International Journal of Game Theory (1996) 25: 437–453

Equilibrium Payoffs of Dynamic Games

JORDI MASSÓ[1]

Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, Edifici B, 08193 Bellaterra (Barcelona), Spain

ALEJANDRO NEME[1,2]

Instituto de Matemátiça Aplicada San Luis, Universidad de San Luis, Ejército de los Andes 950, 5700 (San Luis), Argentina

Abstract: We give a characterization of the equilibrium payoffs of a dynamic game, which is a stochastic game where the transition function is either one or zero and players can only use pure actions in each stage. The characterization is in terms of convex combinations of connected stationary strategies; since stationary strategies are not always connected, the equilibrium set may not be convex. We show that subgame perfection may reduce the equilibrium set.

1 Introduction

Repeated Games have been used in Game and Economic Theory to model conflict situations lasting over time. The study of this kind of model has helped us to increase our understanding of economic phenomena such as cooperation, reputation, collusion, bounded rationality, and so on. However, a restrictive assumption in a repeated game is that the nature of the conflict (actions and payoffs) does not change over time; in particular, current actions have effect only on current payoffs, and thus, do not have any influence over the set of possible future actions and payoffs.

In 1953 Shapley introduced a more general model called a Stochastic Game. There is not a single game repeated over time. Players may be involved in different games. In every period, players have to choose an action (and get a payoff) from a particular game, but in the next period, the game they will play depends on a probability distribution over the set of possible games, which in turn may depend on the joint action taken in the previous period. Therefore, starting from a initial game, the actions players take do not only determine the payoffs they obtain but also the possibilities they will face in the future.

The stochastic and dynamic nature of the model makes it a more apt description of a much larger class of economic situations than the simpler

[1] We are grateful to an anonymous referee for his very careful comments. Financial support from DGICYT grants PB89-0075, PB89-0294, and PB92-0590 and from Programa de Cooperación Científica Iberoamericana grant RD6841 is acknowledged.

repeated game. Nevertheless, less attention has been devoted to stochastic games (probably because their richer structure makes them more difficult to analyze); a vast majority of papers have concentrated on showing the existence of stationary equilibria rather than characterizing the set of equilibria (see for example, besides Shapley's, Bewley and Kohlberg [1976], Jovanovic and Rosenthal [1988], Mertens and Parthasarathy [1987], Duffie, Geanakoplos, McLennan and Mas-Colell [1988], etc).

The main objective of this paper is to show some preliminary results that may help to make clear the structure of the equilibrium set of stochastic games. We analyze a particular type of stochastic game in which the transition function has the property that, given a joint action, the probability distribution over the set of games is a vertex in the simplex. Even though this restriction limits the scope of the paper, we think that, independently of the interest of this particular case, these results are the first steps in providing a characterization of the equilibria for the general case. This will be the content of a future paper.

Our characterization has two important related properties. First of all, it is based upon stationary strategies. In a payoff space, the payoffs of stationary strategies play a similar role to the payoffs of actions of the one-shot game in the Folk Theorem. Points on the convex combinations of stationary strategy payoffs and above the individually rational points emerge as equilibria of the dynamic game. Second, it utilizes the concept of connected strategies. One cannot obtain, even as feasible payoffs, the convex combinations of all stationary strategies; only those that are connected; that is, strategies that have the property that the cycles they generate have common elements. That is why, in general, the equilibrium set is not convex and the relevant individually rational point may depend on the subset of connected stationary strategies one is looking at. The idea of connection captures the influence of the transition function on the equilibrium set.

The paper is organized as follows. Section 2 presents what we call a dynamic game. Sections 3 and 4 characterize the set of feasible and equilibrium payoffs, respectively. Section 5 concludes with general comments and, through an example, shows that, in general, subgame perfection reduces the set of equilibria. Finally, Section 6 consists of an Appendix where the reader will find some proofs omitted in the text.

2 Dynamic Games

The main goal of this paper is to give a full characterization (a full Folk Theorem) of the equilibrium payoffs of a finite stochastic game G, with a deterministic transition function.

The game $G = ((W, w^1), A, T)$ is defined as follows:

i) A finite set of players $I = \{1, 2, \ldots, n\}$. A generic player will be denoted by i.
ii) A finite set of joint actions $A = \times_{i \in I} A_i$, where A_i is the set of pure actions of player $i \in I$.

iii) A finite set of games $W=\{w^1,\dots,w^K\}$. For every $j=1,2,\dots,K$ we denote by $w^j=((I),(A_i^j)_{i\in I},(u_i^j)_{i\in I})$ a finite game in normal form, where $A_i^j\subseteq A_i$ is the finite set of actions for player i in the game j. Without loss of generality we assume that $A_i^j\cap A_i^{j'}=\emptyset$ for every $j\neq j'$. Define $A^j=\times_{i\in I}A_i^j$ as the set of joint actions in the game j. Let u be the function $u:A\to\mathbb{R}^n$ defined by: $u(a)=u^j(a)$ where j is such that $a\in A^j$. Generic elements of these sets will be denoted by $a_i^j\in A_i^j$, $a^j\in A^j$ and $a\in A$.

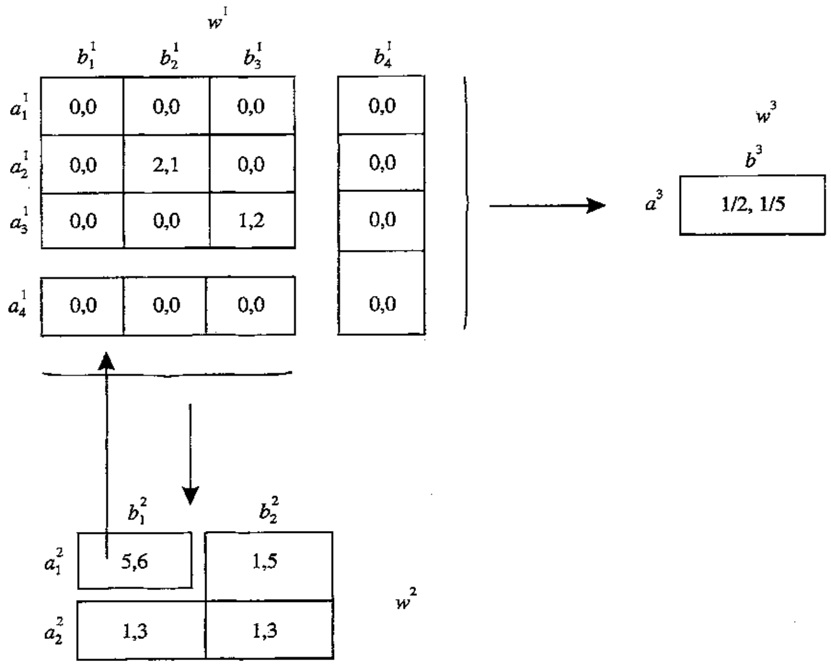
iv) An initial game from W, that we will denote it by w^1.

v) A transition function T,[3] which specifies the new game (state) as a function of the current game (state) and the joint action taken by all players. The domain of T is $B=\{(w^j,a)\in W\times A\,|\,a\in A^j\}$. Therefore, $T:B\to W$. We assume that T is onto. The fact that $T(w,a)$ is an element of W and not a probability distribution on it, makes the dynamic game a particular case of a stochastic game.

