

# Matching

## Marriage Market: Structure

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## 3.1.- Example and Preliminary Remarks

### Example

$$P_{m_1} : \underline{w_1}, w_2, w_3, \overline{w_4}$$

$$P_{m_2} : w_4, \underline{w_2}, w_3, \overline{w_1}$$

$$P_{m_3} : w_4, \underline{w_3}, w_1, \overline{w_2}$$

$$P_{m_4} : w_1, \underline{w_4}, \overline{w_3}, w_2$$

$$P_{m_5} : w_1, w_2, w_4.$$

$$P_{w_1} : \overline{m_2}, m_3, \underline{m_1}, m_4, m_5$$

$$P_{w_2} : \overline{m_3}, m_1, \underline{m_2}, m_4, m_5$$

$$P_{w_3} : \overline{m_4}, m_5, m_1, m_2, \underline{m_3}$$

$$P_{w_4} : \overline{m_1}, \underline{m_4}, m_5, m_2, m_3$$

$$\begin{aligned} \mu_W &= \begin{pmatrix} m_2 & m_3 & m_4 & m_1 & m_5 \\ w_1 & w_2 & w_3 & w_4 & m_5 \end{pmatrix} \\ &\neq \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \\ w_1 & w_2 & w_3 & w_4 & m_5 \end{pmatrix} = \mu_M. \end{aligned}$$

## 3.1.- Example and Preliminary Remarks

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- **Remark 1**  $\mu_W(w)R_w\mu_M(w)$  for all  $w \in W$ .
- **Remark 2**  $\mu_M(m)R_m\mu_W(m)$  for all  $m \in M$ .
- **Remark 3**  $\mu_W(w_5) = \mu_M(m_5) = m_5$ .

## 3.2.- Optimal Matchings

- Let  $(W, M, P)$  be a marriage market.
- Define the binary relation  $\succ_M$  on  $\mathcal{M}$  as follows: given two matchings  $\mu, \mu' \in \mathcal{M}$ , define

$$\mu \succ_M \mu' \quad \text{if and only if} \quad \begin{array}{l} \mu(m) R_m \mu'(m) \text{ for all } m \in M \text{ and} \\ \mu(\hat{m}) P_{\hat{m}} \mu'(\hat{m}) \text{ for at least one } \hat{m} \in M. \end{array}$$

- Fact:  $\succ_M$  is a transitive and antireflexive binary relation on  $\mathcal{M}$ .
- Define the partial order  $\succeq_M$  on  $\mathcal{M}$  as follows: given two matchings  $\mu, \mu' \in \mathcal{M}$ , define

$$\mu \succeq_M \mu' \quad \text{if and only if} \quad \text{either } \mu \succ_M \mu' \text{ or } \mu = \mu'.$$

- Fact:  $\succeq_M$  is a transitive, reflexive and antisymmetric binary relation on  $\mathcal{M}$ .
- Similarly, define the binary relations  $\succ_W$  and  $\succeq_W$  on  $\mathcal{M}$ .

## 3.2.- Optimal Matchings

### Definition

Let  $(W, M, P)$  be a marriage market. A matching  $\mu \in S(P)$  is  $W$ -optimal if for all  $\mu' \in S(P)$ ,  $\mu \succeq_W \mu'$ . Similarly, a matching  $\mu \in S(P)$  is  $M$ -optimal if for all  $\mu' \in S(P)$ ,  $\mu \succeq_M \mu'$ .

### Theorem

*(Gale and Shapley, AMM 1962) Let  $(W, M, P)$  be a marriage market. Then, there exists a  $W$ -optimal stable matching, and a  $M$ -optimal stable matching. Moreover, they are the outcomes of the  $DAA_W$  and  $DAA_M$ , respectively; namely, for all  $\mu \in S(P)$ ,*

$$\mu_W(w) R_w \mu(w) \text{ for all } w \in W$$

and

$$\mu_M(m) R_m \mu(m) \text{ for all } m \in M.$$

## 3.2.- Optimal Matchings: Proof of the Optimality Theorem

### Proof

- Terminology: man  $m$  and woman  $w$  are *achievable* if there is some stable matching  $\mu$  such that  $\mu(m) = w$ .
- We will show that in the  $DAA_M$  no man  $m$  is ever rejected by an achievable woman. Thus,  $\mu_M(m) R_m \mu(m)$  for all  $\mu \in S(P)$ . The proof is by induction
- Induction hypothesis: Up to step  $k$  of the  $DAA_M$  no man has yet been rejected by an achievable woman.
- Assume that at step  $k + 1$ ,  $w$  rejects  $m$ .

## 3.2.- Proof of the Optimality Theorem

$w$  rejects  $m$  at step  $k + 1$ .

- Case 1:  $m$  is unacceptable. Then,  $m$  and  $w$  are unachievable. Thus, the statement holds for step  $k + 1$  (done).
- Case 2:  $w$  rejects  $m$  in favor of  $m'$  who she keeps engaged. Hence,  $m' P_w m$ .

- We show that  $w$  is not achievable to  $m$ .
- Observe that

$$w P_{m'} w' \tag{1}$$

for all  $w'$  except for those who have previously rejected him (at  $k$  or earlier).

- By the induction hypothesis, they are not achievable to  $m'$ .
- To obtain a contradiction, assume  $\mu \in S(P)$  is such that  $m = \mu(w)$ .
- Since  $m' P_w \mu(w) = m$  then,  $(m', w)$  blocks  $\mu$  at  $P$  if  $\mu(m') = m'$ .
- If  $\mu(m') = w'$ , by (1),  $w P_{m'} \mu(m')$ .
- Thus,  $\mu \notin S(P)$ . ■

### 3.3.- Opposition of Interests

#### Example

Let  $M = \{m_1, m_2, m_3, m_4\}$  and  $W = \{w_1, w_2, w_3, w_4\}$ .

$$\begin{array}{ll} P_{m_1} : w_3, w_1, w_2, w_4 & P_{w_1} : m_2, m_1, m_3, m_4 \\ P_{m_2} : w_2, w_4, w_1, w_3 & P_{w_2} : m_3, m_1, m_2, m_4 \\ P_{m_3} : w_3, w_4, w_1, w_2 & P_{w_3} : m_4, m_3, m_1, m_2 \\ P_{m_4} : w_4, w_3, w_1, w_2. & P_{w_4} : m_3, m_4, m_1. \end{array}$$

The set of stable matchings is  $S(P) = \{\mu_M, \mu_W, \tilde{\mu}, \hat{\mu}\}$ , where

	$w_1$	$w_2$	$w_3$	$w_4$
$\mu_W =$	$m_2$	$m_1$	$m_4$	$m_3$
$\tilde{\mu} =$	$m_2$	$m_1$	$m_3$	$m_4$
$\hat{\mu} =$	$m_1$	$m_2$	$m_4$	$m_3$
$\mu_M =$	$m_1$	$m_2$	$m_3$	$m_4$ .

