Matching

Marriage: Paths to Stability

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October 2010
Knuth (1976) asked the following question:
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Starting with an individually rational matching, does there always exist a path of blocking pairs leading to a stable matching?
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Starting with an individually rational matching, does there always exist a path of blocking pairs leading to a stable matching?

Start with an arbitrary (individually rational) matching $\mu$. If it is stable: stop. If it is blocked by a pair $(w, m)$ a new matching $\mu'$ is obtained from $\mu$ by satisfying the blocking pair: $m$ and $w$ are matched to one another at $\mu'$ ($\mu'(m) = w$), their mates (if any) are unmatched ($\mu'(\mu(w)) = \mu(w)$ if $\mu(w) \neq w$ and $\mu'(\mu(m)) = \mu(m)$ if $\mu(m) \neq m$), and $\mu'(x) = \mu(x)$ for all $x \in W \cup M \setminus \{m, w, \mu(w), \mu(m)\}$. 
### Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

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<thead>
<tr>
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<th>$P_{w_1}$</th>
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<td>$P_{w_1}$</td>
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Example

Let \( W = \{ w_1, w_2, w_3 \} \), \( M = \{ m_1, m_2, m_3 \} \), and \( P \) be

\[
\begin{align*}
P_{w_1} & : m_1, m_3, m_2 & P_{m_1} & : w_2, w_1, w_3 \\
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P_{w_3} & : m_1, m_3, m_2 & P_{m_3} & : w_1, w_2, w_3.
\end{align*}
\]

Observe that all possible matchings are individually rational.
Example

Let \( W = \{ w_1, w_2, w_3 \} \), \( M = \{ m_1, m_2, m_3 \} \), and \( P \) be

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Observe that all possible matchings are individually rational.

Consider matching

\[
\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.
\]
3.10.- Paths to Stability: Knuth’s Example

Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$P_{w_1} : m_1, m_3, m_2$ \quad $P_{m_1} : w_2, w_1, w_3$

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$P_{w_3} : m_1, m_3, m_2$ \quad $P_{m_3} : w_1, w_2, w_3$.

Observe that all possible matchings are individually rational. Consider matching

$\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$.

Matching $\mu_1$ is unstable, since $(w_2, m_1)$ blocks it at $P$. 

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Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

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P_{w_3} & : m_1, m_3, m_2 & P_{m_3} & : w_1, w_2, w_3.
\end{align*}
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Observe that all possible matchings are individually rational. Consider matching

\[
\mu_1 = \begin{pmatrix}
w_1 & w_2 & w_3 \\
m_1 & m_2 & m_3
\end{pmatrix}.
\]

Matching $\mu_1$ is unstable, since $(w_2, m_1)$ blocks it at $P$. Obtain $\mu_2$ by satisfying $(w_2, m_1)$. 
Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

\[
P_{w_1} : m_1, m_3, m_2 \quad P_{m_1} : w_2, w_1, w_3 \\
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P_{w_1} : & \quad m_1, m_3, m_2 \\
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P_{m_1} : & \quad w_2, w_1, w_3 \\
P_{m_2} : & \quad w_1, w_3, w_2 \\
P_{m_3} : & \quad w_1, w_2, w_3.
\end{align*}
\]

Blocking pair \((w_2, m_1)\) of \(\mu_1\)

\[
\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ w_1 & m_1 & m_3 & m_2 \end{pmatrix}.
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3.10.- Paths to Stability: Knuth’s Example

**Example**

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\begin{align*}
P_{w_1} & : m_1, m_3, m_2 & P_{m_1} & : w_2, w_1, w_3 \\
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\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ w_1 & m_1 & m_3 & m_2 \end{pmatrix}.
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Matching \(\mu_2\) is unstable, since \((w_1, m_2)\) blocks it at \(P\).
Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

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P_{w_1} : m_1, m_3, m_2 \quad P_{m_1} : w_2, w_1, w_3 \\
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Blocking pair $(w_2, m_1)$ of $\mu_1$

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\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ w_1 & m_1 & m_3 & m_2 \end{pmatrix}.
\]

Matching $\mu_2$ is unstable, since $(w_1, m_2)$ blocks it at $P$.

Obtain $\mu_3 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$ by satisfying $(w_1, m_2)$.
Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$P_{w_1} : m_1, m_3, m_2$ \hspace{1cm} $P_{m_1} : w_2, w_1, w_3$

$P_{w_2} : m_3, m_1, m_2$ \hspace{1cm} $P_{m_2} : w_1, w_3, w_2$

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Blocking pair $(w_2, m_1)$ of $\mu_1$

$$\mu_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ w_1 & m_1 & m_3 & m_2 \end{pmatrix}.$$  

Matching $\mu_2$ is unstable, since $(w_1, m_2)$ blocks it at $P$.

Obtain $\mu_3 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$ by satisfying $(w_1, m_2)$.

Matching $\mu_3$ is unstable, since $(w_2, m_3)$ blocks it at $P$. 
Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

\begin{align*}
P_{w_1} & : m_1, m_3, m_2 & P_{m_1} & : w_2, w_1, w_3 \\
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\end{align*}
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Let \( W = \{w_1, w_2, w_3\} \), \( M = \{m_1, m_2, m_3\} \), and \( P \) be

| \( P_{w_1} \) | \( m_1, m_3, m_2 \) | \( P_{m_1} \) | \( w_2, w_1, w_3 \) |
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Blocking pair \((w_2, m_3)\) of \( \mu_3 \)

\[
\mu_3 = \begin{pmatrix}
w_1 & w_2 & w_3 \\
m_2 & m_1 & m_3
\end{pmatrix}
\]

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\mu_4 = \begin{pmatrix}
w_1 & w_2 & w_3 & m_1 \\
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\end{pmatrix}.
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Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

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P_{w_1} : m_1, m_3, m_2 \quad P_{m_1} : w_2, w_1, w_3 \\
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\]

Blocking pair $(w_2, m_3)$ of $\mu_3$

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\mu_3 = \begin{pmatrix}
w_1 & w_2 & w_3 \\
m_2 & m_1 & m_3
\end{pmatrix} \quad \mu_4 = \begin{pmatrix}
w_1 & w_2 & w_3 & m_1 \\
m_2 & m_3 & w_3 & m_1
\end{pmatrix}.
\]

Matching $\mu_4$ is unstable, since $(w_3, m_1)$ blocks it at $P$. 
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Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$P_{w_1} : m_1, m_3, m_2$ \hspace{1cm} $P_{m_1} : w_2, w_1, w_3$
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Blocking pair $(w_2, m_3)$ of $\mu_3$

$\mu_3 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$ \hspace{1cm} $\mu_4 = \begin{pmatrix} w_1 & w_2 & w_3 & m_1 \\ m_2 & m_3 & w_3 & m_1 \end{pmatrix}$.

Matching $\mu_4$ is unstable, since $(w_3, m_1)$ blocks it at $P$.

Obtain $\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_3 & m_1 \end{pmatrix}$ by satisfying $(w_3, m_1)$.
Example

Let \( W = \{w_1, w_2, w_3\} \), \( M = \{m_1, m_2, m_3\} \), and \( P \) be

\[
\begin{align*}
P_{w_1} & : m_1, m_3, m_2 & P_{m_1} & : w_2, w_1, w_3 \\
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P_{w_3} & : m_1, m_3, m_2 & P_{m_3} & : w_1, w_2, w_3.
\end{align*}
\]

Blocking pair \((w_2, m_3)\) of \(\mu_3\)

\[
\mu_3 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}
\]

\[
\mu_4 = \begin{pmatrix} w_1 & w_2 & w_3 & m_1 \\ m_2 & m_3 & w_3 & m_1 \end{pmatrix}.
\]

Matching \(\mu_4\) is unstable, since \((w_3, m_1)\) blocks it at \(P\).

Obtain \(\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_3 & m_1 \end{pmatrix}\) by satisfying \((w_3, m_1)\).

Matching \(\mu_5\) is unstable, since \((w_1, m_3)\) blocks it at \(P\).
Example

Let \( W = \{ w_1, w_2, w_3 \} \), \( M = \{ m_1, m_2, m_3 \} \), and \( P \) be

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\begin{align*}
P_{w_1} & : m_1, m_3, m_2 \\
P_{w_2} & : m_3, m_1, m_2 \\
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3.10.- Paths to Stability: Knuth’s Example

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Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

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\]

Blocking pair $(w_1, m_3)$ of $\mu_5$

\[
\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_3 & m_1 \end{pmatrix} \quad \mu_6 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ m_3 & w_2 & m_1 & m_2 \end{pmatrix}.
\]
Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$$P_{w_1} : m_1, m_3, m_2 \quad P_{m_1} : w_2, w_1, w_3$$
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Blocking pair $(w_1, m_3)$ of $\mu_5$

$$\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_3 & m_1 \end{pmatrix} \quad \mu_6 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ m_3 & w_2 & m_1 & m_2 \end{pmatrix}.$$

Matching $\mu_6$ is unstable, since $(w_2, m_2)$ blocks it at $P$. 
Example

Let $W = \{ w_1, w_2, w_3 \}$, $M = \{ m_1, m_2, m_3 \}$, and $P$ be

\[
P_{w_1} : m_1, m_3, m_2 \\
P_{w_2} : m_3, m_1, m_2 \\
P_{w_3} : m_1, m_3, m_2 \\
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Blocking pair $(w_1, m_3)$ of $\mu_5$

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\]

Matching $\mu_6$ is unstable, since $(w_2, m_2)$ blocks it at $P$.

Obtain $\mu_7 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_3 & m_2 & m_1 \end{pmatrix}$ by satisfying $(w_2, m_2)$. 
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Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$$
P_{w_1} : m_1, m_3, m_2 \quad P_{m_1} : w_2, w_1, w_3
$$

$$
P_{w_2} : m_3, m_1, m_2 \quad P_{m_2} : w_1, w_3, w_2
$$

$$
P_{w_3} : m_1, m_3, m_2 \quad P_{m_3} : w_1, w_2, w_3.
$$

Blocking pair $(w_1, m_3)$ of $\mu_5$

$$
\mu_5 = \begin{pmatrix}
w_1 & w_2 & w_3 \\
m_2 & m_3 & m_1
\end{pmatrix}
$$

$$
\mu_6 = \begin{pmatrix}
w_1 & w_2 & w_3 & m_2 \\
m_3 & w_2 & m_1 & m_2
\end{pmatrix}
$$

Matching $\mu_6$ is unstable, since $(w_2, m_2)$ blocks it at $P$.

Obtain $\mu_7 = \begin{pmatrix}
w_1 & w_2 & w_3 \\
m_3 & m_2 & m_1
\end{pmatrix}$ by satisfying $(w_2, m_2)$.

Matching $\mu_7$ is unstable, since $(w_1, m_1)$ blocks it at $P$. 
3.10.- Paths to Stability: Knuth’s Example

Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

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\begin{align*}
&P_{w_1} : m_1, m_3, m_2 \\
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&P_{w_3} : m_1, m_3, m_2 \\
&P_{m_1} : w_2, w_1, w_3 \\
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P_{w_3} : m_1, m_3, m_2 \quad P_{m_3} : w_1, w_2, w_3.
\]

Blocking pair \((w_1, m_1)\) of \(\mu_7\)

\[
\mu_7 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_3 & m_2 & m_1 \end{pmatrix} \quad \mu_8 = \begin{pmatrix} w_1 & w_2 & w_3 & m_3 \\ m_1 & m_2 & w_3 & m_3 \end{pmatrix}.
\]
Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$$
\begin{align*}
  P_{w_1} : m_1, m_3, m_2 & & P_{m_1} : w_2, w_1, w_3 \\
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\end{align*}
$$

Blocking pair $(w_1, m_1)$ of $\mu_7$

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\mu_7 = \begin{pmatrix}
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  m_3 & m_2 & m_1
\end{pmatrix} 
$$

$\mu_8 = \begin{pmatrix}
  w_1 & w_2 & w_3 & m_3 \\
  m_1 & m_2 & w_3 & m_3
\end{pmatrix}$.

Matching $\mu_8$ is unstable, since $(w_3, m_3)$ blocks it at $P$. 

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Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

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$$\mu_7 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_3 & m_2 & m_1 \end{pmatrix} \quad \mu_8 = \begin{pmatrix} w_1 & w_2 & w_3 & m_3 \\ m_1 & m_2 & w_3 & m_3 \end{pmatrix}.$$ 

Matching $\mu_8$ is unstable, since $(w_3, m_3)$ blocks it at $P$.

Obtain $\mu_9 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ by satisfying $(w_3, m_3)$. 

3.10.- Paths to Stability: Knuth’s Example

Example

Let \( W = \{ w_1, w_2, w_3 \} \), \( M = \{ m_1, m_2, m_3 \} \), and \( P \) be

\[
\begin{align*}
P_{w_1} & : m_1, m_3, m_2 & P_{m_1} & : w_2, w_1, w_3 \\
P_{w_2} & : m_3, m_1, m_2 & P_{m_2} & : w_1, w_3, w_2 \\
P_{w_3} & : m_1, m_3, m_2 & P_{m_3} & : w_1, w_2, w_3.
\end{align*}
\]

Blocking pair \((w_1, m_1)\) of \(\mu_7\)

\[
\mu_7 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_3 & m_2 & m_1 \end{pmatrix} \quad \mu_8 = \begin{pmatrix} w_1 & w_2 & w_3 & m_3 \\ m_1 & m_2 & w_3 & m_3 \end{pmatrix}.
\]

Matching \(\mu_8\) is unstable, since \((w_3, m_3)\) blocks it at \(P\).

