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An algorithm to compute the full set of many-to-many stable matchings

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Abstract

The paper proposes an algorithm to compute the full set of many-to-many stable matchings when agents have substitutable preferences. The algorithm starts by calculating the two optimal stable matchings using the deferred-acceptance algorithm. Then, it computes each remaining stable matching as the firm-optimal stable matching corresponding to a new preference profile, which is obtained after modifying the preferences of a previously identified sequence of firms.

Keywords: Matching; Stability; Substitutable preferences; Algorithm

JEL classification: C78

1. Introduction

The paper proposes an algorithm to compute the full set of many-to-many stable matchings when agents have substitutable preferences.

Many-to-many matching models have been useful for studying assignment problems with the distinctive feature that agents can be divided from the very beginning into two disjoint subsets: the set of firms and the set of workers.[1] The nature of the assignment

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[1] We will be using as a reference (and as a source of terminology) labor markets with part-time jobs and we will generically refer to these two sets as the two sides of the market.

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problem consists of matching each agent (firms and workers) with a subset of agents from the other side of the market. Thus, each firm will hire a subset of workers, while each worker may work for a number of different firms.

Agents have preferences on the subsets of potential partners. Stability has been considered the main property to be satisfied by any sensible matching. A matching is called stable if all agents are matched to an acceptable subset of partners and there is no unmatched worker–firm pair who both would prefer to add the other to their current subset of partners. To give blocking power to only individual agents and worker–firm pairs seems a very weak requirement in terms of the durability of the matching.[2]

Unfortunately, the set of stable matchings may be empty. Substitutability is the weakest condition that has so far been imposed on agents' preferences under which the existence of stable matchings is guaranteed. An agent has substitutable preferences if he continues to want to be partners with an agent from the other side of the market even if other agents become unavailable.[3]

Surprisingly, the set of stable matchings under substitutable preferences is very-well structured. It contains two distinctive matchings: the firm-optimal stable matching (denoted by μ_F) and the worker-optimal stable matching (denoted by μ_W). The matching μ_F is unanimously considered by all firms to be the best among all stable matchings and by all workers to be the worst among all stable matchings. Symmetrically, the matching μ_W is unanimously considered by all workers to be the best among all stable matchings and by all firms to be the worst among all stable matchings. They can be obtained by the so-called deferred-acceptance algorithm (originally defined by Gale and Shapley, 1962 for the one-to-one case and later adapted by Roth, 1984 to the many-to-many case). Additionally, Blair (1988) shows that the set of stable matchings has a lattice structure.[4] In particular, Roth (1984) and Blair (1988) show that this unanimity and opposition of interests of the two sides of the market is even stronger in the sense that all firms, if they had to choose the best subset from the set of workers made up of the union of the firm-optimal stable matching and any other stable matching, would choose the firm-optimal stable matching. Also, all firms, if they had to choose the best subset from the set of workers made up of the union of the worker-optimal stable matching and any other stable matching, would choose the other stable matching. And symmetrically, the two properties also hold interchanging the roles of firms and workers.[5]

[2] Sotomayor (1999a) uses the name of pairwise stability to refer to this notion of stability. In her paper, she proposes the stronger concept of setwise stability and shows that, in the many-to-many model, the set of pairwise stable matchings, the core, and the set of setwise stable matchings do not coincide. As far as we know, the construction of algorithms using these last two group stability concepts are still open problems.

[3] See Definition 3 for a formal statement of this property. Kelso and Crawford (1982) were the first to use it to show the existence of stable matchings in a many-to-one model with money. Roth (1984) shows that, if all agents have substitutable preferences, the set of many-to-many stable matchings is non-empty.

[4] Roth (1985), Gusfield and Irving (1989), Sotomayor (1999b), Alkan (2001), Baïou and Balinski (2000), and Martínez et al. (2001) also study the lattice structure of the set of stable matchings in different models.

[5] See Remark 1 in Section 2 for a formal statement of these four properties.

Algorithms have played a central role in the matching literature.[6] While there are algorithms designed to compute the *full* set of one-to-one stable matchings as well as the two optimal stable matchings (for the many-to-many model), we are not aware of any algorithm which can compute the full set of matchings for this more general many-to-many case. Our paper contributes to this literature by proposing for the first time an algorithm that computes the full set of stable matchings in a general model with ordinal preferences. In contrast with the marriage model, the structure of this set is not yet fully understood (Alkan (2001), Blair (1988), Martínez et al. (2001), Roth (1985), Sotomayor (1999a), and Sotomayor (1999b) are, among others, examples of papers contributing to this understanding). One of the potential uses of our algorithm is to generate conjectures, counterexamples, and intuitions to make progress in the study of this more general matching model.

McVitie and Wilson (1971) were the first to obtain an algorithm to compute the full set of stable matchings for the one-to-one matching model. Our algorithm extends theirs to the many-to-many matching model with substitutable preferences. Irving and Leather (1986) proposed a different algorithm to compute the set of one-to-one stable matchings based on its lattice structure (see also Roth and Sotomayor, 1990 for an adapted description of this algorithm). In Section 3.3, we briefly describe McVitie and Wilson's algorithm and explain why our algorithm reduces to theirs whenever the matching model is one-to-one; we also briefly explain why this is not the case for the Irving and Leather algorithm.

Roughly, our algorithm works by applying successively the following procedure. First, given as input an original profile of substitutable preferences, it computes by the deferred-acceptance algorithm the two optimal stable matchings μ_F and μ_W. Second, it identifies all firm–worker pairs (f,w) where firm f hires the worker w in μ_F but not in μ_W. Successively, for each of these pairs, it modifies the preference of firm f by declaring all subsets of workers containing worker w unacceptable but leaving the orderings among all subsets not containing w unchanged. This is called an (f,w)-truncation of the original preference. By the deferred-acceptance algorithm, it computes (for each pair) the firm-optimal stable matching corresponding to the preference profile where all agents have the original preferences except that firm f has the (f,w)-truncated preference. Third, although this new firm-optimal stable matching might not be stable relative to the original preference profile it is stable provided that worker w, if he had to choose the best subset from the set of firms made up of the union of the two firm-optimal stable matchings (the original and the new one), he would choose the new one. If it passes this test (and, hence, if it is stable relative to the original profile of preferences), we keep it and proceed again from the very beginning using this modified profile as an input.[7] The algorithm stops when there is no firm–worker pair (f,w) where firm f hires worker w in the firm-optimal stable matching (relative to the truncated preference profile) but not in μ_W.

The paper is organized as follows. In Section 2, we present the preliminary notation, definitions, and results. Section 3 contains the definition of the algorithm, the theorem stating that the outcome of the algorithm is equal to the set of stable matchings, and an

[6] See Gusfield and Irving (1989) for an algorithmic approach to the one-to-one and roommate models.

[7] In the formal definition of the algorithm, the reader will find an additional (but dispensable) step only used to speed up the algorithm.

example illustrating how the algorithm works. In Section 4, we prove the theorem. Section 5 contains two concluding remarks. Finally, an appendix at the end of the paper illustrates by means of an example the deferred-acceptance algorithm of Gale and Shapley adapted to the many-to-many case.

