 0. \end{cases}$$

The following result will be useful in the sequel.

Lemma A *Let $\tilde{f}, f \in \Phi^{NS\cap BB}$ be such that $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$ for all $\alpha \in \mathbb{R}_+^N$. Then, $MWL(\tilde{f}) \leq MWL(f)$.*

Proof First, note that for all α such that $x^f(\alpha) = x^{\tilde{f}}(\alpha)$, $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$ implies that $WL(\tilde{f}, \alpha) \leq WL(f, \alpha)$. Suppose that $x^{\tilde{f}}(\alpha) = 1$ and $x^f(\alpha) = 0$ (the opposite cannot occur since $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$ for all $\alpha \in \mathbb{R}^N_+$). Then, $WL(\tilde{f}, \alpha) = \sum_{i \notin S^{\tilde{f}}(\alpha)} \alpha_i$ and $WL(f, \alpha) = \max\{\sum_{i \in N} \alpha_i - 1, 0\}$. Since $\tilde{f}$ is feasible and individually rational, $\sum_{i \in S^{\tilde{f}}(\alpha)} \alpha_i \geq 1$ and therefore $WL(f, \alpha) = \sum_{i \in N} \alpha_i - 1 = \sum_{i \notin S^{\tilde{f}}(\alpha)} \alpha_i + \sum_{i \in S^{\tilde{f}}(\alpha)} \alpha_i - 1 \geq \sum_{i \notin S^{\tilde{f}}(\alpha)} \alpha_i = WL(\tilde{f}, \alpha)$. ■

5.2 Demand Monotonicity

We state now the property of demand monotonicity, introduced by Ohseto (2000), that will be very useful to prove Theorem 2. Demand monotonicity can be interpreted as a weak efficiency requirement. Its violation, in fact, implies that at some profile the rule excludes some agent who is willing to join the group of users.

Definition 8 A rule $f : \mathbb{R}^N_+ \to FA$ is *demand monotonic* if for all $\alpha, \alpha' \in \mathbb{R}^N_+$ the following two conditions hold:
(DM.1) if $\alpha'_i \geq \alpha_i$ for all $i \in N$ then $S^f(\alpha') \supseteq S^f(\alpha)$; and
(DM.2) if $\alpha'_i \geq \alpha_i$ for all $i \in S^f(\alpha)$ and $\alpha'_j \leq \alpha_j$ for all $j \notin S^f(\alpha)$ then $S^f(\alpha') = S^f(\alpha)$.

Remark 3 Strategy-proofness, individual rationality, non-subsidizingness, and budget balancedness do not imply demand monotonicity (see Example 3 in Ohseto (2000)).

Definition 9 A rule $f : \mathbb{R}^N_+ \to FA$ is a *semiconstant cost sharing rule* if for all $\alpha, \alpha' \in \mathbb{R}^N_+$, $S^f(\alpha) = S^f(\alpha')$ implies $f(\alpha) = f(\alpha')$.

This is a simple class of rules where agents' contributions only depend on the set of users. Ohseto (2000) shows that the following remark holds.

Remark 4 Any strategy-proof, individually rational, non-subsidizing, budget-balanced, and demand monotonic rule is a semiconstant cost sharing rule.

Let $\Phi^{NS \cap BB \cap DM} \subsetneq \Phi$ be the class of strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic rules. Lemma 2 below shows that we can restrict our search of the rule minimizing the maximal welfare loss to the class $\Phi^{NS \cap BB \cap DM}$ without loss of generality, since this class contains all rules which are candidates to minimize the maximal welfare loss.

Lemma 2 *Let $f : \mathbb{R}^N_+ \to FA$ be a strategy-proof, individually rational, non-subsidizing, and budget balanced rule and assume that f is not demand monotonic. Then, there ex-*

ists a strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic rule $\tilde{f} : \mathbb{R}^N_+ \rightarrow FA$ *that has a lower or equal maximal welfare loss than* f*.*

Proof Assume $f \in \Phi \backslash \Phi^{NS \cap BB \cap DM}$. Since f does not satisfy demand monotonicity there exist $\alpha'', \alpha' \in \mathbb{R}^N_+$ such that either
(2.a) $\alpha''_i \geq \alpha'_i$ for all $i \in N$ and $S^f(\alpha') \nsubseteq S^f(\alpha'')$ or
(2.b) $\alpha''_i \geq \alpha'_i$ for all $i \in S^f(\alpha')$ and $\alpha''_i \leq \alpha'_i$ for all $i \notin S^f(\alpha')$ and $S^f(\alpha') \neq S^f(\alpha'')$.

The proof proceeds in 3 steps.

Step 1: Suppose that (2.a) occurs; otherwise, set $\bar{f} = f$ and go to Step 2. Let

$$\begin{aligned} Inf^f_{(1)} = \; & \{\alpha' \in \mathbb{R}^N_+ \mid \text{there exists } \alpha'' \in \mathbb{R}^N_+ \text{ such that } \alpha''_i \geq \alpha'_i \text{ for all } i \in N,\, S^f(\alpha') \nsubseteq S^f(\alpha'') \\ & \text{and for all } \alpha''' \in \mathbb{R}^N_+ \text{ such that } \alpha'_i \geq \alpha'''_i \text{ for all } i \in N,\, S^f(\alpha') \supseteq S^f(\alpha''')\} \end{aligned}$$

be the set of all smallest profiles for which (2.a) occurs. For each $\alpha' \in \mathbb{R}^N_+$, let

$$NoDM^f_{(1)}(\alpha') = \{\alpha'' \in \mathbb{R}^N_+ \mid \alpha''_i \geq \alpha'_i \text{ for all } i \in N \text{ and } S^f(\alpha') \nsubseteq S^f(\alpha'')\}$$

be the set of all profiles for which (2.a) occurs with respect to α'.

Let $\{\phi^f_i\}_{i \in N}$ be the family of functions associated to f identified in Corollary 1. By non-subsidizingness, the corresponding family $\{h^f_i\}_{i \in N}$ has the property that for all $i \in N$, $h^f_i(\alpha_{-i}) = 0$ for all $\alpha_{-i} \in \mathbb{R}^{N \backslash \{i\}}_+$. From $\{\phi^f_i\}_{i \in N}$, define the rule $\bar{f}$ by describing its associated family of functions $\{\phi^{\bar{f}}_i\}_{i \in N}$ as follows.

- For each $\alpha \notin \bigcup_{\alpha' \in Inf^f_{(1)}} NoDM^f_{(1)}(\alpha')$, set for all $i \in N$, $\phi^{\bar{f}}_i(\alpha_{-i}) = \phi^f_i(\alpha_{-i})$.
- For each $\alpha \in \bigcup_{\bar{\alpha} \in Inf^f_{(1)}} NoDM^f_{(1)}(\bar{\alpha})$, set for all $i \in N$,

  $$\phi^{\bar{f}}_i(\alpha_{-i}) = \begin{cases} \phi^f_i(\alpha'_{-i}) & \text{if } i \in S^f(\alpha') \\ 0 & \text{otherwise,} \end{cases}$$

  where α' is such that $\alpha \in NoDM^f_{(1)}(\alpha')$.