Obtain \(\mu_9 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\) by satisfying \((w_3, m_3)\).

Matching \(\mu_9\) is unstable and equal to \(\mu_1\).
Roth and Vande Vate (1990) answer Knuth’s question affirmatively.
Roth and Vande Vate (1990) answer Knuth’s question affirmatively.

Theorem (Roth and Vande Vate, 1990) Let \( \mu \) be an arbitrary unstable matching for the marriage market \((W, M, P)\). Then, there exists a finite sequence of matchings \( \mu_1, \ldots, \mu_K \) such that \( \mu = \mu_1 \), \( \mu_K \) is stable, and for each \( k = 1, \ldots, K \) there is a blocking pair \((w_k, m_k)\) for \( \mu_k \) at \( P \) such that \( \mu_k + 1 \) is obtained from \( \mu_k \) by satisfying the blocking pair \((w_k, m_k)\).

Corollary A random process that begins from an arbitrary matching and continues by satisfying a randomly selected blocking pair must eventually converge with probability one to a stable matching, provided that each blocking pair has a probability of being selected that is bounded away from zero.
3.11.- Roth and Vande Vate Theorem

- Roth and Vande Vate (1990) answer Knuth’s question affirmatively.

**Theorem**

(Roth and Vande Vate, 1990) Let \( \mu \) be an arbitrary unstable matching for marriage market \((W, M, P)\). Then, there exists a finite sequence of matchings \( \mu_1, \ldots, \mu_K \) such that \( \mu = \mu_1, \mu_K \) is stable, and for each \( k = 1, \ldots, K - 1 \) there is a blocking pair \((w_k, m_k)\) for \( \mu_k \) at \( P \) such that \( \mu_{k+1} \) is obtained from \( \mu_k \) by satisfying the blocking pair \((w_k, m_k)\).
Roth and Vande Vate (1990) answer Knuth’s question affirmatively.

**Theorem**

(Roth and Vande Vate, 1990) Let $\mu$ be an arbitrary unstable matching for marriage market $(W, M, P)$. Then, there exists a finite sequence of matchings $\mu_1, ..., \mu_K$ such that $\mu = \mu_1$, $\mu_K$ is stable, and for each $k = 1, ..., K - 1$ there is a blocking pair $(w_k, m_k)$ for $\mu_k$ at $P$ such that $\mu_{k+1}$ is obtained from $\mu_k$ by satisfying the blocking pair $(w_k, m_k)$.

**Corollary**

A random process that begins from an arbitrary matching and continues by satisfying a randomly selected blocking pair must eventually converge with probability one to an stable matching, provided that each blocking pair has a probability of being selected that is bounded away from zero.
3.11.- Roth and Vande Vate Theorem

- References:


3.11.- Roth and Vande Vate Theorem

References:

3.11.- Roth and Vande Vate Theorem

References:

3.11.- The RVV Sequential Algorithm

**Sequential algorithm** (RVV($P, \mu$))

**Inputs:** A marriage market ($W, M, P$), an arbitrary matching $\mu$.

**Initialization:** Consider $\mu$. Select a subset $S$ of agents with the properties:

- $x \in S$ implies $\mu(x) \in S$,
- and there exists no blocking pair of $\mu$ at $P$ containing any agent in $S$ ($S$ could be just a pair of agents or a single agent).

Order the set of agents $W \backslash M$ as follows: $a_{j+1} \in S : \ldots : a_{n+p}$ in such a way that if $x, y / S$ and $\mu(x) = y$ then $a_j = x$ and $a_{j+k+1} = y$ for some $k > 1$.

**Step 1:** Set $A_1 = S$ and $\mu_1 = \mu$. If $j_{A_1} = n+p$, stop and set RVV($P, \mu$) = $\mu$.

**Step $k+1$:** Given $A_k$ and $\mu_k$. Agent $a_{j+S}$ enters the market. Set $A_{k+1} = \{ f_{a_{j+S}} \}$. Suppose $a_{j+S} = w_2 \in W$ and enter the stable room procedure (otherwise, replace $w$ by $m$ in the stable room procedure).
3.11.- The RVV Sequential Algorithm

**Sequential algorithm** \( (RVV(P, \mu)) \)

- **Inputs:** A marriage market \((W, M, P)\), an arbitrary matching \(\mu\).
3.11.- The RVV Sequential Algorithm

**Sequential algorithm** \((RVV(P, \mu))\)

- **Inputs:** A marriage market \((W, M, P)\), an arbitrary matching \(\mu\).
- **Initialization:** Consider \(\mu\). Select a subset \(S\) of agents with the properties:

\[x \in S \implies \mu(x) \in S,\] and there exists no blocking pair of \(\mu\) at \(P\) containing any agent in \(S\) (\(S\) could be just a pair of agents or a single agent).

Order the set of agents \(W\) as follows:

\[a_{j+S}j+1, \ldots, a_{n+p}\] in such a way that if \(x, y \not\in S\) and \(\mu(x) = y\), then \(a_{j+S}k = x\) and \(a_{j+S}k+1 = y\) for some \(k\).

**Step 1:** Set \(A_1 = S\) and \(\mu_1 = \mu\). If \(j\) \(A_1\) \(j = n+p\), Stop and set \(RVV(P, \mu) = \mu\). Otherwise, go to step 2.

**Step \(k+1\):** Given \(A_k\) and \(\mu_k\). Agent \(a_{j+S}k\) enters the market. Set \(A_{k+1} = A_k[f_{a_{j+S}k}g].\) Suppose \(a_{j+S}k = w_2\) \(W\) and enter the stable room procedure (otherwise, replace \(w\) by \(m\) in the stable room procedure).
The RVV Sequential Algorithm

**Sequential algorithm** \( (RVV(P, \mu)) \)

- **Inputs:** A marriage market \((W, M, P)\), an arbitrary matching \(\mu\).
- **Initialization:** Consider \(\mu\). Select a subset \(S\) of agents with the properties:
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3.11.- The RVV Sequential Algorithm

**Sequential algorithm** \( (RVV(P, \mu)) \)

- **Inputs:** A marriage market \((W, M, P)\), an arbitrary matching \(\mu\).
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...
3.11. The RVV Sequential Algorithm

**Sequential algorithm** ($RVV(P, \mu)$)

- **Inputs**: A marriage market $(W, M, P)$, an arbitrary matching $\mu$.
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  - there exists no blocking pair of $\mu$ at $P$ containing any agent in $S$ ($S$ could be just a pair of agents or a single agent).
- Order the set of agents $W \cup M \setminus S$ as follows: $a_{|S|+1}, \ldots, a_{n+p}$ in such a way that if $x, y \notin S$ and $\mu(x) = y$ then $a_{|S|+k} = x$ and $a_{|S|+k+1} = y$ for some $k \geq 1$. 

3.11.- The RVV Sequential Algorithm

**Sequential algorithm** \( (RVV(P, \mu)) \)

- **Inputs:** A marriage market \((W, M, P)\), an arbitrary matching \(\mu\).
- **Initialization:** Consider \(\mu\). Select a subset \(S\) of agents with the properties:
  - \(x \in S\) implies \(\mu(x) \in S\), and
  - there exists no blocking pair of \(\mu\) at \(P\) containing any agent in \(S\) (\(S\) could be just a pair of agents or a single agent).
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- **Step 1:** Set \(A_1 = S\) and \(\mu_1 = \mu\). If \(|A_1| = n + p\), Stop and set \(RVV(P, \mu) = \mu\). Otherwise, go to step 2.
3.11.- The RVV Sequential Algorithm

**Sequential algorithm** \((RVV(P, \mu))\)

- **Inputs:** A marriage market \((W, M, P)\), an arbitrary matching \(\mu\).
- **Initialization:** Consider \(\mu\). Select a subset \(S\) of agents with the properties:
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  - there exists no blocking pair of \(\mu\) at \(P\) containing any agent in \(S\) (\(S\) could be just a pair of agents or a single agent).
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- **Step 1:** Set \(A_1 = S\) and \(\mu_1 = \mu\). If \(|A_1| = n + p\), Stop and set \(RVV(P, \mu) = \mu\). Otherwise, go to step 2.
- **Step \(k+1\):** Given \(A_k\) and \(\mu_k\). Agent \(a_{|S|+k}\) enters the market. Set \(A_{k+1} := A_k \cup \{a_{|S|+k}\}\). Suppose \(a_{|S|+k} = w \in W\) and enter the stable room procedure (otherwise, replace \(w\) by \(m\) in the stable room procedure).
3.11.- The RVV Sequential Algorithm

**Stable room procedure:**

- **Case 1**: There exists no blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\). Stop if \(k + 1 = n + p\) (the number of agents) and define \(RVV(P, \mu) := \mu_k\). Otherwise, set \(\mu_{k+1} := \mu_k\) and go to step \(k + 2\).
3.11.- The RVV Sequential Algorithm

Stable room procedure:

- **Case 1**: There exists no blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\). Stop if \(k + 1 = n + p\) (the number of agents) and define \(RVV(P, \mu) := \mu_k\). Otherwise, set \(\mu_{k+1} := \mu_k\) and go to step \(k + 2\).

- **Case 2**: There exists a blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\).
3.11.- The RVV Sequential Algorithm

**Stable room procedure:**

- **Case 1:** There exists no blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\). Stop if \(k + 1 = n + p\) (the number of agents) and define \(RVV(P, \mu) := \mu_k\). Otherwise, set \(\mu_{k+1} := \mu_k\) and go to step \(k + 2\).

- **Case 2:** There exists a blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\).
3.11.- The RVV Sequential Algorithm

**Stable room procedure:**

- **Case 1:** There exists no blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\). Stop if \(k + 1 = n + p\) (the number of agents) and define \(RVV(P, \mu) := \mu_k\). Otherwise, set \(\mu_{k+1} := \mu_k\) and go to step \(k + 2\).

- **Case 2:** There exists a blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\). Choose the blocking pair \((m^*, w)\) of \(\mu_k\) at \(P\) with \(m^* \in A_{k+1}\) that \(w\) prefers most.
3.11.- The RVV Sequential Algorithm

Stable room procedure:

- Case 1: There exists no blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\). Stop if \(k + 1 = n + p\) (the number of agents) and define \(RVV(P, \mu) := \mu_k\). Otherwise, set \(\mu_{k+1} := \mu_k\) and go to step \(k + 2\).

- Case 2: There exists a blocking pair \((m, w)\) of \(\mu_k\) at \(P\) with \(m \in A_{k+1}\).

  Choose the blocking pair \((m^*, w)\) of \(\mu_k\) at \(P\) with \(m^* \in A_{k+1}\) that \(w\) prefers most.

  If \(\mu_k(m^*) = m^*\) then, define \(\mu_{k+1}\) by setting \(\mu_{k+1}(w) := m^*\) and (i) if \(\mu_k(w) = w\), for all \(x \in W \cup M \setminus \{w, m^*\}\), \(\mu_{k+1}(x) = \mu_k(x)\) and (ii) if \(\mu_k(w) = m''\), for all \(x \in W \cup M \setminus \{w, m^*, m''\}\), \(\mu_{k+1}(x) = \mu_k(x)\) and \(\mu_{k+1}(m'') = m''\). Stop if \(n + p = k + 1\) and define \(RVV(P, \mu) := \mu_{k+1}\). Otherwise, go to step \(k + 2\).
If $\mu_k(m^*) = w'$ then, define $\mu_{k+1}$ by setting $\mu_{k+1}(w) := m^*$, $\mu_{k+1}(w') := w'$, and (i) if $\mu_k(w) = w$, for all $x \in W \cup M \setminus \{w, m^*, w'\}$, $\mu_{k+1}(x) := \mu_k(x)$ and (ii) if $\mu_k(w) = m''$, for all $x \in W \cup M \setminus \{w, m^*, w', m''\}$, $\mu_{k+1}(x) := \mu_k(x)$ and $\mu_{k+1}(m'') = m''$. Set $w := w'$ and repeat the stable room procedure.
3.11.- The RVV Sequential Algorithm

If $\mu_k(m^*) = w'$ then, define $\mu_{k+1}$ by setting $\mu_{k+1}(w) := m^*$, $\mu_{k+1}(w') := w'$, and (i) if $\mu_k(w) = w$, for all $x \in W \cup M \setminus \{w, m^*, w'\}$, $\mu_{k+1}(x) := \mu_k(x)$ and (ii) if $\mu_k(w) = m''$, for all $x \in W \cup M \setminus \{w, m^*, w', m''\}$, $\mu_{k+1}(x) := \mu_k(x)$ and $\mu_{k+1}(m'') = m''$. Set $w := w'$ and repeat the stable room procedure.
If $\mu_k(m^*) = w'$ then, define $\mu_{k+1}$ by setting $\mu_{k+1}(w) := m^*$, $\mu_{k+1}(w') := w'$, and (i) if $\mu_k(w) = w$, for all $x \in W \cup M \setminus \{w, m^*, w'\}$, $\mu_{k+1}(x) := \mu_k(x)$ and (ii) if $\mu_k(w) = m''$, for all $x \in W \cup M \setminus \{w, m^*, w', m''\}$, $\mu_{k+1}(x) := \mu_k(x)$ and $\mu_{k+1}(m'') = m''$. Set $w := w'$ and repeat the stable room procedure.