### 3.3.- Opposition of Interests

#### Example

$P_{m_1} : w_3, w_1, w_2, w_4$	$\mu_W =$	$w_1$	$w_2$	$w_3$	$w_4$
$P_{m_2} : w_2, w_4, w_1, w_3$		$m_2$	$m_1$	$m_4$	$m_3$
$P_{m_3} : w_3, w_4, w_1, w_2$	$\tilde{\mu} =$	$m_2$	$m_1$	$m_3$	$m_4$
$P_{m_4} : w_4, w_3, w_1, w_2.$	$\hat{\mu} =$	$m_1$	$m_2$	$m_4$	$m_3$
	$\mu_M =$	$m_1$	$m_2$	$m_3$	$m_4.$

$$\mu_M(m_1) = \hat{\mu}(m_1) = w_1 P_{m_1} w_2 = \tilde{\mu}(m_1) = \mu_W(m_1)$$

$$\mu_M(m_2) = \hat{\mu}(m_2) = w_2 P_{m_2} w_1 = \tilde{\mu}(m_2) = \mu_W(m_2)$$

$$\mu_M(m_3) = \tilde{\mu}(m_3) = w_3 P_{m_3} w_4 = \hat{\mu}(m_3) = \mu_W(m_3)$$

$$\mu_M(m_4) = \tilde{\mu}(m_4) = w_4 P_{m_4} w_3 = \hat{\mu}(m_4) = \mu_W(m_4).$$

### 3.3.- Opposition of Interests

#### Example

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{w_4} : m_3, m_4, m_1.$$

	$w_1$	$w_2$	$w_3$	$w_4$
$\mu_W =$	$m_2$	$m_1$	$m_4$	$m_3$
$\tilde{\mu} =$	$m_2$	$m_1$	$m_3$	$m_4$
$\hat{\mu} =$	$m_1$	$m_2$	$m_4$	$m_3$
$\mu_M =$	$m_1$	$m_2$	$m_3$	$m_4$ .

$$\mu_W(w_1) = \tilde{\mu}(w_1) = m_2 P_{w_1} m_1 = \hat{\mu}(w_1) = \mu_M(w_1)$$

$$\mu_W(w_2) = \tilde{\mu}(m_2) = m_1 P_{w_2} m_2 = \hat{\mu}(w_2) = \mu_M(w_2)$$

$$\mu_W(w_3) = \hat{\mu}(w_3) = m_4 P_{w_3} m_3 = \tilde{\mu}(w_3) = \mu_M(w_3)$$

$$\mu_W(w_4) = \hat{\mu}(m_4) = m_3 P_{w_4} m_4 = \tilde{\mu}(w_4) = \mu_M(w_4).$$

### 3.3.- Opposition of Interests

#### Theorem

(Knuth, 1976) Let  $(W, M, P)$  be a marriage market and assume  $\mu, \mu' \in S(P)$ . Then,

$\mu(w)R_w\mu'(w)$  for all  $w \in W$  if and only if  $\mu'(m)R_m\mu(m)$  for all  $m \in M$ .

Namely,

$\mu \succeq_W \mu'$  if and only if  $\mu' \succeq_M \mu$ .

#### Corollary

Let  $(W, M, P)$  be a marriage market. Then,

$\mu_W \succeq_W \mu_M$  and  $\mu_M \succeq_M \mu_W$ .

## 3.3.- Opposition of Interests

**Proof of the Theorem** Let  $\mu, \mu' \in S(P)$ .

- Assume  $\mu(w)R_w\mu'(w)$  for all  $w \in W$  and to obtain a contradiction, suppose there exists  $m$  such that  $\mu(m)P_m\mu'(m)$ .
- Then, there is  $w = \mu(m)P_m\mu'(m) \neq w$ .
- By hypothesis,  $m = \mu(w)R_w\mu'(w) \neq m$ .
- Hence,  $mP_w\mu'(w)$ .
- Thus,  $(m, w)$  blocks  $\mu'$  at  $P$ ; a contradiction. ■

### 3.4.- Decomposition Lemma and the “Rural Hospital” Theorem

Let  $(W, M, P)$  be a marriage market and assume  $\mu, \mu' \in \mathcal{M}$ . Define

$$M(\mu) = \{m \in M \mid \mu(m) R_m \mu'(m)\}$$

$$W(\mu') = \{w \in W \mid \mu'(w) R_w \mu(w)\}.$$

Define also, for  $W' \subseteq W$  and  $M' \subseteq M$ ,

$$\mu(W') = \{m \in M \mid \mu(m) \in W'\}$$

$$\mu(M') = \{w \in W \mid \mu(w) \in M'\}.$$

#### Lemma

*Decomposition Lemma (Knuth, 1976)* Let  $(W, M, P)$  be a marriage market and assume  $\mu, \mu' \in S(P)$ . Then,

*Let  $(W, M, P)$  be a marriage*

$$\mu(M(\mu)) = W(\mu') \quad \text{and} \quad \mu'(M(\mu')) = W(\mu).$$

## 3.4.- Decomposition Lemma and the “Rural Hospital” Theorem

### Theorem

*“Rural Hospital” or “Lonely Wolf” Theorem (McVitie and Wilson, 1971)*  
Let  $(W, M, P)$  be a marriage market and let  $\mu \in S(P)$ . Assume  $\mu(x) = x$  for some  $x \in W \cup M$ . Then,  $\mu'(x) = x$  for all  $\mu' \in S(P)$ .  
In particular,  $\mu_M(x) = \mu_W(x) = x$ .

## 3.4.- Proof of the Theorem

**Proof of the Theorem** We will show that for any  $\mu \in S(P)$ ,

$$\begin{aligned}\mu(W) &\equiv \{m \in M \mid \mu(m) \neq m\} \\ &= \{m \in M \mid \mu_M(m) \neq m\} \\ &\equiv \mu_M(W)\end{aligned}$$

and

$$\begin{aligned}\mu(M) &\equiv \{w \in W \mid \mu(w) \neq w\} \\ &= \{w \in W \mid \mu_W(w) \neq w\} \\ &\equiv \mu_W(M).\end{aligned}$$

Note that, by the definition of matching,  $|\mu(W)| = |\mu(M)|$ .

## 3.4.- Proof of the Theorem

$$|\mu(W)| = |\mu(M)|. \quad (2)$$

- Consider matchings  $\mu_M$  and  $\mu_W$ , and sets  $\mu_M(M)$ ,  $\mu_W(M)$ ,  $\mu_M(W)$ , and  $\mu_W(W)$ .
- Since  $\mu_M(m)R_m\mu_W(m)$ ,  $|\mu_M(W)| \geq |\mu_W(W)|$ .
- Since  $\mu_W(w)R_w\mu_M(w)$ ,  $|\mu_W(M)| \geq |\mu_M(M)|$ .
- Thus,

$$|\mu_M(W)| \geq |\mu_W(W)| = |\mu_W(M)| \geq |\mu_M(M)|.$$

- Therefore, by (2),

$$|\mu_M(W)| = |\mu_W(W)| = |\mu_W(M)| = |\mu_M(M)|.$$

## 3.4.- Proof of the Theorem

- We prove  $\mu_M(W) = \mu(W)$ .
  - By optimality and the opposition of interests, for all  $m \in M$  and all  $w \in W$ ,

$$\mu_M(m)R_m\mu(m) \quad \text{and} \quad \mu(w)R_w\mu_M(w). \quad (3)$$

- We first show that

$$|\mu(M)| \geq |\mu_M(M)|. \quad (4)$$

Otherwise, there exists  $w$  (and  $m$ ) such that  $m = \mu_M(w)P_w\mu(w)$ , a contradiction with (3).