Step 2: Consider $\bar{f}$, the outcome of Step 1, and assume (2.b) occurs for $\bar{f}$; otherwise, set $\tilde{f} = \bar{f}$ and go to Step 3. Observe that, by construction, (2.a) does not occur for $\bar{f}$ and it satisfies conditions (C1.a), (C1.b), and (C1.c) of Corollary 1. Let

$$\begin{aligned} Inf^{\bar{f}}_{(2)} = \quad & \{\alpha' \in \mathbb{R}^N_+ \mid \text{there exists } \alpha'' \in \mathbb{R}^N_+ \text{ such that } \alpha''_i \geq \alpha'_i \text{ for all } i \in S^{\bar{f}}(\alpha'), \\ & \alpha''_i \leq \alpha'_i \text{ for all } i \notin S^{\bar{f}}(\alpha'),\ S^{\bar{f}}(\alpha') \neq S^{\bar{f}}(\alpha'') \text{ and for all } \alpha''' \in \mathbb{R}^N_+ \backslash \{\alpha'\} \\ & \text{such that } \alpha'''_i \leq \alpha'_i \text{ for all } i \in S^{\bar{f}}(\alpha''') \text{ and } \alpha'''_i \geq \alpha'_i \text{ for all } i \notin S^{\bar{f}}(\alpha'''), \\ & S^{\bar{f}}(\alpha''') = S^{\bar{f}}(\alpha')\} \end{aligned}$$

be the set of all smallest/largest profiles for which (2.b) occurs. For each $\alpha' \in Inf^{\bar{f}}_{(2)}$, define

$$\begin{aligned} NoDM^{\bar{f}}_{(2)}(\alpha') = \quad & \{\alpha'' \in \mathbb{R}^N_+ \mid \alpha''_i \geq \alpha'_i \text{ for all } i \in S^{\bar{f}}(\alpha'), \alpha''_i \leq \alpha'_i \text{ for all } i \notin S^{\bar{f}}(a), \\ & \text{and } S^{\bar{f}}(\alpha') \neq S^{\bar{f}}(\alpha'')\} \end{aligned}$$

be the set of all profiles for which (2.b) occurs with respect to α'.

Let $\{\phi^{\bar{f}}_i\}_{i\in N}$ be the family of functions associated to $\bar{f}$ identified in Corollary 1. From $\{\phi^{\bar{f}}_i\}_{i\in N}$, define the rule $\tilde{f}$ by describing its associated family of functions $\{\phi^{\tilde{f}}_i\}_{i\in N}$ as follows.

- For each $\alpha \notin \bigcup_{\alpha' \in Inf^{\bar{f}}_{(2)}} NoDM^{\bar{f}}_{(2)}(\alpha')$, set for all $i \in N$, $\phi^{\tilde{f}}_i(\alpha_{-i}) = \phi^{\bar{f}}_i(\alpha_{-i})$.
- For each $\alpha' \in Inf^{\bar{f}}_{(2)}$, set for all $\in N$,

$$\phi^{\tilde{f}}_i(\alpha'_{-i}) = \begin{cases} \phi^{\bar{f}}_i(\alpha'_{-i}) & \text{if } i \in S^{\bar{f}}(\alpha') \\ 0 & \text{otherwise.} \end{cases}$$

- For each $\alpha \in \bigcup_{\bar{\alpha} \in Inf^{\bar{f}}_{(2)}} NoDM^{\bar{f}}_{(2)}(\bar{\alpha}) \backslash Inf^{\bar{f}}_{(2)}$, set for all $i \in N$, $\phi^{\tilde{f}}_i(\alpha_{-i}) = \phi^{\tilde{f}}_i(\alpha'_{-i})$, where $\alpha' \in Inf^{\bar{f}}_2$ is such that $\alpha \in NoDM^{\bar{f}}_{(2)}(\alpha')$.

Step 3: Consider $\tilde{f}$, the outcome of Step 2. It is easy to see that $\tilde{f}$ and $\{\phi^{\tilde{f}}_i\}_{i\in N}$ satisfy properties (C1.a), (C1.b), and (C1.c) of Corollary 1. Hence, $\tilde{f}$ is strategy-proof. Moreover, $\tilde{f}$ is non-subsidizing and individually rational. For all $\alpha \in \mathbb{R}^N_+$ there exists $\alpha' \in \mathbb{R}^N_+$ such that $\sum_{i\in N} p^{\tilde{f}}(\alpha) = \sum_{i\in N} p^f(\alpha')$ and hence, since f is budget balanced, $\tilde{f}$ is budget balanced as well. Furthermore, by construction of $\tilde{f}$, there do not exist $\alpha, \alpha' \in \mathbb{R}^N_+$ for which either property (2.a) or (2.b) of the negation of demand monotonicity holds; thus, $\tilde{f}$ is demand monotonic. Finally, by Lemma A, $\tilde{f}$ has a lower or equal welfare loss than f since, for all $\alpha \in \mathbb{R}^N_+$, $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$ holds. ■

From now on, and without loss of generality, we restrict our search to the class $\Phi^{NS\cap BB\cap DM}$. The following three lemmata will be very useful because they will allow us to pay attention only to profiles where the set of users is empty and define the maximal welfare loss of a rule as its aggregate loss (the maximal sum of non-users' valuations).

Lemma B *For all $f \in \Phi^{NS\cap BB\cap DM}$ and all $\alpha \in \mathbb{R}_+^N$ such that $\#S^f(\alpha) \geq 2$ there exists $\alpha' \in \mathbb{R}_+^N$ such that* (i) $S^f(\alpha') \subsetneq S^f(\alpha)$ *and* (ii) $WL(f, \alpha) < WL(f, \alpha')$.

Proof Let $f \in \Phi^{NS\cap BB\cap DM}$ and assume that $\alpha \in \mathbb{R}_+^N$ is such that $\#S^f(\alpha) \geq 2$. Since f is individually rational there exists at least one user in $S^f(\alpha)$ who pays a strictly positive price. Let $i \in S^f(\alpha)$ be one of such users. Then, by Corollary 1, $\alpha_i \geq \phi_i^f(\alpha_{-i}) > 0$. Since $S^f(\alpha) \neq \varnothing$, $x^f(\alpha) = 1$ and by Remark 2, $WL(f, \alpha) = \sum_{k \notin S^f(\alpha)} \alpha_k$. Let $j \in S^f(\alpha) \backslash \{i\}$ and consider any $\alpha_j' > \max\{1, \alpha_j\}$. Since f is demand monotonic, (DM.2) implies $S^f(\alpha) = S^f(\alpha_j', \alpha_{-j})$. By Remark 4, $f(\alpha) = f(\alpha_j', \alpha_{-j})$. Hence, $p_i^f(\alpha) = p_i^f(\alpha_j', \alpha_{-j})$. Thus, by individual rationality of f and Corollary 1, $\phi_i^f(\alpha_{-i}) = \phi_i^f(\alpha_j', \alpha_{-\{i,j\}})$. Moreover, $WL(f, \alpha) = WL(f, (\alpha_j', \alpha_{-j})) = \sum_{k \notin S^f(\alpha)} \alpha_k$. Consider now any profile $\alpha' = (\alpha_i', \alpha_j', \alpha_{-\{i,j\}})$ such that $0 < \alpha_i' < \phi_i^f(\alpha_{-i}') = \phi_i^f(\alpha_j', \alpha_{-\{i,j\}}) = \phi_i^f(\alpha_{-i})$. By Corollary 1, $i \notin S^f(\alpha')$. By (DM.1) in demand monotonicity, $S^f(\alpha') \subsetneq S^f(\alpha_j', \alpha_{-j}) = S^f(\alpha)$. This proves that (i) holds. If $x^f(\alpha') = 1$ then $WL(f, \alpha') = \sum_{k \notin S^f(\alpha')} \alpha_k' > \sum_{k \notin S^f(\alpha)} \alpha_k = WL(f, \alpha)$. If $x^f(\alpha') = 0$ then $WL(f, \alpha') = \sum_{k \in N} \alpha_k' - 1 = \sum_{k \notin S^f(\alpha)} \alpha_k' + \sum_{k \in S^f(\alpha)} \alpha_k' - 1 > \sum_{k \notin S^f(\alpha)} \alpha_k' = \sum_{k \notin S^f(\alpha)} \alpha_k$, where the strict inequality follows since $\alpha_j' > 1$ and $j \in S^f(\alpha)$ and the last equality holds by definition of α'. This proves that (ii) holds. ■