**Remark** \( RVV(P, \mu) \in S(P) \).
3.11.- The RVV Sequential Algorithm

If \( \mu_k(m^*) = w' \) then, define \( \mu_{k+1} \) by setting

\[
\mu_{k+1}(w) := m^*,
\mu_{k+1}(w') := w',
\mu_{k+1}(x) := \mu_k(x)
\]

for all \( x \in W \cup M \setminus \{w, m^*, w'\} \), (i) if \( \mu_k(w) = w \), for all
\( x \in W \cup M \setminus \{w, m^*, w', m''\} \), \( \mu_{k+1}(x) := \mu_k(x) \)

and (ii) if \( \mu_k(w) = m'' \), for all \( x \in W \cup M \setminus \{w, m^*, w', m''\} \), \( \mu_{k+1}(x) := \mu_k(x) \)

and \( \mu_{k+1}(m'') = m'' \). Set \( w := w' \) and repeat the stable room procedure.

Remark \( \text{RVV}(P, \mu) \in S(P) \).

Remark The matching \( \text{RVV}(P, \mu) \) depends on the initial set of agents \( S \) and the remaining ordering of agents not in \( S \).
3.11.- The RVV Sequential Algorithm

Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$$P_{w_1} : m_1, m_3, m_2 \quad P_{m_1} : w_2, w_1, w_3$$
$$P_{w_2} : m_3, m_1, m_2 \quad P_{m_2} : w_1, w_3, w_2$$
$$P_{w_3} : m_1, m_3, m_2 \quad P_{m_3} : w_1, w_2, w_3.$$
3.11.- The RVV Sequential Algorithm

Example

Let \( W = \{w_1, w_2, w_3\} \), \( M = \{m_1, m_2, m_3\} \), and \( P \) be

\[
P_{w_1} : m_1, m_3, m_2 \quad P_{m_1} : w_2, w_1, w_3 \\
P_{w_2} : m_3, m_1, m_2 \quad P_{m_2} : w_1, w_3, w_2 \\
P_{w_3} : m_1, m_3, m_2 \quad P_{m_3} : w_1, w_2, w_3.
\]

Observe that all possible matchings are individually rational.
Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

$$
\begin{align*}
P_{w_1} & : m_1, m_3, m_2 & P_{m_1} & : w_2, w_1, w_3 \\
P_{w_2} & : m_3, m_1, m_2 & P_{m_2} & : w_1, w_3, w_2 \\
P_{w_3} & : m_1, m_3, m_2 & P_{m_3} & : w_1, w_2, w_3.
\end{align*}
$$

Observe that all possible matchings are individually rational.

Consider the unstable matching

$$
\mu = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix},
$$

$$
\end{pmatrix}
$$
3.11.- The RVV Sequential Algorithm

Example

Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$, and $P$ be

\[
\begin{align*}
P_{w_1} & : m_1, m_3, m_2 & P_{m_1} & : w_2, w_1, w_3 \\
P_{w_2} & : m_3, m_1, m_2 & P_{m_2} & : w_1, w_3, w_2 \\
P_{w_3} & : m_1, m_3, m_2 & P_{m_3} & : w_1, w_2, w_3.
\end{align*}
\]

Observe that all possible matchings are individually rational. Consider the unstable matching

\[
\mu = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix},
\]

the set $S = \{w_1, m_1\}$, and the ordering $w_2, m_2, w_3, m_3$. 
3.11. The RVV Sequential Algorithm

Example

Ordering $w_2, m_2, w_3, m_3$.

$P_{w_1} : m_1, m_3, m_2$ \quad $P_{m_1} : w_2, w_1, w_3$

$P_{w_2} : m_3, m_1, m_2$ \quad $P_{m_2} : w_1, w_3, w_2$

$P_{w_3} : m_1, m_3, m_2$ \quad $P_{m_3} : w_1, w_2, w_3$.

**Step 1:** Set $A_1 = \{ w_1, m_1 \}$ and

$$\mu_1 = \mu = \left( \begin{array}{c|c} w_1 & w_2 \\ \hline m_1 & w_3 \\ \hline m_2 & m_3 \end{array} \right).$$

Since $|A_1| = 2 < 6 = |W \cup M|$, go to Step 2.
Example

Ordering $w_2, m_2, w_3, m_3$.

$P_{w_1} : m_1, m_3, m_2, w_1$  \hspace{1cm}  $P_{m_1} : w_2, w_1, w_3, m_1$

$P_{w_2} : m_3, m_1, m_2, w_2$  \hspace{1cm}  $P_{m_2} : w_1, w_3, w_2, m_2$

$P_{w_3} : m_1, m_3, m_2, w_3$  \hspace{1cm}  $P_{m_3} : w_1, w_2, w_3, m_3$.

Step 2: ($w_2$ enters) Given $A_1 = \{w_1, m_1\}$ and

$$\mu_1 = \mu = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$
Example

Ordering \( w_2, m_2, w_3, m_3 \).

\[
\begin{align*}
P_{w_1} & : m_1, m_3, m_2, w_1 & P_{m_1} & : w_2, w_1, w_3, m_1 \\
P_{w_2} & : m_3, m_1, m_2, w_2 & P_{m_2} & : w_1, w_3, w_2, m_2 \\
P_{w_3} & : m_1, m_3, m_2, w_3 & P_{m_3} & : w_1, w_2, w_3, m_3.
\end{align*}
\]

Step 2: (\( w_2 \) enters) Given \( A_1 = \{ w_1, m_1 \} \) and

\[
\mu_1 = \mu = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.
\]

Set \( A_2 = \{ w_1, m_1, w_2 \} \). Since \( BP(A_2, \mu_1) = \{ (w_2, m_1) \} \), obtain \( \mu_2 \) by satisfying \((w_2, m_1)\):

\[
\mu_2 = \begin{pmatrix} w_1 & w_2 & m_2 & w_3 \\ w_1 & m_1 & m_2 & m_3 \end{pmatrix}.
\]

Set \( w' = w_1 \).
3.11.- The RVV Sequential Algorithm

Example

Ordering $w_2, m_2, w_3, m_3$.

$P_{w_1} : m_1, m_3, m_2, w_1$
$P_{w_2} : m_3, m_1, m_2, w_2$
$P_{w_3} : m_1, m_3, m_2, w_3$

$P_{m_1} : w_2, w_1, w_3, m_1$
$P_{m_2} : w_1, w_3, w_2, m_2$
$P_{m_3} : w_1, w_2, w_3, m_3$

Step 2: ($w_1$ acts)

$$\mu_2 = \begin{pmatrix} w_1 & w_2 \\ w_1 & m_1 \end{pmatrix} \parallel \begin{pmatrix} m_2 & w_3 \\ m_2 & m_3 \end{pmatrix}.$$  

Since $BP(A_2, \mu_2) = \emptyset$, go to step 3.
Example

Ordering $w_2$, $m_2$, $w_3$, $m_3$.

- $P_{w_1} : m_1, m_3, m_2, w_1$
- $P_{w_2} : m_3, m_1, m_2, w_2$
- $P_{w_3} : m_1, m_3, m_2, w_3$
- $P_{m_1} : w_2, w_1, w_3, m_1$
- $P_{m_2} : w_1, w_3, w_2, m_2$
- $P_{m_3} : w_1, w_2, w_3, m_3$

Step 3: ($m_2$ enters) Given $A_2 = \{w_1, m_1, w_2\}$ and

$$
\mu_2 = \begin{pmatrix}
w_1 & w_2 & m_2 & w_3 \\
w_1 & m_1 & m_2 & m_3
\end{pmatrix}.
$$
Example

Ordering $w_2, m_2, w_3, m_3$.

$$P_{w_1} : m_1, m_3, m_2, w_1 \quad P_{m_1} : w_2, w_1, w_3, m_1$$

$$P_{w_2} : m_3, m_1, m_2, w_2 \quad P_{m_2} : w_1, w_3, w_2, m_2$$

$$P_{w_3} : m_1, m_3, m_2, w_3 \quad P_{m_3} : w_1, w_2, w_3, m_3.$$  

**Step 3: (m_2 enters)** Given $A_2 = \{w_1, m_1, w_2\}$ and

$$\mu_2 = \begin{pmatrix} w_1 & w_2 \\ w_1 & m_1 & m_2 & w_3 \\ m_2 & m_3 \end{pmatrix}.$$  

Set $A_3 = \{w_1, m_1, w_2, m_2\}$. Since $BP(A_3, \mu_2, m_2) = \{(w_1, m_2)\}$ obtain $\mu_3$ by satisfying $(w_1, m_2)$:

$$\mu_3 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}.$$  

Since $BP(A_3, \mu_3) = \emptyset$, go to Step 4.
Example

Ordering $w_2, m_2, w_3, m_3$.

$P_{w_1}: m_1, m_3, m_2, w_1$  \hspace{1cm}  $P_{m_1}: w_2, w_1, w_3, m_1$

$P_{w_2}: m_3, m_1, m_2, w_2$  \hspace{1cm}  $P_{m_2}: w_1, w_3, w_2, m_2$

$P_{w_3}: m_1, m_3, m_2, w_3$  \hspace{1cm}  $P_{m_3}: w_1, w_2, w_3, m_3$. 

**Step 4:** ($w_3$ enters) Given $A_3 = \{w_1, m_1, w_2, m_2\}$ and

$$
\mu_3 = \begin{pmatrix}
  w_1 & w_2 & | & w_3 \\
  m_2 & m_1 & | & m_3
\end{pmatrix}.
$$
Example

Ordering \( w_2, m_2, w_3, m_3 \).

\[
\begin{align*}
P_{w_1} & : m_1, m_3, m_2, w_1 & P_{m_1} & : w_2, w_1, w_3, m_1 \\
P_{w_2} & : m_3, m_1, m_2, w_2 & P_{m_2} & : w_1, w_3, w_2, m_2 \\
P_{w_3} & : m_1, m_3, m_2, w_3 & P_{m_3} & : w_1, w_2, w_3, m_3.
\end{align*}
\]

Step 4: \( (w_3 \text{ enters}) \) Given \( A_3 = \{w_1, m_1, w_2, m_2\} \) and

\[
\mu_3 = \begin{pmatrix} w_1 & w_2 \\ m_2 & m_1 \\ w_3 & m_3 \end{pmatrix}.
\]

Set \( A_4 = \{w_1, m_1, w_2, m_2, w_3\} \). Since \( BP(A_4, \mu_3) = \emptyset \) set

\[
\mu_4 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \\ w_3 & m_3 \end{pmatrix}
\]

and go to Step 5.
Example

Ordering $w_2, m_2, w_3, m_3$.

$$P_{w_1} : m_1, m_3, m_2, w_1 \quad P_{m_1} : w_2, w_1, w_3, m_1$$
$$P_{w_2} : m_3, m_1, m_2, w_2 \quad P_{m_2} : w_1, w_3, w_2, m_2$$
$$P_{w_3} : m_1, m_3, m_2, w_3 \quad P_{m_3} : w_1, w_2, w_3, m_3.$$ 

**Step 5**: ($m_3$ enters) Given $A_4 = \{w_1, m_1, w_2, m_2, w_3\}$ and

$$\mu_4 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & w_3 \end{pmatrix} \parallel m_3.$$
Example

Ordering $w_2, m_2, w_3, m_3$.

$P_{w_1} : m_1, m_3, m_2, w_1$  \quad $P_{m_1} : w_2, w_1, w_3, m_1$

$P_{w_2} : m_3, m_1, m_2, w_2$  \quad $P_{m_2} : w_1, w_3, w_2, m_2$

$P_{w_3} : m_1, m_3, m_2, w_3$  \quad $P_{m_3} : w_1, w_2, w_3, m_3$.

**Step 5:** ($m_3$ enters) Given $A_4 = \{w_1, m_1, w_2, m_2, w_3\}$ and

$$\mu_4 = \begin{pmatrix} w_1 & w_2 & w_3 & m_3 \\ m_2 & m_1 & w_3 & m_3 \end{pmatrix}.$$  

Set $A_5 = \{w_1, m_1, w_2, m_2, w_3, m_3\}$.  

$BP(A_5, \mu_4, m_3) = \{(w_1, m_3), (w_2, m_3), (w_3, m_3)\}$. Since $w_1 P_{m_3} w_2 P_{m_3} w_3$, satisfy the blocking pair $(w_1, m_3)$:

$$\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ m_3 & m_1 & w_3 & m_2 \end{pmatrix}.$$  

Set $m' = m_2$.  