2. Preliminaries

There are two disjoint sets of *agents*, the set of *n firms* $F=\{f_1,\ldots,f_n\}$ and the set of *m workers* $W=\{w_1,\ldots,w_m\}$. Generic elements of both sets will be denoted, respectively, by f, $f_i, f_{i_k}, \bar{f}$, and $\tilde{f}$, and by w, w_j, w_{j_k}, $\bar{w}$, and $\tilde{w}$. A generic agent will be denoted by a, and we will refer to a *set of partners of a* as a subset of agents of the set not containing a. Associated with each agent $a\in F\cup W$ is a strict linear ordering $P(a)$, called a preference relation, over the set of all subsets of partners (over 2^F if a is a worker and over 2^W if a is a firm). *Preference profiles* are $(n+m)$-tuples of preference relations, and they are represented by $P=(P(f_1),\ldots,P(f_n); P(w_1),\ldots,P(w_m))$. Given a preference relation of an agent $P(a)$, the sets of partners preferred to the empty set by a are called *acceptable*; therefore, we are allowing for the possibility that firm f may prefer not hiring any worker rather than hiring unacceptable sets of workers and that worker w may prefer to remain unemployed rather than working for an unacceptable set of firms.

To express preference relations in a concise manner, and since only acceptable sets of partners will matter, we will represent preference relations as lists of acceptable partners. For instance,

$$P(f_i) = w_1w_3, w_2, w_1, w_3$$

$$P(w_j) = f_1f_3, f_1, f_3$$

indicate that $\{w_1,w_3\}P(f_i)\{w_2\}P(f_i)\{w_1\}P(f_i)\{w_3\}P(f_i)\emptyset$ and $\{f_1,f_3\}P(w_j)\{f_1\}P(w_j)\{f_3\}P(w_j)\emptyset$.

The assignment problem consists of matching workers with firms keeping the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

Definition 1. A *matching* μ is a mapping from the set $F\cup W$ into the set of all subsets of $F\cup W$ such that for all $w\in W$ and $f\in F$:

1. $\mu(w)\in 2^F$
2. $\mu(f)\in 2^W$
3. $f\in\mu(w)$ if and only if $w\in\mu(f)$.

We say that an agent a is *single* if $\mu(a)=\emptyset$, otherwise he is matched. A matching μ is said to be *one-to-one* if firms can hire at most one worker and workers can work for at most one firm. The model in which all matchings are one-to-one is also known in the literature as the *marriage model*. A matching μ is said to be *many-to-one* if workers can work for at most one firm but firms may hire many workers. The model in which all matchings are many-to-one, and firms have responsive preferences,[8] is also known in the literature as the *college admissions model*.

Let P be a preference profile. Given a set of partners S, let $\mathrm{Ch}(S, P(a))$ denote agent a's *most-preferred* subset of S according to a's preference ordering $P(a)$. A matching μ is *blocked by agent a* if $\mu(a) \neq \mathrm{Ch}(\mu(a), P(a))$. We say that a matching is *individually rational* if it is not blocked by any agent. We will denote by $IR(P)$ as the set of all individually rational matchings. A matching μ is *blocked by a worker–firm pair* (w,f) if $w \notin \mu(f)$, $w \in \mathrm{Ch}(\mu(f) \cup \{w\}, P(f))$, and $f \in \mathrm{Ch}(\mu(w) \cup \{f\}, P(w))$.

Definition 2. A matching μ is *stable* if it is blocked neither by an individual agent nor by a worker–firm pair.

Given a preference profile P, denote the set of stable matchings by $S(P)$. It is easy to construct examples of preference profiles with the property that the set of stable matchings is empty (see, for instance, Example 2.7 in Roth and Sotomayor, 1990). Those examples share the feature that at least one agent regards two partners as being complements, in the sense that the desirability of a partner might depend on the presence of the other one. This is the reason why the literature has focused on the restriction where partners are regarded as substitutes. Here, the assumption that preference profiles are substitutable will be essential.

Definition 3. An agent a's preference ordering $P(a)$ satisfies *substitutability* if for any set S of partners containing agents b and c $(b \neq c)$, if $b \in \mathrm{Ch}(S, P(a))$ then $b \in \mathrm{Ch}(S \setminus \{c\}, P(a))$.

A preference profile P is *substitutable* if for each agent a, the preference ordering $P(a)$ satisfies substitutability. Observe that this many-to-many model with substitutable preferences includes, as particular cases, the marriage model and the college admissions model.

Roth (1984) shows that if all agents have substitutable preferences then: (1) the set of stable matchings is non-empty, (2) firms (workers) unanimously agree that a stable matching $\mu_F(\mu_W)$ is the best stable matching, and (3) the optimal stable matching for one side is the worst stable matching for the other side. The matchings μ_F and μ_W are called, respectively, the firm-optimal stable matching and the worker-optimal stable matching. We are following the convention of extending preferences from the original sets (2^W and 2^F) to the set of matchings. However, we now have to consider weak orderings since the matchings μ and μ' may associate the same set of partners to an agent. These orderings will be denoted by $R(f)$ and $R(w)$. For instance, to say that all firms prefer μ_F to any stable μ means that for every $f \in F$ we have that $\mu_F R(f) \mu$ for all stable μ (that is, either $\mu_F(f) = \mu(f)$ or else $\mu_F(f) P(f) \mu(f)$).

The deferred-acceptance algorithm, originally defined by Gale and Shapley (1962) for the one-to-one case, produces either μ_F or μ_W depending on who makes the offers. At any

[8] Namely, for any two subsets of workers that differ in only one worker, a firm prefers the subset containing the most-preferred worker. See Roth and Sotomayor (1990) for a precise and formal definition of responsive preferences as well as for a masterful and illuminating analysis of these models and an exhaustive bibliography.

step of the algorithm in which firms make offers, a firm proposes itself to the most-preferred subset of the set of workers that have not already rejected it during the previous steps, while a worker accepts the choice set of the union of the set consisting of the firms provisionally matched to him in the previous step (if any) and the set of current proposals. The algorithm stops at the step at which all offers are accepted; the (provisional) matching then becomes definite and is the stable matching μ_F. Symmetrically, if workers make offers, the outcome of the algorithm is the stable matching μ_W. Appendix A at the end of the paper illustrates by means of an example how the deferred-acceptance algorithm works for the many-to-many case.

Our algorithm will consist of applying the deferred-acceptance algorithm where firms make offers to preference profiles that are obtained after modifying the preference of a firm by making all sets containing a particular worker unacceptable.[9] Formally,

Definition 4. We say that the preference $P^{(f,w)}(f)$ is the (f,w)-*truncation of* $P(f)$ if:

1. All sets containing w are unacceptable to f according to $P^{(f,w)}(f)$, that is, if $w \in S$ then $\emptyset P^{(f,w)}(f)S$.
2. The preferences $P(f)$ and $P^{(f,w)}(f)$ coincide on all sets that do not contain w, that is, if $w \notin S_1 \cup S_2$ then $S_1 P(f) S_2$ if and only if $S_1 P^{(f,w)}(f) S_2$.
3. The preferences $P(f)$ and $P^{(f,w)}(f)$ coincide on all sets that contain w, that is, if $w \in S_1 \cap S_2$ then $S_1 P(f) S_2$ if and only if $S_1 P^{(f,w)}(f) S_2$.
4. All sets "artificially" made unacceptable in $P^{(f,w)}(f)$ are preferred to the original unacceptable sets, that is, if S_1 and S_2 are such that $w \in S_1$ and $S_1 P(f) \emptyset P(f) S_2$ then $S_1 P^{(f,w)}(f) S_2$.

