Strategy-proofness and Equal-cost Sharing for a Binary Excludable Public Good with Fixed Cost*

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For the provision of a binary excludable public good with fixed cost we show that the equal-cost sharing social choice function minimizes the maximal welfare loss among the class of all strategy-proof and individually rational social choice functions.

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1 Introduction

In this paper we revisit a standard problem in incentive theory, that of the provision of an excludable binary public good.[1] A group of agents has to make a joint decision regarding the provision of a public good that costs a fixed amount to produce. The good can be used by several agents but an agent can be excluded from its consumption. Each agent has a valuation for the good but it is private information. Three decisions have to be made: (i) whether or not to provide the public good, (ii) the set of its users if it is provided, and (iii) a list of agents' contributions (or prices).

These decisions will typically depend on agents' valuations; however since these are private information they have to be *elicited* from the agents. In other words, the *social choice function* that maps profiles of valuations into allocations or decisions must be *incentive-compatible*. We impose the requirement of *strategy-proofness* that guarantees that no agent can gain by misrepresenting her true valuation irrespective of her beliefs about the valuations of other agents. A second basic requirement is that the social choice function is *individually rational*, i.e. an agent who is a user cannot be charged more than her valuation while a non-user cannot be charged at all. This requirement ensures that all agents participate voluntarily in the decision-making process.

Unfortunately, it is well-known that strategy-proofness and individual rationality are incompatible with *efficiency*. Since there is no rivalry in consumption, excluding an agent with a strictly positive valuation is not efficient. Since individual rationality requires that the contribution paid by each agent is not larger than his valuation of the public good, agents will have incentives to misreport their own valuation in order to lower their contribution. In view of this incompatibility, we adopt a second-best approach that is increasingly popular in the literature on mechanism design. We aim to identify a social choice function that minimizes the *maximal welfare loss* in the class of all strategy-proof and individually rational social choice functions.[2]

[1] See for instance, Deb and Razzolini (1999a, 1999b), Dobzinski et al. (2008), Moulin and Shenker (2001), Mutuswami (2005, 2008), Ohseto (2000, 2005), Olszewski (2004), and Yu (2007).

[2] The worst-case welfare objective function is a well-established and widely-used criterion. For applications in related areas, see Moulin and Shenker (2001), Moulin (2008), and Juarez (2008a, 2008b) in the context of public good provision. See also Koutsoupias and Papadimitriou (1999), Roughgarden (2002), and Roughgarden and Tardos (2002) in the computer science literature on the price of anarchy, introduced

The welfare loss of a social choice function at a profile of valuations is the difference between the aggregate welfare of the first-best and the aggregate welfare of the social choice function evaluated at the profile. The maximal welfare loss of a social choice function is the supremum, taken over all profiles of valuations, of its welfare loss. Then, each social choice function is evaluated according to its maximal welfare loss and the goal is to select a social choice function that minimizes it. Since we are interested in social choice functions satisfying individual rationality, it turns out that inefficiencies arise from the exclusion (as users) of some (or all) agents who have strictly positive valuations. The maximal welfare loss of a social choice function is then the sum of the valuations of all non-users of the good at the preference profile which maximizes this sum.

Our result is to show that the *equal-cost sharing* social choice function minimizes the maximum welfare loss among all strategy-proof and individually rational social choice functions. At every profile of valuations, the equal-cost sharing social choice function selects the allocation that maximizes (in the set-inclusion sense) the group of users subject to the following requirements: (i) each user's contribution is the cost of provision divided by the number of users, i.e. pays an equal-cost share, (ii) each user's contribution is no greater than her valuation, and (iii) all non-users pay zero. It is easy to verify that this social choice function is well-defined at every profile. It is a simple and attractive social choice function that satisfies several properties (such as *budget-balancedness*) in addition to strategy-proofness and individual rationality. Indeed, we use this fact in an important way to prove our result.