Example

Ordering \textit{w}_2, \textit{m}_2, \textit{w}_3, \textit{m}_3.

\begin{align*}
P_{w_1} : & \quad m_1, m_3, m_2, w_1 \\
P_{w_2} : & \quad m_3, m_1, m_2, w_2 \\
P_{w_3} : & \quad m_1, m_3, m_2, w_3
\end{align*}

\begin{align*}
P_{m_1} : & \quad w_2, w_1, w_3, m_1 \\
P_{m_2} : & \quad w_1, w_3, w_2, m_2 \\
P_{m_3} : & \quad w_1, w_2, w_3, m_3.
\end{align*}

**Step 5:** (\textit{m}_2 acts) Given \( A_5 = \{ w_1, m_1, w_2, m_2, w_3, m_3 \} \) and

\[
\mu_5 = \begin{pmatrix}
w_1 & w_2 & w_3 & m_2 \\
m_3 & m_1 & w_3 & m_2
\end{pmatrix}.
\]
Example

Ordering $w_2, m_2, w_3, m_3$.

$$P_{w_1} : m_1, m_3, m_2, w_1 \quad P_{m_1} : w_2, w_1, w_3, m_1$$

$$P_{w_2} : m_3, m_1, m_2, w_2 \quad P_{m_2} : w_1, w_3, w_2, m_2$$

$$P_{w_3} : m_1, m_3, m_2, w_3 \quad P_{m_3} : w_1, w_2, w_3, m_3.$$ 

Step 5: ($m_2$ acts) Given $A_5 = \{w_1, m_1, w_2, m_2, w_3, m_3\}$ and

$$\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 & m_2 \\ m_3 & m_1 & w_3 & m_2 \end{pmatrix}.$$ 

$$BP(A_5, \mu_5, m_2) = \{(w_3, m_2)\}.$$ Satisfy the blocking pair $(w_3, m_2)$:

$$\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_3 & m_1 & m_2 \end{pmatrix}.$$ 

Since $BP(A_5, \mu_5) = \emptyset$ go to Step 6.
3.11.- The RVV Sequential Algorithm

Example

Ordering $w_2, m_2, w_3, m_3$.

$$P_{w_1} : m_1, m_3, m_2, w_1$$
$$P_{m_1} : w_2, w_1, w_3, m_1$$

$$P_{w_2} : m_3, m_1, m_2, w_2$$
$$P_{m_2} : w_1, w_3, w_2, m_2$$

$$P_{w_3} : m_1, m_3, m_2, w_3$$
$$P_{m_3} : w_1, w_2, w_3, m_3$$.

**Step 6:** Given $A_6 = \{w_1, m_1, w_2, m_2, w_3, m_3\}$ and

$$\mu_5 = \begin{pmatrix}
w_1 & w_2 & w_3 \\
m_3 & m_1 & m_2
\end{pmatrix}.$$
Example

Ordering $w_2, m_2, w_3, m_3$.

$P_{w_1} : m_1, m_3, m_2, w_1$
$P_{w_2} : m_3, m_1, m_2, w_2$
$P_{w_3} : m_1, m_3, m_2, w_3$

$P_{m_1} : w_2, w_1, w_3, m_1$
$P_{m_2} : w_1, w_3, w_2, m_2$
$P_{m_3} : w_1, w_2, w_3, m_3$

Step 6: Given $A_6 = \{w_1, m_1, w_2, m_2, w_3, m_3\}$ and

$$\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_3 & m_1 & m_2 \end{pmatrix}.$$ 

Since $BP(A_5, \mu_5) = \emptyset$ and $6 = n + p$, Stop and set:

$$RVV(P, \mu) = \mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_3 & m_1 & w_2 \end{pmatrix}.$$
Another algorithm, based on the Roth and Vande Vate algorithm to find stable matchings (and to define random mechanism by assigning a probability distribution on the set of all orderings on the set of agents) [We will describe random mechanisms later].
3.12.- Another Algorithm to Find Stable Matchings

- Another algorithm, based on the Roth and Vande Vate algorithm to find stable matchings (and to define random mechanism by assigning a probability distribution on the set of all orderings on the set of agents) [We will describe random mechanisms later].

- References:
Another algorithm, based on the Roth and Vande Vate algorithm to find stable matchings (and to define random mechanism by assigning a probability distribution on the set of all orderings on the set of agents) [We will describe random mechanisms later].

References:

Another algorithm, based on the Roth and Vande Vate algorithm to find stable matchings (and to define random mechanism by assigning a probability distribution on the set of all orderings on the set of agents) [We will describe random mechanisms later].

References:

Another algorithm, based on the Roth and Vande Vate algorithm to find stable matchings (and to define random mechanism by assigning a probability distribution on the set of all orderings on the set of agents) [We will describe random mechanisms later].

References:

Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$ \hspace{1cm} $P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1$

$P_{m_2} : w_4, w_2, w_3, w_1, m_2$ \hspace{1cm} $P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2$

$P_{m_3} : w_4, w_3, w_1, w_2, m_3$ \hspace{1cm} $P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3$

$P_{m_4} : w_1, w_4, w_3, w_2, m_4$ \hspace{1cm} $P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4$

$P_{m_5} : w_1, w_2, w_4, m_5$

**Step 1:** ($m_1$ enters) $A_1 = \{m_1\}$ and

$$\mu_0 = \left( \begin{array}{cccccccc} w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 & m_5 \\ w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 & m_5 \end{array} \right).$$
Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[
\begin{align*}
P_{m_1} & : w_1, w_2, w_3, w_4, m_1 \\
P_{m_2} & : w_4, w_2, w_3, w_1, m_2 \\
P_{m_3} & : w_4, w_3, w_1, w_2, m_3 \\
P_{m_4} & : w_1, w_4, w_3, w_2, m_4 \\
P_{m_5} & : w_1, w_2, w_4, m_5
\end{align*}
\]

\[
P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1 \\
P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2 \\
P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3 \\
P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4
\]

**Step 1:** (\( m_1 \) enters) \( A_1 = \{ m_1 \} \) and

\[
\mu_0 = \begin{pmatrix}
  w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 & m_5 \\
  w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 & m_5
\end{pmatrix}
\]

\[
BP(A_1, \mu_0, m_1) = \emptyset.\] Set

\[
\mu_1 = \begin{pmatrix}
  m_1 & w_1 & w_2 & w_3 & w_4 & m_2 & m_3 & m_4 & m_5 \\
  m_1 & w_1 & w_2 & w_3 & w_4 & m_2 & m_3 & m_4 & m_5
\end{pmatrix}
\]
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[ P_{m_1} : w_1, w_2, w_3, w_4, m_1 \]

\[ P_{m_2} : w_4, w_2, w_3, w_1, m_2 \]

\[ P_{m_3} : w_4, w_3, w_1, w_2, m_3 \]

\[ P_{m_4} : w_1, w_4, w_3, w_2, m_4 \]

\[ P_{m_5} : w_1, w_2, w_4, m_5 \]

Step 2: \((m_3 \text{ enters})\) \( A_2 = \{m_1, m_3\} \) and

\[ \mu_1 = \left( \begin{array}{ccccccccc}
  m_1 & w_1 & w_2 & w_3 & w_4 & m_2 & m_3 & m_4 & m_5 \\
  m_1 & w_1 & w_2 & w_3 & w_4 & m_2 & m_3 & m_4 & m_5 \\
\end{array} \right). \]
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4. \)

\[
\begin{align*}
P_{m_1} &: w_1, w_2, w_3, w_4, m_1 \\
P_{m_2} &: w_4, w_2, w_3, w_1, m_2 \\
P_{m_3} &: w_4, w_3, w_1, w_2, m_3 \\
P_{m_4} &: w_1, w_4, w_3, w_2, m_4 \\
P_{m_5} &: w_1, w_2, w_4, m_5
\end{align*}
\]

Step 2: \((m_3 \text{ enters})\) \( A_2 = \{m_1, m_3\} \) and

\[
\mu_1 = \begin{pmatrix}
m_1 & | & w_1 & w_2 & w_3 & w_4 & m_2 & m_3 & m_4 & m_5 \\
m_1 & | & w_1 & w_2 & w_3 & w_4 & m_2 & m_3 & m_4 & m_5
\end{pmatrix}.
\]

\[BP(A_2, \mu_1, m_3) = \emptyset.\] Set

\[
\mu_2 = \begin{pmatrix}
m_1 & m_3 & | & w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \\
m_1 & m_3 & | & w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5
\end{pmatrix}.
\]
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$
$P_{m_2} : w_4, w_2, w_3, w_1, m_2$
$P_{m_3} : w_4, w_3, w_1, w_2, m_3$
$P_{m_4} : w_1, w_4, w_3, w_2, m_4$
$P_{m_5} : w_1, w_2, w_4, m_5$

$P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1$
$P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2$
$P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3$
$P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4$

Step 3: ($w_2$ enters) $A_3 = \{m_1, m_3, w_2\}$ and

$\mu_2 = \begin{pmatrix} m_1 & m_3 & w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \\ m_1 & m_3 & w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \end{pmatrix}$. 
Example

Order: \(m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4\).

\[
\begin{align*}
P_{m_1} & : w_1, w_2, w_3, w_4, m_1 \\
P_{m_2} & : w_4, w_2, w_3, w_1, m_2 \\
P_{m_3} & : w_4, w_3, w_1, w_2, m_3 \\
P_{m_4} & : w_1, w_4, w_3, w_2, m_4 \\
P_{m_5} & : w_1, w_2, w_4, m_5
\end{align*}
\]

\[
P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1 \\
P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2 \\
P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3 \\
P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4.
\]

Step 3: (\(w_2\) enters) \(A_3 = \{m_1, m_3, w_2\}\) and

\[
\mu_2 = \left( \begin{array}{cccccccc}
m_1 & m_3 & | & w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \\
m_1 & m_3 & | & w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5
\end{array} \right).
\]

\[
BP(A_3, \mu_2, w_2) = \{(m_3, w_2), (m_1, w_2)\}. \text{ Since } m_3P_{w_2}m_1, \text{ set}
\]

\[
\mu_3 = \left( \begin{array}{cccccccc}
m_1 & w_2 & | & w_1 & w_3 & w_4 & m_2 & m_4 & m_5 \\
m_1 & m_3 & | & w_1 & w_3 & w_4 & m_2 & m_4 & m_5
\end{array} \right).
\]

\[
BP(A_3, \mu_3) = \emptyset.
\]
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4. \)

\[
\begin{align*}
P_{m_1} & : w_1, w_2, w_3, w_4, m_1 \\
P_{m_2} & : w_4, w_2, w_3, w_1, m_2 \\
P_{m_3} & : w_4, w_3, w_1, w_2, m_3 \\
P_{m_4} & : w_1, w_4, w_3, w_2, m_4 \\
P_{m_5} & : w_1, w_2, w_4, m_5 \\
P_{w_1} & : m_2, m_3, m_1, m_4, m_5, w_1 \\
P_{w_2} & : m_3, m_1, m_2, m_4, m_5, w_2 \\
P_{w_3} & : m_4, m_5, m_1, m_2, m_3, w_3 \\
P_{w_4} & : m_1, m_4, m_5, m_2, m_3, w_4.
\end{align*}
\]

Step 4: (\( w_3 \) enters) \( A_4 = \{ m_1, m_3, w_2, w_3 \} \) and

\[
\mu_3 = \begin{pmatrix}
m_1 & w_2 & \parallel & w_1 & w_3 & w_4 & m_2 & m_4 & m_5 \\
m_1 & m_3 & \parallel & w_1 & w_3 & w_4 & m_2 & m_4 & m_5
\end{pmatrix}.
\]
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$

$P_{m_2} : w_4, w_2, w_3, w_1, m_2$

$P_{m_3} : w_4, w_3, w_1, w_2, m_3$

$P_{m_4} : w_1, w_4, w_3, w_2, m_4$

$P_{m_5} : w_1, w_2, w_4, m_5$

Step 4: (w_3 enters) $A_4 = \{m_1, m_3, w_2, w_3\}$ and

$\mu_3 = \begin{pmatrix}
    m_1 & w_2 & \ | & w_1 & w_3 & w_4 & m_2 & m_4 & m_5 \\
    m_1 & m_3 & \ | & w_1 & w_3 & w_4 & m_2 & m_4 & m_5 \\
\end{pmatrix}$.

$BP(A_4, \mu_3, w_3) = \{(m_3, w_3), (m_1, w_3)\}$. Since $m_1 P_{w_3} m_3$, set

$\mu_4 = \begin{pmatrix}
    w_2 & w_3 & \ | & w_1 & w_4 & m_2 & m_4 & m_5 \\
    m_3 & m_1 & \ | & w_1 & w_4 & m_2 & m_4 & m_5 \\
\end{pmatrix}$.

$BP(A_4, \mu_4) = \emptyset$. 
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[ P_{m_1} : w_1, w_2, w_3, w_4, m_1 \quad P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1 \]

\[ P_{m_2} : w_4, w_2, w_3, w_1, m_2 \quad P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2 \]

\[ P_{m_3} : w_4, w_3, w_1, w_2, m_3 \quad P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3 \]

\[ P_{m_4} : w_1, w_4, w_3, w_2, m_4 \quad P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4. \]

\[ P_{m_5} : w_1, w_2, w_4, m_5 \]

Step 5: \((w_1 \text{ enters})\) \( A_5 = \{ m_1, m_3, w_2, w_3, w_1 \} \) and

\[ \mu_4 = \begin{bmatrix}
  w_2 & w_3 \\
  m_3 & m_1
\end{bmatrix} \quad \begin{bmatrix}
  w_1 & w_4 & m_2 & m_4 & m_5 \\
  w_1 & w_4 & m_2 & m_4 & m_5
\end{bmatrix}. \]
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$
$P_{m_2} : w_4, w_2, w_3, w_1, m_2$
$P_{m_3} : w_4, w_3, w_1, w_2, m_3$
$P_{m_4} : w_1, w_4, w_3, w_2, m_4$
$P_{m_5} : w_1, w_2, w_4, m_5$

$P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1$
$P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2$
$P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3$
$P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4$

Step 5: ($w_1$ enters) $A_5 = \{m_1, m_3, w_2, w_3, w_1\}$ and

$\mu_4 = \begin{pmatrix}
            w_2 & w_3 & | & w_1 & w_4 & m_2 & m_4 & m_5 \\
            m_3 & m_1 & | & w_1 & w_4 & m_2 & m_4 & m_5 
\end{pmatrix}$.