- Assume  $m \in \mu(W) \setminus \mu_M(W)$ . Then,  $\mu_M(m) = m$  and  $\mu(m)P_m m$ . Hence,  $\mu(m)P_m\mu_M(m)$ , contradicting that  $\mu_M$  is the  $M$ -optimal stable matching (i.e., (3)).
- Thus,  $\mu(W) \subseteq \mu_M(W)$ . Assume  $\mu(W) \subsetneq \mu_M(W)$ . Then

$$\begin{aligned} |\mu_M(W)| &> |\mu(W)| \\ &= |\mu(M)| && \text{by (2)} \\ &\geq |\mu_M(M)| && \text{by (4)}. \end{aligned}$$

A contradiction with (2). ■

## 3.5.- Preliminaries on Lattices

- We first introduce the notion of a lattice.
- Let  $X$  be a non-empty and finite set.
- A *partial order*  $\succeq$  on  $X$  is a reflexive, transitive, and antisymmetric binary relation on  $X$ .
- Given  $x, y \in X$  we write  $x \succ y$  to denote that  $x \succeq y$  and  $x \neq y$ .
- Then,  $\succ$  is a transitive and antireflexive binary relation on  $X$ .
- Given a partial order  $\succeq$  on  $X$  and a subset  $Y \subseteq X$  define the set of *upper bounds of  $Y$*  as  $ub_{\succeq} Y = \{x \in X \mid x \succeq y \text{ for all } y \in Y\}$  and *the least upper bound of  $Y$*  as  $lub_{\succeq} Y = y$ , where  $y \in ub_{\succeq} Y$  and, for all  $y' \in ub_{\succeq} Y$ ,  $y' \succeq y$ .
- Similarly, given a partial order  $\succeq$  on  $X$  and a subset  $Y \subseteq X$  define the set of *lower bounds of  $Y$*  as  $lb_{\succeq} Y = \{x \in X \mid y \succeq x \text{ for all } y \in Y\}$  and *the greatest lower bound of  $Y$*  as  $glb_{\succeq} Y = y$ , where  $y \in lb_{\succeq} Y$  and, for all  $y' \in lb_{\succeq} Y$ ,  $y \succeq y'$ .

## 3.5.- Preliminaries on Lattices

- Given a partial order  $\preceq$  on  $X$ , define the binary operation  $\vee$  on  $X$  (*the joint*) as follows: for  $x, y \in X$ ,  $x \vee y = \text{lub}_{\preceq}\{x, y\}$ .
- Given a partial order  $\preceq$  on  $X$ , define the binary operation  $\wedge$  on  $X$  (*the meet*) as follows: for  $x, y \in X$ ,  $x \wedge y = \text{glb}_{\preceq}\{x, y\}$ .
- In general,  $x \vee y$  and  $x \wedge y$  may not exist.
- For instance, consider  $X = \{x, y, z\}$  and  $\preceq$  let be such that  $x \succ z$  and  $y \succ z$ . Then, the joint  $x \vee y$  does not exist because  $\text{lub}_{\preceq}\{x, y\} = \emptyset$  and the meet  $x \wedge y = z$  because  $\text{glb}_{\preceq}\{x, y\} = \{z\}$ . The other way around for  $\preceq'$  where  $x \succ' y$  and  $x \succ' z$ .
- However, by the antisymmetry of  $\preceq$ , if  $\text{lub}_{\preceq}$  and  $\text{glb}_{\preceq}$  do exist then, they are unique.

### Definition

The quadruple  $L = (X, \preceq, \vee, \wedge)$  is a *lattice* if, for all  $x, y \in X$ ,  $\text{lub}_{\preceq}\{x, y\}$  and  $\text{glb}_{\preceq}\{x, y\}$  do exist.

## 3.5.- Preliminaries on Lattices

- Observe that although the binary operations  $\vee$  and  $\wedge$  follow from the partial order  $\preceq$ , it is useful to refer to both in the notation of the lattice.
- This is because there is an (equivalent) algebraic approach where, instead of starting from the partial order  $\preceq$ , one can start from two binary operations as follows.
- A *join*  $\vee$  on  $X$  is an idempotent, commutative, and associative binary operation on  $X$ ; that is, for all  $x, y, z \in X$ ,  $x \vee x = x$ ,  $x \vee y = y \vee x$ , and  $x \vee (y \vee z) = (x \vee y) \vee z$ .
- Similarly for the *meet*  $\wedge$  on  $X$ .

## 3.5.- Preliminaries on Lattices

- Given a join  $\vee$  and a meet  $\wedge$  on  $X$ , define the partial order  $\succeq$  on  $X$  as follows: for all  $x, y \in X$ ,

$$x \succeq y \text{ if and only if } x = x \vee y \text{ and } y = x \wedge y.$$

- Indeed, both approaches are equivalent in the sense that the partial order obtained from  $\vee$  and  $\wedge$  is  $\succeq$  (i.e., the partial order from which  $\vee$  and  $\wedge$  are defined).

## 3.5.- Preliminaries on Lattices

### Definition

The quadruple  $(X, \succeq, \vee, \wedge)$  is a *complete lattice* if, for all non-empty  $Y \subseteq X$ ,  $\text{lub}_{\succeq} Y$  and  $\text{glb}_{\succeq} Y$  do exist.

### Definition

The lattice  $(X, \succeq, \vee, \wedge)$  is *distributive* if, for all  $x, y, z \in X$ ,  
 $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ .

**Remark** Let  $(X, \succeq, \vee, \wedge)$  be a distributive lattice. Then, for all  $x, y, z \in X$ ,

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  if and only if  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ .

## 3.5.- Preliminaries on Lattices

### Definition

The triple  $(X, \succeq, \vee)$  is a *join semilattice* if, for all  $x, y \in X$ ,  $\text{lub}_{\succeq}\{x, y\}$  do exist.

### Definition

The triple  $(X, \succeq, \vee)$  is a *meet semilattice* if, for all  $x, y \in X$ ,  $\text{glb}_{\succeq}\{x, y\}$  do exist.

- **Reference:**

- D. Knuth. *Marriages Stables*. Montreal: Les Presses de l'Université de Montreal (1976).

## 3.6.- Lattice Theorem

- Let  $(W, M, P)$  be a marriage market and let  $\mu, \mu' \in S(P)$ .
  - Define the mapping  $\nu : W \cup M \longrightarrow W \cup M$  as follows: for each  $w \in W$ ,

$$\nu(w) = \begin{cases} \mu(w) & \text{if } \mu(w)P_w\mu'(w) \\ \mu'(w) & \text{if } \mu'(w)R_w\mu(w) \end{cases}$$

and for each  $m \in M$ ,

$$\nu(m) = \begin{cases} \mu'(m) & \text{if } \mu(m)P_m\mu'(m) \\ \mu(m) & \text{if } \mu'(m)R_m\mu(m). \end{cases}$$

- Define the mapping  $\lambda : W \cup M \longrightarrow W \cup M$  as follows: for each  $w \in W$ ,

$$\lambda(w) = \begin{cases} \mu'(w) & \text{if } \mu(w)P_w\mu'(w) \\ \mu(w) & \text{if } \mu'(w)R_w\mu(w) \end{cases}$$

and for each  $m \in M$ ,

$$\lambda(m) = \begin{cases} \mu(m) & \text{if } \mu(m)P_m\mu'(m) \\ \mu'(m) & \text{if } \mu'(m)R_m\mu(m). \end{cases}$$

## 3.6.- Lattice Theorem

### Theorem

*(Knuth, 1976. [Conway]) Let  $(W, M, P)$  be a marriage market and assume  $\mu, \mu' \in S(P)$ . Then,  $\nu, \lambda \in \mathcal{M}$ . Moreover,  $\nu, \lambda \in S(P)$ .*

## Example

Let  $M = \{m_1, m_2, m_3, m_4\}$  and  $W = \{w_1, w_2, w_3, w_4\}$ .

$$\begin{array}{ll} P_{m_1} : w_3, w_1, w_2, w_4 & P_{w_1} : m_2, m_1, m_3, m_4 \\ P_{m_2} : w_2, w_4, w_1, w_3 & P_{w_2} : m_3, m_1, m_2, m_4 \\ P_{m_3} : w_3, w_4, w_1, w_2 & P_{w_3} : m_4, m_3, m_1, m_2 \\ P_{m_4} : w_4, w_3, w_1, w_2 & P_{w_4} : m_3, m_4, m_1. \end{array}$$

Let  $\mu, \mu' \in \mathcal{M}$  be such that

$$\begin{array}{cccc} & w_1 & w_2 & w_3 & w_4 \\ \mu = & \hline m_1 & m_4 & m_2 & m_3 \\ \mu' = & m_4 & m_2 & m_3 & m_1. \end{array}$$

Observe that  $(w_1, m_2)$  blocks  $\mu$  at  $P$  and  $(w_2, m_1)$  blocks  $\mu'$  at  $P$ . Thus,  $\mu, \mu' \notin S(P)$ .