The game G is played as follows. At the beginning players are in the initial game w^1 and they have to choose simultaneously an action a^1 in A^1. The next game will be $T(w^1,a^1)=w^j$ from which they have to choose simultaneously an action a^j in A^j, and so on. A similar model has been used in Kalai, Samet and Stanford [1986] to study issues of bounded complexity.

The following example illustrates a dynamic game.

Example:



[3] Confusion with the use of T as a period of time shoud not arise. The context will always make clear which is the correct one.

The set of games is $W=\{w^1,w^2,w^3\}$ and the set of players is $I=\{1,2\}$. The transition function is:

$$T(w^1,(a^1_m,b^1_n))=w^1 \quad \text{for } m,n=1,2,3$$
$$T(w^1,(a^1_m,b^1_4))=w^3 \quad \text{for } m=1,2,3,4$$
$$T(w^1,(a^1_4,b^1_n))=w^2 \quad \text{for } n=1,2,3$$
$$T(w^2,(a^2_1,b^2_1))=w^1$$
$$T(w^2,(a^2_1,b^2_2))=T(w^2,(a^2_2,b^2_1))=T(w^2,(a^2_2,b^2_2))=w^2$$
$$T(w^3,(a^3,b^3))=w^3.$$

For every $t\in\mathbb{N}$, define H^t as the cartesian product of A t-times, i.e. an element $h\in H^t$ is a history of length t. We denote by $H^0=\{e\}$ the set of histories of length 0, with the convention that e stands for the empty history. Let $H=\bigcup_{t=0}^{\infty}H^t$ be the set of all histories.

Definition 1: A history consistent with G, $d=(d_1,\ldots,d_{t+1})$, is an element of H such that: $d_1\in A^1$; $d_2\in A^j$ where $w^j=w(d_1)=T(w^1,d_1)$, and recursively, for $2\leq\tau\leq t$, $d_{\tau+1}\in A^j$ where:

$$w^j=w(d_1,\ldots,d_\tau)=T(w(d_1,\ldots,d_{\tau-1}),d_\tau). \tag{1.1}$$

Denote by $D^0=\{e\}$; for every $t\geq 1$, D^t is the set of all consistent histories with G of length t and $D=\bigcup_{t=0}^{\infty}D^t$. We assume that the game $G=((W,w^1),A,T)$ has the property that for every $j=1,\ldots,K$ there exists a consistent history $d\in D$ such that $w(d)=w^j$.

A strategy of player $i\in I$ for the game G is a function $f_i:D\rightarrow A_i$ such that if $d\in D$ has the property that $w(d)=w^j$, then $f_i(d)\in A^j_i$. Notice that we are assuming here what has been known in the literature as perfect monitoring; i.e. players decide current actions knowing all the previous actions and only pure actions are allowed.

Let F_i be the set of all these functions, and denote the set of aggregate strategies by $F=\times_{i\in I}F_i$.

Let $f\in F$, define recursively the history of actions generated by f, by $a(f)=\{a^t(f)\}_{t=1}^{\infty}$ where $a^1(f)=f(e)$ and for every $t\geq 1$, $a^{t+1}(f)=f(a^1(f),\ldots,a^t(f))$. Define, also recursively, the sequence of games generated by f by, $w(f)=\{w^t(f)\}_{t=1}^{\infty}$ where $w^1(f)=w(e)=w^1$ and for every $t\geq 1$ $w^{t+1}(f)=w(a^1(f),\ldots,a^t(f))$.

Definition 2: A strategy $f_i\in F_i$ is stationary if for every $d,d'\in D$ such that $w(d)=w(d')$, we have that $f_i(d)=f_i(d')$.

Denote by S_i the set of stationary strategies for player i. Given $s_i\in S_i$ and $j=1,2,\ldots,K$, define $s_i(j)$ as the action prescribed by s_i in all $d\in D$ such that $w(d)=w^j$. Since we assumed that all games were reachable, $s_i(j)$ is well defined

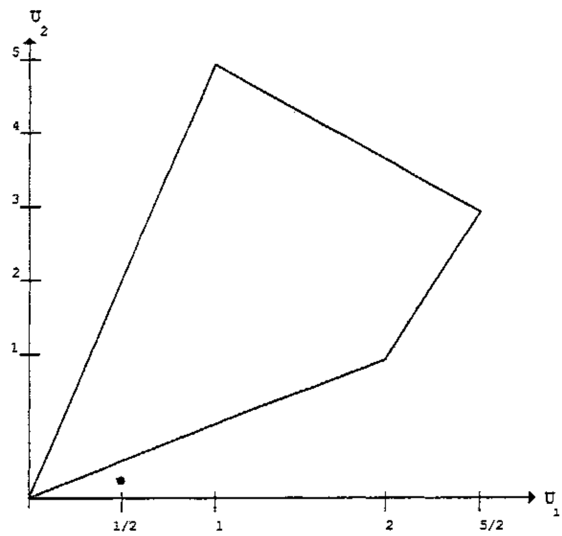


Fig. 1.

Figure 1 shows the set of feasible payoffs of the example.

The idea behind the proof is simple. For sufficiently, given $v \in \mathbb{R}^n$ feasible, there exists $f \in F$ such that $v = U(f)$. For every large T, one looks at the path of average payoffs generated by f up to period T. One shows that it can be approximated by a convex combination of payoffs of connected stationary strategies (Lemma 2). Finally, one has to take care of the limit. In this construction, the assumption that W is finite is used. For necessity, one starts with rational convex combinations and constructs f from the set of connected stationary strategies using them in the "right" proportion. One has to make sure that the stationary strategies used to connect them are used in a proportion that goes to zero. Non-rational convex combinations are obtained as a limit of rational ones. The proof of Theorem 1 is in the Appendix at the end of the paper.

4 Equilibrium Payoffs

Definition 5: $f^* \in F$ is a *Nash equilibrium of* G if for every $i \in I$ $U_i(f^*) \geq U_i(g_i, f^*_{-i})$ for every $g_i \in F_i$. $v \in \mathbb{R}^n$ is an *equilibrium payoff of* G if there exists a Nash equilibrium of G, $f \in F$, such that $U(f) = v$.

Let F^* be the set of Nash equilibria of G.

For every $j = 1, \ldots, K$ one can define a stochastic game $G(j)$ with initial state j by $((W, w^j), A, T)$. A strategy for player $i \in I$ for the game $G(j)$ will be denoted by f^j_i. We can define the other concepts in the same way.