Let $f \in \Phi^{NS\cap BB\cap DM}$ and $\alpha \in \mathbb{R}_+^N$ be such that $S^f(\alpha) = \{i\}$. Define

$$A^\alpha(f) = \{\alpha' \in \mathbb{R}_+^N \mid \alpha_j' = \alpha_j \text{ for all } j \neq i \text{ and } \alpha_i' < 1\}.$$

Lemma C *For all $f \in \Phi^{NS\cap BB\cap DM}$ and all $\alpha \in \mathbb{R}_+^N$ such that $S^f(\alpha) = \{i\}$, $WL(f, \alpha) = \sup_{\alpha' \in A^\alpha(f)} WL(f, \alpha')$.*

Proof Let $f \in \Phi^{NS\cap BB\cap DM}$ and assume that $\alpha \in \mathbb{R}_+^N$ is such that $S^f(\alpha) = \{i\}$. By individual rationality of f and Corollary 1, $\phi_i^f(\alpha_{-i}) = 1 \leq \alpha_i$. By Remark 2, and since $x^f(\alpha) = 1$, $WL(f, \alpha) = \sum_{j \neq i} \alpha_j$. Now, consider any $\alpha' \in A^\alpha$. By (DM.1) in demand monotonicity and Corollary 1, $S^f(\alpha') = \varnothing$ and $x^f(\alpha') = 0$. Thus, by Remark 2, $WL(f, \alpha') =$

$\max\{\sum_{j\neq i}\alpha'_j+\alpha'_i-1,0\}$. Hence,

$$\sup_{\alpha'\in A^{\alpha}(f)} WL(f,\alpha') = \sup_{\alpha'\in A^{\alpha}(f)} \max\{\textstyle\sum_{j\neq i}\alpha'_j+\alpha'_i-1,0\} = \textstyle\sum_{j\neq i}\alpha'_j = \sum_{j\neq i}\alpha_j = WL(f,\alpha).$$

■

Given $f\in\Phi^{NS\cap BB\cap DM}$, define the set of profiles where the set of users is empty as $A^{\varnothing}(f)\equiv\{\alpha\in\mathbb{R}^N_+ \mid S^f(\alpha)=\varnothing\}$. By Corollary 1,

$$A^{\varnothing}(f)=\{\alpha\in\mathbb{R}^N_+ \mid \alpha_i<\phi_i^f(\alpha_{-i}) \text{ for all } i\in N\}. \tag{3}$$

Observe that for any $\alpha\in\mathbb{R}^N_+$ such that $S^f(\alpha)=\{i\}$,

$$A^{\alpha}(f)\subseteq A^{\varnothing}(f). \tag{4}$$

Lemma D *For all* $f\in\Phi^{NS\cap BB\cap DM}$, $\sup_{\alpha\in\mathbb{R}^N_+} WL(f,\alpha)=\sup_{\alpha\in\mathbb{R}^N_+}\sum_{i\in N}\phi_i^f(\alpha_{-i})$.

Proof Let $f\in\Phi^{NS\cap BB\cap DM}$. Then,

$$\begin{aligned}
\sup_{\alpha\in\mathbb{R}^N_+} WL(f,\alpha) &= \sup_{\alpha\in\{\alpha'\in\mathbb{R}^N_+ \mid \#S^f(\alpha')\geq 2\}\cup\{\alpha'\in\mathbb{R}^N_+ \mid \#S^f(\alpha')=1\}\cup A^{\varnothing}(f)} WL(f,\alpha)\\
&= \sup_{\alpha\in\{\alpha'\in\mathbb{R}^N_+ \mid \#S^f(\alpha')=1\}\cup A^{\varnothing}(f)} WL(f,\alpha)\\
&= \sup_{\alpha\in A^{\varnothing}(f)} WL(f,\alpha)\\
&= \sup_{\alpha\in\mathbb{R}^N_+}\sum_{i\in N}\phi_i^f(\alpha_{-i}),
\end{aligned}$$

where the second equality follows from Lemma B, the third one from Lemma C and condition (4), and the fourth equality follows from condition (4). ■

We now define the aggregate loss of a rule $f\in\Phi^{NS\cap BB\cap DM}$ as

$$AL(f)\equiv\sup_{\alpha\in\mathbb{R}^N_+}\sum_{i\notin S^f(\alpha)}\alpha_i.$$

Then, for any $f\in\Phi^{NS\cap BB\cap DM}$,

$$\begin{aligned}
MWL(f) &= \sup_{\alpha\in\mathbb{R}^N_+} WL(f,\alpha)\\
&= \sup_{\alpha\in\mathbb{R}^N_+}\sum_{i\in N}\phi_i^f(\alpha_{-i})\\
&= \sup_{\alpha\in\mathbb{R}^N_+}\sum_{i\notin S^f(\alpha)}\alpha_i\\
&= AL(f),
\end{aligned}$$

where the second equality follows from Lemma D. Hence, the following remark holds.

Remark 5 To find a rule $\hat{f} \in \Phi^{NS \cap BB \cap DM}$ such that $MWL(\hat{f}) \leq MWL(f)$ for all $f \in \Phi^{NS \cap BB \cap DM}$ is equivalent to find a rule $\hat{f} \in \Phi^{NS \cap BB \cap DM}$ such that $AL(\hat{f}) \leq AL(f)$ for all $f \in \Phi^{NS \cap BB \cap DM}$.

Finally, it is straightforward to check that $f^{EC} \in \Phi^{NS \cap BB \cap DM}$ and that the aggregate loss for the equal-cost sharing rule is equal to

$$AL(f^{EC}) = 1 + \frac{1}{2} + ... + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}. \tag{5}$$

5.3 Consumer sovereignty

The next property imposes to rules a minimal requirement of being sensible to agents' valuations: each agent has a valuation that guarantees her that the public good is provided and she is one of its users.

Definition 10 A rule $f : \mathbb{R}_+^N \to FA$ satisfies *consumer sovereignty* if for all $i \in N$ there exists $\overline{\alpha}_i \in \mathbb{R}_+$ such that for all $\alpha_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$, $i \in S^f(\overline{\alpha}_i, \alpha_{-i})$.