## 3.6.- Lattice Theorem

### Example

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{w_4} : m_3, m_4, m_1.$$

$$\begin{array}{cccc} & w_1 & w_2 & w_3 & w_4 \\ \mu = & \frac{}{m_1} & m_4 & m_2 & m_3 \\ \mu' = & m_4 & m_2 & m_3 & m_1. \end{array}$$

$$\begin{array}{cccccccc} & w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 \\ \nu = & \frac{}{} & & & & & & & \\ \lambda = & & & & & & & & \end{array}$$

## 3.6.- Lattice Theorem

### Example

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_4} : m_3, m_4, m_1.$$

$$\mu = \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ \hline m_1 & m_4 & m_2 & m_3 \end{array}$$

$$\mu' = \begin{array}{cccc} m_4 & m_2 & m_3 & m_1. \end{array}$$

$$v = \begin{array}{cccc|cccc} w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 \\ \hline m_1 & m_2 & m_3 & m_3 & w_4 & w_3 & w_4 & w_2 \end{array}$$

$$\lambda =$$

## 3.6.- Lattice Theorem

### Example

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_4} : m_3, m_4, m_1.$$

$$\mu = \begin{array}{cccc} & w_1 & w_2 & w_3 & w_4 \\ \hline & m_1 & m_4 & m_2 & m_3 \end{array}$$

$$\mu' = \begin{array}{cccc} & m_4 & m_2 & m_3 & m_1. \end{array}$$

$$v = \begin{array}{ccccccccc} & w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 \\ \hline & m_1 & m_2 & m_3 & m_3 & w_4 & w_3 & w_4 & w_2 \end{array}$$

$$\lambda = \begin{array}{ccccccccc} & m_4 & m_4 & m_2 & m_1 & w_1 & w_2 & w_3 & w_1 \end{array}$$

## Example

Let  $M = \{m_1, m_2, m_3, m_4\}$  and  $W = \{w_1, w_2, w_3, w_4\}$ .

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{w_4} : m_3, m_4, m_1.$$

Consider  $\tilde{\mu}, \hat{\mu} \in S(P)$

	$w_1$	$w_2$	$w_3$	$w_4$
$\tilde{\mu} =$	$m_2$	$m_1$	$m_3$	$m_4$
$\hat{\mu} =$	$m_1$	$m_2$	$m_4$	$m_3$ .

## 3.6.- Lattice Theorem

### Example

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{w_4} : m_3, m_4, m_1.$$

$$\begin{array}{r} \tilde{\mu} = \\ \hat{\mu} = \end{array} \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \hline m_2 \quad m_1 \quad m_3 \quad m_4 \\ m_1 \quad m_2 \quad m_4 \quad m_3. \end{array}$$

$$\begin{array}{l} v = \\ \lambda = \end{array} \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \quad m_1 \quad m_2 \quad m_3 \quad m_4 \\ \hline \end{array}$$

## 3.6.- Lattice Theorem

### Example

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_4} : m_3, m_4, m_1.$$

$$\begin{array}{r} \tilde{\mu} = \\ \hat{\mu} = \end{array} \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \hline m_2 \quad m_1 \quad m_3 \quad m_4 \\ m_1 \quad m_2 \quad m_4 \quad m_3. \end{array}$$

$$\begin{array}{r} v = \\ \lambda = \end{array} \begin{array}{cccccccc} w_1 & w_2 & w_3 & w_4 & m_1 & m_2 & m_3 & m_4 \\ \hline m_2 & m_1 & m_4 & m_3 & w_2 & w_1 & w_4 & w_3 \end{array}$$

## 3.6.- Lattice Theorem

### Example

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_4} : m_3, m_4, m_1.$$

$$\begin{array}{r} \tilde{\mu} = \\ \hat{\mu} = \end{array} \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \hline m_2 \quad m_1 \quad m_3 \quad m_4 \\ m_1 \quad m_2 \quad m_4 \quad m_3. \end{array}$$

$$\begin{array}{r} v = \\ \lambda = \end{array} \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \quad m_1 \quad m_2 \quad m_3 \quad m_4 \\ \hline m_2 \quad m_1 \quad m_4 \quad m_3 \quad w_2 \quad w_1 \quad w_4 \quad w_3 \\ m_1 \quad m_2 \quad m_3 \quad m_4 \quad w_1 \quad w_2 \quad w_3 \quad w_4 \end{array}$$

## 3.6.- Lattice Theorem

### Example

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2.$$

$$P_{w_4} : m_3, m_4, m_1.$$

$$\begin{array}{r} \tilde{\mu} = \\ \hat{\mu} = \end{array} \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \hline m_2 \quad m_1 \quad m_3 \quad m_4 \\ m_1 \quad m_2 \quad m_4 \quad m_3. \end{array}$$

$$\begin{array}{l} v = \\ \lambda = \end{array} \begin{array}{c} w_1 \quad w_2 \quad w_3 \quad w_4 \\ \hline m_2 \quad m_1 \quad m_4 \quad m_3 \\ m_1 \quad m_2 \quad m_3 \quad m_4 \end{array} \begin{array}{c} m_1 \quad m_2 \quad m_3 \quad m_4 \\ w_2 \quad w_1 \quad w_4 \quad w_3 \\ w_1 \quad w_2 \quad w_3 \quad w_4 \end{array} = \begin{array}{l} \mu_W \\ \mu_M \end{array}$$

## 3.6.- Lattice Theorem

### Theorem

(Knuth, 1976. [Conway]) Let  $(W, M, P)$  be a marriage market and assume  $\mu, \mu' \in S(P)$ . Then,

(a)  $\nu = \mu \vee_W \mu' = \text{lub}_{\succeq_W} \{\mu, \mu'\}$ .

(b)  $\lambda = \mu \wedge_W \mu' = \text{glb}_{\succeq_W} \{\mu, \mu'\}$ .

### Theorem

(Knuth, 1976. [Conway]) Let  $(W, M, P)$  be a marriage market. Then,  $(S(P), \succeq_W, \vee_W, \wedge_W)$  is a complete and distributive lattice.

## 3.6.- Lattice Theorem

**Remark** Let  $(W, M, P)$  be a marriage market. Then,

- for all  $\mu \in S(P)$ ,
  - $\mu_W = \mu_W \vee_W \mu$ .
  - $\mu = \mu_W \wedge_W \mu$ .
  - $\mu_M = \mu_M \wedge_M \mu$ .
  - $\mu = \mu_M \vee_M \mu$ .
- it is possible to find  $\mu, \mu' \in S(P)$  such that  $\mu \not\leq_W \mu'$  and  $\mu' \not\leq_W \mu$ ;  
for instance,  $\tilde{\mu}$  and  $\hat{\mu}$  in the previous example.

## 3.6.- Lattice Theorem

### Theorem

(Knuth, 1976. [Conway]) Let  $(W, M, P)$  be a marriage market and assume  $\mu, \mu' \in S(P)$ . Then,

(a)  $\nu = \mu \wedge_M \mu' = \text{glb}_{\succeq_M} \{\mu, \mu'\}$ .

(b)  $\lambda = \mu \vee_M \mu' = \text{lub}_{\succeq_M} \{\mu, \mu'\}$ .

