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Strategy-Proofness and the Strict Core in a Market with Indivisibilities[1]

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Abstract: We show that, in markets with indivisibilities (typified by the Shapley-Scarf housing market), the strict core mechanism is categorically determined by three assumptions: individual rationality, Pareto optimality and strategy-proofness.

Key words: Shapley-Scarf Housing Market, strict core mechanism, individual rationality, Pareto optimality and strategy-proofness

1 Introduction

The main objective of this paper is to provide a noncooperative foundation of the strict core in a market with indivisibilities (typified by the Shapley-Scarf (1974) housing market).

Let us recall the model in Shapley-Scarf (1974). In a housing market with n traders, each trader owns a house, and strictly ranks all the n houses (including his own). This strict order is called his preference; and the set of all traders' preferences is called a profile. Clearly we can identify a housing market with its profile.

An allocation in such a market is simply a permutation of the houses among the traders. For any fixed profile, we say that a coalition can "improve upon" an allocation if the members of that coalition can trade their own houses among themselves so as to make at least one member strictly better off without making any other member worse off (compared to what the allocation gives them). An allocation is called (a) individually rational (IR) if no trader can, on his own, improve upon it; (b) Pareto-optimal (PO) if the coalition of all traders cannot improve upon it; (c) in the strict core if no coalition of traders can improve upon it. (An allocation that is in the strict core is also obviously individually rational and Pareto-optimal). It was shown by Shapley and Scarf (1974) (also see Postlewaite and Roth (1977)) that the strict core of any housing market is nonempty and consists of exactly one allocation.

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A mechanism is a map from profiles to allocations. Any solution, which prescribes allocations in housing markets, can be viewed as a mechanism. We will say a mechanism satisfies the IR or PO properties if, for every profile, the corresponding allocation is IR or PO with respect to that profile. The IR and PO properties are so basic that almost all solutions satisfy them.

Once a specific mechanism is instituted, and traders fully know it, a well-known fundamental issue comes up: could it be that traders will have incentive to strategically misrepresent their true preferences? To avoid this situation, a key desirable property of mechanisms is strategy-proofness. Precisely, a mechanism is (individually) *strategy-proof* (SP) if, given an arbitrary profile, no trader can (by unilaterally misrepresenting his preference while others stay put) obtain a house that is better for him compared to the house he gets when he reveals his true preference.

The main result of this paper is that a mechanism satisfies IR, PO and SP properties if, and only if, it is the strict core mechanism.

The paper is organized as follows: section 2 introduces notation and definitions, section 3 proves the main result, section 4 gives some remarks and section 5 provides a variant of the main result for the case when preferences are not strict.

2 Definitions

Let $N=\{1, 2, \ldots, n\}$ denote the set of traders. Each trader i possesses an initial endowment e_i of one unit of a "personalized" indivisible commodity (e.g. a house). Also he wishes to consume no more than one unit of any such commodity. Abusing notation slightly, let N represent the set of commodities as well[2]. Denote the set of all permutations of N by Ω. An element in Ω corresponds to a strict preference of N. A profile of preferences $P=(P_1, \ldots, P_n)\in\Omega^n$, where P_i denotes the preference of the trader i, determines a housing market $\epsilon(P)$.

An element x in Ω also represents an *allocation*[2] of $\epsilon(P)$. Similarly, for a coalition $T\subset N$ with $T\neq\emptyset$, a *T-allocation* y^T is defined by a permutation of the set T. (Thus an N-allocation is simply an allocation.) A T-allocation y^T *weakly dominates* an allocation x of $\epsilon(P)$ if for all[3,4] $i\in T$, $\neg x_i\, P_i\, y_i^T$ and for some $j\in T$, $y_j^T P_j x_j$. A T-allocation y^T *dominates* an allocation x of $\epsilon(P)$ if $y_i^T P_i x_i$ for all $i\in T$.

The core $C\epsilon(P)$ and the strict core $SC\epsilon(P)$ of a housing market $\epsilon(P)$ are defined as follows.

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[2] It will be always clear from the context whether the element in N (in Ω) is a trader or a commodity (a preference or an allocation).

[3] $\neg$ = not.