We state in Lemma 3 below the observation that a rule that minimizes the maximal welfare loss has to satisfy consumer sovereignty.

Lemma 3 *Let $f : \mathbb{R}_+^N \to FA$ be a rule that minimizes the maximal welfare loss. Then, f satisfies consumer sovereignty.*

Proof Immediate from the definition of consumer sovereignty. ■

5.4 Cross monotonicity

The last additional property on rules that we consider is cross monotonicity. It imposes conditions on the price vector chosen by the rule at two profiles for which the set of users at one profile is contained in the set of users at the other profile: the price paid by the users can not increase if more agents become users at the new profile.[9]

[9]This property has already been used in a more general public good setting by Moulin and Shenker (2001). Similar notions have also been used in different setings under the name of population monotonicity (see for instance Sprumont (1990)).

Definition 11 A rule $f : \mathbb{R}^N_+ \to FA$ satisfies *cross monotonicity* if for all $\alpha, \alpha' \in \mathbb{R}^N_+$ such that $S^f(\alpha) \subset S^f(\alpha')$, $p^f_i(\alpha) \geq p^f_i(\alpha')$ for all $i \in S^f(\alpha)$.

Lemma 4 below says that cross monotonicity is satisfied by all strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic rules for which consumer sovereignty holds.

Lemma 4 *Let $f : \mathbb{R}^N_+ \to FA$ be a strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic rule that satisfies consumer sovereignty. Then, f is cross monotonic.*

Proof Let f be a rule satisfying the hypothesis of Lemma 4 and let $\alpha, \alpha' \in \mathbb{R}^N_+$ be such that $S^f(\alpha) \subset S^f(\alpha')$. We want to show that $p^f_i(\alpha) \geq p^f_i(\alpha')$ for all $i \in S^f(\alpha)$. If $S^f(\alpha) = N$ then, by Remark 4, f is cross monotonic. Let $\widetilde{\alpha} \in \mathbb{R}^N_+$ be such that

$$\widetilde{\alpha}_j = \begin{cases} \alpha_j & \text{if } j \in S^f(\alpha) \\ 0 & \text{if } j \notin S^f(\alpha). \end{cases}$$

By (DM.2), $S^f(\alpha) = S^f(\widetilde{\alpha})$. By Remark 4, $p^f(\alpha) = p^f(\widetilde{\alpha})$. By Theorem 1, for all $j \in S^f(\widetilde{\alpha})$, $p^f_j(\widetilde{\alpha}) = \phi^f_j(\widetilde{\alpha}_{-j})$ and by individual rationality $\widetilde{\alpha}_j \geq \phi^f_j(\widetilde{\alpha}_{-j})$. Take any $j^1 \in S^f(\widetilde{\alpha})$ and let $\widetilde{\alpha}^1 = (\phi^f_{j^1}(\widetilde{\alpha}_{-j^1}), \widetilde{\alpha}_{-j^1})$. By Corollary 1, $j^1 \in S^f(\widetilde{\alpha}^1)$. By (DM.2), $S^f(\widetilde{\alpha}^1) = S^f(\widetilde{\alpha})$. By Remark 4, $p^f(\widetilde{\alpha}^1) = p^f(\widetilde{\alpha})$. Iterating this argument for all $j \in S^f(\widetilde{\alpha})$ we obtain a profile $\widehat{\alpha}$, where

$$\widehat{\alpha}_j = \begin{cases} \phi^f_j(\widetilde{\alpha}_{-j}) & \text{if } j \in S^f(\widetilde{\alpha}) \\ 0 & \text{if } j \notin S^f(\widetilde{\alpha}) \end{cases}$$

such that $S^f(\widehat{\alpha}) = S^f(\alpha)$ and $p^f(\widehat{\alpha}) = p^f(\alpha)$. By consumer sovereignty, for each $j \in S^f(\alpha') \backslash S^f(\alpha) \equiv T$, there exists $\overline{\alpha}_j$ such that $\overline{\alpha}$ defined by

$$\overline{\alpha}_j = \begin{cases} \phi^f_j(\overline{\alpha}_{-j}) & \text{if } j \in S^f(\widetilde{\alpha}) \\ \overline{\alpha}_j & \text{if } j \in T \\ 0 & \text{if } j \notin S^f(\widetilde{\alpha}) \cup T \end{cases}$$

is such that, by (DM.1) in demand monotonicity, $S^f(\overline{\alpha}) \supseteq S^f(\alpha') = S^f(\alpha) \cup T$. We want to show that $S^f(\overline{\alpha}) = S^f(\alpha')$. Define $\alpha'' \in \mathbb{R}^N_+$ by

$$\alpha''_j = \begin{cases} \alpha'_j & \text{if } j \in S^f(\alpha) \\ 0 & \text{if } j \notin S^f(\alpha). \end{cases}$$

By (DM.2) in demand monotonicity, $S^f(\alpha'') = S^f(\alpha')$. Consider any profile α^* such that $\alpha_j^* > \max\{\alpha_j'', \overline{\alpha}_j\}$ for all $j \in S^f(\alpha')$ and $\alpha_j^* = 0$ for all $j \notin S^f(\alpha')$. By (DM.2) in demand monotonicity, $S^f(\alpha^*) = S^f(\alpha'') = S^f(\alpha')$. Again, by (DM.2), $S^f(\alpha^*) = S^f(\overline{\alpha})$. Thus, $S^f(\overline{\alpha}) = S^f(\alpha')$. By Remark 4, $p^f(\overline{\alpha}) = p^f(\alpha')$. Since for all $i \in S^f(\alpha)$, $\overline{\alpha}_i = p^f(\alpha)$ and, by individual rationality $\overline{\alpha}_i \geq p^f(\overline{\alpha})$, we have that $p_i^f(\alpha') = p_i^f(\overline{\alpha}) \leq \overline{\alpha}_i = p_i^f(\alpha)$ ■

Let Φ^* be the class of all strategy-proof, individually rational, non-subsidizing, budget balanced, demand and cross monotonic rules that satisfy consumer sovereignty.