For every $j = 1, \ldots, K$ we denote by $v_i(j) = \inf_{f^j_{-i} \in F^j_{-i}} \sup_{f^j_i \in F^j_i} U_i(f^j_i, f^j_{-i})$ the "minimax" payoff of player i in the game $G(j)$. We let $-i$ denote the n-tuple

Let $\mathscr{H}$ be a Banach limit on l_∞, let $f \in F$ and $i \in I$; we define the payoff for G by

$$U_i(f) = \mathscr{H}\left(\left\{\frac{1}{T}\sum_{t=1}^{T} u_i^{j(t)}(a^t(f))\right\}_{T=1}^{\infty}\right),$$

where $j(t)$ is such that $w^{j(t)} = w^t(f)$ for all $t \geq 1$.

Lemma 1: Let $s \in S$. Then, there exist $M, R \in \mathbb{N}$ such that $w^{t+R}(s) = w^{t+R+M}(s)$ for every $t \geq 1$. That is, a stationary strategy produces a finite cycle of length M after R periods.

Proof: Since W is finite and s is stationary, the result follows easily. □

For every $s \in S$, define: $b_w(s) = \{w^1(s), \ldots, w^R(s)\}$ and $c_w(s) = \{w^{R+1}(s), \ldots, w^{R+M}(s)\}$ as the initial path and cycle of states generated by s, where R and M are the smallest natural numbers of Lemma 1. Obviously, $w(s) = \{b_w(s), c_w(s), c_w(s), \ldots\}$. Define $b_a(s) = \{a^1(s), \ldots, a^R(s)\}$ and $c_a(s) = \{a^{R+1}(s), \ldots, a^{R+M}(s)\}$. Denote by $|s|$ the cardinality of $c_a(s)$ (that is M).

Remark 1: Let $s \in S$. Then, for every $i \in I$, $U_i(s) = (1/|s|)\sum_{r=1}^{M} u_i^{j(r)}(a^{R+r}(s))$, where $j(r)$ is such that $w^{j(r)} = w^{R+r}(s)$. That means that the payoff of a stationary strategy is just the mean of the cycle payoffs.

Definition 3: Let $s^l, s^{l'} \in S$. *i)* We say that s^l and $s^{l'}$ are *directly connected*, $s^l \sim s^{l'}$, if $c(s^l) \cap c(s^{l'}) \neq \emptyset$. *ii)* We say that s^l and $s^{l'}$ are *connected*, $s^l \approx s^{l'}$, if there exist $s^1, \ldots, s^m \in S$ such that $s^l \sim s^1 \sim \cdots \sim s^m \sim s^{l'}$. In this case, we will say that s^l and $s^{l'}$ are connected through $s^1, \ldots, s^m$.

Two stationary strategies are directly connected if from the cycle of states of one of them, players have direct access to the cycle of states of the other and vice versa. If they are connected, players can access, through a path, from the cycle of states of one of them to the other, and vice versa; those paths may be different. The transition function T determines the connections between stationary strategies.

Those concepts will allow us to construct strategies in G with the property that their paths have cycles of connected stationary strategies. Hence, our characterization will rely on convex combinations of payoffs of stationary connected strategies. But first, Theorem 1 in Section 3 gives a characterization of the set of feasible payoffs of G.

3 Feasible Payoffs

Definition 4: A vector $v \in \mathbb{R}^n$ is feasible if these exists a strategy $f \in F$ such that $v = U(f)$.

Theorem 1: A vector $v \in \mathbb{R}^n$ is feasible if and only if there exists $S(v) = \{s^1, \ldots, s^{\bar{k}}\} \subseteq S$ such that for every $s^r, s^{r'} \in S(v)$ $s^r \approx s^{r'}$ and there exists $(\alpha^1, \ldots, \alpha^{\bar{k}}) \in \bar{\Delta}$ (the $\bar{k}$-dimensional unit simplex) such that

$$v = \sum_{k=1}^{\bar{k}} \alpha^k U(s^k).$$

without the ith element, i.e. $f^j_{-i}=(f^j_1,\dots,f^j_{i-1},f^j_{i+1},\dots,f^j_n)$, as well as sets obtained by taking a cartesian product, i.e. $F^j_{-i}=\times_{i'\in I, i'\neq i}F^j_{i'}$.

The next Proposition gives us two important properties related to the minimax strategies. The first one (Lemma 3) is that the strategy by which player i is punished, as well as his defense strategy, is stationary. The second one is that even though the minimax payoff depends on the initial state, the punishment and defense strategies are independent of the initial state (the rest of the proof).

Proposition 1: For every $i\in I$, there eixist $m_{-i}\in S_{-i}$ and $m_i\in S_i$ such that for every $j=1,\dots,K$, $U_i(m^j_i,m^j_{-i})=v_i(j)$ and $U_i(m^j_i,m^j_{-i})\geq U_i(f^j_i,m^j_{-i})$ for every $f^j_i\in F^j_i$.

Proof: See the Appendix at the end of the paper. □

Let $s\in S$ and $i\in I$; we define $b^i_w(s)$ and $c^i_w(s)$ as the set of games available to player i through a deviation of s from the initial path and cycle, respectively, of s. Formally,

$$b^i_w(s)=\{w^j\in W|\exists w^{j'}\in b_w(s),\exists a^{j'}_i\in A^{j'}_i \text{ s.t. } T(w^{j'},(s_{-i}(j'),a^{j'}_i))=w^j\},$$

notice that $b_w(s)\subseteq b^i_w(s)$, and,

$$c^i_w(s)=\{w^j\in W|\exists w^{j'}\in c_w(s),\exists a^{j'}_i\in A^{j'}_i \text{ s.t. } T(w^{j'},(s_{-i}(j'),a^{j'}_i))=w^j\}.$$

Let $v_i(s)=\max_{w^j\in b^i_w(s)\cup c^i_w(s)}v_i(j)$ be the highest payoff that player i can guarantee by a deviation from s. Let $v_i(b_w(s))=\max_{w^j\in b^i_w(s)}v_i(j)$, and $v_i(c_w(s))=\max_{w^j\in c^i_w(s)}v_i(j)$ be, respectively, the highest payoff that player i can guarantee by a deviation from s on the initial path and the cycle. The notation of $v_i(j)$, $v_i(s)$, $v_i(b_w(s))$ and $v_i(c_w(s))$ as a function with different domains should not confuse the reader. It means the highest payoff that player i can be guarantee by a deviation from j, s, $b_w(s)$ and $c_w(s)$ respectively.