$BP(A_5, \mu_4, w_1) = \{(m_3, w_1), (m_1, w_1)\}$. Since $m_3 P_{w_1} m_1$, set

$\mu_5 = \begin{pmatrix}
            w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\
            m_3 & w_2 & m_1 & | & w_4 & m_2 & m_4 & m_5 
\end{pmatrix}$.

Set $w' = w_2$. 
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$  $P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1$
$P_{m_2} : w_4, w_2, w_3, w_1, m_2$  $P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2$
$P_{m_3} : w_4, w_3, w_1, w_2, m_3$  $P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3$
$P_{m_4} : w_1, w_4, w_3, w_2, m_4$  $P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4$.
$P_{m_5} : w_1, w_2, w_4, m_5$

Step 5: ($w_2$ acts) $A_5 = \{m_1, m_3, w_2, w_3, w_1\}$ and

$$\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \\
 m_3 & w_2 & m_1 & w_4 & m_2 & m_4 & m_5
\end{pmatrix}.$$
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

- $P_{m_1} : w_1, w_2, w_3, w_4, m_1$
- $P_{m_2} : w_4, w_2, w_3, w_1, m_2$
- $P_{m_3} : w_4, w_3, w_1, w_2, m_3$
- $P_{m_4} : w_1, w_4, w_3, w_2, m_4$
- $P_{m_5} : w_1, w_2, w_4, m_5$

**Step 5:** ($w_2$ acts) $A_5 = \{ m_1, m_3, w_2, w_3, w_1 \}$ and

$$
\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\
  m_3 & w_2 & m_1 & | & w_4 & m_2 & m_4 & m_5 \\
\end{pmatrix}.
$$

$BP(A_5, \mu_5, w_2) = \{ (m_1, w_2) \}$. Set

$$
\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\
  m_3 & w_2 & m_1 & | & w_4 & m_2 & m_4 & m_5 \\
\end{pmatrix}.
$$

Set $w' = w_3$. 
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[
\begin{align*}
\text{Step 5: (} w_3 \text{ acts)} & \quad A_5 = \{ m_1, m_3, w_2, w_3, w_1 \} \quad \text{and} \\
\mu_5 &= \begin{pmatrix}
  w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\
  m_3 & m_1 & w_3 & | & w_4 & m_2 & m_4 & m_5
\end{pmatrix}.
\end{align*}
\]
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[
P_{m_1} : w_1, w_2, w_3, w_4, m_1 \quad P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1
\]

\[
P_{m_2} : w_4, w_2, w_3, w_1, m_2 \quad P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2
\]

\[
P_{m_3} : w_4, w_3, w_1, w_2, m_3 \quad P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3
\]

\[
P_{m_4} : w_1, w_4, w_3, w_2, m_4 \quad P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4.
\]

\[
P_{m_5} : w_1, w_2, w_4, m_5
\]

**Step 5:** (\( w_3 \) acts) \( A_5 = \{ m_1, m_3, w_2, w_3, w_1 \} \) and

\[
\mu_5 = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \\
m_3 & m_1 & w_3 & w_4 & m_2 & m_4 & m_5
\end{pmatrix}.
\]

\[BP(A_5, \mu_5, w_3) = \{(m_3, w_3)\}. \text{ Set}
\]

\[
\mu_5 = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \\
w_1 & m_1 & m_3 & w_4 & m_2 & m_4 & m_5
\end{pmatrix}.
\]

Set \( w' = w_1 \).
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$
$P_{m_2} : w_4, w_2, w_3, w_1, m_2$
$P_{m_3} : w_4, w_3, w_1, w_2, m_3$
$P_{m_4} : w_1, w_4, w_3, w_2, m_4$
$P_{m_5} : w_1, w_2, w_4, m_5$

$P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1$
$P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2$
$P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3$
$P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4$.

Step 5: ($w_1$ acts) $A_5 = \{ m_1, m_3, w_2, w_3, w_1 \}$ and

$$\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\ w_1 & m_1 & m_3 & | & w_4 & m_2 & m_4 & m_5 \end{pmatrix}.$$

Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1}: w_1, w_2, w_3, w_4, m_1$

$P_{m_2}: w_4, w_2, w_3, w_1, m_2$

$P_{m_3}: w_4, w_3, w_1, w_2, m_3$

$P_{m_4}: w_1, w_4, w_3, w_2, m_4$

$P_{m_5}: w_1, w_2, w_4, m_5$

Step 5: ($w_1$ acts) $A_5 = \{ m_1, m_3, w_2, w_3, w_1 \}$ and

$\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\
  w_1 & m_1 & m_3 & | & w_4 & m_2 & m_4 & m_5
\end{pmatrix}$.

$BP(A_5, \mu_5, w_1) = \{(m_1, w_1)\}$. Set

$\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\
  m_1 & w_2 & m_3 & | & w_4 & m_2 & m_4 & m_5
\end{pmatrix}$.

Set $w' = w_2$. 
Example

Order: \(m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4\).

- \(P_{m_1}: w_1, w_2, w_3, w_4, m_1\)
- \(P_{m_2}: w_4, w_2, w_3, w_1, m_2\)
- \(P_{m_3}: w_4, w_3, w_1, w_2, m_3\)
- \(P_{m_4}: w_1, w_4, w_3, w_2, m_4\)
- \(P_{m_5}: w_1, w_2, w_4, m_5\)

Step 5: (\(w_2\) acts) \(A_5 = \{m_1, m_3, w_2, w_3, w_1\}\) and

\[
\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 & w_4 & m_2 & m_4 & m_5 \\
  m_1 & w_2 & m_3 & w_4 & m_2 & m_4 & m_5
\end{pmatrix}.
\]
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

\[ P_{m_1} : w_1, w_2, w_3, w_4, m_1 \]
\[ P_{m_2} : w_4, w_2, w_3, w_1, m_2 \]
\[ P_{m_3} : w_4, w_3, w_1, w_2, m_3 \]
\[ P_{m_4} : w_1, w_4, w_3, w_2, m_4 \]
\[ P_{m_5} : w_1, w_2, w_4, m_5 \]

Step 5: (w_2 acts) $A_5 = \{ m_1, m_3, w_2, w_3, w_1 \}$ and

$\mu_5 = \left( \begin{array}{ccccc} w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\ m_1 & w_2 & m_3 & | & w_4 & m_2 & m_4 & m_5 \end{array} \right)$.

$BP(A_5, \mu_5) = \emptyset$. Set

$\mu_5 := \left( \begin{array}{ccccc} w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\ m_1 & w_2 & m_3 & | & w_4 & m_2 & m_4 & m_5 \end{array} \right)$. 
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$ \quad $P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1$

$P_{m_2} : w_4, w_2, w_3, w_1, m_2$ \quad $P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2$

$P_{m_3} : w_4, w_3, w_1, w_2, m_3$ \quad $P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3$

$P_{m_4} : w_1, w_4, w_3, w_2, m_4$ \quad $P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4$.

$P_{m_5} : w_1, w_2, w_4, m_5$

Step 6: ($m_5$ enters) \quad $A_6 = \{ m_1, m_3, w_2, w_3, w_1, m_5 \}$ and

$$\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 \\
  m_1 & w_2 & m_3 \\
\end{pmatrix}; \quad
\begin{pmatrix}
  w_4 & m_2 & m_4 & m_5 \\
  w_4 & m_2 & m_4 & m_5 \\
\end{pmatrix}.$$
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1}: w_1, w_2, w_3, w_4, m_1$
$P_{m_2}: w_4, w_2, w_3, w_1, m_2$
$P_{m_3}: w_4, w_3, w_1, w_2, m_3$
$P_{m_4}: w_1, w_4, w_3, w_2, m_4$
$P_{m_5}: w_1, w_2, w_4, m_5$

$P_{w_1}: m_2, m_3, m_1, m_4, m_5, w_1$
$P_{w_2}: m_3, m_1, m_2, m_4, m_5, w_2$
$P_{w_3}: m_4, m_5, m_1, m_2, m_3, w_3$
$P_{w_4}: m_1, m_4, m_5, m_2, m_3, w_4$.

Step 6: ($m_5$ enters) $A_6 = \{ m_1, m_3, w_2, w_3, w_1, m_5 \}$ and

$\mu_5 = \begin{pmatrix}
  w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 & m_5 \\
  m_1 & w_2 & m_3 & | & w_4 & m_2 & m_4 & m_5
\end{pmatrix}$.

$BP(A_6, \mu_5, m_5) = \{ (w_2, m_5) \}$. Set

$\mu_6 = \begin{pmatrix}
  w_1 & w_2 & w_3 & | & w_4 & m_2 & m_4 \\
  m_1 & m_5 & m_3 & | & w_4 & m_2 & m_4
\end{pmatrix}$.

$BP(A_6, \mu_6) = \emptyset$. 
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1}: w_1, w_2, w_3, w_4, m_1$  
$P_{m_2}: w_4, w_2, w_3, w_1, m_2$  
$P_{m_3}: w_4, w_3, w_1, w_2, m_3$  
$P_{m_4}: w_1, w_4, w_3, w_2, m_4$  
$P_{m_5}: w_1, w_2, w_4, m_5$

Step 7: ($w_4$ enters) $A_7 = \{ m_1, m_3, w_2, w_3, w_1, m_5, w_4 \}$ and

$\mu_6 = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & m_2 & m_4 \\
m_1 & m_5 & m_3 & w_4 & m_2 & m_4
\end{pmatrix}$. 

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Example
Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1}: w_1, w_2, w_3, w_4, m_1$  
$P_{m_2}: w_4, w_2, w_3, w_1, m_2$  
$P_{m_3}: w_4, w_3, w_1, w_2, m_3$  
$P_{m_4}: w_1, w_4, w_3, w_2, m_4$  
$P_{m_5}: w_1, w_2, w_4, m_5$

Step 7: ($w_4$ enters) $A_7 = \{ m_1, m_3, w_2, w_3, w_1, m_5, w_4 \}$ and

$$\mu_6 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_2 & m_4 \\ m_1 & m_5 & m_3 & w_4 & m_2 & m_4 \end{pmatrix}.$$

$BP(A_7, \mu_6, w_4) = \{(m_3, w_4)\}$. Set

$$\mu_7 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_2 & m_4 \\ m_1 & m_5 & w_3 & m_3 & m_2 & m_4 \end{pmatrix}.$$

Set $w' = w_3$. Observe that $BP(A_7, \mu_7) = \emptyset$. 

Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

- \( P_{m_1} : w_1, w_2, w_3, w_4, m_1 \)
- \( P_{m_2} : w_4, w_2, w_3, w_1, m_2 \)
- \( P_{m_3} : w_4, w_3, w_1, w_2, m_3 \)
- \( P_{m_4} : w_1, w_4, w_3, w_2, m_4 \)
- \( P_{m_5} : w_1, w_2, w_4, m_5 \)

**Step 8:** (\( m_2 \) enters) \( A_8 = \{ m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2 \} \) and

\[
\mu_7 = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & m_2 & m_4 \\
m_1 & m_5 & w_3 & m_3 & m_2 & m_4 \\
\end{pmatrix}.
\]
Example

Order:  $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1} : w_1, w_2, w_3, w_4, m_1$  $P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1$

$P_{m_2} : w_4, w_2, w_3, w_1, m_2$  $P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2$

$P_{m_3} : w_4, w_3, w_1, w_2, m_3$  $P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3$

$P_{m_4} : w_1, w_4, w_3, w_2, m_4$  $P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4$.

$P_{m_5} : w_1, w_2, w_4, m_5$

Step 8: ($m_2$ enters) $A_8 = \{ m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2 \}$ and

$$\mu_7 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_2 & m_4 \\ m_1 & m_5 & w_3 & m_3 & m_2 & m_4 \end{pmatrix}.$$  

$BP(A_8, \mu_7, m_2) = W \times \{ m_2 \}$. Since $w_4 P_{m_2} w$ for all $w \neq w_4$, set

$$\mu_8 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_3 & m_4 \\ m_1 & m_5 & w_3 & m_2 & m_3 & m_4 \end{pmatrix}.$$  

Set $w' = m_3$. Observe that $BP(A_8, \mu_8, m_3) \neq \emptyset$. 
Example

Order: $m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4$.

$P_{m_1}: w_1, w_2, w_3, w_4, m_1$
$P_{m_2}: w_4, w_2, w_3, w_1, m_2$
$P_{m_3}: w_4, w_3, w_1, w_2, m_3$
$P_{m_4}: w_1, w_4, w_3, w_2, m_4$
$P_{m_5}: w_1, w_2, w_4, m_5$

$P_{w_1}: m_2, m_3, m_1, m_4, m_5, w_1$
$P_{w_2}: m_3, m_1, m_2, m_4, m_5, w_2$
$P_{w_3}: m_4, m_5, m_1, m_2, m_3, w_3$
$P_{w_4}: m_1, m_4, m_5, m_2, m_3, w_4$.