Notice that conditions 3 and 4, although irrelevant for stability of matchings, guarantee that given $P(f)$ and w, the corresponding truncation $P^{(f,w)}(f)$ is unique. Given a preference profile P and the (f,w)-truncation of $P(f)$, we denote by $P^{(f,w)}$ the preference profile obtained by replacing $P(f)$ in P by $P^{(f,w)}(f)$, that is, $P(a)=P^{(f,w)}(a)$ for all agents $a \neq f$ and $P(f)$ and $P^{(f,w)}(f)$ differ, essentially, in that $P^{(f,w)}(f)$ eliminates, as acceptable, all sets of workers that contain w. We denote by $\mu_F^{(f,w)}$ and $\mu_W^{(f,w)}$ the firm and worker-optimal stable matchings corresponding to the preference profile $P^{(f,w)}$. Moreover, given a preference profile P and a sequence of pairs $(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})$, we will represent by $P^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})}$ the preference profile obtained from P after successively truncating the corresponding preference(s); we will also denote by $\mu_F^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})}$ and $\mu_W^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})}$ its corresponding optimal stable matchings. The following lemma states that the property of substitutability is preserved by truncations, and therefore $\mu_F^{(f,w)}$ and $\mu_W^{(f,w)}$ exist provided that P is substitutable.

Lemma 1. *If $P(f)$ is substitutable, then $P^{(f,w)}(f)$ is substitutable.*

Proof. *Let $\bar{w}, w' \in S$ be arbitrary and assume that $\bar{w} \in Ch(S, P^{(f,w)}(f))$. If $w \notin S$, then $\bar{w} \in Ch(S \setminus \{w'\}, P^{(f,w)}(f))$ because $Ch(S, P^{(f,w)}(f)) = Ch(S, P(f))$, $Ch(S \setminus \{w'\}, P^{(f,w)}(f)) =$*

[9] Given the symmetric role of firms and workers, it will become clear that the construction that follows could be equivalently done by interchanging the roles of workers and firms.

$Ch(S \setminus \{w'\}, P(f))$, and because of the substitutability of $P(f)$. If $w \in S$, then we have that $Ch(S, P^{(f,w)}(f)) = Ch(S \setminus \{w\}, P(f))$; therefore, by assumption $\bar{w} \in Ch(S \setminus \{w\}, P(f))$. By the substitutability of $P(f)$, we have that $\bar{w} \in Ch([S \setminus \{w\}] \setminus \{w'\}, P(f))$ but the equality $Ch([S \setminus \{w\}] \setminus \{w'\}, P(f)) = Ch(S \setminus \{w'\}, P^{(f,w)}(f))$ implies that worker $\bar{w} \in Ch(S \setminus \{w'\}, P^{(f,w)}(f))$. □

Before finishing this section, we present, as a remark below, four properties of stable matchings.

Remark 1. *Assume P is substitutable, and let $\mu \in S(P)$. Then, for all f and w:*

1. $\mathrm{Ch}(\mu_F(f) \cup \mu(f), P(f)) = \mu_F(f)$.
2. $\mathrm{Ch}(\mu_W(w) \cup \mu(w), P(w)) = \mu_W(w)$.
3. $\mathrm{Ch}(\mu_W(f) \cup \mu(f), P(f)) = \mu(f)$.
4. $\mathrm{Ch}(\mu_F(w) \cup \mu(w), P(w)) = \mu(w)$.

Properties 1 and 2 are due to Roth (1984), while properties 3 and 4 follow from 1, 2, and Theorem 4.5 in Blair (1988). They can be interpreted as a strengthening of the optimality of μ_F and μ_W. Example 1 below shows that, although necessary, they are far from being a characterization of stable matchings.

Example 1. Let $F = \{f_1, f_2, f_3, f_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the preference profile P, where

$$P(f_1) = w_1, w_2, w_3, w_4$$

$$P(f_2) = w_2, w_4, w_1$$

$$P(f_3) = w_3, w_1, w_2$$

$$P(f_4) = w_4, w_2, w_3$$

$$P(w_1) = f_2, f_3, f_1$$

$$P(w_2) = f_3, f_1, f_4, f_2$$

$$P(w_3) = f_4, f_1, f_3$$

$$P(w_4) = f_1, f_2, f_4.$$

The matching

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3 & w_4 & w_1 & w_2 \end{pmatrix}$$

is not stable since (w_2, f_1) blocks it.[10] However, it can be verified that

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix},$$

$$\mu_W = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_4 & w_1 & w_2 & w_3 \end{pmatrix},$$

and μ satisfies the four properties of Remark 1.

The fact that whether or not property 4 in Remark 1 holds *only* for a particular worker w will play a crucial role in the construction of our algorithm, we will sometimes refer to it as the Choice Property for w relative to P. More precisely, given P and its corresponding μ_F, we say that a matching μ satisfies the *Choice Property for w relative to P* if

$$\mathrm{Ch}(\mu_F(w) \cup \mu(w), P(w)) = \mu(w).$$

3. An algorithm to compute the set of stable matchings

3.1. The algorithm and the theorem

Given a preference profile P, we define an algorithm to compute the set of stable matchings $S(P)$.

begin Set $T^0(P) := P$, $S^0(P) := \{\mu_F\}$, and $k := 0$.
repeat
Step 1: Define

$$\tilde{T}(T^k(P)) = \left\{ \begin{array}{l} P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)} \mid w \in \mu_F^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})}(f) \setminus \mu_W(f), \\ P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})} \in T^k(P), \text{ and } f \in F \end{array} \right\}.$$

Step 2: **if** $\tilde{T}(T^k(P)) = \emptyset$ set $T^{k+1}(P) = \emptyset$, $S^{k+1}(P) = S^k(P)$,
else, for each truncation $P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)} \in \tilde{T}(T^k(P))$ obtain $\mu_F^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)}$, which exists by Lemma 1.

[10] To represent matchings concisely, we follow the widespread notation where for instance here, in matching μ, f_1 is matched to w_3, f_2 is matched to w_4, and so on.

Step 3: Define

$$T^*(T^k(P)) = \left\{ \begin{array}{l} P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)} \in \tilde{T}(T^k(P)) \mid \\ \mathrm{Ch}(\mu_{\mathrm{F}}^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)}(w) \cup \mu_{\mathrm{F}}^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})}(w), P(w)) = \\ = \mu_{\mathrm{F}}^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)}(w) \end{array} \right\}.$$

Order the set $T^*(T^k(P))$ in an arbitrary way and let $\prec^{k+1}$ denote this ordering.

Step 4: Define

$$\hat{T}(T^k(P)) = \left\{ \begin{array}{l} P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)} \in T^*(T^k(P)) \mid \\ \forall P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f',w')} \in T^*(T^k(P)) \text{ such that} \\ P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)} \prec^{k+1} P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f',w')}, \\ w' \in \mu_{\mathrm{F}}^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)}(f') \end{array} \right\}.$$

Set

$$T^{k+1}(P) := \hat{T}(T^k(P)),$$

$$S^{k+1}(P) := S^k(P) \cup \left\{ \mu_{\mathrm{F}}^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)} \mid P^{(f_{i_1},w_{j_1})\dots(f_{i_k},w_{j_k})(f,w)} \in T^{k+1}(P) \right\},$$

and

$$k := k + 1.$$

until $T^k(P)$ is empty.
end.

Let K be the stage where the algorithm stops, i.e., $T^K(P)=\emptyset$. We can now state our main result.

Theorem 1. *Assume P is substitutable. Then $S^K(P)=S(P)$.*

3.2. An example

We illustrate how the algorithm works with the following example.