### Theorem

(Knuth, 1976. [Conway]) Let  $(W, M, P)$  be a marriage market. Then,  $(S(P), \succeq_M, \vee_M, \wedge_M)$  is a complete and distributive lattice. Moreover,  $(S(P), \succeq_W, \vee_W, \wedge_W)$  and  $(S(P), \succeq_M, \vee_M, \wedge_M)$  are dual; namely, for all  $\mu, \mu' \in S(P)$ ,

$\mu \succeq_W \mu'$  if and only if  $\mu' \succeq_M \mu$ ,

$\mu \vee_W \mu'$  if and only if  $\mu \wedge_M \mu'$ , and

$\mu \wedge_W \mu'$  if and only if  $\mu \vee_M \mu'$ .

## 3.6.- Lattice Theorem

### Example

Let  $M = \{m_1, m_2, m_3, m_4\}$  and  $W = \{w_1, w_2, w_3, w_4\}$ .

$$P_{m_1} : w_3, w_1, w_2, w_4 \quad P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3 \quad P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{m_3} : w_3, w_4, w_1, w_2 \quad P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2. \quad P_{w_4} : m_3, m_4, m_1.$$

The set of stable matchings is  $S(P) = \{\mu_M, \mu_W, \tilde{\mu}, \hat{\mu}\}$ , where

	$w_1$	$w_2$	$w_3$	$w_4$
$\mu_W =$	$m_2$	$m_1$	$m_4$	$m_3$
$\tilde{\mu} =$	$m_2$	$m_1$	$m_3$	$m_4$
$\hat{\mu} =$	$m_1$	$m_2$	$m_4$	$m_3$
$\mu_M =$	$m_1$	$m_2$	$m_3$	$m_4$ .

## 3.6.- Lattice Theorem

### Example

Let  $M = \{m_1, m_2, m_3, m_4\}$  and  $W = \{w_1, w_2, w_3, w_4\}$ .

$$\begin{array}{ll}
 P_{m_1} : w_3, w_1, w_2, w_4 & P_{w_1} : m_2, m_1, m_3, m_4 \\
 P_{m_2} : w_2, w_4, w_1, w_3 & P_{w_2} : m_3, m_1, m_2, m_4 \\
 P_{m_3} : w_3, w_4, w_1, w_2 & P_{w_3} : m_4, m_3, m_1, m_2 \\
 P_{m_4} : w_4, w_3, w_1, w_2. & P_{w_4} : m_3, m_4, m_1.
 \end{array}$$

The set of stable matchings is  $S(P) = \{\mu_M, \mu_W, \tilde{\mu}, \hat{\mu}\}$ , where

	$w_1$	$w_2$	$w_3$	$w_4$		$\mu_W$		$\succeq_W$	$\succeq_M$
$\mu_W =$	$m_2$	$m_1$	$m_4$	$m_3$				$\downarrow$	$\uparrow$
$\tilde{\mu} =$	$m_2$	$m_1$	$m_3$	$m_4$	$\tilde{\mu}$	$\nearrow$		$\downarrow$	$\uparrow$
$\hat{\mu} =$	$m_1$	$m_2$	$m_4$	$m_3$		$\nwarrow$		$\downarrow$	$\uparrow$
$\mu_M =$	$m_1$	$m_2$	$m_3$	$m_4$ .		$\mu_M$		$\downarrow$	$\uparrow$

## 3.6.- Lattice Theorem

### Example

Let  $M = \{m_1, m_2, m_3, m_4\}$  and  $W = \{w_1, w_2, w_3, w_4\}$ .

$P_{m_1} : w_1, w_2, w_3, w_4$

$P_{w_1} : m_4, m_3, m_2, m_1$

$P_{m_2} : w_2, w_1, w_4, w_3$

$P_{w_2} : m_3, m_4, m_1, m_2$

$P_{m_3} : w_3, w_4, w_1, w_2$

$P_{w_3} : m_2, m_1, m_4, m_3$

$P_{m_4} : w_4, w_3, w_2, w_1$

$P_{w_4} : m_1, m_2, m_3, m_4$ .

$S(P) = \{\mu_1, \dots, \mu_{10}\}$  where

	$w_1$	$w_2$	$w_3$	$w_4$
$\mu_1 = \mu_M$	$m_1$	$m_2$	$m_3$	$m_4$
$\mu_2$	$m_2$	$m_1$	$m_3$	$m_4$
$\mu_3$	$m_1$	$m_2$	$m_4$	$m_3$
$\mu_4$	$m_2$	$m_1$	$m_4$	$m_3$
$\mu_5$	$m_3$	$m_1$	$m_4$	$m_2$

	$w_1$	$w_2$	$w_3$	$w_4$
$\mu_6$	$m_2$	$m_4$	$m_1$	$m_3$
$\mu_7$	$m_3$	$m_4$	$m_1$	$m_2$
$\mu_8$	$m_4$	$m_3$	$m_1$	$m_2$
$\mu_9$	$m_3$	$m_4$	$m_2$	$m_1$
$\mu_{10} = \mu_W$	$m_4$	$m_3$	$m_2$	$m_1$

## 3.6.- Lattice Theorem

### Example

Let  $M = \{m_1, m_2, m_3, m_4\}$  and  $W = \{w_1, w_2, w_3, w_4\}$ .

$$P_{m_1} : w_1, w_2, w_3, w_4$$

$$P_{w_1} : m_4, m_3, m_2, m_1$$

$$P_{m_2} : w_2, w_1, w_4, w_3$$

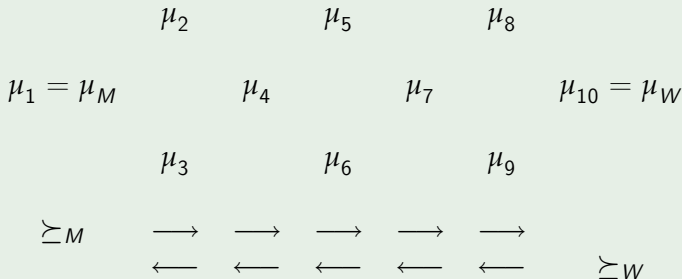
$$P_{w_2} : m_3, m_4, m_1, m_2$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{w_3} : m_2, m_1, m_4, m_3$$

$$P_{m_4} : w_4, w_3, w_2, w_1$$

$$P_{w_4} : m_1, m_2, m_3, m_4.$$



## 3.7.- An Algorithm to Compute all Stable Matchings

- **Objective:** Given a marriage market  $(W, M, P)$ , define an algorithm to obtain the set of all stable matchings  $S(P)$ .
- **Idea:** Use the lattice structure of  $S(P)$ .
  - By the duality, we will have two symmetric algorithms. We will describe the one using the point of view of men.
  - Given  $\mu, \mu' \in S(P)$ , we say that  $\mu$  and  $\mu'$  are *consecutive* if  $\mu \succ_M \mu'$  and there does not exist  $\mu'' \in S(P)$  such that  $\mu \succ_M \mu'' \succ_M \mu'$ .
  - Assume that  $\mu$  and  $\mu'$  are consecutive. Then, the set  $\{m \in M \mid \mu(m)P_m\mu'(m)\} \equiv \{m_1, \dots, m_r\}$  forms a *cycle* in the sense that for all  $i = 1, \dots, r - 1$ ,  $\mu'(m_i) = \mu(m_{i+1})$  and  $\mu'(m_r) = \mu(m_1)$ . For example,

$$\begin{array}{cccc}
 P_{m_1} & P_{m_2} & P_{m_3} & P_{m_4} \\
 \hline
 \dots & \dots & \dots & \dots \\
 \mu(m_1) = w_4 & \mu(m_2) = w_1 & \mu(m_3) = w_2 & \mu(m_4) = w_3 \\
 \dots & \dots & \dots & \dots \\
 \longrightarrow \mu'(m_1) = w_1 & \mu'(m_2) = w_2 & \mu'(m_3) = w_3 & \mu'(m_4) = w_4 \\
 \dots & \dots & \dots & \dots
 \end{array}$$

## 3.7.- An Algorithm to Compute all Stable Matchings

- To make the algorithm easier, it is better to first “clean” the preference profile of “irrelevant entries” ( some of the “...” in the example) from the point of view of stability.
- **References:**
  - D. McVitie and L. Wilson. “The stable marriage problem,” *Communications of the ACM* 14, 486-492 (1971).
  - R. Irving and P. Leather. “The complexity of counting stable marriages,” *SIAM Journal of Computing* 15, 655-667 (1986).