Step 8: ($m_3$ acts) $A_8 = \{m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2\}$ and

$\mu_8 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_3 & m_4 \\ m_1 & m_5 & w_3 & m_2 & m_3 & m_4 \end{pmatrix}$. 
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[
\begin{align*}
P_{m_1} & : w_1, w_2, w_3, w_4, m_1 \\
P_{m_2} & : w_4, w_2, w_3, w_1, m_2 \\
P_{m_3} & : w_4, w_3, w_1, w_2, m_3 \\
P_{m_4} & : w_1, w_4, w_3, w_2, m_4 \\
P_{m_5} & : w_1, w_2, w_4, m_5
\end{align*}
\]

\[
\begin{align*}
P_{w_1} & : m_2, m_3, m_1, m_4, m_5, w_1 \\
P_{w_2} & : m_3, m_1, m_2, m_4, m_5, w_2 \\
P_{w_3} & : m_4, m_5, m_1, m_2, m_3, w_3 \\
P_{w_4} & : m_1, m_4, m_5, m_2, m_3, w_4.
\end{align*}
\]

Step 8: (\( m_3 \) acts) \( A_8 = \{ m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2 \} \) and

\[
\mu_8 = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & m_3 & \parallel & m_4 \\
m_1 & m_5 & w_3 & m_2 & m_3 & \parallel & m_4
\end{pmatrix}.
\]

\( BP(A_8, \mu_8, m_3) = \{ (w_3, m_3) \} \). Set

\[
\mu_8 := \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & \parallel & m_4 \\
m_1 & m_5 & m_3 & m_2 & \parallel & m_4
\end{pmatrix}.
\]

\( BP(A_8, \mu_8) = \emptyset. \)
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[ P_{m_1} : w_1, w_2, w_3, w_4, m_1 \]  \[ P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1 \]

\[ P_{m_2} : w_4, w_2, w_3, w_1, m_2 \]  \[ P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2 \]

\[ P_{m_3} : w_4, w_3, w_1, w_2, m_3 \]  \[ P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3 \]

\[ P_{m_4} : w_1, w_4, w_3, w_2, m_4 \]  \[ P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4. \]

\[ P_{m_5} : w_1, w_2, w_4, m_5 \]

**Step 9:** (\( m_4 \) enters) \( A_9 = \{ m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \} \) and \( \mu_8 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_4 \\ m_1 & m_5 & m_3 & m_2 & m_4 \end{pmatrix} \).
Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\( P_{m_1} : w_1, w_2, w_3, w_4, m_1 \)
\( P_{m_2} : w_4, w_2, w_3, w_1, m_2 \)
\( P_{m_3} : w_4, w_3, w_1, w_2, m_3 \)
\( P_{m_4} : w_1, w_4, w_3, w_2, m_4 \)
\( P_{m_5} : w_1, w_2, w_4, m_5 \)

\( P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1 \)
\( P_{w_2} : m_3, m_1, m_2, m_4, m_5, w_2 \)
\( P_{w_3} : m_4, m_5, m_1, m_2, m_3, w_3 \)
\( P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4 \).

**Step 9:** (\( m_4 \) enters) \( A_9 = \{ m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \} \) and

\[ \mu_8 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_4 \\ m_1 & m_5 & m_3 & m_2 & m_4 \end{pmatrix}. \]

\( BP(A_9, \mu_8, m_4) = \{(w_2, m_4), (w_3, m_4), (w_4, m_4)\} \). Since \( w_4 P_{m_4} w_3 P_{m_4} w_2 \), set

\[ \mu_9 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_2 \\ m_1 & m_5 & m_3 & m_4 & m_2 \end{pmatrix}. \]

Set \( w' = m_2 \). Observe that \( BP(A_9, \mu_9, m_2) \neq \emptyset \).
Example

Order: \(m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4\).

\[P_{m_1} : w_1, w_2, w_3, w_4, m_1\]
\[P_{m_2} : w_4, w_2, w_3, w_1, m_2\]
\[P_{m_3} : w_4, w_3, w_1, w_2, m_3\]
\[P_{m_4} : w_1, w_4, w_3, w_2, m_4\]
\[P_{m_5} : w_1, w_2, w_4, m_5\]

\[P_{w_1} : m_2, m_3, m_1, m_4, m_5, w_1\]
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\[P_{w_4} : m_1, m_4, m_5, m_2, m_3, w_4.\]

\textbf{Step 9: (}m_2 \textit{acts}) \(A_9 = \{m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4\}\) and

\[\mu_9 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_2 \\ m_1 & m_5 & m_3 & m_4 & m_2 \end{pmatrix}.\]
Example

Order: \(m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4\).

\[
P_{m_1} : w_1, w_2, w_3, w_4, m_1 \quad P_{w_1} : \underline{m_2}, m_3, \underline{m_1}, m_4, m_5, w_1
\]

\[
P_{m_2} : w_4, w_2, w_3, w_1, m_2 \quad P_{w_2} : \underline{m_3}, m_1, \underline{m_2}, m_4, m_5, w_2
\]

\[
P_{m_3} : w_4, w_3, w_1, w_2, m_3 \quad P_{w_3} : m_4, m_5, m_1, \underline{m_2}, m_3, w_3
\]

\[
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\]

\[
P_{m_5} : w_1, w_2, w_4, m_5
\]

Step 9: (\(m_2\) acts) \(A_9 = \{m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4\}\) and

\[
\mu_9 = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & m_2 \\
m_1 & m_5 & m_3 & m_4 & m_2
\end{pmatrix}.
\]

\(BP(A_9, \mu_9, m_2) = \{(w_1, m_2), (w_2, m_2), (w_3, m_2)\}\). Since \(w_2 P_{m_2} w_3 P_{m_2} w_1\), set

\[
\mu_9 = \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 & m_5 \\
m_1 & m_2 & m_3 & m_4 & m_5
\end{pmatrix}.
\]

Set \(w' = m_5\). Observe that \(BP(A_9, \mu_9) = \emptyset\).
3.12.- Another Algorithm to Find Stable Matchings

Example

Order: \( m_1, m_3, w_2, w_3, w_1, m_5, w_4, m_2, m_4 \).

\[
\begin{align*}
P_{m_1} &: w_1, w_2, w_3, w_4, m_1 \\
P_{m_2} &: w_4, w_2, w_3, w_1, m_2 \\
P_{m_3} &: w_4, w_3, w_1, w_2, m_3 \\
P_{m_4} &: w_1, w_4, w_3, w_2, m_4 \\
P_{m_5} &: w_1, w_2, w_4, m_5
\end{align*}
\]

\[
\begin{align*}
P_{w_1} &: m_2, m_3, m_1, m_4, m_5, w_1 \\
P_{w_2} &: m_3, m_1, m_2, m_4, m_5, w_2 \\
P_{w_3} &: m_4, m_5, m_1, m_2, m_3, w_3 \\
P_{w_4} &: m_1, m_4, m_5, m_2, m_3, w_4.
\end{align*}
\]

Since \( 9 = k + 1 = n + p = 0 + 9 = 9 \), stop with the stable matching output

\[
\mu_9 = \mu_M = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_5 \\ m_1 & m_2 & m_3 & m_4 & m_5 \end{pmatrix}
\]

as output of the algorithm.
3.13.- Vacancy Chains

- Reference:

In contrast to markets in which candidates and positions become available at the same time (for instance, once a year), in senior-level labor markets positions become available when either a worker retires or a new position in a firm is created. But then, when a position of these is filled a new vacancy is created elsewhere. Thus, instabilities are resolved through a decentralized process of offers and acceptances.

Question: Does this process converge to stability?
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- Thus, instabilities are resolved through a decentralized process of offers and acceptances.

- Question: Does this process converge to stability?
3.13.- Vacancy Chains

\[ W = \text{set of workers.} \]
\[ F = \text{set of firms.} \]

**Definition**

Let \( (W, F, P) \) be a market and let \( \mu \) be a matching. We say that \( \mu \) is **firm-quasi stable** at \( P \) if it is stable or if \( (w, f) \) blocks \( \mu \) at \( P \) then, \( \mu(f) = f \).

- An execution of the (firms proposing) Deferred Acceptance Algorithm starting at a matching \( \mu \),
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- An execution of the (firms proposing) Deferred Acceptance Algorithm starting at a matching \(\mu\),
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- An execution of the (firms proposing) Deferred Acceptance Algorithm starting at a matching \(\mu\),
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  - \(f\) approaches its most preferred workers (in order of preference) checking whether they form a blocking pair.
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  - If they do, the blocking pair is satisfied and a new matching is formed.
3.13.- Vacancy Chains

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**Definition**

Let $(W,F,P)$ be a market and let $\mu$ be a matching. We say that $\mu$ is *firm-quasi stable* at $P$ if it is stable or if $(w,f)$ blocks $\mu$ at $P$ then, $\mu(f) = f$.

- An execution of the (firms proposing) Deferred Acceptance Algorithm starting at a matching $\mu$,
  - selects a firm $f$ (randomly or otherwise) whose position is vacant,
  - $f$ approaches its most preferred workers (in order of preference) checking whether they form a blocking pair.
  - If they do, the blocking pair is satisfied and a new matching is formed.
  - This process is then iterated until there is no firm with a vacant position that is part of a blocking pair.
Let $DAA_F(\mu)$ be a particular execution of the $DAA_F$ starting at $\mu$. 
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**Theorem**

Let $\mu$ be a firm-quasi stable matching at $P$. Then, every execution of the $DAA_F(\mu)$ is stable at $P$. 
3.14.- On the Number of Stable Matchings

**Question:** How fast does the maximum number of stable matchings grow when the number of agents becomes larger?

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J. Massó (Università degli Studi di Padova)

Matching: Paths to Stability

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3.14.- On the Number of Stable Matchings

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- An *instance of size* $n$ is a marriage market $(W, M, P^n)$ with $n$ men and $n$ women.
3.14.- On the Number of Stable Matchings

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- **Reference:**

- An *instance of size* $n$ is a marriage market $(W, M, P^n)$ with $n$ men and $n$ women.

**Theorem**

*(Irving and Leather)* For each $k \in \mathbb{N}$ there exists an instance of size $n = 2^k$ with at least $2^{n-1}$ stable matchings.
On the other hand, the literature has identified sufficient conditions on profiles under which the set of stable matchings is a singleton.
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3.14.- On the Number of Stable Matchings

Definition  We say that market \((W, M, P)\) satisfies condition \((\text{SUFF})\) if:

1. \(n = p\); i.e., \(W = \{w_1, ..., w_n\}\) and \(M = \{m_1, ..., m_n\}\).
3.14.- On the Number of Stable Matchings

**Definition** We say that market \((W, M, P)\) satisfies condition \((S_{\text{UFF}})\) if:

1. \(n = p\); i.e., \(W = \{w_1, ..., w_n\}\) and \(M = \{m_1, ..., m_n\}\).
2. For all \(w \in W\), \(A(P_w) = M\) and for all \(m \in M\), \(A(P_m) = W\).
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3. There exist two one-to-one mappings \(i: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}\) and \(j: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}\) such that for all \(1 \leq k < n\):
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Definition. We say that market $(W, M, P)$ satisfies condition $(\text{SUFF})$ if:

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Namely,
3.14.- On the Number of Stable Matchings

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\[ m_i(1), \ldots, m_i(k), m_i(k+1), \ldots, m_i(t), \ldots, m_i(n) \]
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worse than \( w_j(k) \) by \( m_i(k) \) (in any order)
3.14.- On the Number of Stable Matchings

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worse than \( m_i(k) \) by \( w_j(k) \) (in any order)
Theorem

(Eeckhout, 2000) Let \((W, M, P)\) be a market satisfying condition \((S^{\text{UFF}})\). Then, \(S(p) = \{\mu\}\), where \(\mu\) is such that \(\mu(m_{i(k)}) = w_{j(k)}\) for all \(1 \leq k \leq n\).
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Two particular families of profiles satisfying \((S^{\text{UFF}})\).
All men (women) share the same ranking.
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Definition  We say that market \((W, M, P)\) satisfies \textit{vertical heterogeneity} if:

1. \(n = p\); i.e., \(W = \{w_1, \ldots, w_n\}\) and \(M = \{m_1, \ldots, m_n\}\).
All men (women) share the same ranking.