The paper most closely related to ours is Dobzinski et al. (2008). They show that the equal-cost sharing social choice function is a maximal welfare loss minimizer in the class of social choice functions that are strategy-proof, budget-balanced and satisfy an additional axiom called *equal treatment*. The authors write: "An interesting research problem is to characterize the class of mechanisms obtained by dropping the (admittedly strong) equal treatment condition." We do not offer such a characterization because we do not claim that the equal-cost sharing social choice function is the unique maximal welfare loss minimizer. However we are able to show that the equal-cost sharing social choice function is a maximal welfare loss minimizer in the larger class of social choice functions when the equal treatment

to measure the effects of selfish routing in a congested network.

axiom is dropped. Budget-balancedness is also replaced by the more innocuous condition of individual rationality.

Another related paper is Moulin and Shenker (2001). They consider the provision of a binary, excludable public good when the cost function is a submodular function of the set of users. They show that the social choice function associated with the Shapley value cost sharing formula (which corresponds to the equal-cost sharing method for the case of a binary public good with fixed cost of provision) is the unique social choice function that minimizes maximum welfare loss in the class of social choice functions that are defined from a *cross monotonic* cost sharing method and are *group strategy-proof*, individually rational, *non-subsidizing* (the cost shares are non negative), budget balanced, and satisfy *consumer sovereignty*. Cross monotonicity requires the price paid by a user to weakly decrease when the set of users expands.[3] Group strategy-proofness is an incentive-compatibility requirement when coalitions of agents are allowed to coordinate messages for mutual benefit. Consumer sovereignty ensures that every agent has a valuation that guarantees her participation irrespective of the valuations of other agents. Thus our result applies to a more specialized setting than that in Moulin and Shenker (2001) but establishes the optimality of the equal-cost sharing social choice function amongst a much broader class of social choice functions. We also note that strategy-proofness is a more compelling axiom than group strategy-proofness from a decision-theoretic perspective. If agents are ignorant of the valuations of other agents, assumptions about the ability of coalitions to coordinate their messages for mutual benefit require stronger justification. Group strategy-proofness can also be a demanding requirement in this setting−see Juarez (2008b).

The critical steps in our proof consist in showing that while searching for optimal strategy-proof and individually rational social choice functions, there is no loss of generality in restricting attention to social choice functions that satisfy the additional properties of being non-subsidizing, budged balanced, demand monotonic, cross monotonic and satisfying consumer sovereignty. It is then relatively straightforward to show that the equal-cost sharing social choice function is optimal in the restricted class of social choice functions.

The paper is organized as follows. Section 2 introduces the basic model and definitions,

[3]Cross monotonicity is related to the axiom of *population monotonicity* introduced and analyzed in Thomson (1983a, 1983b).

presents the main properties of social choice functions and states a general characterization result for strategy-proof social choice functions (Proposition 1). Section 3 describes the efficiency criterion of minimizing the maximal welfare loss, defines the equal-cost sharing social choice function, and presents the result (Theorem 1). Section 4 contains the proof of Theorem 1 while Section 5 contains some final remarks.

2 Preliminaries

Consider a finite set of agents $N = \{1, ..., n\}$ that has to decide on the provision of a binary public good. The public good is *binary* because it can either be provided (denoted by 1) or not provided (denoted by 0). Let $X = \{0, 1\}$ be the set of the two binary choices and let $x \in X$ be a generic choice. The public good is *excludable* whenever a subset of agents (called *non-users*) can be excluded from its use, even when $x = 1$. The set of agents that are not excluded are called *users*. A generic subset of users will be denoted by S. A public good is *pure* if no agent can be excluded from its consumption when the public good is produced; namely, when $x = 1$ the set of users S is the entire set of agents N. We postpone discussion of the cost of the public good till after Proposition 1. This proposition applies to all binary goods independently of cost specifications.