## 3.7.- An Algorithm to Compute all Stable Matchings

PRELIMINARY ALGORITHM: REDUCTION PROCEDURE OF  $P$  GIVEN  $\mu_M$

- **Initialization** Given  $P$ ,  $\mu_M$ , and  $\mu_W$ , set  $P^0(\mu_M) = P$ .
- **Step 1** Input: profile  $P^0(\mu_M) \equiv P^0$ . For each  $m$ , remove from  $P_m^0$  all  $w$  such that  $wP_m^0\mu_M(m)$ . For each  $w$ , remove from  $P_w^0$  all  $m$  such that  $mP_w^0\mu_W(w)$ . Output: profile  $P^1(\mu_M)$ .
- **Step 2** Input: profile  $P^1(\mu_M) \equiv P^1$ . For each  $m$ , remove from  $P_m^1$  all  $w$  such that  $\mu_W(m)P_m^1w$ . For each  $w$ , remove from  $P_w^1$  all  $m$  such that  $\mu_M(w)P_w^1m$ . Output:  $P^2(\mu_M)$ .
- **Step 3** Input: profile  $P^2(\mu_M) \equiv P^2$ . If  $m$  is not acceptable for  $w$  (i.e.,  $m$  is not on  $P_w^2$ ), remove  $w$  from  $P_m^2$ . If  $w$  is not acceptable for  $m$  (i.e.,  $w$  is not on  $P_m^2$ ), remove  $m$  from  $P_w^2$ . Output:  $P(\mu_M)$ .

### Lemma

$$S(P) = S(P(\mu_M)).$$

## 3.7.- An Algorithm to Compute all Stable Matchings

REDUCTION PROCEDURE OF  $P$  GIVEN ANY  $\mu \in S(P)$

- **Initialization** Given  $P$ ,  $\mu$ , and  $\mu_W$ , set  $P^0(\mu) = P$ .
- **Step 1** Input: profile  $P^0(\mu) \equiv P^0$ . For each  $m$ , remove from  $P_m^0$  all  $w$  such that  $wP_m^0\mu(m)$ . For each  $w$ , remove from  $P_w^0$  all  $m$  such that  $mP_w^0\mu_W(w)$ . Output: profile  $P^1(\mu)$ .
- **Step 2** Input: profile  $P^1(\mu) \equiv P^1$ . For each  $m$ , remove from  $P_m^1$  all  $w$  such that  $\mu_W(m)P_m^1w$ . For each  $w$ , remove from  $P_w^1$  all  $m$  such that  $\mu(w)P_w^1m$ . Output:  $P^2(\mu)$ .
- **Step 3** Input: profile  $P^2(\mu) \equiv P^2$ . If  $m$  is not acceptable for  $w$  (i.e.,  $m$  is not on  $P_w^2$ ), remove  $w$  from  $P_m^2$ . If  $w$  is not acceptable for  $m$  (i.e.,  $w$  is not on  $P_m^2$ ), remove  $m$  from  $P_w^2$ . Output:  $P(\mu)$ .

### Lemma

$$S(P(\mu)) \subseteq S(P).$$

## 3.7.- An Algorithm to Compute all Stable Matchings

### Definition

Let  $\mu \in S(P)$ . A subset  $\sigma = \{m_1, \dots, m_r\} \subseteq M$  defines a *cycle* for  $P(\mu)$  if:

- (1) for  $i = 1, \dots, r - 1$ , the second woman in  $P(\mu)_{m_i}$  is  $\mu(m_{i+1})$ , the first woman in  $P(\mu)_{m_{i+1}}$ , and
- (2) the second woman in  $P(\mu)_{m_r}$  is  $\mu(m_1)$ , the first woman in  $P(\mu)_{m_1}$ .

### Example

$P(\mu)_{m_1}$	$P(\mu)_{m_2}$	$P(\mu)_{m_3}$	$P(\mu)_{m_4}$
$\mu(m_1)$	$\mu(m_2)$	$\mu(m_3)$	$\mu(m_4)$
$\mu(m_2)$	$\mu(m_3)$	$\mu(m_4)$	$\mu(m_1)$
...	...	...	...

## 3.7.- An Algorithm to Compute all Stable Matchings

- Given  $P(\mu)$ , let  $\sigma = \{m_1, \dots, m_r\}$  be a cycle for  $P(\mu)$ . Define a matching  $\mu^\sigma$  as follows:
  - $\mu^\sigma(m_i) = \mu(m_{i+1})$ , for all  $i = 1, \dots, r - 1$ ;
  - $\mu^\sigma(m_r) = \mu(m_1)$ ; and
  - $\mu^\sigma(m) = \mu(m)$ , for all  $m \notin \sigma$ .
- We will refer to  $\mu^\sigma$  as the  $\sigma$ -cycle matching under  $P(\mu)$ .

### Example

Let  $M = \{m_1, \dots, m_6\}$  and let  $\sigma = \{m_1, m_2, m_3, m_4\}$  be a cycle for  $P(\mu)$ .

	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$
$\mu$	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$
$\mu^\sigma$	$w_2$	$w_3$	$w_4$	$w_1$	$w_5$	$w_6$

## 3.7.- An Algorithm to Compute all Stable Matchings

ALGORITHM: Let  $(W, M, P)$  be a marriage market.

- **Step 1** Input  $P$ . Find  $\mu_M$  and  $\mu_W$  (by the  $DAA_M$  and  $DAA_W$ , respectively), and  $P(\mu_M)$  (by the reduction procedure of  $P$  given  $\mu_M$ . Output:  $P(\mu_M)$ .
- **Step 2** Input  $P(\mu_M)$ .
  - Find the set  $\Sigma^{P(\mu_M)}$  of all cycles for  $P(\mu_M)$ .
  - For each  $\sigma \in \Sigma^{P(\mu_M)}$  obtain the corresponding  $\sigma$ -cycle matching  $\mu^\sigma$  under  $P(\mu_M)$ .
  - For each  $\mu^\sigma$ , for  $\sigma \in \Sigma^{P(\mu_M)}$ , obtain the reduced profile  $P(\mu^\sigma)$ .
  - Output: The set of reduced profiles  $P(\mu^\sigma)$ , one for each  $\sigma \in \Sigma^{P(\mu_M)}$ .

## 3.7.- An Algorithm to Compute all Stable Matchings

- **Step**  $k + 1$  Input: a set of reduced profiles, output of step  $k$ . For each  $P(\mu^\sigma)$  of reduced profiles.
  - Find the set  $\Sigma^{P(\mu^\sigma)}$  of all cycles for  $P(\mu^\sigma)$ .
  - For each  $\sigma' \in \Sigma^{P(\mu^\sigma)}$  obtain the corresponding  $\sigma'$ -cycle matching  $\mu^{\sigma'}$  under  $P(\mu^{\sigma'})$ .
  - For each  $\mu^{\sigma'}$ , for  $\sigma' \in \Sigma^{P(\mu^{\sigma'})}$ , obtain the reduced profile  $P(\mu^{\sigma'})$ .
  - Output: The family of sets of reduced profiles  $P(\mu^{\sigma'})$ , one for each  $\sigma' \in \Sigma^{P(\mu^\sigma)}$ .
- **Stop** at stage  $K$  when there are no cycles for any reduced profile in the set of profiles obtained as output of stage  $K - 1$ .