[4] We use the notation $x_i(y_i^T)$ instead of $x(i)$ $(y^T(i))$ to represent the commodity assigned to trader i under the allocation x (T-allocation y^T).

Definition 1. The Core $C\epsilon(P)$ (the Strict core $SC\epsilon(P)$) consists of all allocations x of $\epsilon(P)$ which are not dominated (not weakly dominated) by any T-allocation $(T \subset N)$.

It suffices to consider the direct revelation allocation mechanism because of the well-known revelation principle.

Definition 2. A mechanism ϕ is a map $\phi: \Omega^n \to \Omega$ from profiles in Ω^n to allocations in Ω.

We will impose three properties on a mechanism, namely: Individual Rationality, Pareto Optimality and Strategy-proofness.

Definition 3. Individual Rationality (IR). An allocation $x \in \Omega$ is IR w.r.t[5] a profile P if $\neg e_i P_i x_i$ for all $i \in N$.

Denote the set of all allocations that are IR w.r.t the profile P by $IR(P)$.

Definition 4. A mechanism ϕ satisfies IR if $\phi(P) \in IR(P)$ for all $P \in \Omega^n$.

Definition 5. Pareto Optimality (PO). An allocation x is PO w.r.t the profile P if it is not weakly dominated by any N-allocation.

Denote the set of all allocations that are PO w.r.t the profile P by $PO(P)$.

Definition 6. A mechanism ϕ satisfies PO if $\phi(P) \in PO(P)$ for all $P \in \Omega^n$.

Denote $P_{-i} := (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$ and $(P_{-i}, P_i') \equiv (P_{-i} | P_i') := (P_1, \ldots, P_{i-1}, P_i', P_{i+1}, \ldots, P_n)$.

Definition 7. Strategy Proofness (SP). A mechanism ϕ satisfies SP if for all $i \in N$, all $Q \in \Omega^n$, all $P_i \in \Omega$, all $P_i' \in \Omega$, we have $\neg \phi_i(Q_{-i}, P_i')\ P_i\ \phi_i(Q_{-i}, P_i)$.

3 The Main Result

It was shown by Postlewaite and Roth (1977) that $|SC\epsilon(P)| = 1$ for all $P \in \Omega^n$.

Definition 8. The map $P \mapsto SC\epsilon(P)$ is called the strict core mechanism and is denoted by φ.

Theorem 1. A mechanism ψ satisfies IR, PO and SP on Ω^n if, and only if, $\psi = \varphi$.

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[5] with respect to

We prepare for the proof with some lemmas.

Given $P\in\Omega^n$ and two allocations $x, y\in\Omega$, define

$$J(x, y, P):=\{j\in N : x_j\, P_j\, y_j\}. \tag{1}$$

Clearly the three sets $J(x, y, P), J(y, x, P)$ and $N\setminus(J(x, y, P)\cup J(y, x, P))$ form a partition of N.

Lemma 1. Let $x, y\in PO(P)$ be two PO allocations w.r.t P, and suppose $x\neq y$. Then $J(x, y, P)\neq\emptyset$.

Proof. If $J(x, y, P)=\emptyset$, then $\neg x_i\, P_i\, y_i$ for all $i\in N$. We must have either (a). $y_i\, P_i\, x_i$ for some $i\in N$; or (b). $\neg y_i\, P_i\, x_i$ for all $i\in N$. (a) implies that x is not in $PO(P)$ and (b) implies $x=y$. Both cases lead to a contradiction. □

Lemma 2. Let $x\in SC\epsilon(P)$ and $y\in IR(P)\cap PO(P)$ with $x\neq y$. Then $\exists j\in J(x, y, P)$ such that $x_j\, P_j\, y_j\, P_j\, e_j$.