Define the set of all cycles generated by all the elements of S as $C(S)$, i.e., $C(S)=\{c_w(s)|s\in S\}$, and let $\mathscr{P}(C(S))$ be the power set of $C(S)$. Let $s^l,s^{l'}\in S$, define the *set of connections* of s^l with $s^{l'}$ by

$$C_{l,l'}=\{c_{l,l'}\in\mathscr{P}(C(S))|c_{l,l'}=\{c_w(s^1),\dots,c_w(s^{k'})\}$$
$$\text{such that } s^l\approx s^{l'} \text{ through } s^1,\dots,s^{k'}\}.$$

For every $i\in I$, define:

$$c^i_w(s^l,s^{l'})=\{w^j\in W|\exists w^{j'}\in c_w(s^r) \text{ and } c_{l,l'}\in C_{l,l'} \text{ s.t. } c_w(s^r)\in c_{l,l'} \text{ and}$$
$$\exists a^{j'}_i\in A^{j'}_i \text{ s.t. } T(w^{j'},(a^{j'}_i,s^r_{-i}(j')))=w^j\}$$

and $v_i(c_{l,l'})=\max_{w^j\in c^i_w(s^l,s^{l'})}v_i(j)$.

Given $s^l \approx s^{l'}$ and $i \in I$, $c^i_w(s^l, s^{l'})$ is the set of games that player i has the power to reach from the set of all cycles that connect s^l and $s^{l'}$, and $v_i(c_{l,l'})$ is the highest payoff that player i can guarantee from those games.

Let $s^1, \ldots, s^{k'} \in S$ be such that $s^1 \approx s^2 \approx \cdots \approx s^{k'}$ and $i \in I$; for every $l = 1, 2, \ldots, k'$ define

$$v_i(s^1, \ldots, s^{k'}, c_{1,2}, c_{2,3}, \ldots, c_{k'-1,k'}, c_{k',1}, l) = \max\{v_i(c_w(s^1)), \ldots, v_i(c_w(s^{k'})), v_i(b_w(s^l)), v_i(c_{1,2}), \ldots, v_i(c_{k',1})\}.$$

$v_i(s^1, \ldots, s^{k'}, c_{1,2}, c_{2,3}, \ldots, c_{k'-1,k'}, c_{k',1}, l)$ is the highest payoff that player i can guarantee by himself by deviating while in the cycles of $s^1, s^2, \ldots, s^{k'}$, the cycles that connect them and the initial path of s^l (one of those strategies).

The next theorem gives us the characterization of equilibrium payoffs based upon stationary strategies.

Theorem 2: Let v be a feasible payoff of G. Then, v is an equilibrium payoff of G if and only if there exists $s^1, \ldots, s^{k'}$ $(k' \leq n+1)$, $(\alpha^1, \ldots, \alpha^{k'}) \in \Delta^{k'}$ such that $v = \sum_{r=1}^{k'} \alpha^r U(s^r)$ and there exists $c_{1,2}, \ldots, c_{k',1}$ connections such that $v_i \geq v_i(s^1, \ldots, s^{k'}, c_{1,2}, \ldots, c_{k',1}, l)$ for some $l = 1, \ldots, k'$ and for every $i \in I$.

Proof: See the Appendix at the end of the paper. □

See Figure 2 for a geometric representation of Theorem 2 for our example, where α_j is the payoff of a stationary strategy and β_1 is the minimax payoff of

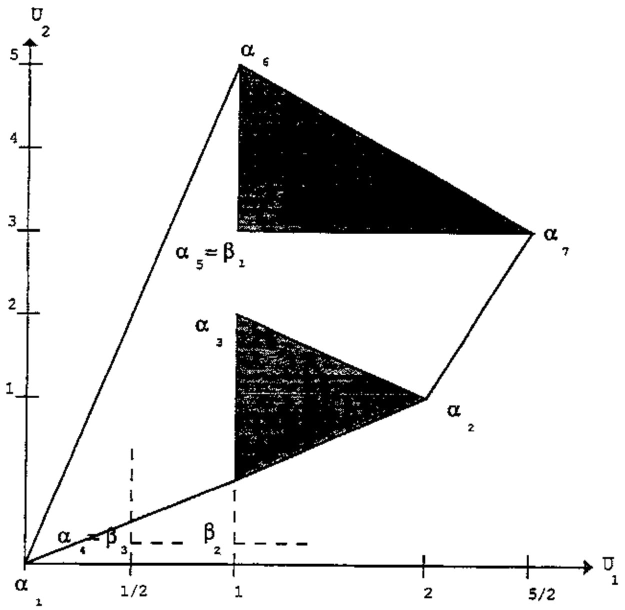


Fig. 2.

combining strategies producing α_5, α_6 and α_7, β_2 is the minimax payoff of combining strategies producing α_1, α_2 and α_3, and finally, β_3 is the minimax of playing the strategy that gives the payoff of α_4.

5 Comments and Conclusions

Theorem 2 in the previous section gives a characterization of the equilibrium payoffs of a Dynamic Game based upon convex combinations of connected stationary strategies. However, we illustrated with an example that the set of feasible and individually rational payoffs might be disconnected. Nevertheless, Theorem 1 implies that if the Stochastic Game has the property that every pair of stationary strategies are connected, then the set of feasible payoffs is connected, but the set of individually rational payoffs may still be disconnected. This is due to the fact that payoffs of minimax strategies are not independent of the initial state.

Most of the earlier papers on Stochastic Games dealt with the existence of equilibrium. However, the papers providing a characterization of the equilibrium set, for example Friedman [1989] and Lockwood [1990], contain different results to the ones established here concerning the non-convexity of the equilibrium set.

The Time-Dependent supergame models, studied by Friedman, are games in which the payoff in each period depends upon the actions taken in the present period as well as those actions taken in one or more previous periods. In Friedman's model the set of actions available for each player is compact and convex. The time-dependent model with a finite number of one shot actions is an interesting example, which satisfies the conditions that every pair of strategies are connected and that for every $j,\ j' = 1, \ldots, K$ and every $i \in I,\ v_i(j) = v_i(j')$. Theorems 1 and 2 imply that, under these two conditions, the set of Nash equilibrium payoffs is convex.

Lockwood characterized the equilibrium payoffs with little or no discounting in a class of stochastic games known as scrambling games. These are games where no matter what actions are chosen by the players, the transition function has the property that given any pair of initial states, there is always some positive probability of moving to a common third state. This property implies that the payoff of any stationary strategy is independent of the initial state. If one requires Lockwood's condition in our model (the transition function has only zeros and ones), then the resulting stochastic game has the following property: There exists a one shot game $w \in W$ such that for any $w' \in W$ and $a \in A$, the transition function is $T(w', a) = w$. Clearly the scrambling games can be reduced to games which repeat the game w infinitely many times. Therefore, all the results about repeated games may be applied in this setting.

The Subgame Perfect Folk Theorem for infinitely repeated games (e.g. Rubinstein [1977], Aumann and Shapley [1976], etc.) states that every individually rational payoff can be supported as a subgame perfect equilibrium strategy.