**Definition** We say that market \((W, M, P)\) satisfies *vertical heterogeneity* if:

1. \(n = p\); i.e., \(W = \{w_1, \ldots, w_n\}\) and \(M = \{m_1, \ldots, m_n\}\).
2. For all \(w \in W\), \(A(P_w) = M\) and for all \(m \in M\), \(A(P_m) = W\).
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   - for all \(w \in W\), \(m_{i(t)}(t) P_w m_{i(t')}\) for all \(t < t'\).
3.14.- On the Number of Stable Matchings

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   - \(w_{j(k)} P_{m_{i(k)}} w_{j(k')}\) for all \(k' \neq k\),
   - \(m_{i(k)} P_{w_{j(k)}} m_{i(k')}\) for all \(k' \neq k\).
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3.15.- Indifferences

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- In particular, the existence of the $W$—optimal and $M$—optimal stable matchings and the lattice structure of the set of stable matchings.
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**Reference:**

Example
Let $W = \{w_1, w_2, w_3\}$, $M = \{m_1, m_2, m_3\}$ and the profile $P$ where

\begin{align*}
P_{w_1} & : m_1, m_2, m_3 & P_{m_1} & : [w_2, w_3], w_1 \\
P_{w_2} & : m_1, m_2 & P_{m_2} & : w_2, w_1 \\
P_{w_3} & : m_1, m_3 & P_{m_3} & : w_3, w_1.
\end{align*}
3.15.- Indifferences

Example

Let \( W = \{w_1, w_2, w_3\} \), \( M = \{m_1, m_2, m_3\} \) and the profile \( P \) where

\[
\begin{align*}
P_{w_1} : m_1, m_2, m_3 & \quad P_{m_1} : [w_2, w_3], w_1 \\
P_{w_2} : m_1, m_2 & \quad P_{m_2} : w_2, w_1 \\
P_{w_3} : m_1, m_3 & \quad P_{m_3} : w_3, w_1.
\end{align*}
\]

The stable matchings are

\[
\begin{array}{ccc}
\mu_1 & w_1 & w_2 & w_3 \\
& m_2 & m_1 & m_3 \\
\mu_2 & m_3 & m_2 & m_1,
\end{array}
\]
### Example

Let \( W = \{w_1, w_2, w_3\} \), \( M = \{m_1, m_2, m_3\} \) and the profile \( P \) where

\[
\begin{align*}
P_{w_1} &: m_1, m_2, m_3 & P_{m_1} &: [w_2, w_3], w_1 \\
P_{w_2} &: m_1, m_2 & P_{m_2} &: w_2, w_1 \\
P_{w_3} &: m_1, m_3 & P_{m_3} &: w_3, w_1.
\end{align*}
\]

The stable matchings are

<table>
<thead>
<tr>
<th>( \mu_1 )</th>
<th>( w_1 )</th>
<th>( w_2 )</th>
<th>( w_3 )</th>
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</thead>
<tbody>
<tr>
<td>( m_2 )</td>
<td>( m_1 )</td>
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<tr>
<td>( m_3 )</td>
<td>( m_2 )</td>
<td>( m_1 ),</td>
<td></td>
</tr>
</tbody>
</table>

but there are no optimal-stable matchings since \( \mu_1 P_{w_1} \mu_2 \) but \( \mu_2 P_{w_3} \mu_1 \), and \( \mu_2 P_{m_2} \mu_1 \) but \( \mu_1 P_{m_3} \mu_2 \).
References:


3.16.- Axiomatic Characterizations of the Core

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- Now, we will consider situations in which the set of agents (men and women) may change.
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Let $\mathcal{P} = \{\pi = (W, M, P) \mid (W, M, P) \text{ is a marriage market}\}$ be the set of problems and let $\mathcal{M} = \{\mu \in \mathcal{M}(\pi) \mid \pi \in \mathcal{P}\}$ be the set of all possible matchings.
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**Definition**

A *solution* is a correspondence $\Phi : \mathcal{P} \rightarrow \mathcal{M}$ such that for all $\pi \in \mathcal{P}$, $\Phi(\pi) \subseteq \mathcal{M}(\pi)$. 
3.16.- Axiomatic Characterizations of the Core

- Fix \( \pi = (W, M, P) \in \mathcal{P} \). A matching \( \mu \in \mathcal{M}(\pi) \) is pareto optimal at \( \pi \) if there is no \( \mu' \in \mathcal{M}(\pi) \) such that \( \mu'(x) \) \( P \) \( \mu(x) \) for all \( x \in W \cup M \) such that \( \mu'(x) \neq \mu(x) \).
Fix $\pi = (W, M, P) \in \Pi$. A matching $\mu \in \mathcal{M}(\pi)$ is Pareto optimal at $\pi$ if there is no $\mu' \in \mathcal{M}(\pi)$ such that $\mu'(x)P_x\mu(x)$ for all $x \in W \cup M$ such that $\mu'(x) \neq \mu(x)$.

Denote by $PO(\pi)$ the set of Pareto Optimal matchings at $\pi$. 
Fix $\pi = (W, M, P) \in \mathbb{P}$. A matching $\mu \in \mathcal{M}(\pi)$ is **Pareto optimal** at $\pi$ if there is no $\mu' \in \mathcal{M}(\pi)$ such that $\mu'(x) P_x \mu(x)$ for all $x \in W \cup M$ such that $\mu'(x) \neq \mu(x)$.

Denote by $PO(\pi)$ the set of **Pareto Optimal** matchings at $\pi$. 
3.16.- Axiomatic Characterizations of the Core

- Fix $\pi = (W, M, P) \in \mathcal{I}$. A matching $\mu \in \mathcal{M}(\pi)$ is Pareto optimal at $\pi$ if there is no $\mu' \in \mathcal{M}(\pi)$ such that $\mu'(x) P x \mu(x)$ for all $x \in W \cup M$ such that $\mu'(x) \neq \mu(x)$.

- Denote by $PO(\pi)$ the set of Pareto Optimal matchings at $\pi$.

**Definition**

A solution $\Phi : \mathcal{I} \rightarrow \mathcal{M}$ is Pareto Optimal if for all $\pi \in \mathcal{I}$, $\Phi(\pi) \subseteq PO(\pi)$. 

J. Massó

Matching: Paths to Stability

October 2010
A matching $\mu \in \mathcal{M}(W, M, P)$ is complete if for all $x \in W \cup M$, $
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A matching $\mu \in \mathcal{M}(W, M, P)$ is complete if for all $x \in W \cup M$, $\mu(x) \neq x$. 

Remark: All Pareto Optimal solutions satisfy Weak Unanimity.
A matching \( \mu \in \mathcal{M}(W, M, P) \) is complete if for all \( x \in W \cup M \), \( \mu(x) \neq x \).

**Definition**

A solution \( \Phi : \mathcal{P} \rightarrow \mathcal{M} \) satisfies *Weak Unanimity* if for all problem \( \pi \in \mathcal{P} \) such that there exists a complete matching \( \mu \in \mathcal{M}(\pi) \) with the property that \( \mu(x) \mathcal{P}_x y \) for all \( y \neq \mu(x) \) then, \( \Phi(\pi) = \{ \mu \} \).
A matching $\mu \in \mathcal{M}(W, M, P)$ is complete if for all $x \in W \cup M$, $\mu(x) \neq x$.

**Definition**

A solution $\Phi : P \to \mathcal{M}$ satisfies *Weak Unanimity* if for all problem $\pi \in P$ such that there exists a complete matching $\mu \in \mathcal{M}(\pi)$ with the property that $\mu(x)P_x y$ for all $y \neq \mu(x)$ then, $\Phi(\pi) = \{\mu\}$.

**Remark** All Pareto Optimal solutions satisfy Weak Unanimity.
A problem $\pi' = (W', M', P') \in \Pi$ is an extension of $\pi = (W, M, P) \in \Pi$ if
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A problem \( \pi' = (W', M', P') \in \mathcal{P} \) is an *extension* of \( \pi = (W, M, P) \in \mathcal{P} \) if

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3.16.- Axiomatic Characterizations of the Core

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A solution \( \Phi : \Pi \rightarrow \mathbb{M} \) is *Population Monotonic* if, for all \( \pi = (W, M, P) \in \Pi \), the following two properties hold.
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**Definition**

A solution \( \Phi : \Pi \rightarrow \mathcal{M} \) is *Population Monotonic* if, for all \( \pi = (W, M, P) \in \Pi \), the following two properties hold:

1. For each \( M \)–extension \( \pi' \) of \( \pi \), if \( \mu \in \Phi(\pi) \) then there exists \( \mu' \in \Phi(\pi') \) such that \( \mu(m)R_m\mu'(m) \) for all \( m \in M \).
3.16.- Axiomatic Characterizations of the Core

- A problem \( \pi' = (W', M', P') \in \Pi \) is an extension of \( \pi = (W, M, P) \in \Pi \) if
  - \( W \subseteq W' \),
  - \( M \subseteq M' \), and
  - \( P' \big|_{W \cup M} = P \).

- If \( M = M' \) and \( W \not\subseteq W' \) then we say that the extension \( \pi' \) of \( \pi \) is a \( W \)–extension of \( \pi \).
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2. For each \( W \)–extension \( \pi' \) of \( \pi \), if \( \mu \in \Phi(\pi) \) then there exists \( \mu' \in \Phi(\pi') \) such that \( \mu(w)R_w\mu'(w) \) for all \( w \in W \).
3.16.- Axiomatic Characterizations of the Core

Let \( \pi = (W, M, P) \in \mathcal{P} \) be a problem, \( x \in W \cup M \) be an agent, and \( \mu \in \mathcal{M}(\pi) \) be a matching. Define the the lower contour set at \( \mu \) and \( P_x \) as the set

\[
LCS(\mu, P_x) = \{ y \in W \cup M \mid \mu(x)P_xy \}.
\]
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Let $\pi = (W, M, P) \in \mathcal{P}$ be a problem, $x \in W \cup M$ be an agent, and $\mu \in \mathcal{M}(\pi)$ be a matching. Define the lower contour set at $\mu$ and $P_x$ as the set

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**Definition**

A solution $\Phi : \mathcal{P} \rightarrow \mathcal{M}$ is *Maskin Monotonic* if for each $\pi \in \mathcal{P}$ and $\mu \in \Phi(\pi)$ if $\pi'$ is obtained from $\pi$ by a monotonic transformation at $\mu$ then, $\mu \in \Phi(\pi')$. 
3.16.- Axiomatic Characterizations of the Core

- Let \( \pi = (W, M, P) \in \Pi \) be a problem, \( x \in W \cup M \) be an agent, and \( \mu \in \mathcal{M}(\pi) \) be a matching. Define the the lower contour set at \( \mu \) and \( P_x \) as the set

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\[
LCS(\mu, P_x) \subseteq LCS(\mu, P'_x) \text{ for all } x \in W \cup M.
\]

**Definition**

A solution \( \Phi : \Pi \to \mathcal{M} \) is Maskin Monotonic if for each \( \pi \in \Pi \) and \( \mu \in \Phi(\pi) \) if \( \pi' \) is obtained from \( \pi \) by a monotonic transformation at \( \mu \) then, \( \mu \in \Phi(\pi') \).

**Remark**  Maskin Monotonicity is a necessary condition for Nash implementation.
Let $\pi = (W, M, P) \in \mathcal{P}$ be a problem, $\mu \in \mathcal{M}(\pi)$ be a matching, and consider $\emptyset \neq M' \subseteq M$ and $\emptyset \neq W' \subseteq W$. If $\mu(W' \cup M') \subseteq W' \cup M'$ and $\pi$ is an extension of $\pi' = (W', M', P')$ then, $\pi'$ is a reduced problem of $\pi$ at $\mu$. 


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We denote by \( \mu \mid_{W' \cup M'} \) the restriction of \( \mu \) to \( W' \cup M' \).
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We denote by $\mu \mid_{W' \cup M'}$ the restriction of $\mu$ to $W' \cup M'$.

**Definition**

A solution $\Phi : \mathcal{P} \rightarrow \mathcal{M}$ is *Consistent* if for each $\pi \in \mathcal{P}$ and each $\mu \in \Phi(\pi)$, if $\pi' = (W', M', P')$ is a reduced problem of $\pi$ at $\mu$ then, $\mu \mid_{W' \cup M'} \in \Phi(\pi')$. 
(Toda, 2006) The Core is the unique solution satisfying Weak Unanimity, Population Monotonicity, and Maskin Monotonicity.
### Theorem

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3.16.- Axiomatic Characterizations of the Core

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- **Remark** In both Theorems, Weak Unanimity can be replaced by Pareto Optimality.
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Theorem

(Toda, 2006) Suppose that agents may have preferences with indifferences. Then, the Core is the unique solution satisfying Weak Unanimity, Population Monotonicity, Maskin Monotonicity, and Consistency.
3.17.- Von Neumann-Morgenstern Stable Sets of Matchings

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- Suppose the players consider a certain set of matchings (without knowing which one will be ultimately chosen) to be reasonable solutions.

- A coalition credibly objects a matching if it can suggest another matching, attainable by the coalition, that is better for all members of the coalition.
Then, any vN-M stable set is a set of matchings having the following two robustness conditions:
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- **Internal stability**: no coalition credibly objects any matching in the vN-M stable set by suggesting another matching also in the set.
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The Core is always internally stable but it may violate external stability. There may be matchings outside the core which are not objected by a coalition through a core matching.
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There are no general theorems of its existence.

It is very difficult to work with vN-M stable sets.
Fix a market \((W, M, P)\).
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- Let \(\mu, \mu' \in \mathcal{M}\) be two matchings and \(S \subseteq W \cup M\) a coalition. We say that \(\mu\) dominates \(\mu'\) via \(S\) under \(P\), denoted by \(\mu \succ_S \mu'\), if (i) \(\mu(S) = S\) and (ii) for all \(x \in S\), \(\mu(x) P_x \mu'(x)\).
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- Coalition \(S\) blocks \(\mu'\) if \(\mu \succ_S \mu'\) for some \(\mu\).
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- Matching \(\mu\) dominates \(\mu'\), denoted by \(\mu \succ \mu'\), if there exists a coalition \(S\) such that \(\mu \succ_S \mu'\).
Fix a market $(W, M, P)$.