5.5 Proof of Theorem 2

By Lemmata 1 to 4 we can restrict our search of a rule in Φ (all strategy-proof and individually rational rules) minimizing the maximal welfare loss to the class of rules in Φ^*. Thus, to proceed with the proof of Theorem 2, consider any $f \in \Phi^*$ such that $f \neq f^{EC}$. By Remark 5, We have to show that $AL(f) > AL(f^{EC})$. By Remark 4, f is a semiconstant cost sharing rule. Hence, for all $\alpha, \alpha' \in \mathbb{R}_+^N$ such that $S^f(\alpha) = S^f(\alpha')$, $f(\alpha) = f(\alpha')$. To show that $AL(f) > AL(f^{EC})$, pick $\bar{\alpha} \in \mathbb{R}_+^N$ with the property that for each $i \in N$, $i \in S^f(\overline{\alpha}_i, \alpha_{-i})$ for all $\alpha_{-i} \in \mathbb{R}_+^{N\backslash\{i\}}$. Observe that by consumer sovereignty such $\bar{\alpha}$ exists. Then $S^f(\bar{\alpha}) = N$. Let $i_1 \in N$ be the agent such that $p_{i_1}^f(\bar{\alpha}) = \max_{j\in N} p_j^f(\bar{\alpha})$. Observe that, by Remark 4, $p^f(\alpha) = p^f(\bar{\alpha})$ for all α such that $S^f(\alpha) = N$. Since, by budget balancedness, $\sum_{j\in N} p_j^f(\bar{\alpha}) = 1$, then $p_{i_1}^f(\bar{\alpha}) \geq \frac{1}{n}$. By Theorem 1, $\phi_{i_1}^f(\bar{\alpha}_{-i_1}) = p_{i_1}^f(\bar{\alpha}) > 0$. Pick now $\bar{\alpha}^1 = (0, \bar{\alpha}_{-i_1})$. Observe that by consumer sovereignty, $S^f(\bar{\alpha}^1) = N\backslash\{i_1\}$. Let $p_{i_2}^f(\bar{\alpha}^1) = \max_{j\in S^f(\bar{\alpha}^1)} p_j^f(\bar{\alpha}^1)$. Then $p_{i_2}(\bar{\alpha}^1) \geq \frac{1}{n-1}$. Iterating the above argument we get a profile $(p_{i_1}^f(\bar{\alpha}), ..., p_{i_n}^f(\bar{\alpha}^{n-1}))$ with the property that $p_{i_j}^f(\bar{\alpha}) \geq \frac{1}{n-j+1}$ holds for all $j = 1, ..., n$, with at least one strict inequality since $f \neq f^{EC}$ and Remark 4. By cross monotonicity, for sufficiently small $\varepsilon > 0$, $S(p_{i_1}^f(\bar{\alpha}) - \frac{\varepsilon}{n}, ..., p_{i_n}^f(\bar{\alpha}^{n-1}) - \frac{\varepsilon}{n}) = \varnothing$. Since this holds for all sufficiently small $\varepsilon > 0$, $AL(f) \equiv \sup_{\alpha\in\mathbb{R}_+^N} \sum_{i\notin S^f(\alpha)} \alpha_i \geq \sum_{j=1}^n p_{i_j}^f(\bar{\alpha}) > AL(f^{EC}) = \sum_{i=1}^n \frac{1}{i}$, where the strict inequality follows because $f \neq f^{EC}$. Hence, Theorem 2 holds. ■

6 Final Remark

We have shown that the equal-cost sharing rule minimizes the maximal welfare loss in the class of strategy-proof and individually rational rules. With respect to previous characterizations in the literature we have considered a much broader set of rules and therefore our result cannot be derived from already existing ones. In particular, in Example 1 below we show that the set of rules we consider is strictly larger than the set of rules considered in Moulin and Shenker's (2001) because there are rules which are strategy-proof and individually rational (and they also satisfy budget balanced) that are neither cross monotonic nor group strategy-proof (it is also easy to exhibit rules which satisfy the three above mentioned properties and are cross monotonic but not group strategy-proof).

Example 1 A strategy-proof, individually rational, and budget balanced rule f that it is neither group strategy-proof nor cross monotonic.

Let $N = \{1, 2, 3\}$. Define f as follows. Let $\alpha \in \mathbb{R}_+^N$:
(i) if $\alpha_1 < \frac{1}{3}$ and $\alpha_2, \alpha_3 \geq \frac{1}{2}$ then, $f(\alpha) = (1, \{2, 3\}, (0, \frac{1}{2}, \frac{1}{2}))$;
(ii) if $\alpha_1 < \frac{1}{3}$ and $\min\{\alpha_2, \alpha_3\} < \frac{1}{2}$ then, $f(\alpha) = (0, \varnothing, (0, 0, 0))$;
(iii) if $\alpha_1 \geq \frac{1}{3}$, $\alpha_2 \geq \frac{1}{9}$ and $\alpha_3 \geq \frac{5}{9}$ then, $f(\alpha) = (1, N, (\frac{1}{3}, \frac{1}{9}, \frac{5}{9}))$;
(iv) if $\alpha_1 \geq \frac{1}{3}$, $\alpha_2 \geq \frac{1}{9}$, and $\alpha_3 < \frac{5}{9}$ then, $f(\alpha) = (0, \varnothing, (0, 0, 0))$;
(v) if $\alpha_1 \geq \frac{1}{3}$, $\alpha_2 < \frac{1}{9}$, and $\alpha_3 \geq \frac{5}{9}$ then, $f(\alpha) = (0, \varnothing, (0, 0, 0))$; and
(vi) if $\alpha_1 \geq \frac{1}{3}$, $\alpha_2 < \frac{1}{9}$, and $\alpha_3 < \frac{5}{9}$ then, $f(\alpha) = (0, \varnothing, (0, 0, 0))$.

Observe that, by Theorem 1, f is strategy-proof. To see that f is not cross monotonic, consider $\alpha = (\frac{1}{3}, 1, 1)$ and $\alpha' = (0, 1, 1)$. Then, $f(\alpha) = (1, N, (\frac{1}{3}, \frac{1}{9}, \frac{5}{9}))$ and $f(\alpha') = (1, \{2, 3\}, (0, \frac{1}{2}, \frac{1}{2}))$. Moreover, $\{1, 3\}$ can manipulate f at α by declaring $(\alpha_1', \alpha_3') = (0, 1)$. $\square$

References

[1] Clarke, E. "Multipart pricing of public goods," *Public Choice* 11, 17-33 (1971).

[2] Deb, R. and Razzolini, L. "Auction-like mechanisms for pricing excludable public goods," *Journal Economic Theory* 88, 340-368 (1999a).

[3] Deb, R. and Razzolini, L. "Voluntary cost sharing for an excludable public project," *Mathematical Social Sciences* 37, 123-138 (1999b).

[4] Green, J. and Laffont, J.J. "Characterization of satisfactory mechanisms for the revelation of preferences for public goods," *Econometrica* 45, 427-438 (1977).

[5] Green, J. and Laffont, J.J. *Incentives in Public Decision Making*. North-Holland, Amsterdam (1979).

[6] Groves, T. "Incentives in teams," *Econometrica* 41, 617-663 (1973).

[7] Juarez, R. "The worst absolute loss in the problem of the commons: random priority versus average cost," *Economic Theory* 34, 69-84 (2008a).

[8] Juarez, R. "Optimal group strategyproof cost sharing: budget-balance vs. efficiency," mimeo. University of Hawaii. (2008b).

[9] Koutsoupias, E. and Papadimitriou, C. "Worst-case equilibria," in the *16th Annual Symposium on Theoretical Aspects of Computer Science* 404–413 (1999).

[10] Krishna, V. *Auction Theory*, Academic Press (2002).

[11] Moulin, H. "Serial cost sharing of excludable public goods," *Review of Economic Studies* 61, 305-325 (1994).

[12] Moulin H. "The price of anarchy of serial, average and incremental cost sharing," *Economic Theory* 36, 379-405 (2008).

[13] Moulin, H. "Auctioning or assigning an object: some remarkable VCG mechanisms," *Social Choice and Welfare* 34, 193-216 (2010).

[14] Moulin, H. and Shenker, S. "Strategyproof sharing of submodular costs: budget balance versus efficiency," *Economic Theory* 18, 511–533 (2001).

[15] Mussa, M. and Rosen, S. "Monopoly and product quantity," *Journal of Economic Theory* 18, 301-317 (1978).