For each agent $i \in N$, let $\alpha_i \in \mathbb{R}_+$ be the (monetary) *valuation* that i assigns to the public good if it is produced and i is a user. By requiring that α_i be independent of the set of users we are implicitly assuming that there is no rivalry in the consumption of the public good. A *profile* $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}_+^N$ is a vector of valuations, one for each agent. For each subset of agents $S \subseteq N$, let $\mathbf{1}_S : N \to \{0, 1\}$ be the indicator function where for all $i \in N$,

$$\mathbf{1}_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

To simplify notation we write $\mathbf{1}_S^i$ instead of $\mathbf{1}_S(i)$. Let $p = (p_i)_{i \in N} \in \mathbb{R}^N$ be a vector of *prices* (or contributions).[4]

The set of agents N has to decide whether or not to provide the public good ($x \in X$), its set of users $S \in 2^N$, and the vector of contributions $p \in \mathbb{R}^N$. An *allocation* is a triple

[4] We are admitting the possibility of negative prices.

$(x, S, p) \in X \times 2^N \times \mathbb{R}^N$ with the property that $x = 0$ implies $S = \emptyset$.[5] Denote by $A \equiv \{(x, S, p) \in X \times 2^N \times \mathbb{R}^N \mid x = 0 \text{ implies } S = \emptyset\}$ the set of all allocations. Agent i's preferences on the set of allocations A depend on i's valuation $\alpha_i \in \mathbb{R}_+$ and are represented by the utility function $v_i : A \times \mathbb{R}_+ \to \mathbb{R}$, where for each $(x, S, p, \alpha_i) \in A \times \mathbb{R}_+$,

$$v_i(x, S, p, \alpha_i) = \mathbf{1}_S^i \cdot x \cdot \alpha_i - p_i.$$

Since the society N will remain fixed, a profile $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}_+^N$ completely describes a *problem*. We will write (α_i, α_{-i}) to emphasize the role of agent i in the profile α, and (α_S, α_{-S}) to emphasize the role of the subset of agents S.

A *social choice function* $f : \mathbb{R}_+^N \to A$ selects, for each profile $\alpha \in \mathbb{R}_+^N$, an allocation $f(\alpha) \in A$. Hence, a social choice function f can be identified by its three components $f = (x^f, S^f, p^f)$, where $x^f : \mathbb{R}_+^N \to \{0, 1\}$, $S^f : \mathbb{R}_+^N \to 2^N$, and $p^f : \mathbb{R}_+^N \to \mathbb{R}^N$. Namely, for each $\alpha \in \mathbb{R}_+^N$, $f(\alpha) = (x^f(\alpha), S^f(\alpha), p^f(\alpha))$; obviously, for all $\alpha \in \mathbb{R}_+^N$, if $x^f(\alpha) = 0$ then $S^f(\alpha) = \emptyset$. When no confusion arises we omit the superscript f and write $f = (x, S, p)$. Given a social choice function $f : \mathbb{R}_+^N \to A$ and an agent $i \in N$, let $\beta_i^f : \mathbb{R}_+^N \to \{0, 1\}$ be the function such that, for all $\alpha \in \mathbb{R}_+^N$, $\beta_i^f(\alpha) = \mathbf{1}_{S^f(\alpha)}^i \cdot x^f(\alpha)$; namely, $\beta_i^f(\alpha) = 1$ if i is a user at α and $\beta_i^f(\alpha) = 0$ if i is not a user at α.

A social choice function is *strategy-proof* if reporting truthfully is a dominant strategy for all agents at all profiles,. To state it formally, we need the notion of manipulation. Agent $i \in N$ *successfully manipulates* $f : \mathbb{R}_+^N \to A$ *at profile* $\alpha \in \mathbb{R}_+^N$ if there exists $\alpha_i' \in \mathbb{R}_+$ such that

$$v_i(x^f(\alpha_i', \alpha_{-i}), S^f(\alpha_i', \alpha_{-i}), p^f(\alpha_i', \alpha_{-i}), \alpha_i) > v_i(x^f(\alpha), S^f(\alpha), p^f(\alpha), \alpha_i).$$

In this case we say that i successfully manipulates f at α via α_i'.