### Theorem

*If  $\mu \in S(P) \setminus \{\mu_M\}$  then,  $\mu$  is a  $\sigma$ -cycle matching under some reduced profile.*

**Remark** Using the algorithm we can compute all stable matchings. A stable matching  $\mu$  can be computed more than once.

## 3.7.- An Algorithm to Compute all Stable Matchings

### Example

**Step 1** Input:  $P$

$$P_{m_1} : w_3, w_1, w_2, w_4$$

$$P_{m_2} : w_2, w_4, w_1, w_3$$

$$P_{m_3} : w_3, w_4, w_1, w_2$$

$$P_{m_4} : w_4, w_3, w_1, w_2$$

$$P_{w_1} : m_2, m_1, m_3, m_4$$

$$P_{w_2} : m_3, m_1, m_2, m_4$$

$$P_{w_3} : m_4, m_3, m_1, m_2$$

$$P_{w_4} : m_3, m_4, m_1.$$

	$w_1$	$w_2$	$w_3$	$w_4$
$\mu_W =$	$m_2$	$m_1$	$m_4$	$m_3$
$\mu_M =$	$m_1$	$m_2$	$m_3$	$m_4$ .

## 3.7.- An Algorithm to Compute all Stable Matchings

### Example

**Step 1** Input:  $P$

$$\begin{array}{ll} P_{m_1} : w_3, \underline{w_1}, \overline{w_2}, w_4 & P_{w_1} : \overline{m_2}, \underline{m_1}, m_3, m_4 \\ P_{m_2} : \underline{w_2}, w_4, \overline{w_1}, w_3 & P_{w_2} : m_3, \overline{m_1}, \underline{m_2}, m_4 \\ P_{m_3} : \underline{w_3}, \overline{w_4}, w_1, w_2 & P_{w_3} : \overline{m_4}, \underline{m_3}, m_1, m_2 \\ P_{m_4} : \underline{w_4}, \overline{w_3}, w_1, w_2 & P_{w_4} : \overline{m_3}, \underline{m_4}, m_1. \end{array}$$

$$\begin{array}{rcccc} & w_1 & w_2 & w_3 & w_4 \\ \mu_W = & \underline{m_2} & m_1 & m_4 & m_3 \\ \mu_M = & m_1 & m_2 & m_3 & \underline{m_4}. \end{array}$$

Output:  $P(\mu_M)$ .

## 3.7.- An Algorithm to Compute all Stable Matchings

### Example

**Step 1** Input:  $P$

$$P(\mu_M)_{m_1} : \underline{w_1}, \overline{w_2}$$

$$P(\mu_M)_{m_2} : \underline{w_2}, \mathbf{w_4}, \overline{w_1}$$

$$P(\mu_M)_{m_3} : \underline{w_3}, \overline{w_4}$$

$$P(\mu_M)_{m_4} : \underline{w_4}, \overline{w_3}$$

$$P(\mu_M)_{w_1} : \overline{m_2}, \underline{m_1}$$

$$P(\mu_M)_{w_2} : \overline{m_1}, \underline{m_2}$$

$$P(\mu_M)_{w_3} : \overline{m_4}, \underline{m_3}$$

$$P(\mu_M)_{w_4} : \overline{m_3}, \underline{m_4}.$$

# Algorithm to Compute the Full Set of Stable Matchings

## Example

**Step 2** Input:  $P(\mu_M)$ .

$$P(\mu_M)_{m_1} : w_1, w_2$$

$$P(\mu_M)_{m_2} : w_2, w_1$$

$$P(\mu_M)_{m_3} : w_3, w_4$$

$$P(\mu_M)_{m_4} : w_4, w_3$$

$$P(\mu_M)_{w_1} : m_2, m_1$$

$$P(\mu_M)_{w_2} : m_1, m_2$$

$$P(\mu_M)_{w_3} : m_4, m_3$$

$$P(\mu_M)_{w_4} : m_3, m_4.$$

Set  $\Sigma^{P(\mu_M)} = \{\sigma^1 = \{m_1, m_2\}, \sigma^2 = \{m_3, m_4\}\}$ .

	$m_1$	$m_2$	$m_3$	$m_4$		$m_1$	$m_2$	$m_3$	$m_4$
$\mu_M =$	$w_1$	$w_2$	$w_3$	$w_4$	$\mu^{\sigma^1} =$	$w_2$	$w_1$	$w_3$	$w_4$
$\mu_W =$	$w_2$	$w_1$	$w_4$	$w_3.$	$\mu^{\sigma^2} =$	$w_1$	$w_2$	$w_4$	$w_3$
					$\mu_W =$	$w_2$	$w_1$	$w_4$	$w_3.$

## Example

**Step 2:** Input  $P(\mu_M)$ .

$$P(\mu_M)_{m_1} : w_1, w_2$$

$$P(\mu_M)_{w_1} : m_2, m_1$$

$$P(\mu_M)_{m_2} : w_2, w_1$$

$$P(\mu_M)_{w_2} : m_1, m_2$$

$$P(\mu_M)_{m_3} : w_3, w_4$$

$$P(\mu_M)_{w_3} : m_4, m_3$$

$$P(\mu_M)_{m_4} : w_4, w_3$$

$$P(\mu_M)_{w_4} : m_3, m_4.$$

	$m_1$	$m_2$	$m_3$	$m_4$
$\mu^{\sigma^1} =$	$w_2$	$w_1$	$w_3$	$w_4$
$\mu^{\sigma^2} =$	$w_1$	$w_2$	$w_4$	$w_3$
$\mu_W =$	$w_2$	$w_1$	$w_4$	$w_3.$

$$P(\mu^{\sigma^1})_{m_1} : w_2$$

$$P(\mu^{\sigma^1})_{w_1} : m_2$$

$$P(\mu^{\sigma^1})_{m_2} : w_1$$

$$P(\mu^{\sigma^1})_{w_2} : m_1$$

$$P(\mu^{\sigma^1})_{m_3} : w_3, w_4$$

$$P(\mu^{\sigma^1})_{w_3} : m_4, m_3$$

$$P(\mu^{\sigma^1})_{m_4} : w_4, w_3$$

$$P(\mu^{\sigma^1})_{w_4} : m_3, m_4.$$

## Example

**Step 2:** Input  $P(\mu_M)$ .

$$\begin{array}{ll} P(\mu_M)_{m_1} : w_1, w_2 & P(\mu_M)_{w_1} : m_2, m_1 \\ P(\mu_M)_{m_2} : w_2, w_1 & P(\mu_M)_{w_2} : m_1, m_2 \\ P(\mu_M)_{m_3} : w_3, w_4 & P(\mu_M)_{w_3} : m_4, m_3 \\ P(\mu_M)_{m_4} : w_4, w_3 & P(\mu_M)_{w_4} : m_3, m_4. \end{array}$$

$$\begin{array}{rcccc} & m_1 & m_2 & m_3 & m_4 \\ \mu^{\sigma^1} = & w_2 & w_1 & w_3 & w_4 \\ \mu^{\sigma^2} = & w_1 & w_2 & w_4 & w_3 \\ \mu_W = & w_2 & w_1 & w_4 & w_3. \end{array}$$

$$\begin{array}{ll} P(\mu^{\sigma^2})_{m_1} : w_1, w_2 & P(\mu^{\sigma^2})_{w_1} : m_1, m_2 \\ P(\mu^{\sigma^2})_{m_2} : w_2, w_1 & P(\mu^{\sigma^2})_{w_2} : m_2, m_1 \\ P(\mu^{\sigma^2})_{m_3} : w_4 & P(\mu^{\sigma^2})_{w_3} : m_4 \\ P(\mu^{\sigma^2})_{m_4} : w_3 & P(\mu^{\sigma^2})_{w_4} : m_3. \end{array}$$