- Let $\mu, \mu' \in \mathcal{M}$ be two matchings and $S \subseteq W \cup M$ a coalition. We say that $\mu$ dominates $\mu'$ via $S$ under $P$, denoted by $\mu \succ_S \mu'$, if (i) $\mu(S) = S$ and (ii) for all $x \in S$, $\mu(x) P_x \mu'(x)$.
- Coalition $S$ blocks $\mu'$ if $\mu \succ_S \mu'$ for some $\mu$.
- Matching $\mu$ dominates $\mu'$, denoted by $\mu \succ \mu'$, if there exists a coalition $S$ such that $\mu \succ_S \mu'$.

**Remark**

$C(P) = S(P) = \{ \mu \in \mathcal{M} \mid \text{for all } \emptyset \neq S \text{ and all } \mu' \in \mathcal{M}, \mu' \not\succ_S \mu \}$. 
Definition

A set \( V \subseteq \mathcal{M} \) is called a \( vN-M \) stable set if two stability conditions hold.

Internal stability: for all \( \mu, \mu_0 \in V \), \( \mu \neq \mu_0 \).

External stability: for all \( \mu_0 \in \mathcal{M} \setminus V \), there exists \( \mu \in V \) such that \( \mu \neq \mu_0 \).

Properties:

Let \( P \) be a profile and \( V \subseteq \mathcal{M} \).

(a) \( C(P) \subseteq V \).

(b) \( (V, \subseteq) \) is a distributive lattice.

(c) The set of unmatched agents is the same for all matchings belonging to \( V \).

Theorem (Ehlers, 2007)

Let \( P \) be a profile and \( V \subseteq \mathcal{M} \).

(1) Let \( V \) be a \( vN-M \) stable set. Then, \( V \) is a (set inclusion) maximal set satisfying properties (a), (b), and (c).

(2) Let \( V \) be the unique (set inclusion) maximal set satisfying properties (a), (b), and (c). Then, \( V \) is a \( vN-M \) stable set.
A set $V \subseteq M$ is called a *VNM stable set* if two stability conditions hold. Internal stability: for all $\mu, \mu' \in V$, $\mu \not\sim \mu'$. 

*Properties:*

(a) $C(P) \subseteq V$.

(b) $(V, \sim)$ is a distributive lattice.

(c) The set of unmatched agents is the same for all matchings belonging to $V$. 

*Theorem* (Ehlers, 2007) Let $P$ be a profile and $V \subseteq M$.

(1) Let $V$ be a VNM stable set. Then, $V$ is a (set inclusion) maximal set satisfying properties (a), (b), and (c).

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Definition

A set $V \subseteq \mathcal{M}$ is called a *vN-M stable set* if two stability conditions hold.

Internal stability: for all $\mu, \mu' \in V$, $\mu \not\succ \mu'$.

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**Theorem** (Ehlers, 2007) Let $P$ be a profile and $V \subseteq \mathcal{M}$.

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A set $V \subseteq \mathcal{M}$ is called a *vN-M stable set* if two stability conditions hold.

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3.17.-Von Neumann-Morgenstern Stable Sets of Matchings

**Definition**

A set \( V \subseteq \mathcal{M} \) is called a *vN-M stable set* if two stability conditions hold.

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Definition

A set \( V \subseteq \mathcal{M} \) is called a *vN-M stable set* if two stability conditions hold.

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3.17.- Von Neumann-Morgenstern Stable Sets of Matchings

**Definition**

A set $V \subseteq \mathcal{M}$ is called a *vN-M stable set* if two stability conditions hold.

**Internal stability:** for all $\mu, \mu' \in V$, $\mu \not\sim \mu'$.

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**Properties:** Let $P$ be a profile and $V \subseteq \mathcal{M}$.

- (a) $C(P) \subseteq V$.
- (b) $(V, \succ)$ is a distributive lattice.
- (c) The set of unmatched agents is the same for all matchings belonging to $V$. 

Theorem (Ehlers, 2007) Let $P$ be a profile and $V \subseteq \mathcal{M}$.

1. Let $V$ be a vN-M stable set. Then, $V$ is a (set inclusion) maximal set satisfying properties (a), (b), and (c).

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3.17.-Von Neumann-Morgenstern Stable Sets of Matchings

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Properties: Let $P$ be a profile and $V \subseteq \mathcal{M}$.

- (a) $C(P) \subseteq V$.
- (b) $(V, \succ)$ is a distributive lattice.
- (c) The set of unmatched agents is the same for all matchings belonging to $V$.
A set $V \subseteq M$ is called a vN-M stable set if two stability conditions hold. Internal stability: for all $\mu, \mu' \in V$, $\mu \not\sim \mu'$. External stability: for all $\mu' \in M \setminus V$, there exists $\mu \in V$ such that $\mu \succ \mu'$.

**Properties:** Let $P$ be a profile and $V \subseteq M$.
- (a) $C(P) \subseteq V$.
- (b) $(V, \succ)$ is a distributive lattice.
- (c) The set of unmatched agents is the same for all matchings belonging to $V$.

**Theorem**

*(Ehlers, 2007)* Let $P$ be a profile and $V \subseteq M$. 


A set $V \subseteq \mathcal{M}$ is called a von Neumann-Morgenstern stable set if two stability conditions hold. 

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**Properties:** Let $P$ be a profile and $V \subseteq \mathcal{M}$.

- (a) $C(P) \subseteq V$.
- (b) $(V, \succ)$ is a distributive lattice.
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**Theorem**

(Ehlers, 2007) Let $P$ be a profile and $V \subseteq \mathcal{M}$.

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3.17.-Von Neumann-Morgenstern Stable Sets of Matchings

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A set $V \subseteq \mathcal{M}$ is called a *vN-M stable set* if two stability conditions hold.

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- (a) $C(P) \subseteq V$.
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**Theorem**

*(Ehlers, 2007)* Let $P$ be a profile and $V \subseteq \mathcal{M}$.

1. Let $V$ be a vN-M stable set. Then, $V$ is a (set inclusion) maximal set satisfying properties (a), (b), and (c).
2. Let $V$ be the unique (set inclusion) maximal set satisfying properties (a), (b), and (c). Then, $V$ is a vN-M stable set.
Properties (a), (b), and (c) are independent.
Properties (a), (b), and (c) are independent.

Ehlers (2007) contains an example of a market \((W, M, P)\) (with \(n = p = 4\)) for which \(C(P) = \{\mu\}\) and there are two other matchings \(\mu'\) and \(\mu''\) such that \(V \equiv \{\mu, \mu'\}\) is a vN-M stable set because \(\mu \not\succ \mu', \mu' \not\succ \mu, \mu' \succ \mu''\) and any \(\mu''' \in M \backslash \{\mu, \mu', \mu''\}\), \(\mu \succ \mu''\) (and \(V\) satisfies properties (a), (b), and (c)). However, \(V' \equiv \{\mu, \mu''\}\) is a maximal set satisfying properties (a), (b), and (c) but \(V'\) is not a vN-M stable set because \(\mu \not\succ \mu'\) and \(\mu'' \not\succ \mu'\) (it is not externally stable); observe that \(V'\) is not the unique maximal set satisfying (a), (b), and (c), since \(V\) does too.
Example

(Ehlers, 2007) Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let $P$ be such that

\begin{align*}
P_{m_1} & : w_2, w_3, w_1, w_4 \\
P_{m_2} & : w_1, w_2, w_3, w_4 \\
P_{m_3} & : w_1, w_3, w_2, w_4 \\
P_{m_4} & : w_1, w_4, w_2, w_3
\end{align*}

\begin{align*}
P_{w_1} & : m_1, m_4, m_2, m_3 \\
P_{w_2} & : m_2, m_1, m_3, m_4 \\
P_{w_3} & : m_3, m_1, m_1, m_4 \\
P_{w_4} & : m_4, m_1, m_2, m_3.
\end{align*}

Let

\[
\begin{array}{c|cccc}
\mu & m_1 & m_2 & m_3 & m_4 \\
\hline
w_1 & w_2 & w_3 & w_4 \\
\mu' & w_2 & w_1 & w_3 & w_4 \\
\mu'' & w_3 & w_2 & w_1 & w_4.
\end{array}
\]
3.18.- A Bargaining Set

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Fix a matching market \((W, M, P)\).
3.18.- A Bargaining Set

- **References:**

Fix a matching market $(W, M, P)$.

**Definition**

A blocking pair $(m, w)$ for $\mu$ is called weak if there is a woman $w' \in W$ such that $w' P_m w$ and $(m, w')$ is a blocking pair for $\mu$, or a man $m' \in M$ such that $m' P_w m$ and $(m', w)$ is a blocking pair for $\mu$. 
3.18.- A Bargaining Set

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**Definition**

A blocking pair $(m, w)$ for $\mu$ is called weak if there is a woman $w' \in W$ such that $w'P_m w$ and $(m, w')$ is a blocking pair for $\mu$, or a man $m' \in M$ such that $m'P_w m$ and $(m', w)$ is a blocking pair for $\mu$.

**Definition**

A matching $\mu$ is *weakly stable* if it is individually rational and if all blocking pairs are weak.
Remark 1  Any stable matching is a weakly stable matching. Hence, the set of weakly stable matchings is non-empty. There are, however, weakly stable matchings that are not stable.
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The concept of weak stability reflects the idea that agents are not myopic. Thus, weak blocking pairs are ruled out, since they are not credible in the sense that one of the partners may decide to form another blocking pair, leaving the former blocking pair partner behind with a painful illusion.
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The concept of weak stability reflects the idea that agents are not myopic. Thus, weak blocking pairs are ruled out, since they are not credible in the sense that one of the partners may decide to form another blocking pair, leaving the former blocking pair partner behind with a painful illusion.

A coalition \( S \) is a subset of the set of agents \( W \cup M \).
Definition

Given a matching \( \mu \), a coalition \( S \) is said to be able to enforce a matching \( \mu' \) over \( \mu \), if the following condition holds:

for all \( x \in S \), if \( \mu'(x) \neq \mu(x) \), then \( \mu'(x) \in S \).
3.18.- A Bargaining Set

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\text{for all } x \in S, \text{ if } \mu'(x) \neq \mu(x), \text{ then } \mu'(x) \in S.
\]

Definition

(Zhou, 1994) An objection against a matching \( \mu \) is a pair \((S, \mu')\) where \( S \) is a coalition and \( \mu' \) a matching that can be enforced over \( \mu \) by \( S \) and in which all agents in \( S \) are strictly better off than in \( \mu \); i.e., \( \mu'(x) \mathrel{P} \mu(x) \) for all \( x \in S \).
3.18.- A Bargaining Set

Definition

(Zhou, 1994) A *counterobjection* against an objection \((S, \mu')\) (against \(\mu\)) is a pair \((T, \mu'')\) where \(T\) is a coalition and \(\mu''\) a matching that can be enforced over \(\mu\) by \(T\) such that:

\(\begin{align*}
(C_1) & \quad T \cap S = \emptyset, \quad T \cap T = \emptyset, \quad T \cap S = \emptyset; \\
(C_2) & \quad \mu''(x) \geq x \quad \forall x \in T \setminus S, \quad \mu''(x) \leq x \quad \forall x \in T \cap S.
\end{align*}\)

An objection \((S, \mu')\) against a matching \(\mu\) is justified if there is no counterobjection against \((S, \mu')\).

Definition

The bargaining set is the set of matchings that have no justified objections.

Remark 2

The Core is a subset of the bargaining set.
3.18.- A Bargaining Set

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(Zhou, 1994) A *counterobjection* against an objection \((S, \mu')\) (against \(\mu\)) is a pair \((T, \mu'')\) where \(T\) is a coalition and \(\mu''\) a matching that can be enforced over \(\mu\) by \(T\) such that:

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1. \(T \setminus S \neq \emptyset, S \setminus T \neq \emptyset, \text{ and } T \cap S \neq \emptyset;\)
2. \(\mu''(x) R_x \mu(x)\) for all \(x \in T \setminus S\) and \(\mu''(x) R_x \mu'(x)\) for all \(x \in T \cap S\).
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2. \((C2)\) \(\mu''(x) R_x \mu(x)\) for all \(x \in T \setminus S\) and \(\mu''(x) R_x \mu'(x)\) for all \(x \in T \cap S\).

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### Definition

The *bargaining set* is the set of matchings that have no justified objections.

### Remark 2

The Core is a subset of the bargaining set.
A matching $\mu$ is said to be *weakly efficient* if there is no other matching $\mu'$ in which all agents are strictly better off; i.e., $\mu'(x)P_x\mu(x)$ for all $x \in W \cup M$. 

Remark 3: Any stable matching is weakly efficient. There are, however, weakly stable matchings that are not stable (Knuth’s (1976) example) and weakly stable matchings that are not weakly efficient.

Remark 4: There are weakly stable matchings that are not in the bargaining set.
3.18.- A Bargaining Set

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- **Remark 4** There are weakly stable matchings that are not in the bargaining set.
Theorem

In a marriage market, the bargaining set coincides with the set of weakly stable and weakly efficient matchings.
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