Example 2. Let $F=\{f_1, f_2, f_3, f_4\}$ and $W=\{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the substitutable profile of preferences P, where

$$P(f_1) = w_1w_2, w_1w_3, w_2w_4, w_3w_4, w_1w_4, w_2w_3, w_1, w_2, w_3, w_4$$

$$P(f_2) = w_1w_2, w_2w_3, w_1w_4, w_3w_4, w_1w_3, w_2w_4, w_1, w_2, w_3, w_4$$

$$P(f_3) = w_3w_4, w_2w_3, w_1w_4, w_1w_2, w_2w_4, w_1w_3, w_1, w_2, w_3, w_4$$

$$P(f_4) = w_3w_4, w_2w_4, w_1w_3, w_1w_2, w_2w_3, w_1w_4, w_1, w_2, w_3, w_4$$

$$P(w_1) = f_3f_4, f_2f_3, f_2f_4, f_1f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4$$

$$P(w_2) = f_3f_4, f_2f_3, f_1f_4, f_2f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4$$

$$P(w_3) = f_1f_2, f_2f_3, f_1f_3, f_2f_4, f_1f_4, f_3f_4, f_1, f_2, f_3, f_4$$

$$P(w_4) = f_1f_2, f_1f_3, f_1f_4, f_2f_3, f_2f_4, f_3f_4, f_1, f_2, f_3, f_4.$$

By the deferred-acceptance algorithm, we obtain the two optimal stable matchings

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1w_2 & w_1w_2 & w_3w_4 & w_3w_4 \end{pmatrix}$$

$$\mu_W = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2 \end{pmatrix}.$$

Set $T^0(P)=P$, $S^0(P)=\{\mu_F\}$, and $k=0$.

Stage 1: The set $\tilde{T}(T^0(P))$ of Step 1 consists of the following truncations of P:

$$\tilde{T}(T^0(P)) = \{P^{(f_1,w_1)},\ P^{(f_1,w_2)},\ P^{(f_2,w_1)},\ P^{(f_2,w_2)},\ P^{(f_3,w_3)},\ P^{(f_3,w_4)},\ P^{(f_4,w_3)},\ P^{(f_4,w_4)}\}$$

where in all profiles firms and workers have the same preference as in P, except

$$P^{(f_1,w_1)}(f_1) = w_2w_4, w_3w_4, w_2w_3, w_2, w_3, w_4$$

$$P^{(f_1,w_2)}(f_1) = w_1w_3, w_3w_4, w_1w_4, w_1, w_3, w_4$$

$$P^{(f_2,w_1)}(f_2) = w_2w_3, w_3w_4, w_2w_4, w_2, w_3, w_4$$

$$P^{(f_2,w_2)}(f_2) = w_1w_4, w_3w_4, w_1w_3, w_1, w_3, w_4$$

$$P^{(f_3,w_3)}(f_3) = w_1w_4, w_1w_2, w_2w_4, w_1, w_2, w_4$$

$$P^{(f_3,w_4)}(f_3) = w_2w_3, w_1w_2, w_1w_3, w_1, w_2, w_3$$

$$P^{(f_4,w_3)}(f_4) = w_2w_4, w_1w_2, w_1w_4, w_1, w_2, w_4$$

$$P^{(f_4,w_4)}(f_4) = w_1w_3, w_1w_2, w_2w_3, w_1, w_2, w_3.$$

In Step 2, and since the set $\tilde{T}(T^0(P))$ is non-empty, we obtain for each of its truncations the corresponding firm-optimal stable matching

$$\mu_F^{(f_1,w_1)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_1w_2 & w_3w_4 & w_1w_3 \end{pmatrix}$$

$$\mu_F^{(f_1,w_2)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1w_3 & w_1w_2 & w_3w_4 & w_2w_4 \end{pmatrix}$$

$$\mu_F^{(f_2,w_1)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1w_2 & w_3w_4 & w_3w_4 & w_1w_2 \end{pmatrix}$$

$$\mu_F^{(f_2,w_2)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_1w_4 & w_2w_3 & w_1w_3 \end{pmatrix}$$

$$\mu_F^{(f_3,w_3)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_2w_3 & w_1w_4 & w_1w_3 \end{pmatrix}$$

$$\mu_F^{(f_3,w_4)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1w_3 & w_1w_4 & w_2w_3 & w_2w_4 \end{pmatrix}$$

$$\mu_{\mathrm{F}}^{(f_4,w_3)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_1w_4 & w_2w_3 & w_1w_2 \end{pmatrix}$$

$$\mu_{\mathrm{F}}^{(f_4,w_4)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_1w_2 & w_3w_4 & w_1w_3 \end{pmatrix}.$$

Notice that $\mu_{\mathrm{F}}^{(f_1,w_1)} = \mu_{\mathrm{F}}^{(f_4,w_4)}$. In Step 3, we obtain the set $T^*(T^0(P))=\{P^{(f_1,w_1)}, P^{(f_4,w_3)}, P^{(f_4,w_4)}\}$. For instance, the truncation $P^{(f_1,w_2)}$ does not belong to this set because

$$\begin{aligned} \mathrm{Ch}(\mu_{\mathrm{F}}(w_2) \cup \mu_{\mathrm{F}}^{(f_1,w_2)}(w_2), P(w_2)) &= \mathrm{Ch}(f_1,f_2 \cup f_2,f_4, P(w_2)) \\ &= \mathrm{Ch}(\{f_1,f_2,f_4\}, P(w_2)) \\ &= \{f_1,f_4\} \\ &\neq \{f_2,f_4\} \\ &= \mu_{\mathrm{F}}^{(f_1,w_2)}(w_2), \end{aligned}$$

but this is not a problem since $\mu_{\mathrm{F}}^{(f_1,w_2)}$ is not stable because the pair (w_2, f_1) blocks it. Considering the ordering $P^{(f_1,w_1)} \prec^1 P^{(f_4,w_3)} \prec^1 P^{(f_4,w_4)}$, we have that $\hat{T}(T^0(P))=\{P^{(f_4,w_4)}\}$ since $P^{(f_1,w_1)}$ does not belong to it because $w_4 \notin \mu_{\mathrm{F}}^{(f_1,w_1)}(f_4)$ and $P^{(f_1,w_1)} \prec^1 P^{(f_4,w_4)}$ and $P^{(f_4,w_3)}$ does not belong to it either because $w_4 \notin \mu_{\mathrm{F}}^{(f_4,w_3)}(f_4)$ and $P^{(f_4,w_3)} \prec^1 P^{(f_4,w_4)}$. Set $T^1(P)=\{P^{(f_4,w_4)}\}$ and $S^1(P)=\{\mu_{\mathrm{F}}, \mu_1\}$ where $\mu_1=\mu_{\mathrm{F}}^{(f_1,w_1)}=\mu_{\mathrm{F}}^{(f_4,w_4)}$. This finishes Stage 1.

Stage 2: In Step 1, we obtain for the truncation $P^{(f_4,w_4)}$ (the unique one belonging to the set $T^1(P)$) the corresponding set of truncations using $\mu_{\mathrm{F}}^{(f_4,w_4)}$ and μ_{W}:

$$\tilde{T}(T^1(P)) = \left\{ \begin{matrix} P^{(f_4,w_4)(f_1,w_2)}, & P^{(f_4,w_4)(f_2,w_1)}, & P^{(f_4,w_4)(f_2,w_2)}, \\ P^{(f_4,w_4)(f_3,w_3)}, & P^{(f_4,w_4)(f_3,w_4)}, & P^{(f_4,w_4)(f_4,w_3)} \end{matrix} \right\}.$$

Now, in Step 2 and since $\tilde{T}(T^1(P)) \neq \emptyset$, we compute for each truncation in $\tilde{T}(T^1(P))$ its corresponding firm-optimal stable matching

$$\mu_{\mathrm{F}}^{(f_4,w_4)(f_1,w_2)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_1w_2 & w_3w_4 & w_1w_2 \end{pmatrix}$$

$$\mu_{\mathrm{F}}^{(f_4,w_4)(f_2,w_1)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1w_2 & w_3w_4 & w_3w_4 & w_1w_2 \end{pmatrix}$$

$$\mu_{\mathrm{F}}^{(f_4,w_4)(f_2,w_2)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_1w_4 & w_2w_3 & w_1w_3 \end{pmatrix}$$

$$\mu_{\mathrm{F}}^{(f_4,w_4)(f_3,w_3)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_2w_3 & w_1w_4 & w_1w_3 \end{pmatrix}$$

$$\mu_{\mathrm{F}}^{(f_4,w_4)(f_3,w_4)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_1w_4 & w_2w_3 & w_1w_2 \end{pmatrix}$$

$$\mu_{\mathrm{F}}^{(f_4,w_4)(f_4,w_3)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_1w_4 & w_2w_3 & w_1w_2 \end{pmatrix}.$$