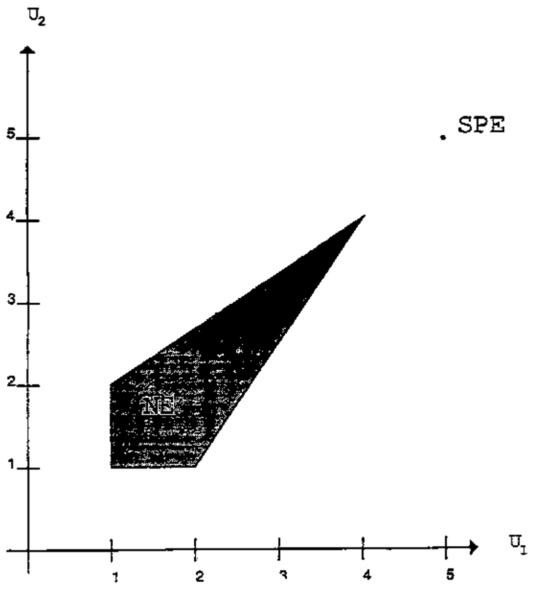
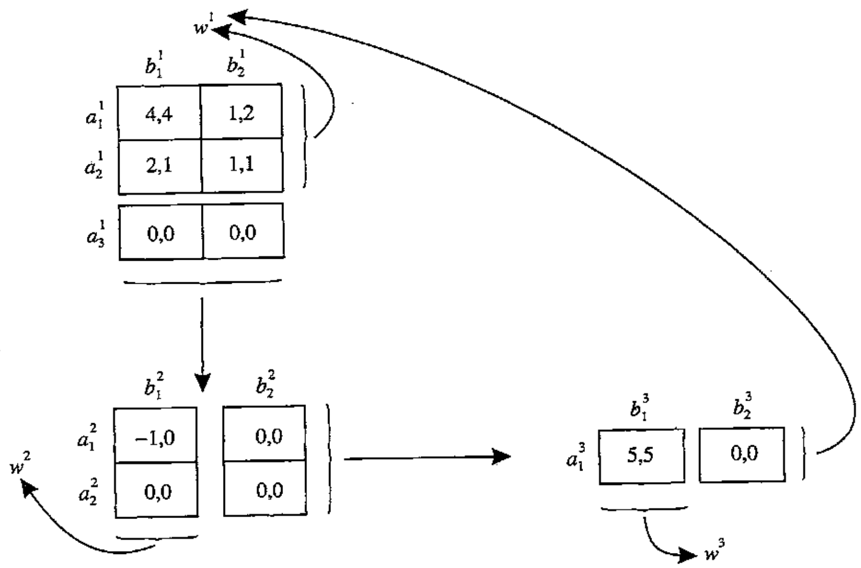


Fig. 3.

Similar results have been shown for the class of stochastic games mentioned above. Nevertheless, the following example shows that, in general, subgame perfection reduces the set of Nash equilibria (see Figure 3).

Example:



In the example above, any Nash equilibrium generating a payoff in the convex hull of $\{(4,4),(2,1),(1,2),(1,1)\}$ is not a subgame perfect equilibrium since player 1 has a deviation that game w^2 is reached, but from game w^2 the unique Nash equilibrium payoff is $(5,5)$.

By an argument similar to the one used by Ben-Porath and Peleg [1989] and Fudenberg and Maskin [1986], we can show that, if the stochastic game has the property that every pair of stationary strategy are connected and for every $j=1,\ldots,K$ and every $i\in I$, $v_i(j)=v_i(j')$, then every individually rational payoff can be supported as a subgame perfect equilibrium strategy.

6 Appendix

Theorem 1: A vector $v\in\mathbb{R}^n$ is feasible if and only if there exists $S(v)=\{s^1,\ldots,s^{\bar{k}}\}\subseteq S$ such that for every $s^r,s^{r'}\in S(v)$ $s^r\approx s^{r'}$ and there exists $(\alpha^1,\ldots,\alpha^{\bar{k}})\in\Delta^{\bar{k}}$ (the $\bar{k}$-dimensional unit simplex) such that $v=\sum_{k=1}^{\bar{k}}\alpha^k U(s^k)$.

Proof: Sufficiency:

Let v be a feasible payoff. Therefore, there exists a strategy $f\in F$ such that $v=U(f)$.

Consider the sequence of actions $a(f)=\{a^t(f)\}_{t=1}^{\infty}$ generated by f. Since the game G is finite there exists a point in the sequence with the property that all the following actions have already been taken previously, that is, there exists $\tau\geq 1$ such that for every $t\geq 1$, there exists $\tau_t\leq\tau$ such that $a^t(f)=a^{\tau_t}(f)$.

For every $i\in I$, define

$$p_i=\min\{u_i(a^t(f))\,|\,1\leq t\leq\tau\}\quad\text{and}\quad P_i=\max\{u_i(a^t(f))\,|\,1\leq t\leq\tau\}.$$

Lemma 2: For every $T\geq\tau$, there exists a set of stationary strategies (depending of T) $s^1,\ldots,s^{k_T}$ such that:

i) For every $l,l'=1,\ldots,k_T$ we have that $s^l\approx s^{l'}$.
ii) There exists $\alpha_1^T,\ldots,\alpha_{k_T}^T$ such that $\sum_{k=1}^{k_T}\alpha_k^T+\tau_T-1=T$ and for every $i\in I$

$$-\frac{|p_i|}{T}\tau+\frac{1}{T}\sum_{k=1}^{k_T}\alpha_k^T U_i(s^k)\leq\frac{1}{T}\sum_{t=1}^{T}u_i(a^t(f))\leq\frac{|P_i|}{T}\tau+\frac{1}{T}\sum_{k=1}^{k_T}\alpha_k^T U_i(s^k).$$

Proof of Lemma 2: Let $T\geq\tau$ be given. There exists $\tau_T\leq\tau$ such that $w^{T+1}(f)=w^{\tau_T}(f)$. Define $\hat{T}=T-\tau_T+1$ and for every $1\leq t\leq\hat{T}+1$: $\bar{w}^t(f)=w^{\tau_T+t-1}(f)$. Hence, we have that $\bar{w}^1(f)=\bar{w}^{\hat{T}+1}(f)$.

Let t_1' be the smallest index that $1\leq t_1\leq t_1'\leq\hat{T}+1$ such that $\bar{w}^{t_1}(f)=\bar{w}^{t_1'}(f)$. There exists $s^1\in S$ such that $c_w(s^1)=\{\bar{w}^{t_1}(f),\ldots,\bar{w}^{t_1'-1}(f)\}$. Consider $T_1=\{1,2,\ldots,t_1-1,t_1',\ldots,\hat{T}+1\}$.