[16] Myerson, R. "Optimal auction design," *Mathematics of Operations Research* 6, 58-73 (1981).

[17] Myerson, R. and Satterthwaite, M. "Efficient mechanisms for bilateral trading," *Journal of Economic Theory* 29, 265-281 (1983).

[18] Ohseto, S. "Characterizations of strategy-proof mechanisms for excludable versus nonexcludable public projects," *Games and Economic Behavior* 32, 51-66 (2000).

[19] Ohseto, S. "Augmented serial rules for an excludable public good," *Economic Theory* 26, 589-606 (2005).

[20] Roughgarden, T. "The price of anarchy is independent of the network topology," *Symposium on Theory of Computing* 428-437 (2002).

[21] Roughgarden, T. and Tardos, E. "How bad is selfish routing?," *Journal of the Association for Computing Machinery* 49, 236-259 (2002).

[22] Sprumont, Y. "Population monotonic allocation schemes for cooperative games with transferable utility," *Games and Economic Behavior* 2, 378-394 (1990).

Appendix: Proof of Theorem 1

$\Longrightarrow$) Assume $f = (x, S, p)$ is strategy-proof. Fix $i \in N$. We first show that (T1.a) holds. Let $\alpha_i, \alpha_i' \in \mathbb{R}_+$ and $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$ be arbitrary. By strategy-proofness,

$$\mathbf{1}^i_{S(\alpha_i,\alpha_{-i})} x(\alpha_i, \alpha_{-i})\alpha_i - p_i(\alpha_i, \alpha_{-i}) \geq \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i})\alpha_i - p_i(\alpha_i', \alpha_{-i}) \tag{6}$$

and

$$\mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i})\alpha_i' - p_i(\alpha_i', \alpha_{-i}) \geq \mathbf{1}^i_{S(\alpha_i,\alpha_{-i})} x(\alpha_i, \alpha_{-i})\alpha_i' - p_i(\alpha_i, \alpha_{-i}). \tag{7}$$

Adding (6) and (7),

$$(\mathbf{1}^i_{S(\alpha_i,\alpha_{-i})} x(\alpha_i, \alpha_{-i}) - \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i}))(\alpha_i - \alpha_i') \geq 0. \tag{8}$$

Assume without loss of generality that $\alpha_i > \alpha_i'$. By (8),

$$\mathbf{1}^i_{S(\alpha_i,\alpha_{-i})} x(\alpha_i, \alpha_{-i}) \geq \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i}).$$

Namely, for all $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$, $\mathbf{1}^i_{S(\alpha_i,\alpha_{-i})} x(\alpha_i, \alpha_{-i})$ is an increasing function of α_i. Thus, (T1.a) holds.

We next show that (T1.b) holds. By (6), for all $\alpha_i, \alpha_i' \in \mathbb{R}_+$ and all $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$,

$$v_i(x(\alpha_i, \alpha_{-i}), S(\alpha_i, \alpha_{-i}), p(\alpha_i, \alpha_{-i}), \alpha_i) \geq \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i})\alpha_i - p_i(\alpha_i', \alpha_{-i}). \tag{9}$$

Since

$$\begin{aligned} \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i})\alpha_i - p_i(\alpha_i', \alpha_{-i}) &= v_i(x(\alpha_i', \alpha_{-i}), S(\alpha_i', \alpha_{-i}), p(\alpha_i', \alpha_{-i}), \alpha_i') \\ &\quad + \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i})(\alpha_i - \alpha_i'), \end{aligned}$$

(9) can be written as

$$v_i(x(\alpha), S(\alpha), p(\alpha), \alpha_i) \geq v_i(x(\alpha_i', \alpha_{-i}), S(\alpha_i', \alpha_{-i}), p(\alpha_i', \alpha_{-i}), \alpha_i') + \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i})(\alpha_i - \alpha_i'). \tag{10}$$

Similarly, by (7),

$$v_i(x(\alpha_i', \alpha_{-i}), S(\alpha_i', \alpha_{-i}), p(\alpha_i', \alpha_{-i}), \alpha_i') \geq v_i(x(\alpha), S(\alpha), p(\alpha), \alpha_i) + \mathbf{1}^i_{S(\alpha)} x(\alpha)(\alpha_i' - \alpha_i). \tag{11}$$

Fix $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$ and assume without loss of generality that $\alpha_i' > \alpha_i$. Then, by (10) and (11),

$$\begin{aligned}\mathbf{1}^i_{S(\alpha_i',\alpha_{-i})}x(\alpha_i',\alpha_{-i}) &\geq \frac{v_i(x(\alpha_i',\alpha_{-i}),S(\alpha_i',\alpha_{-i}),p(\alpha_i',\alpha_{-i}),\alpha_i') - v_i(x(\alpha),S(\alpha),p(\alpha),\alpha_i)}{(\alpha_i'-\alpha_i)} \\ &\geq \mathbf{1}^i_{S(\alpha_i,\alpha_{-i})}x(\alpha_i,\alpha_{-i}).\end{aligned} \tag{12}$$

By (T1.a), $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})$ is continuous and differentiable almost everywhere; in fact, and since $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i}) \in \{0,1\}$ for all $t \in \mathbb{R}_+$, it has at most one point of discontinuity. Call it $\alpha_i^*(\alpha_{-i})$. We first consider the case where $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})$ is continuous at $\hat{\alpha}_i$; *i.e.*, either $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})$ is a continuous (and constant) function or else $\hat{\alpha}_i \neq \alpha_i^*(\alpha_{-i})$. Let $\{\alpha_i'^k\}_{k=1}^\infty \to \hat{\alpha}_i$ be such that for all $k \geq 1$, $\alpha_i'^k > \hat{\alpha}_i$. Since $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})$ is continuous at $\hat{\alpha}_i$, $\{\mathbf{1}^i_{S(\alpha_i'^k,\alpha_{-i})}x(\alpha_i'^k,\alpha_{-i})\}_{k=1}^\infty \to \mathbf{1}^i_{S(\hat{\alpha}_i,\alpha_{-i})}x(\hat{\alpha}_i,\alpha_{-i})$. By (12), $\frac{\partial v_i(x(\hat{\alpha}_i,\alpha_{-i}),S(\hat{\alpha}_i,\alpha_{-i}),p(\hat{\alpha}_i,\alpha_{-i}),\hat{\alpha}_i)}{\partial \alpha_i}$ exists and it is equal to $\mathbf{1}^i_{S(\hat{\alpha}_i,\alpha_{-i})}x(\hat{\alpha}_i,\alpha_{-i})$. Hence,

$$\frac{\partial v_i(x(\hat{\alpha}_i,\alpha_{-i}),S(\hat{\alpha}_i,\alpha_{-i}),p(\hat{\alpha}_i,\alpha_{-i}),\hat{\alpha}_i)}{\partial \alpha_i} = \mathbf{1}^i_{S(\hat{\alpha}_i,\alpha_{-i})}x(\hat{\alpha}_i,\alpha_{-i})$$

for all $\hat{\alpha}_i \in \mathbb{R}_+$ at which $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})$ is continuous at $\hat{\alpha}_i$. By the Fundamental Theorem of Calculus,