In Step 3, we obtain the set

$$T^*(T^1(P)) = \{P^{(f_4,w_4)(f_3,w_4)}, P^{(f_4,w_4)(f_4,w_3)}\}$$

and consider the ordering $P^{(f_4,w_4)(f_3,w_4)} \prec^2 P^{(f_4,w_4)(f_4,w_3)}$. In Step 4, the set $\hat{T}(T^1(P))$ is the singleton set $\{P^{(f_4,w_4)(f_4,w_3)}\}$ since $w_3 \notin \mu_{\mathrm{F}}^{(f_4,w_4)(f_3,w_4)}(f_4)$. Set $T^2(P)=\{P^{(f_4,w_4)(f_4,w_3)}\}$ and $S^2(P)=\{\mu_{\mathrm{F}},\ \mu_1,\ \mu_2\}$ where $\mu_2=\mu_{\mathrm{F}}^{(f_4,w_4)(f_4,w_3)}$.

Stage 3: In Step 1, we obtain for the truncation $P^{(f_4,w_4)(f_4,w_3)}$ its corresponding truncations using $\mu_{\mathrm{F}}^{(f_4,w_4)(f_4,w_3)}$ and μ_{W}:

$$\tilde{T}(T^2(P)) = \{P^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}, P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}\}.$$

Since it is non-empty we compute, in Step 2, the corresponding firm-optimal stable matchings

$$\mu_{\mathrm{F}}^{(f_4,w_4)(f_4,w_3)(f_2,w_1)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1w_2 & w_3w_4 & w_3w_4 & w_1w_2 \end{pmatrix}$$

$$\mu_F^{(f_4,w_4)(f_4,w_3)(f_3,w_3)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2 \end{pmatrix}.$$

In Step 3, we obtain the set

$$T^*(T^2(P)) = \{P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}\}.$$

Notice that $P^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}$ does not belong to it because

$$\begin{aligned}\mathrm{Ch}(\mu_F^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}(w_1) \cup \mu_F^{(f_4,w_4)(f_4,w_3)}(w_1), P(w_1)) &= \mathrm{Ch}(\{f_1,f_4\} \cup \{f_2,f_4\}, P(w_1)) \\ &= \{f_2,f_4\} \\ &\neq \{f_1,f_4\} \\ &= \mu_F^{(f_4,w_4)(f_4,w_3)(f_2,w_1)}(w_1).\end{aligned}$$

Since $T^*(T^2(P))$ is a singleton set, we set $T^3(P)=\hat{T}(T^2(P))=\{P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}\}$ and $S^3(P)=\{\mu_F, \mu_1, \mu_2, \mu_W\}$ because $\mu_F^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}=\mu_W$.

Stage 4: Finally, the algorithm stops (that is, $K=4$) because $\tilde{T}(T^3(P))=\emptyset$. Therefore, $S(P)=\{\mu_F, \mu_1, \mu_2, \mu_W\}$, where

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1w_2 & w_1w_2 & w_3w_4 & w_3w_4 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_1w_2 & w_3w_4 & w_1w_3 \end{pmatrix}$$

$$\mu_2 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_1w_4 & w_2w_3 & w_1w_2 \end{pmatrix}$$

$$\mu_W = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2 \end{pmatrix}.$$

3.3. Comments

Before moving to the next section to prove Theorem 1, a few comments about the algorithm are in order.

First, for all truncations the worker-optimal stable matching coincides with the worker-optimal stable matching of the original preference profile P; that is, $\mu_W = \mu_W^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})}$ for all $P^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})}$. To see this, consider the following modification of the deferred-acceptance algorithm in which workers make offers. At any step of the algorithm, and after firm f rejects the offer of worker w, the preference of worker w is changed by declaring all sets of firms containing f as unacceptable. Now, the output of this modified algorithm is the same matching μ_W and a new preference profile $\hat{P}$ (which has the property that for all $w \in W$, $Ch(F, \hat{P}(w)) = \mu_W(w)$). Denote by $\hat{\mu}_W$ the output of the original deferred-acceptance algorithm applied to the preference profile $\hat{P}$. Obviously,

$$\mu_W = \hat{\mu}_W \tag{1}$$

and therefore,

$$\mu_W^{(f,w)} = \hat{\mu}_W^{(f,w)} \tag{2}$$

for any (f,w)-truncation. Moreover,

$$\hat{\mu}_W^{(f,w)} = \hat{\mu}_W \tag{3}$$

since $w \in \mu_F(f) \backslash \mu_W(f)$ implies that in both cases each worker w proposes himself to the set $\mu_W(w)$, all offers are accepted, and the algorithm terminates just after Step 1. Hence, by (1), (2), and (3),

$$\mu_W = \mu_W^{(f,w)}.$$

This fact is used in Step 1, and it guarantees that as the iteration process proceeds, "the end point" stays the same and the iteration process will terminate once μ_W is reached.

Second, to make sure that the firm-optimal stable matching corresponding to an (f,w)-truncation is indeed stable it is sufficient to check *only* that Property 4 of Remark 1 holds for worker w; that is, worker w would choose it if confronted with the union of itself and the firm-optimal stable matching of the original profile. This is what Step 3 does in each stage. In the light of Example 1 this is surprising, although Lemma 2 in Section 4 states that this is the case. However, the fact that a truncation only changes one firm's preference guarantees that the other properties of Remark 1 also hold.

Third, the algorithm would also work without Step 4. However, it helps very much to speed up the algorithm (see Corollary 1 in Section 4) because, by adding it, we avoid carrying to subsequent stages all truncations (and all others obtained from them) whose corresponding firm-optimal stable matching will be identified later on.

Fourth, the particular ordering on the set $T^*(T^k(P))$ is irrelevant but necessary. Namely, it is necessary because we cannot ask for individual rationality of each truncation against all other truncations. To see this consider in Stage 1 of Example 2, the set

$T^*(T^0(P))=\{P^{(f_1,w_2)}, P^{(f_2,w_3)}, P^{(f_4,w_4)}\}$. If we had defined it without the restriction of the ordering, i.e.

$$\hat{T}_{\nprec}(T^0(P)) = \{P^{(f,w)}\in T^*(T^0(P)) \mid \forall P^{(f',w')}\in T^*(T^0(P)), w'\in\mu_F^{(f,w)}(f')\}$$

this set would have been empty since $P^{(f_1,w_2)}\notin\hat{T}_{\nprec}(T^0(P))$ because $w_4\notin\mu_F^{(f_1,w_1)}(f_4)$, $P^{(f_1,w_2)}\notin\hat{T}_{\nprec}(T^0(P))$ because $w_4\notin\mu_F^{(f_4,w_3)}(f_4)$, and (in contrast with the correct definition of $\hat{T}(T^0(P))$ $P^{(f_4,w_4)}\notin\hat{T}_{\nprec}(T^0(P))$ because $w_1\notin\mu_F^{(f_4,w_4)}(f_1)$. Moreover, this ordering is irrelevant because the outcome of the algorithm does not depend on the specific ordering on the set $T^*(T^k(P))$. For instance, in Stage 1 of Example 2, we could have used (instead of $\prec^1$) the ordering $P^{(f_4,w_4)}\prec^{1'}P^{(f_4,w_3)}\prec^{1'}P^{(f_1,w_1)}$ without altering the final outcome of the algorithm.