Let t_2' be the smallest index in T_1 that $1 \leq t_2 \leq t_2' \leq \hat{T}+1$ such that $t_2 \in T_1$ and $\bar{w}^{t_2}(f) = \bar{w}^{t_2'}(f)$. Then, there exists $s^2 \in S$ such that $c_w(s^2) = \{\bar{w}^{t_2}(f), \ldots, \bar{w}^{t_2'-1}(f)\} - c_w(s^1)$.

Let $T_2 = T_1 - \{t_2, \ldots, t_2' - 1\}$, and continue in a similar manner. This process stops at some point such that $t_r = 1$ and $t_r' = \hat{T}+1$. We have identified $\hat{S}(T) = \{s^1, \ldots, s^r\}$ (some of the s^k may be repeated).

Let $S(T) = \{s^1, \ldots, s^{k_T}\}$ be the set of stationary strategies obtained by deleting the repeated elements of $\hat{S}(T)$. For every $1 \leq k \leq k_T$ denote by β_k^T the number of times that s^k is in $\hat{S}(T)$. Define $\alpha_k^T = \beta_k^T \cdot |s^k| \leq T$ and notice that:

$$\sum_{k=1}^{k_T} \alpha_k^T = \sum_{k=1}^{k_T} \beta_k^T \cdot |s^k| = \hat{T} = T - \tau_T + 1.$$

Therefore,

$$\sum_{k=1}^{k_T} \alpha_k^T + \tau_k - 1 = T.$$

It is easy to show that for every $i \in I$ we have that:

$$\sum_{t=1}^{\hat{T}} u_i(\vec{a}^t(f)) = \sum_{k=1}^{k_T} \alpha_k^T U_i(s^k).$$

Therefore ii) follows. By construction i) also follows because it is possible to find a connection between any pair of strategies in $S(T)$ since $\bar{w}^1(f) = \bar{w}^{\hat{T}+1}(f)$. This proves Lemma 2. □

Since for every $T \geq \tau$ ii) of Lemma 2 is true, we have that for every $i \in I$:

$$\mathscr{H}\left[\left\{-\frac{|p_i|}{T}\tau + \frac{1}{T}\sum_{k=1}^{k_T} \alpha_k^T U_i(s^k)\right\}_{T=\tau}^{\infty}\right] \leq U_i(f) \leq \mathscr{H}\left[\left\{\frac{|P_i|}{T}\tau + \frac{1}{T}\sum_{k=1}^{k_T} \alpha_k^T U_i(s^k)\right\}_{T=\tau}^{\infty}\right]$$

and therefore

$$U_i(f) = \mathscr{H}\left[\left\{\frac{1}{T}\sum_{k=1}^{k_T} \alpha_k^T U_i(s^k)\right\}_{T=\tau}^{\infty}\right].$$

By the finiteness assumption it is easy to show that there exists $\bar{T} \geq \tau$ such that for every $T > \bar{T}$, $S(T) \subseteq S(\bar{T})$. Denote by $\bar{S}$ the maximal set $S(\bar{T}) = \{s^1, \ldots, s^{\bar{k}}\}$.

Therefore

$$U_i(f) = \mathscr{H}\left[\left\{\frac{1}{T}\sum_{k=1}^{\bar{k}} \bar{\alpha}_k^T U_i(s^k)\right\}_{T=\tau}^{\infty}\right]$$

for every $i\in I$, where

$$\bar{\alpha}_k^T = \begin{cases} \alpha_k^T & \text{if } s^k\in S(T) \\ 0 & \text{if } s^k\notin S(T) \end{cases}.$$

For every $i\in I$, $v_i = U_i(f) = \sum_{k=1}^{\bar{k}} \mathscr{H}[\{(1/T)\bar{\alpha}_k^T\}_{T=\tau}^{\infty}]U_i(s^k)$, by linearity of the Banach limit. Now, for every $1\leq k\leq \bar{k}: 0\leq \mathscr{H}[\{(1/T)\bar{\alpha}_k^T\}_{T=\tau}^{\infty}]\leq 1$, since $0\leq \bar{\alpha}_k^T\leq T$. Letting $\alpha^k = \mathscr{H}[\{(1/T)\bar{\alpha}_k^T\}_{T=\tau}^{\infty}]$, it remains to be shown that $\sum_{k=1}^{\bar{k}} \mathscr{H}[(1/T)\bar{\alpha}_k^T\}_{T=\tau}^{\infty}] = 1$.

Notice that $\sum_{k=1}^{\bar{k}} \mathscr{H}[\{(1/T)\bar{\alpha}_k^T\}_{T=\tau}^{\infty}] = \mathscr{H}[\{(1/T)\sum_{k=1}^{\bar{k}}\bar{\alpha}_k^T\}_{T=\tau}^{\infty}]$ and $\mathscr{H}[\{(1/T)\sum_{k=1}^{\bar{k}}\bar{\alpha}_k^T + (1/T)\tau_T\}_{T=1}^{\infty}] = 1$ because $\sum_{k=1}^{\bar{k}}\bar{\alpha}_k^T + \tau_T = T$ and $\tau_T\leq\tau$. Hence the equality follows.

Proof: Necessity:

Suppose that there exists a subset of S, $\{s^1,\ldots,s^{\bar{k}}\}$, such that for every $s^r, s^{r'}$ we have that $s^r\approx s^{r'}$ and there exists $(\alpha^1,\ldots,\alpha^{\bar{k}})\in\Delta^{\bar{k}}$ such that $v=\sum_{k=1}^{\bar{k}}\alpha^k U(s^k)$. We want to show that there exists $f\in F$ such that $U(f)=v$. Assume $(\alpha^1,\ldots,\alpha^{\bar{k}})\in\mathbb{Q}^{\bar{k}}$. Therefore there exists integer numbers $(r_1,\ldots,r_{\bar{k}})$ and $(q_1,\ldots,q_{\bar{k}})$ such that $\alpha^k = r_k/q_k$ for every $k=1,\ldots,\bar{k}$. Let $q=\prod_{k=1}^{\bar{k}} q_k$ and $q_{-j}=\prod_{k\neq j} q_k$. Therefore $\alpha^k = r_k q_{-k}/q$ for every $k=1,\ldots,\bar{k}$.

Let $s^l, s^{l'}\in S$ be such that $s^l\approx s^{l'}$, define $c_{l,l'}^a = \{a^1,\ldots,a^m\}$ where $a^k\in A^k$ for every $k=1,\ldots,m$ and $T(w^k,a^k)=w^{k+1}$ for every $k=1,\ldots,m-1$; $w^1\in c_w(s^l)$ and $T(w^m,a^m)\in c_w(s^{l'})$.