$$v_i(x(\hat{\alpha}_i,\alpha_{-i}),S(\hat{\alpha}_i,\alpha_{-i}),p(\hat{\alpha}_i,\alpha_{-i}),\hat{\alpha}_i) = \int_0^{\hat{\alpha}_i} \mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})dt + h_i(\alpha_{-i}),$$

where $h_i(\alpha_{-i})$ is a constant (*i.e.*, it does not depend on α_i). Since

$$v_i(x(\hat{\alpha}_i,\alpha_{-i}),S(\hat{\alpha}_i,\alpha_{-i}),p(\hat{\alpha}_i,\alpha_{-i}),\hat{\alpha}_i) = \mathbf{1}^i_{S(\hat{\alpha}_i,\alpha_{-i})}x(\hat{\alpha}_i,\alpha_{-i})\hat{\alpha}_i - p_i(\hat{\alpha}_i,\alpha_{-i}),$$

$$p_i(\hat{\alpha}_i,\alpha_{-i}) = \mathbf{1}^i_{S(\hat{\alpha}_i,\alpha_{-i})}x(\hat{\alpha}_i,\alpha_{-i})\hat{\alpha}_i - \int_0^{\hat{\alpha}_i} \mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})dt - h_i(\alpha_{-i}).$$

Thus, (T1.b) holds for all $\hat{\alpha}_i \in \mathbb{R}_+$ at which $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})$ is continuous at $\hat{\alpha}_i$. We now consider the case where $\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i})$ is not continuous at $\bar{\alpha}_i \equiv \alpha_i^*(\alpha_{-i})$. We distinguish between two cases.

Case 1: $\bar{\alpha}_i = 0$. That is,

$$\mathbf{1}^i_{S(t,\alpha_{-i})}x(t,\alpha_{-i}) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0. \end{cases}$$

By (12), for any $\alpha_i' > \bar{\alpha}_i = 0$,

$$1 \geq \frac{\alpha_i' - p_i(\alpha_i', \alpha_{-i}) + p_i(0, \alpha_{-i})}{\alpha_i'} \geq 0. \tag{13}$$

Since $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})$ is continuous at α_i',

$$\begin{aligned} p_i(\alpha_i', \alpha_{-i}) &= \mathbf{1}^i_{S(\alpha_i',\alpha_{-i})} x(\alpha_i', \alpha_{-i})\alpha_i' - \int_0^{\alpha_i'} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})dt - h_i(\alpha_{-i}) \\ &= \alpha_i' - \alpha_i' - h_i(\alpha_{-i}) \\ &= -h_i(\alpha_{-i}). \end{aligned}$$

Hence, by (13), $\alpha_i' \geq \alpha_i' + h_i(\alpha_{-i}) + p_i(0, \alpha_{-i})$, which implies that $0 \geq h_i(\alpha_{-i}) + p_i(0, \alpha_{-i})$. Also, by (13), $\alpha_i' + h_i(\alpha_{-i}) + p_i(0, \alpha_{-i}) \geq 0$ holds for all $\alpha_i' > 0$. Thus, $h_i(\alpha_{-i}) + p_i(0, \alpha_{-i}) = 0$. Hence,

$$p_i(0, \alpha_{-i}) = \mathbf{1}^i_{S(0,\alpha_{-i})} x(0, \alpha_{-i})0 - \int_0^0 \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})dt - h_i(\alpha_{-i});$$

namely, (T1.b) holds for $\bar{\alpha}_i = 0$.

Case 2: $\bar{\alpha}_i > 0$. We consider two subcases, depending on whether $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})$ is left or right continuous at $\bar{\alpha}_i$. First, assume it is left continuous; *i.e.*,

$$\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) = \begin{cases} 0 & \text{if } t \leq \bar{\alpha}_i \\ 1 & \text{if } t > \bar{\alpha}_i. \end{cases}$$

By (12), for any $\alpha_i < \bar{\alpha}_i$,

$$0 \geq \frac{-p_i(\bar{\alpha}_i, \alpha_{-i}) + p_i(\alpha_i, \alpha_{-i})}{\bar{\alpha}_i - \alpha_i} \geq 0.$$

Hence, $p_i(\bar{\alpha}_i, \alpha_{-i}) = p_i(\alpha_i, \alpha_{-i})$. Since $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})$ is continuous at α_i and $p_i(\alpha_i, \alpha_{-i}) = -h_i(\alpha_{-i})$ because $\mathbf{1}^i_{S(\alpha_i,\alpha_{-i})} x(\alpha_i, \alpha_{-i})\alpha_i - \int_0^{\alpha_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})dt = 0$, $p_i(\bar{\alpha}_i, \alpha_{-i}) = -h_i(\alpha_{-i})$. Thus, since $\mathbf{1}^i_{S(\bar{\alpha}_i,\alpha_{-i})} x(\bar{\alpha}_i, \alpha_{-i})\bar{\alpha}_i - \int_0^{\bar{\alpha}_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})dt = 0$,

$$p_i(\bar{\alpha}_i, \alpha_{-i}) = \mathbf{1}^i_{S(\bar{\alpha}_i,\alpha_{-i})} x(\bar{\alpha}_i, \alpha_{-i})\bar{\alpha}_i - \int_0^{\bar{\alpha}_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})dt - h_i(\alpha_{-i});$$

namely, (T1.b) holds for $\bar{\alpha}_i > 0$ if $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})$ is left continuous at $\bar{\alpha}_i$. Assume $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})$ is right continuous at $\bar{\alpha}_i$; *i.e.*,

$$\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) = \begin{cases} 0 & \text{if } t < \bar{\alpha}_i \\ 1 & \text{if } t \geq \bar{\alpha}_i. \end{cases}$$

By (12), for any $\alpha_i > \bar{\alpha}_i$,

$$1 \geq \frac{\alpha_i - p_i(\alpha_i, \alpha_{-i}) - \bar{\alpha}_i + p_i(\bar{\alpha}_i, \alpha_{-i})}{\alpha_i - \bar{\alpha}_i} \geq 1.$$

Hence, $p_i(\bar{\alpha}_i, \alpha_{-i}) = p_i(\alpha_i, \alpha_{-i})$. Since $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})$ is continuous at α_i,

$$\begin{aligned}
p_i(\alpha_i, \alpha_{-i}) &= \mathbf{1}^i_{S(\alpha_i,\alpha_{-i})} x(\alpha_i, \alpha_{-i})\alpha_i - \int_0^{\alpha_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt - h_i(\alpha_{-i}) \\
&= \alpha_i - \int_0^{\bar{\alpha}_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt - \int_{\bar{\alpha}_i}^{\alpha_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt - h_i(\alpha_{-i}) \\
&= \alpha_i - 0 - \alpha_i + \bar{\alpha}_i - h_i(\alpha_{-i}) \\
&= \mathbf{1}^i_{S(\bar{\alpha}_i,\alpha_{-i})} x(\bar{\alpha}_i, \alpha_{-i})\bar{\alpha}_i - \int_0^{\bar{\alpha}_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt - h_i(\alpha_{-i}).
\end{aligned}$$

Thus, $p_i(\bar{\alpha}_i, \alpha_{-i}) = \mathbf{1}^i_{S(\bar{\alpha}_i,\alpha_{-i})} x(\bar{\alpha}_i, \alpha_{-i})\bar{\alpha}_i - \int_0^{\bar{\alpha}_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt - h_i(\alpha_{-i})$; namely (T1.b) holds for $\bar{\alpha}_i$ if $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i})$ is right continuous at $\bar{\alpha}_i$.