Fifth, unfortunately we do not know how to use, in the design of the algorithm, the lattice structure of the set of stable matchings. The problem is that, in contrast with the marriage model, the lattice structure of the set of many-to-many stable matchings identified by Blair (1988) is built upon a very complex least upper bound. The proof of Theorem 4.11 in Blair (1988) shows that this least upper bound has to be obtained as the limit of a sequence of matchings constructed in a very indirect way.

Sixth, McVitie and Wilson's (1971) algorithm computes the full set of one-to-one stable matchings roughly as follows: (1) Compute μ_F by a version of the deferred-acceptance algorithm in which firms make offers sequentially. (2) Break the marriage of any matched pair (f,w) at μ_F forcing f to take a poorer worker in his preference ordering along the new application of the deferred-acceptance algorithm (whose outcome is $\mu_F^{(f,w)}$). (3) Check the stability of $\mu_F^{(f,w)}$ by checking that $\mu_F^{(f,w)}(w)P(w)\mu_F(w)$. (4) Avoid multiple identifications of the same stable matching by restricting the pairs whose marriage is broken (according to point (2) above); this restriction is based on the arbitrary order of firms used in the sequential version of the deferred-acceptance algorithm. In contrast, our extension applied to the one-to-one matching model requires the following: (1) Compute μ_W, as well as μ_F. (2) Break only the marriage of matched pairs (f,w) at μ_F such that $w\in\mu_F(f)\backslash\mu_W(f)$. (3) To check the stability of $\mu_F^{(f,w)}$, it is not enough, in our many-to-many setting, to check that $\mu_F^{(f,w)}(w)P(w)\mu_F(w)$ holds; instead, we have to insure that the stronger condition $\mathrm{Ch}(\mu_F^{(f,w)}(w)\cup\mu_F(w), P(w))=\mu_F^{(f,w)}(w)$ is satisfied (observe that both conditions coincide in the one-to-one case). (4) Avoid multiple identifications of the same stable matching (Step 4 of the algorithm) by using an arbitrary order on the set of profiles that have successfully passed the choice property test (Step 3 of the algorithm). Finally, in contrast to McVitie and Wilson's (1971) algorithm, the one of Irving and Leather (1986) does not obtain all stable matchings by successive application of the deferred-acceptance algorithm (to truncated preferences); instead, a stable matching is obtained after breaking a marriage and satisfying a subset of identified agents that form a cycle. The difficulty of extending their algorithm from the one-to-one case to the many-to-many one is that there is no an unambiguous (and useful) extension of the cycle generated by breaking a particular marriage.

4. The proof of the theorem

Let P be a substitutable preference profile and let μ_F and μ_W be its corresponding optimal stable matchings. Given an (f,w)-truncation of P where $w\in\mu_F(f)\mu_W(f)$, denote by $S^{(f,w)}(P)$ the set of stable matchings (with respect to the truncated profile $P^{(f,w)}$) that satisfy the Choice Property for w relative to P; namely,

$$S^{(f,w)}(P) = \{\mu \in S(P^{(f,w)}) \mid \mathrm{Ch}(\mu_{\mathrm{F}}(w) \cup \mu(w), P(w)) = \mu(w)\}. \tag{4}$$

Lemma 2 below says that $S^{(f,w)}(P)$ is a subset of $S(P)$. Hence, the Choice Property for w relative to P is sufficient to guarantee stability of a matching, which is stable with respect to a truncation.

Lemma 2. *For $w \in \mu_F(f) \backslash \mu_W(f)$, let μ be a matching such that $\mu \in S^{(f,w)}(P)$. Then $\mu \in S(P)$.*

Proof. *By $\mu \in S^{(f,w)}(P)$, we have $w \notin \mu(f)$. Thus, μ is individually rational for P. Suppose $(\tilde{w}, \tilde{f})$ blocks μ under P; namely,*

$$\tilde{f} \notin \mu(\tilde{w}), \tag{5}$$

$$\tilde{w} \in \mathrm{Ch}(\mu(\tilde{f}) \cup \{\tilde{w}\}, P(\tilde{f})), \text{ and} \tag{6}$$

$$\tilde{f} \in \mathrm{Ch}(\mu(\tilde{w}) \cup \{\tilde{f}\}, P(\tilde{w})). \tag{7}$$

If $\tilde{f} \neq f$ then the pair $(\tilde{w}, \tilde{f})$ also blocks μ under $P^{(f,w)}$, a contradiction. Thus, $\tilde{f} = f$. Then by conditions (6) and (7)

$$\tilde{w} \in \mathrm{Ch}(\mu(f) \cup \{\tilde{w}\}, P(f)) \text{ and} \tag{8}$$

$$f \in \mathrm{Ch}(\mu(\tilde{w}) \cup \{f\}, P(\tilde{w})). \tag{9}$$

Since $\mu \in S^{(f,w)}(P)$, then $\mu \in S(P^{(f,w)})$; hence,

$$\tilde{w} \notin \mathrm{Ch}(\mu(f) \cup \{\tilde{w}\}, P^{(f,w)}(f)), \tag{10}$$

otherwise $(\tilde{w}, f)$ is a blocking pair for μ under $P^{(f,w)}$. The definition of $P^{(f,w)}(f)$ and conditions (8) and (10) imply

$$w \in \mu(f) \cup \{\tilde{w}\}.$$

But, by the definition of $P^{(f,w)}(f)$, $w \notin \mu(f)$. Thus, $\tilde{w} = w$. Now, we can rewrite conditions (8) and (9) as

$$w \in \mathrm{Ch}(\mu(f) \cup \{w\}, P(f)) \text{ and}$$

$$f \in \mathrm{Ch}(\mu(w) \cup \{f\}, P(w)). \tag{11}$$

Since $\mu \in S^{(f,w)}(P)$, it follows by definition that

$$\mathrm{Ch}(\mu_{\mathrm{F}}(w) \cup \mu(w), P(w)) = \mu(w). \tag{12}$$

By condition (5), and the fact that $\tilde{f}=f$ *and* $\tilde{w}=w$, $f\notin\mu(w)$ *which, together with conditions (11) and (12) imply that*

$$\mathrm{Ch}(\mu(w)\cup\{f\},P(w))P(w)\mu(w)=\mathrm{Ch}(\mu_{\mathrm{F}}(w)\cup\mu(w),P(w)). \tag{13}$$

Therefore, since $f\in\mu_F(w)$, $\{f\}\cup\mu(w)\subseteq\mu_F(w)\cup\mu(w)$. *Hence,*

$$\mathrm{Ch}(\mu_{\mathrm{F}}(w)\cup\mu(w),P(w))R(w)\mathrm{Ch}(\mu(w)\cup\{f\},P(w)).$$

But this contradicts condition (13). □

The next lemma establishes two useful properties of the choice set.

Lemma 3. *For all subsets of partners A, B, and C of agent* $a\in F\cup W$:

(a) $Ch(A\cup B, P(a))=Ch(Ch(A, P(a))\cup B, P(a))$.
(b) $Ch(A\cup B, P(a))=A$ *and* $Ch(B\cup C, P(a))=B$ *imply* $Ch(A\cup C, P(a))=A$.