Construct the following (constant) aggregate strategy in F:

$$a(f) = \{b_a(s^1), \underbrace{c_a(s^1),\ldots,c_a(s^1)}_{(r_1q_{-1})\text{-times}}, c_{1,2}^a, \underbrace{c_a(s^2),\ldots,c_a(s^2)}_{(r_2q_{-2})\text{-times}}, c_{2,3}^a,\ldots,c_{\bar{k}-1,\bar{k}}^a,$$
$$\underbrace{c_a(s^{\bar{k}}),\ldots,c_a(s^{\bar{k}})}_{(r_{\bar{k}}q_{-\bar{k}})\text{-times}}, c_{1,2}^a,\ldots,c_{\bar{k}-1,\bar{k}}^a, \underbrace{c_a(s^{\bar{k}}),\ldots,c_a(s^{\bar{k}})}_{(r_{\bar{k}}q_{-\bar{k}})\text{-times}}, c_{\bar{k},1}^a,\ldots,c_{\bar{k},1}^a,$$
$$c_a(s^1),\ldots,c_a(s^1),\ldots\}.$$

It is easy to show that $\lim_{T\to\infty}(1/T)\sum_{t=1}^{T}u_i(a^t(f)) = v_i$ for every $i\in I$.

If $(\alpha^1,\ldots,\alpha^{\bar{k}})\notin\mathbb{Q}^{\bar{k}}$ take a sequence $\{v^m\}_{m=1}^{\infty}\to v$ where every v^m may be obtained as a rational convex combination of stationary strategy payoffs. As in Massó [1993] it is possible to construct a strategy converging to v. □

Lemma 3: For every $j=1,\ldots,K$, there exists $[sj]_i\in S_i$ and $[sj]_{-i}\in S_{-i}$ such that $v_i(j) = U_i([sj]_i^j,[sj]_{-i}^j)$.

Proof: Let $j=1,\ldots,K$ and $f_{-i}^j\in F_{-i}^j$ be given. Then $\sup_{f_i^j\in F_i^j} U_i(f_i^j,f_{-i}^j)\geq \max_{[sj]_i\in S_i} U_i([sj]_i^j,f_{-i}^j)$. Therefore,

$$\inf_{f_{-i}^j\in F_{-i}^j}\ \sup_{f_i^j\in F_i^j} U_i(f_i^j,f_{-i}^j)\geq \inf_{f_{-i}^j\in F_{-i}^j}\ \max_{[sj]_i\in S_i} U_i([sj]_i^j,f_{-i}^j) \tag{1}$$

It is easy to show that for every $([sj]_i^j, f^j_{-i})$ there exists a subsequence $\{T_m\}_{m=1}^{\infty}$ of $\{T\}_{T=1}^{\infty}$ such that:

i) for every $m, \bar{m} \geq 1$, $w^{T_m}([sj]_i^j, f^j_{-i}) = w^{T_{\bar{m}}}([sj]_i^j, f^j_{-i})$, and

ii) $\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} u_i(a^t([sj]_i^j, f^j_{-i})) = \lim_{m\to\infty} \frac{1}{T_m}\sum_{t=1}^{T_m} u_i(a^t([sj]_i^j, f^j_{-i}))$.

Therefore, for every $m \geq 1$, there exist $[sj]_{-i}^{m,1}, \ldots, [sj]_{-i}^{m,k(m)}$ such that

$$\frac{1}{T_m}\sum_{t=1}^{T_m} u_i(a^t([sj]_i^j, f^j_{-i})) = \frac{1}{T_m}\sum_{k=1}^{k(m)} U_i([sj]_i^j, [sj]_{-i}^{m,k})\#([sj]_i^j, [sj]_{-i}^{m,k}) + \frac{1}{T_m}\delta_i$$
$$\geq \frac{1}{T_m}\min\{U_i([sj]_i^j, [sj]_{-i}^{m,k}) \mid 1 \leq k \leq k(m)\}\sum_{k=1}^{k(m)} \#([sj]_i^j, [sj]_{-i}^{m,k}) + \frac{1}{T_m}\delta_i$$
$$\geq \frac{1}{T_m} U_i([sj]_i^j, [sj]_{-i}^j)\sum_{k=1}^{k(m)} \#([sj]_i^j, [sj]_{-i}^j) + \frac{1}{T_m}\delta_i,$$

where $[sj]_{-i}^j \in \operatorname{argmin}\{U_i([sj]_i^j, [sj]_{-i}^j) \mid [sj]_{-i}^j \in S_{-i}^j\}$, δ_i is the sum of payoffs for player i, coming from the path of actions starting in game j until the cycle of one of the stationary strategies $([sj]_i^j, [sj]_{-i}^{m,k})$ is reached, and $\#([sj]_i^j, [sj]_{-i}^{m,k})$ means the number of times that the cycle of $([sj]_i^j, [sj]_{-i}^{m,k})$ is played from T_1 to T_m. Notice that for every $m \geq 1$ we are decomposing the actual play generated by $([sj]_i^j, f^j_{-i})$ as an initial path (from 1 to T_1) and a collection of cycles of stationary strategies (from T_1 to T_m).

$$U_i([sj]_i^j, f^j_{-i}) \geq \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} u_i(a^t([sj]_i^j, f^j_{-i}))$$
$$= \mathscr{H}\left(\left\{\frac{1}{T_m}\sum_{t=1}^{T_m} u_i(a^t([sj]_i^j, f^j_{-i}))\right\}_{m=1}^{\infty}\right)$$
$$\geq U_i([sj]_i^j, [sj]_{-i}^j).$$

Therefore, for every f^j_{-i}

$$\max_{[sj]_i \in S_i} U_i([sj]_i^j, f^j_{-i}) \geq \max_{[sj]_i \in S_i} U_i([sj]_i^j, [sj]_{-i}^j)$$

implying that

$$\inf_{f^j_{-i} \in F^j_{-i}} \max_{[sj]_i \in S_i} U_i([sj]_i^j, f^j_{-i}) \geq \min_{[sj]_{-i} \in S_{-i}} \max_{[sj]_i \in S_i} U_i([sj]_i^j, [sj]_{-i}^j).$$

Therefore, by (1),

$$v_i(j) = \inf_{f^i_{-i} \in F^j_{-i}} \sup_{f_i^j \in F_i^j} U_i(f_i^j, f^j_{-i}) \geq \min_{[sj]_{-i} \in S_{-i}} \max_{[sj]_i \in S_i} U_i([sj]_i^j, [sj]_{-i}^j).$$

By the same type of argument, one can show that

$$\min_{[sj]_{-i}\in S_{-i}} \max_{[sj]_i\in S_i} U_i([sj]^j_i, [sj]^j_{-i}) \geq v_i(j).$$

Thus, there exist $[sj]_i \in S_i$ and $[sj]_{-i} \in S_{-i}$ such that $v_i(j) = U_i([sj]^j_i, [sj]^j_{-i})$, which proves Lemma 3. □

Proposition 1: For every $i \in I$, there exist $m_{-i} \in S_{-i}$ and $m_i \in S_i$ such that for every $j = 1, \ldots, K$, $U_i(m^j_i, m^j_{-i}) = v_i(j)$ and $U_i(m^j_i, m^j_{-i}) \geq U_i(f^j_i, m^j_{-i})$ for every $f^j_i \in F^j_i$.