Proof. By Lemma 1, $J(x, y, P)\neq\emptyset$. Suppose $x_j\, P_j\, y_j\, P_j\, e_j$ is false for all $j\in J(x, y, P)$. Then, since y is IR, $y_j=e_j$ for all $j\in J(x, y, P)$. Let $S=N\setminus(J(x, y, P)\cup J(y, x, P))$. Also let $T=S\bigcup J(y, x, P)$ and note that the union is disjoint. Clearly the restriction of the allocation y to the coalition T is a T-allocation, and since $J(y, x, P)\neq\emptyset$ by Lemma 1, y weakly dominates the allocation x, contradicting that $x\in SC\epsilon(P)$. □

For $P\in\Omega^n$, define

$$T_P=\{i\in N : \exists \text{ a house } h\in N \quad \text{such that } \varphi_i(P)P_i\, hP_i\, e_i\}; \tag{2}$$

and the profile of preferences $P'=(P_1', \ldots, P_n')\in\Omega^n$ as follows:

$$P_i' = \begin{cases} (\overbrace{\ldots\ldots, \varphi_i(P), e_i,}^{\text{truncation of } P_i} \quad \overbrace{\ldots\ldots\ldots\ldots}^{\text{ranked according to } P_i}) & \text{if } i\in T_P \\ P_i & \text{if } i\in N\setminus T_P \end{cases} \tag{3}$$

Notation: Let $T\subset N$ be any subset of N. Denote $P_T=(P_i)_{i\in T}$ and $P_{-T}=P_{N\setminus T}$.

Lemma 3. $\varphi(P)=\varphi(P')=\varphi(P'_{-T}, P_T)$ for all subsets $T\subset N$.

Proof. Obvious. □

Lemma 4. $\varphi(P')=\psi(P')$.

Proof. Suppose $\varphi(P')\neq\psi(P')$. Then by Lemma 2, $\exists j\in J(\varphi(P'), \psi(P'), P')$ such that

$$\varphi_j(P')\, P_j'\, \psi_j(P')\, P_j'\, e_j\,. \tag{4}$$

But by the construction of P' in (3) and by the fact (see Lemma 3) that $\varphi(P')=\varphi(P)$, we have for each $j\in N$ either (a). e_j follows $\varphi_j(P')$ in P_j' immediately or (b). $e_j=\varphi_j(P')$ in P_j'. In any case we contradict (4). □

Lemma 5. $\varphi(P'_{-T}, P_T)=\psi(P'_{-T}, P_T)$ *for any subset* $T\subset N$.

Proof. Because of Lemma 4, it is sufficient to prove Lemma 5 for all subsets $T\subset T_P$. This is done by induction. When $|T|=0$, Lemma 4 gives us the desired conclusion. Now assume $\varphi(P'_{-T}, P_T)=\psi(P'_{-T}, P_T)$ for any $|T|=k$.

Suppose $\varphi(P'_{-T}, P_T)\neq\psi(P'_{-T}, P_T)$ for some $|T|=k+1$. For convenience, denote $Q=(P'_{-T}, P_T)$. Then by Lemma 2, $\exists j\in J(\varphi(Q), \psi(Q), Q)$ such that

$$\varphi_j(Q)\; Q_j\; \psi_j(Q)\; Q_j\; e_j. \tag{5}$$

If $j\in N\backslash T$, then by Lemma 3 we get from (5):

$$\varphi_j(P)\, P_j'\, \psi_j(Q)\, P_j'\, e_j, \tag{6}$$

which is impossible by the construction of the P_j' (since either e_j follows $\varphi_j(P)$ immediately or $e_j=\varphi_j(P)$ in P_j' for all $j\in N\backslash T$).

If $j\in T$, then by Lemma 3 we have:

$$\varphi_j(Q)=\varphi_j(Q_{-j}, P_j'), \tag{7}$$

and by our induction hypothesis, we have:

$$\varphi_j(Q_{-j}, P_j')=\psi_j(Q_{-j}, P_j'). \tag{8}$$

Thus we get from (5), (7) and (8):

$$\psi_j(Q_{-j}, P_j')\; P_j\; \psi_j(Q). \tag{9}$$

Substituting for Q, we get from (9):

$$\psi_j(P'_{-T}, P_T | P'_j)\ P_j\ \psi_j(P'_{-T}, P_T), \tag{10}$$

which contradicts that the mechanism ψ satisfies SP. □

Proof of Theorem 1:

The *only if* part of the theorem follows by setting $T=N$ in Lemma 5.