## 3.7.- An Algorithm to Compute all Stable Matchings

### Example

**Step 3** Input:  $P(\mu^{\sigma^1})$  and  $P(\mu^{\sigma^2})$ .

$$P(\mu^{\sigma^1})_{m_1} : w_2$$

$$P(\mu^{\sigma^1})_{m_2} : w_1$$

$$P(\mu^{\sigma^1})_{m_3} : w_3, w_4$$

$$P(\mu^{\sigma^1})_{m_4} : w_4, w_3$$

$$P(\mu^{\sigma^1})_{w_1} : m_2$$

$$P(\mu^{\sigma^1})_{w_2} : m_1$$

$$P(\mu^{\sigma^1})_{w_3} : m_4, m_3$$

$$P(\mu^{\sigma^1})_{w_4} : m_3, m_4.$$

$$P(\mu^{\sigma^2})_{m_1} : w_1, w_2$$

$$P(\mu^{\sigma^2})_{m_2} : w_2, w_1$$

$$P(\mu^{\sigma^2})_{m_3} : w_4$$

$$P(\mu^{\sigma^2})_{m_4} : w_3$$

$$P(\mu^{\sigma^2})_{w_1} : m_1, m_2$$

$$P(\mu^{\sigma^2})_{w_2} : m_2, m_1$$

$$P(\mu^{\sigma^2})_{w_3} : m_4$$

$$P(\mu^{\sigma^2})_{w_4} : m_3.$$

$$\Sigma^{P(\mu^{\sigma^1})} = \{\sigma^3 = \{m_3, m_4\}\} \text{ and } \Sigma^{P(\mu^{\sigma^2})} = \{\sigma^4 = \{m_1, m_2\}\}.$$

## 3.7.- An Algorithm to Compute all Stable Matchings

### Example

**Step 3** Input:  $P(\mu^{\sigma^1})$  and  $P(\mu^{\sigma^2})$ .

$$\begin{array}{ll} P(\mu^{\sigma^3})_{m_1} = P(\mu^{\sigma^4})_{m_1} : w_2 & P(\mu^{\sigma^3})_{w_1} = P(\mu^{\sigma^4})_{w_1} : m_2 \\ P(\mu^{\sigma^3})_{m_2} = P(\mu^{\sigma^4})_{m_2} : w_1 & P(\mu^{\sigma^3})_{w_2} = P(\mu^{\sigma^4})_{w_2} : m_1 \\ P(\mu^{\sigma^3})_{m_3} = P(\mu^{\sigma^4})_{m_3} : w_4 & P(\mu^{\sigma^3})_{w_3} = P(\mu^{\sigma^4})_{w_3} : m_4 \\ P(\mu^{\sigma^3})_{m_4} = P(\mu^{\sigma^4})_{m_4} : w_3 & P(\mu^{\sigma^3})_{w_4} = P(\mu^{\sigma^4})_{w_4} : m_3. \end{array}$$

$$\mu_W = \mu^{\sigma^3} = \mu^{\sigma^4} \quad \begin{array}{cccc} m_1 & m_2 & m_3 & m_4 \\ \hline w_2 & w_1 & w_4 & w_3. \end{array}$$

There are not more cycles: Stop.  $S(P) = \{\mu_W, \mu^{\sigma^1}, \mu^{\sigma^2}, \mu_M\}$ .

## 3.8.- The Blocking Lemma

### Lemma

(The Blocking Lemma) Let  $(W, M, P)$  be a marriage market. Assume  $\mu \in IR(P)$ . Let  $M'$  be the set of men who strictly prefer  $\mu$  to  $\mu_M$ ; i.e.,  $M' = \{m \in M \mid \mu(m) P_m \mu_M(m)\}$ . Assume  $M' \neq \emptyset$ . Then, there exist  $m \in M \setminus M'$  and  $w \in \mu(M')$  such that the pair  $(w, m)$  blocks  $\mu$  at  $P$ .

### References:

- J. Hwang. "Modeling on college admissions in terms of stable marriages," *Academia Sinica*, mimeo (n.d.) [statement].
- D. Gale and M. Sotomayor. "Some remarks on the stable matching problem," *Discrete Applied Mathematics* 11, 223-232 (1985) [two proofs].

## 3.9.- The Weak Pareto Optimality Theorem

### Theorem

Let  $(W, M, P)$  be a marriage market. Then, there is no individually rational matching  $\mu$  such that  $\mu(m)P_m\mu_M(m)$  for all  $m \in M$ . Similarly, there is no individually rational matching  $\mu$  such that  $\mu(w)P_w\mu_W(w)$  for all  $w \in W$ .

**Remark** The optimal stable matching is weakly Pareto optimal for the set of agents on that side of the market.

- **Reference:**

- A. Roth. "The economics of matching: stability and incentives," *Mathematics of Operations Research* 7, 617-628 (1982).

## 3.9.- The Weak Pareto Optimality Theorem

**Remark** The optimal stable matching is not *strongly* Pareto optimal for the set of agents on that side of the market. Namely, there are individually rational matchings  $\mu$  such that  $\mu(m)R_m\mu_M(m)$  for all  $m \in M$  and  $\mu(m')P_{m'}\mu_M(m')$  for at least one  $m' \in M$ .

### Example

Let  $M = \{m_1, m_2, m_3\}$ ,  $W = \{w_1, w_2, w_3, w_4\}$ , and  $P$  be such that

$$P_{m_1} : w_2, w_1, w_3$$

$$P_{m_2} : w_1, w_2, w_3$$

$$P_{m_3} : w_1, w_2, w_3$$

$$P_{w_1} : m_1, m_2, m_3$$

$$P_{w_2} : m_3, m_1, m_2$$

$$P_{w_3} : m_1, m_2, m_3.$$

$$\begin{array}{r} \mu_M = \\ \mu = \end{array} \begin{array}{ccc} w_1 & w_2 & w_3 \\ \hline m_1 & m_3 & m_2 \\ m_3 & m_1 & m_2 \end{array}$$

$$\begin{aligned} \mu(m_1) &= w_2 P_{m_1} w_1 = \mu_M(m_1) \\ \mu(m_2) &= w_3 = \mu_M(m_1) \\ \mu(m_3) &= w_1 P_{m_3} w_2 = \mu_M(m_3). \end{aligned}$$

## 3.8.- and 3.9.- Example

**Remark** The statements of the Weak Pareto Optimal Theorem and the Blocking Lemma do not hold if  $\mu$  is not individually rational.

### Example

Let  $M = \{m_1, m_2, m_3\}$ ,  $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ , and  $P$  be such that

$$\begin{array}{lll} P_{m_1} : w_4, w_2, w_1, w_3 & P_{w_1} : m_1, m_2, m_3 & P_{w_4} : m_2 \\ P_{m_2} : w_5, w_1, w_2, w_3 & P_{w_2} : m_3, m_1, m_2 & P_{w_5} : m_3 \\ P_{m_3} : w_6, w_1, w_2, w_3 & P_{w_3} : m_1, m_2, m_3 & P_{w_6} : m_1. \end{array}$$

$$\begin{array}{ll} \mu_M = \begin{array}{ccc} m_1 & m_2 & m_3 \\ \hline w_1 & w_3 & w_2 \end{array} & \mu(m_1) = w_4 P_{m_1} w_1 = \mu_M(m_1) \\ \mu = \begin{array}{ccc} w_4 & w_5 & w_6 \end{array} & \mu(m_2) = w_5 P_{m_2} w_3 = \mu_M(m_2) \\ & \mu(m_3) = w_6 P_{m_3} w_2 = \mu_M(m_3). \end{array}$$

Since  $\emptyset \neq M' = \{m_1, m_2, m_3\} = M$ , the statement of the Blocking Lemma does not hold.