Proof. Property (a) follows from Proposition 2.3 in Blair (1988). To prove (b), consider the following equalities:

$$\begin{aligned}\mathrm{Ch}(A\cup C,P(a)) &= \mathrm{Ch}(\mathrm{Ch}(A\cup B,P(a))\cup C,P(a)) \text{ by hypothesis}\\ &= \mathrm{Ch}(A\cup B\cup C,P(a)) \text{ by } (a)\\ &= \mathrm{Ch}(A\cup \mathrm{Ch}(B\cup C,P(a)),P(a)) \text{ by } (a)\\ &= \mathrm{Ch}(A\cup B,P(a)) \text{ by hypothesis}\\ &= A \text{ by hypothesis.}\end{aligned}$$ □

Lemma 4 below can be understood as a strengthening of Lemma 2. It says that checking the Choice Property (for w relative to P) for *only* the firm-optimal stable matching is sufficient to guarantee that all stable matchings relative to the truncated profile are indeed stable for the original profile.

Lemma 4. *Let* $P^{(f,w)}$ *be a truncation such that*

$$\mathrm{Ch}(\mu_{\mathrm{F}}(w)\cup\mu_{\mathrm{F}}^{(f,w)}(w),P(w))=\mu_{\mathrm{F}}^{(f,w)}(w)$$

holds. Then, $\mu\in S(P^{(f,w)})$ *implies* $\mu\in S(P)$.

Proof. *Let* μ *be a matching such that* $\mu\in S(P^{(f,w)})$. *By Lemma 1 and the Choice Property for w relative to* $P^{(f,w)}$,

$$\mathrm{Ch}(\mu(w)\cup\mu_{\mathrm{F}}^{(f,w)}(w),P^{(f,w)}(w))=\mu(w).$$

However, preferences $P^{(f,w)}(w)$ *and* $P(w)$ *coincide. Therefore,*

$$\mathrm{Ch}(\mu(w)\cup\mu_{\mathrm{F}}^{(f,w)}(w),P(w))=\mu(w) \tag{14}$$

also holds. By hypothesis,

$$\mathrm{Ch}(\mu_{\mathrm{F}}(w) \cup \mu_{\mathrm{F}}^{(f,w)}(w), P(w)) = \mu_{\mathrm{F}}^{(f,w)}(w). \tag{15}$$

By Lemma 3, we have that conditions (14) and (15) imply

$$\mathrm{Ch}(\mu_{\mathrm{F}}(w) \cup \mu(w), P(w)) = \mu(w).$$

Hence, by definition, $\mu \in S^{(f,w)}(P)$ and by Lemma 2, $\mu \in S(P)$. □

Lemma 5 says that, for a given stable matching, adding the individual rationality condition relative to a truncation ensures that the matching is stable relative to the truncated profile. This will immediately imply Corollary 1, which will be crucial to the justification of Step 4 in the algorithm.

Lemma 5. *Let μ be a matching such that $\mu \in S(P) \cap IR(P^{(f,w)})$. Then $\mu \in S(P^{(f,w)})$.*

Proof. Assume $\mu \notin S(P^{(f,w)})$. Since $\mu \in IR(P^{(f,w)})$, there must exist a blocking pair $(\tilde{w}, \tilde{f})$ of μ, namely, $\tilde{w} \notin \mu(\tilde{f})$,

$$\tilde{w} \in \mathrm{Ch}(\mu(\tilde{f}) \cup \{\tilde{w}\}, P^{(f,w)}(\tilde{f})), \text{ and} \tag{16}$$

$$\tilde{f} \in \mathrm{Ch}(\mu(\tilde{w}) \cup \{\tilde{f}\}, P^{(f,w)}(\tilde{w})). \tag{17}$$

Consider the following two cases:

1. $\tilde{f} \neq f$. Since $P^{(f,w)}(\tilde{w}) = P(\tilde{w})$ and $P^{(f,w)}(\tilde{f}) = P(\tilde{f})$ the pair $(\tilde{w}, \tilde{f})$ also blocks the matching μ in the preference profile P. Hence, $\mu \notin S(P)$.
2. $\tilde{f} = f$. Then by conditions (16) and (17)

$$\tilde{w} \in \mathrm{Ch}(\mu(f) \cup \{\tilde{w}\}, P^{(f,w)}(f)), \tag{18}$$

and hence $\tilde{w} \neq w$, and

$$f \in \mathrm{Ch}(\mu(\tilde{w}) \cup \{f\}, P(\tilde{w})).$$

The hypothesis that $\mu \in IR(P^{(f,w)})$ implies that $\mu(f) = \mathrm{Ch}(\mu(f), P^{(f,w)}(f))$. Thus, $w \notin \mu(f)$. Consequently, condition (18) can be rewritten as $\tilde{w} \in \mathrm{Ch}(\mu(f) \cup \{\tilde{w}\}, P(f))$, implying that the pair $(\tilde{w}, f)$ blocks μ in the preference profile P. Hence, $\mu \notin S(P)$. □

As we have just said, Corollary 1 below justifies the insertion of Step 4 at each stage of the algorithm. If we have two truncations $P^{(f,w)}$ and $P^{(f',w')}$ with the properties that (1) their corresponding firm-optimal stable matchings $\mu_{\mathrm{F}}^{(f,w)}$ and $\mu_{\mathrm{F}}^{(f',w')}$ satisfy the Choice Property for w and w', respectively (that is, they are stable relative to the original profile), and (2) the matching $\mu_{\mathrm{F}}^{(f',w')}$ is individually rational relative to $P^{(f,w)}$ (that is, $w \notin \mu_{\mathrm{F}}^{(f',w')}(f)$), then we need not add $\mu_{\mathrm{F}}^{(f',w')}$ at this stage (with the subsequent computational savings) because we will find it later on (and add it to the provisional set of stable matchings) as a firm-optimal stable matching of a subsequent truncation of $P^{(f,w)}$.

Corollary 1. *Let $P^{(f,w)}$, $P^{(f',w')}$ be two truncations such that $\mu_F^{(f',w')}\in S(P)$. If $w\notin\mu_F^{(f',w')}(f)$, then $\mu_F^{(f',w')}\in S(P^{(f,w)})$.*

Proof. *The hypothesis $\mu_F^{(f',w')}\in S(P)$ implies that $\mu_F^{(f',w')}\in IR(P)$. Since $w\notin\mu_F^{(f',w')}(f)$, it follows that $\mu_F^{(f',w')}\in IR(P^{(f,w)})$. Hence, by Lemma 5, $\mu_F^{(f',w')}\in S(P^{(f,w)})$.* □

The next lemma establishes a useful fact about the set of stable matchings: a worker who is matched to the same firm in the two optimal stable matchings has also to be matched to the same firm in all stable matchings.

Lemma 6. *Assume $w\in\mu_F(f)\cap\mu_W(f)$. Then, $w\in\mu(f)$ for all $\mu\in S(P)$.*

Proof. Suppose there exist w, f, and $\mu\in S(P)$ such that $w\in\mu_F(f)\cap\mu_W(f)$ and $w\notin\mu(f)$. By Remark 1,

$$\mathrm{Ch}(\mu_F(f)\cup\mu(f), P(f)) = \mu_F(f)$$

and

$$\mathrm{Ch}(\mu_W(w)\cup\mu(w), P(w)) = \mu_W(w).$$

Since $w\in\mu_F(f)$, this implies $w\in\mathrm{Ch}(\mu_F(f)\cup\mu(f), P(f))$; hence, $w\in\mathrm{Ch}(\{w\}\cup\mu(f), P(f))$ since P is a substitutable preference profile. Now, $w\in\mu_W(f)$ implies $f\in\mu_W(w)$. It follows that $f\in\mathrm{Ch}(\mu_W(w)\cup\mu(w), P(w))$, which means that $f\in\mathrm{Ch}(\{f\}\cup\mu(w), P(w))$. Since $w\notin\mu(f)$, this implies that (w, f) is a blocking pair for μ which contradicts $\mu\in S(P)$. Thus, $w\in\mu(f)$. □

Lemma 7 and its Corollary 2 guarantee that any non-optimal stable matching μ will eventually be identified and selected as the firm-optimal stable matching corresponding to a preference profile which will be obtained after truncating the preferences of a sequence of firms.