The *if* part of the theorem is well-known. Since $\varphi(P) \in SC\epsilon(P)$ for all $P \in \Omega^n$, it is clear that φ satisfies IR and PO. Roth (1982) has shown that φ satisfies SP. □

4 Some Remarks

(1). The property SP is tantamount to the requirement that truth-telling be a dominant strategy Nash Equilibrium (NE) in the game Γ_ψ induced by the mechanism ψ. But in fact, the strict core mechanism Γ_φ has a stronger property: truth-telling is a dominant strategy *strong* NE.

Definition 9. Coalition strategy proofness (CSP). A mechanism ϕ satisfies CSP if for all $T \subset N$ (with $T \neq \emptyset$), all $i \in T$, all $Q \in \Omega^n$, all $P_T \in \Omega^T$, all $P'_T \in \Omega^T$, we have[6] $\neg \phi_i(Q_{-T}, P'_T)\ \mathrm{P}_i\ \phi_i(Q_{-T}, P_T)$.

It was shown by Bird (1984) that φ satisfies CSP. This result, in conjunction with Theorem 1, immediately implies

Corollary 1. φ is the unique mechanism that satisfies IR, PO and CSP.

(2). It is worth noting that competitive allocations and stable sets (defined via weak domination) turn out to be equivalent to the strict core (see Postlewaite and Roth (1977) and Wako (1991)). However, the core itself is not equivalent to the strict core and often may contain several allocations (see Shapley and Scarf (1974)).

(3). It is easy to check, via the following examples, that Theorem 1 is "tight".

Example 1. Consider a market with $N=\{1, 2, 3\}$ and $P_1=(2\ 3\ 1)$, $P_2=(1\ 3\ 2)$, $P_3=(1\ 3\ 2)$. Then both $\varphi(P)=(2\ 1\ 3)$ and $\psi(P)=(2\ 3\ 1)$ satisfy IR and PO at P. Define ψ on Ω^3 by $\psi(P)=(2\ 3\ 1)$; $\psi(Q)=\varphi(Q)$ for all $Q \in \Omega^3 \backslash \{P\}$. Now ψ satisfies IR and PO but not SP.

[6] Recall $-T := N \backslash T$.

Example 2. Consider the mechanism in which each trader is assigned his initial endowment. Clearly this mechanism satisfies IR and SP, but not PO.

Example 3. Consider a market with $N=\{1\ 2\}$ and the mechanism in which trader 1 is always assigned the house he likes most. This mechanism satisfies PO and SP. But at the profile given by $P_1=(2\ 1)$ and $P_2=(2\ 1)$, it is different from φ and does not satisfy IR.

5 Extensions

Let Ω_* be the set of all *weak* preferences on N. Thus an element in Ω_* is a ranking of houses in N as before, except that we allow also for indifferences. Suppose $Q_i \in \Omega_*$ is a weak preference for trader i. Then Q_i induces a strict preference $P(Q_i)$ and an indifference $I(Q_i)$ in the standard manner ($jP(Q_i)k$ if jQ_ik and $\neg kQ_ij$; and $jI(Q_i)k$ if jQ_ik and kQ_ij).

Let $\Omega_I=\{Q\in\Omega_*^N : SC\epsilon(Q)\neq\emptyset\}$. We now consider the possibility that, to each market $Q\in\Omega_I$, there might correspond a nonempty set of allocations in Ω. To this end, we make:

Definition 10. A correspondence mechanism is a map from Ω_I to $2^{\Omega}\backslash\{\emptyset\}$.

Definition 11. A correspondence mechanism F is a strict core correspondence mechanism if, for any Q in Ω_I, $F(Q)\subset SC\epsilon(Q)$.

Notation: Denote the set of all strict core correspondence mechanisms by SCM.

Definition 12. Let F be a correspondence mechanism. Then $\Omega_I \xrightarrow{f} \Omega$ is called a selection from F if $f(Q)\in F(Q)$ for all $Q\in\Omega_I$.

Definition 13. A correspondence mechanism F satisfies IR, PO and SP if each selection f from F satisfies IR, PO and SP.

Notation: Denote the set of all correspondence mechanisms which satisfy IR, PO and SP by Ξ.

Any strict core mechanism i.e. any $F\in SCM$, clearly satisfies IR and PO. Furthermore by Wako's result (Theorem 3 in Appendix) and the work of Roth (1982) and Bird (1984), it is easy to see that F satisfies SP (and indeed CSP). In the spirit of Theorem 1 (the only if part) we will show the converse.