$\Longleftarrow$) Let $f = (x, S, p)$ be a social choice function and assume that, for all $i \in N$, (T1.a) and (T1.b) hold. We will show that f is strategy-proof. Fix $i \in N$ and consider $\alpha_i, \alpha'_i \in \mathbb{R}_+$ and $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$. By (T1.b), the (truth-telling) payoff $v_i(x(\alpha), S(\alpha), p(\alpha), \alpha_i)$ is equal to

$$\int_0^{\alpha_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt + h_i(\alpha_{-i}),$$

and the (lying) payoff $v_i(x(\alpha'_i, \alpha_{-i}), S(\alpha'_i, \alpha_{-i}), p(\alpha'_i, \alpha_{-i}), \alpha_i)$ is equal to

$$\mathbf{1}^i_{S(\alpha'_i,\alpha_{-i})} x(\alpha'_i, \alpha_{-i})\alpha_i - \mathbf{1}^i_{S(\alpha'_i,\alpha_{-i})} x(\alpha'_i, \alpha_{-i})\alpha'_i + \int_0^{\alpha'_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt + h_i(\alpha_{-i}).$$

That is, $v_i(x(\alpha'_i, \alpha_{-i}), S(\alpha'_i, \alpha_{-i}), p(\alpha'_i, \alpha_{-i}), \alpha_i)$ is equal to

$$\mathbf{1}^i_{S(\alpha'_i,\alpha_{-i})} x(\alpha'_i, \alpha_{-i})(\alpha_i - \alpha'_i) + \int_0^{\alpha'_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt + h_i(\alpha_{-i}).$$

Hence, the difference between the (truthtelling) payoff $v_i(x(\alpha), S(\alpha), p(\alpha), \alpha_i)$ and the (lying) payoff $v_i(x(\alpha'_i, \alpha_{-i}), S(\alpha'_i, \alpha_{-i}), p(\alpha'_i, \alpha_{-i}), \alpha_i)$ is equal to

$$\Delta(\alpha_i, \alpha'_i, \alpha_{-i}) \equiv \int_0^{\alpha_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt - \int_0^{\alpha'_i} \mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) dt - \mathbf{1}^i_{S(\alpha'_i,\alpha_{-i})} x(\alpha'_i, \alpha_{-i})(\alpha_i - \alpha'_i). \tag{14}$$

We want to show that for all $\alpha_i, \alpha_i' \in \mathbb{R}_+$ and all $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$, $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) \geq 0$. By (T1.a), $\mathbf{1}^i_{S(t,\alpha_{-i})} x(t, \alpha_{-i}) \equiv \psi_i(t)$ is an increasing function of t; hence, it has at most one point of discontinuity, denote it by $\bar{\alpha}_i \equiv \alpha_i^*(\alpha_{-i})$. Obviously, $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = 0$ whenever $\alpha_i = \alpha_i'$ or ψ_i is continuous (and thus, it is constant and equal to either 0 or 1). We now distinguish between the other two remaining cases.

Case 1: $\alpha_i' < \alpha_i$ and ψ_i is discontinuous at $\bar{\alpha}_i$. If $\alpha_i' < \alpha_i \leq \bar{\alpha}_i$ then ψ_i is equal to 0 in the interval $[0, \alpha_i)$; thus, by (14), $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = 0$. If $\alpha_i' < \bar{\alpha}_i < \alpha_i$ then ψ_i is equal to 0 in the interval $[0, \bar{\alpha}_i)$ and equal to 1 in the interval $(\bar{\alpha}_i, \alpha_i]$; thus, by (14),

$$\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = \int_{\bar{\alpha}_i}^{\alpha_i} \psi_i(t)dt - \psi_i(\alpha_i')(\alpha_i - \alpha_i') = \alpha_i - \bar{\alpha}_i - 0 > 0.$$

If $\bar{\alpha}_i \leq \alpha_i' < \alpha_i$ then, ψ_i is equal to 1 in the interval $(\alpha_i', \alpha_i]$; thus, by (14),

$$\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = \int_{\alpha_i'}^{\alpha_i} \psi_i(t)dt - \psi_i(\alpha_i')(\alpha_i - \alpha_i') = (\alpha_i - \alpha_i') - \psi_i(\alpha_i')(\alpha_i - \alpha_i').$$

Hence, if $\psi_i(\alpha_i') = 1$ (because either $\bar{\alpha}_i < \alpha_i'$ or $\bar{\alpha}_i = \alpha_i'$ and $\psi_i(\bar{\alpha}_i) = 1$) then $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = (\alpha_i - \alpha_i') - (\alpha_i - \alpha_i') = 0$ and if $\psi_i(\alpha_i') = 0$ (because $\bar{\alpha}_i = \alpha_i'$ and $\psi_i(\bar{\alpha}_i) = 0$) then $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = \alpha_i - \alpha_i' > 0$.

Case 2: $\alpha_i < \alpha_i'$ and ψ_i is discontinuous at $\bar{\alpha}_i$. If $\bar{\alpha}_i < \alpha_i < \alpha_i'$ then, ψ_i is equal to 1 in the interval $[\alpha_i, \alpha_i']$; thus, by (14),

$$\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = -\int_{\alpha_i}^{\alpha_i'} \psi_i(t)dt - \psi_i(\alpha_i')(\alpha_i - \alpha_i') = -(\alpha_i' - \alpha_i) - (\alpha_i - \alpha_i') = 0.$$

If $\alpha_i \leq \bar{\alpha}_i < \alpha_i'$ then ψ_i is equal to 0 in the interval $[0, \bar{\alpha}_i)$ and equal to 1 in the interval $(\bar{\alpha}_i, \alpha_i']$; thus, by (14),

$$\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = -\int_{\bar{\alpha}_i}^{\alpha_i'} \psi_i(t)dt - \psi_i(\alpha_i')(\alpha_i - \alpha_i') = -(\alpha_i' - \bar{\alpha}_i) - (\alpha_i - \alpha_i') = \bar{\alpha}_i - \alpha_i \geq 0.$$

If $\alpha_i < \alpha_i' \leq \bar{\alpha}_i$ then ψ_i is equal to 0 in the interval $[0, \alpha_i')$. Thus, by (14), if $\alpha_i' = \bar{\alpha}_i$ and $\psi_i(\bar{\alpha}_i) = 0$ then $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = 0$. If $\alpha_i' = \bar{\alpha}_i$ and $\psi_i(\bar{\alpha}_i) = 1$ then $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = \alpha_i' - \alpha_i > 0$. If $\alpha_i < \alpha_i' < \bar{\alpha}_i$ then $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = 0$. Thus, for all $i \in N$, $\Delta(\alpha_i, \alpha_i', \alpha_{-i}) = v_i(x(\alpha), S(\alpha), p(\alpha), \alpha_i) - v_i(x(\alpha_i', \alpha_{-i}), S(\alpha_i', \alpha_{-i}), p(\alpha_i', \alpha_{-i}), \alpha_i) \geq 0$ for all $\alpha_i, \alpha_i' \in \mathbb{R}_+$ and all $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$. Hence, f is strategy-proof. ∎