Proof: Consider the set of stationary strategies $\{[s1], \ldots, [sK]\}$ obtained by Lemma 3. Since $v_i(j) = U_i([sj])$ it must be the case that $v_i(j) = v_i(\hat{j})$ for every $\hat{j} \in c_w([sj])$. Assume that $1 \leq j \leq j' \leq K$ are such that $c_w([sj]) \cap c_w([sj']) \neq \emptyset$. This implies that, for every $i \in I$, $v_i(j) = v_i(j')$. Therefore, define $\bar{s} \in S$ as follows:

$$\bar{s}(r) = \begin{cases} [sj](r) & \text{if } r \in b_w([sj]) \cup c_w([sj]). \\ [sj'](r) & \text{otherwise.} \end{cases}$$

Notice that $c_w(\bar{s}^j) = c_w(\bar{s}^{j'}) = c_w([sj])$, which implies that $U_i(\bar{s}^j) = U_i(\bar{s}^{j'}) = v_i(j) = v_i(j')$. Construct the new set of strategies $(\{[s1], \ldots, [sK]\} - \{[sj], [sj']\}) \cup \{\bar{s}\}$. Repeat this process until one obtains a set of stationary strategies $\{[\bar{s}1], \ldots, [\bar{s}k']\}$ such that for every $j, j' = 1, \ldots, k'$ $(j \neq j')$ one has $c_w([\bar{s}j]) \cap c_w([\bar{s}j']) = \emptyset$. Pick as m of the Proposition 1 one of the stationary strategies that satisfies $m(r) = [\bar{s}j](r)$ for every $w^r \in c_w([\bar{s}j])$ and for every $j = 1, \ldots, k'$. This proves Proposition 1. □

Theorem 2: Let v be a feasible payoff of G. Then, v is an equilibrium payoff of G if and only if there exists $s^1, \ldots, s^{k'}$ $(k' \leq n+1)$, $(\alpha^1, \ldots, a^{k'}) \in \Delta^{k'}$ such that $v = \sum_{r=1}^{k'} \alpha^r U(s^r)$ and there exist $c_{1,2}, \ldots, c_{k',1}$ connections such that $v_i \geq v_i(s^1, \ldots, s^{k'}, c_{1,2}, \ldots, c_{k',1}, l)$ for some $l = 1, \ldots, k'$ and for every $i \in I$.

Proof: Necessity:

Let v be an equilibrium payoff of G. Then there exists $f \in F^*$ such that $v = U(f)$. Obviously, v is feasible, therefore by Theorem 1, there exist $s^1, \ldots, s^{k'}$, $(\alpha^1, \ldots, \alpha^{k'}) \in \Delta^{k'}$ and $(c_{1,2}, \ldots, c_{k',1})$ such that $v = \sum_{r=1}^{k'} \alpha^r U(s^r)$. Those are the ones obtained in the constructive proof of Theorem 1. Consider $a(f) = \{a^1(f), \ldots\}$. Let w^j be the game $w^j = w^t(f)$ such that for every $1 \leq \tau < t$, $w^\tau(f) \notin \bigcup_{r=1}^{k'} c_w(s^r)$ and let $1 \leq l \leq k'$ be such that $w^j \in c_w(s^l)$ (If it does not exist, let $w^j = w^1$.) There exists $\bar{s}^l \in S$ such that $c_a(s^l) = c_a(\bar{s}^l)$ and $b_a(\bar{s}^l) \subseteq \{a^1(f), \ldots, a^{t-1}(f)\}$. (This can be proved by an argument similar to the one used in the proof of Theorem 1).

Change s^l by $\bar{s}^l$ in the set $\{s^1, \ldots, s^{k'}\}$. Notice that the set of actions $\{c_a(s^1), \ldots, c_a(s^{k'}), a(c_{1,2}), \ldots, a(c_{k',1}), b_a(\bar{s}^l)\}$ (where $a(c_{k,\bar{k}})$ is the path of actions used in the connection of k with $\bar{k}$) is a subset of $\{a^t(f) | t \in \mathbb{N}\}$. Since $f \in F^*$ is an equilibrium, we must have $v_i \geq v_i(.)$ for every $i \in I$, otherwise there would exist $i \in I$ who would have a successful deviation. Notice, that by the Charatheodory Theorem, one could express v as a convex combination of at most $n+1$ points. Hence, $k' \leq n+1$ (see Figure 2).

Proof: Sufficiency:

We have to construct an strategy $f \in F$ such that:

i) $v = U(f)$
ii) $f \in F^*$.

Without loss of generality consider that $l = 1$. Define $f \in F$ as the strategy constructed in the proof of Theorem 1, but now players monitor the other players in such a way that if player $i \in I$ deviates from this path of actions in the game w^j, all the other players use m_{-i} (Proposition 1 guarantees that this is independent of j). Hence, if player $i \in I$ does not deviate he gets v_i, and if he deviates he gets $v_i(j)$ which is smaller or equal than $v_i(s^1, \ldots, s^{k'}, c_{1,2}, \ldots, c_{k',1}, l)$. Therefore this strategy is in F^*. □

References

Aumann R, Shapley L (1976) Long term competition: A game theoretic analysis. Mimeo, Hebrew University

Ben-Porath E, Peleg B (1989) On the folk theorem and finite automata. Mimeo, Hebrew University

Bewley T, Kohlberg E (1976) The asymptotic theory of stochastic games. Mathematics of Operations Research 1

Duffie D, Geanakoplos J, Mas-Colell A, McLennan A (1994) Stationary Markov equilibrium. Econometrica 62

Friedman J (1989) A modified folk theorem for time-dependent supergames. Mimeo, University of North Carolina

Fudenberg D, Maskin E (1986) The folk theorem in repeated games with discounting or with incomplete information. Econometrica 54

Jovanovic B, Rosenthal R (1988) Anonymous sequential games. Journal of Mathematical Economics 17

Kalai E, Samet D, Stanford W (1986) Complexity in stochastic games. Northwestern University discussion paper

Lockwood B (1990) The folk theorem in stochastic games with and without discounting. Mimeo, Birkbeck College

Massó J (1993) Undiscounted equilibrium payoffs of repeated games with a continuum of players. Journal of Mathematical Economics 22

Mertens J-F, Parthasarathy T (1987) Equilibria for discounted stochastic games. Discussion Paper 8750, Center for Operations Research and Econometrics, Louvain

Rubinstein A (1977) Equilibrium in supergames. Mimeo, The Hebrew University

Shapley L (1953) Stochastic games. Proceedings of the National Academy of Sciences USA 39: 1095–1100

Received August 1993
Revised version August 1994
Final version January 1995