Lemma 7. *Let $\mu\in S(P)$ be such that $\mu_F\neq\mu$ Then there exists $P^{(f,w)}$ with $w\in\mu_F(f)\backslash\mu_W(f)$ and $w\notin\mu(f)$ such that $\mu\in S(P^{(f,w)})$.*

Proof. Since $\mu_F\neq\mu$, there exist w and f such that $w\in\mu_F(f)\backslash\mu(f)$. If this were not so, then $\mu_F(f)\subseteq\mu(f)$ for all f. By Property 1 of Remark 1, and since $\mu\in IR(P)$, then

$$\mu_F(f) = \mathrm{Ch}(\mu_F(f)\cup\mu(f), P(f)) = \mathrm{Ch}(\mu(f), P(f)) = \mu(f) \text{ for all } f.$$

Thus, $\mu_F=\mu$ which is a contradiction. Since $w\notin\mu(f)$, it follows from Lemma 6 that $w\notin\mu_W(f)$. Consider the preference profile $P^{(f,w)}$. Because $w\notin\mu(f)$, we have that $\mu\in IR(P^{(f,w)})$, since $\mu\in S(P)$ and $P^{(f,w)}(a)=P(a)$ for all $a\neq f$. By Lemma 5, $\mu\in S(P^{(f,w)})$. □

Remark 2. Let $P^{(f,w)}$ be a preference profile such that its corresponding $\mu_F^{(f,w)}$ satisfies the Choice Property for w relative to P. By Lemma 4, $S(P)\geq S(P^{(f,w)})$.[11] Then $w\in\mu_F(f)\backslash\mu_W(f)$ implies $\mu_F\notin S(P^{(f,w)})$, and $S(P)>S(P^{(f,w)})$.

Corollary 2. *Let $\mu\in S(P)$ be such that $\mu_F\neq\mu$ Then there exists a sequence of pairs $(f_{i_1}, w_{j_1})\ldots(f_{i_k}, w_{j_k})$ such that $\mu_W = \mu_W^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})}\in S(P^{(f_{i_1},w_{j_1})\ldots(f_{i_k},w_{j_k})})$.*

[11] The notation $|S(P)|$ means the number of stable matchings under preference profile P.

Proof. Let $\mu \in S(P)$ be such that $\mu_F \neq \mu$. By Lemma 7, there exists $P^{(f,w)}$ such that $\mu \in S(P^{(f,w)})$. If $\mu = \mu_F^{(f,w)}$, the statement follows. Otherwise (in which case, by Remark 2 we have that $S(P) > S(P^{(f,w)})$), we apply again Lemma 7 replacing the roles of P and μ_F by $P^{(f,w)}$ and $\mu_F^{(f,w)}$, respectively. Since $S(P) < \infty$, the statement of Corollary 2 follows. □

Now, we are ready to show that the outcome of the algorithm is the set of stable matchings.

Proof of Theorem 1. First, from Lemma 4, we have $\boldsymbol{S}^1(P) \subseteq S(P)$. Applying iteratively Lemma 4 to successive stages, we obtain

$$S^K(P) \subseteq S(P).$$

Second, assume that $\mu \in S(P)$. By Corollary 2, there exists $k \leq K$ such that $\mu \in S^k(P)$. Therefore,

$$S(P) \subseteq S^K(P). \quad \square$$

5. Concluding remark

Our contribution is three-fold. First, we come to understand that the firm-optimal stable matching of truncated preference profiles might be stable in the original profile. Second, we discover that the Choice Property for w relative to P is the only thing that has to be checked to guarantee the stability of this matching (Lemmas 2, 3, and 4). Third, and more importantly, we show that all stable matchings are identified in this way (Lemmas 5, 6, and 7).

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Appendix A

To illustrate the deferred-acceptance algorithm in which firms make offers, we use the preference profile $P^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}$ of Example 2 to compute $\mu_F^{(f_4,w_4)(f_4,w_3)(f_3,w_3)}$; that is, $F=\{f_1, f_2, f_3, f_4\}$ and $W=\{w_1, w_2, w_3, w_4\}$ are the two sets of agents with the following substitutable profile of preferences

$$P(f_1) = w_1w_2, w_1w_3, w_2w_4, w_3w_4, w_1w_4, w_2w_3, w_1, w_2, w_3, w_4$$

$$P(f_2) = w_1w_2, w_2w_3, w_1w_4, w_3w_4, w_1w_3, w_2w_4, w_1, w_2, w_3, w_4$$

$$P(f_3) = w_1w_4, w_1w_2, w_2w_4, w_1, w_2, w_4$$

$$P(f_4) = w_1w_2, w_1, w_2$$

$$P(w_1) = f_3f_4, f_2f_3, f_2f_4, f_1f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4$$

$$P(w_2) = f_3f_4, f_2f_3, f_1f_4, f_2f_4, f_1f_3, f_1f_2, f_1, f_2, f_3, f_4$$

$$P(w_3) = f_1f_2, f_2f_3, f_1f_3, f_2f_4, f_1f_4, f_3f_4, f_1, f_2, f_3, f_4$$

$$P(w_4) = f_1f_2, f_1f_3, f_1f_4, f_2f_3, f_2f_4, f_3f_4, f_1, f_2, f_3, f_4.$$

The offers made by firms, and received and accepted by workers, in Step 1 are:

$$\begin{array}{cccccccc} f_1 & f_2 & f_3 & f_4 & w_1 & w_2 & w_3 & w_4 \\ w_1w_2 & w_1w_2 & w_1w_4 & w_1w_2 & f_1f_2f_3f_4 & f_1f_2f_4 & \emptyset & f_3 \\ & & & & f_3f_4 & f_1f_4 & \emptyset & f_3. \end{array}$$

The provisional matching μ^1 after Step 1 is:

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2 & \emptyset & w_1w_4 & w_1w_2 \end{pmatrix}.$$

The offers made by firms, and received and accepted by workers, in Step 2 are:

$$\begin{array}{cccccccc} f_1 & f_2 & f_3 & f_4 & w_1 & w_2 & w_3 & w_4 \\ w_2w_4 & w_3w_4 & w_1w_4 & w_1w_2 & f_3f_4 & f_1f_4 & f_2 & f_1f_2f_3 \\ & & & & f_3f_4 & f_1f_4 & f_2 & f_1f_2. \end{array}$$

The provisional matching μ^2 after Step 2 is:

$$\mu^2 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2w_4 & w_3w_4 & w_1 & w_1w_2 \end{pmatrix}.$$

The offers made by firms, and received and accepted by workers, in Step 3 are:

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_2w_4	w_3w_4	w_1w_2	w_1w_2	f_3f_4	$f_1f_3f_4$	f_2	f_1f_2
				f_3f_4	f_3f_4	f_2	f_1f_2.

The provisional matching μ^3 after Step 3 is:

$$\mu^3 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_4 & w_3w_4 & w_1w_2 & w_1w_2 \end{pmatrix}.$$

The offers made by firms, and received and accepted by workers, in Step 4 are:

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_3w_4	w_3w_4	w_1w_2	w_1w_2	f_3f_4	f_3f_4	f_1f_2	f_1f_2
				f_3f_4	f_3f_4	f_1f_2	f_1f_2.

The provisional matching μ^4 after Step 4 is:

$$\mu^4 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3w_4 & w_3w_4 & w_1w_2 & w_1w_2 \end{pmatrix}.$$

The algorithm stops after Step 4 because all offers have been accepted. The provisional matching μ^4 becomes definite, and it is the firm-optimal stable matching.

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