Theorem 2. A correspondence mechanism F satisfies IR, PO and SP (or CSP) if and only if $F\in SCM$.

The following two lemmas will be useful in the proof of Theorem 2.

Lemma 7. Let $Q=(Q_1, Q_2, \ldots, Q_n)\in\Omega_I$ and $x\in SC\epsilon(Q)$. $x_i I(Q_i) y_i$ for all $i\in N$ implies $y\in SC\epsilon(Q)$.

Proof. Trivial. □

Lemma 8. $\forall Q\in\Omega_I$, $\forall F\in\Xi$ and $\forall f, g\in F(Q)$, $f_i I(Q_i) g_i$ for all $i\in N$.

Proof. Implicit in the proof of Theorem 1. □

Proof of Theorem 2. Let $F\in\Xi$. By Lemma 8, we have $f_i I(Q_i) g_i$ for all $i\in N$, all $Q\in\Omega^I$ and all $f, g\in F(Q)$. If either $f\in SC\epsilon(Q)$ or $g\in SC\epsilon(Q)$, then $F\in SCM$ by Lemma 7. Thus assume that both f and g are not in the strict core $SC\epsilon(Q)$. Let h be an allocation in the strict core $SC\epsilon(Q)$. By Lemmas 2 and 7, $J(h, f, Q)\neq\emptyset$ (since f satisfies IR and PO), and h is not indifferent to f in the profile Q. By repeating the proof of Theorem 1, the selection f can be shown not to satisfy SP at Q. Thus $\Xi\subset SCM$. As we mentioned before, $SCM\subset\Xi$ by the work of Wako (1991), Bird (1984) and Roth (1982). □

Remark. Theorem 2 does not apply to the competitive allocation mechanisms since, on Ω_I, the set of competitive allocations may strictly include the strict core. Indeed one can find examples with the peculiar feature that some of the competitive allocations are not even PO.

Example 4. Let $N=(1\ 2\ 3)$, $Q_1=(1=2,\ 3)$ (i.e. trader 1 is indifferent between houses 1 and 2 but prefers each of them to house 3), $Q_2=(3\ 1\ 2)$ and $Q_3=(1\ 2\ 3)$. Consider the two allocations $x=(2\ 3\ 1)$ and $y=(1\ 3\ 2)$. Then x is competitive with prices $\pi_1=\pi_2=\pi_3>0$; and y is competitive with prices $\pi_1>\pi_2=\pi_3>0$. But observe that y is not PO w.r.t the profile Q.

Appendix

In this appendix, we will give a new and simple proof of a result in Wako (1991), which we have invoked in Section 5.

Theorem 3 [Wako (1991)]. *$x_i I(Q_i) y_i$ for all $i\in N$, all $Q\in\Omega_I$, all $x\in SC\epsilon(Q)$, all $y\in SC\epsilon(Q)$.*

Proof. Suppose there are two different allocations x and y that are in the strict core $SC\epsilon(Q)$ at the profile $Q\in\Omega_I$, and that x is not indifferent to y in Q. Then $\exists j\in J(x, y, Q)$ with $x_j P(Q_j) y_j P(Q_j) e_j$ by Lemma 2. As in the proof of Theorem 1, construct a truncated preference $Q_j'=(\ldots, x_j, e_j, \ldots)$. Let $Q'=(Q_{-j}, Q_j')$. Now

$x \in SC\epsilon(Q')$, so $Q' \in \Omega_I$. But observe that y is not in $SC\epsilon(Q')$ since it does not satisfy IR.

Now construct the selection f, from the strict core correspondence mechanism, as follows:

$$f(R) = \begin{cases} y & \text{if } R = Q \\ x & \text{if } R = Q' \\ \text{any } z \in SC\epsilon(R) & \text{otherwise} \end{cases}$$

By Roth (1982), f satisfies SP. But since $f_j(Q) = y_j$ and $f_j(Q') = x_j$, we deduce that $f_j(Q_{-j}, Q_j') P(Q_j) f_j(Q)$. It then follows that f does not satisfy SP at Q, a contradiction. □

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