Definition 1 A social choice function $f : \mathbb{R}_+^N \to A$ is *strategy-proof* if no agent successfully manipulates f at any profile.

We state without proof[6] a basic characterization of all strategy-proof social choice functions. Results of this nature are well-known in the literature; see for instance Dobzinski et

[5]Observe that we are not imposing yet any condition on the vector of prices p nor excluding the possibility that $x = 1$ and $S = \emptyset$.

[6]A complete proof can be found in the working paper version at the authors' websites.

al. (2008), Mehta et al. (2007), and Nisan (2008). It follows from an application of Myerson (1981)'s fundamental result.

Proposition 1 *A social choice function* $f : \mathbb{R}_+^N \longrightarrow A$ *is strategy-proof if and only if, for each* $i \in N$ *there exist two functions* $\phi_i^f : \mathbb{R}_+^{N\setminus\{i\}} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ *and* $h_i^f : \mathbb{R}_+^N \longrightarrow \mathbb{R}$ *such that*
(P1.a) *if* $\alpha_i > \phi_i^f(\alpha_{-i})$ *then* $\beta_i^f(\alpha) = 1$ *and* $p_i^f(\alpha) = \phi_i^f(\alpha_{-i}) - h_i^f(\alpha_{-i})$;
(P1.b) *if* $\alpha_i < \phi_i^f(\alpha_{-i})$ *then* $\beta_i^f(\alpha) = 0$ *and* $p_i^f(\alpha) = -h_i^f(\alpha_{-i})$;
(P1.c) *if* $\alpha_i = \phi_i^f(\alpha_{-i})$ *then either* $[\beta_i^f(\alpha) = 1$ *and* $p_i^f(\alpha) = \phi_i^f(\alpha_{-i}) - h_i^f(\alpha_{-i})]$ or $[\beta_i^f(\alpha) = 0$ *and* $p_i^f(\alpha) = -h_i^f(\alpha_{-i})]$.

A social choice function is *individually rational* at a profile if no agent obtains a lower utility than the utility she would have obtained by not participating.

Definition 2 A social choice function $f : \mathbb{R}_+^N \to A$ is *individually rational at* $\alpha \in \mathbb{R}_+^N$ if for all $i \in N$, $v_i(x^f(\alpha), S^f(\alpha), p^f(\alpha), \alpha_i) \geq 0$. A social choice function $f : \mathbb{R}_+^N \to A$ is *individually rational* if it is individually rational at all profiles.

Consider a strategy-proof and individually rational social choice function $f = (x, S, p)$. Fix $i \in N$ and $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$. By individual rationality,

$$v_i(x(0, \alpha_{-i}), S(0, \alpha_{-i}), p(0, \alpha_{-i}), 0) \geq 0.$$

Hence, $\beta_i^f(0, \alpha_{-i}) \cdot 0 - p_i(0, \alpha_{-i}) \geq 0$; namely, $p_i(0, \alpha_{-i}) \leq 0$. Then, by (P1.b) if $\phi_i(\alpha_{-i}) > 0$ or by (P1.c) if $\phi_i(\alpha_{-i}) = 0$, $p_i(0, \alpha_{-i}) = -h_i(\alpha_{-i}) \leq 0$. Hence, $h_i(\alpha_{-i}) \geq 0$. We state this fact as Remark 1 below.

Remark 1 Let $f : \mathbb{R}_+^N \to A$ be a strategy-proof and individually rational social choice function. Then, for all $i \in N$ and all $\alpha_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$, $h_i^f(\alpha_{-i}) \geq 0$.

From now on we will only consider cases where the cost of providing the binary public good is constant, and independent on the set of users; we normalize the cost of providing the public good to be equal to 1 while the cost of non-provision is 0. An allocation $(x, S, p) \in A$ satisfies the *no-deficit* condition if $x = 0$ implies $\sum_{i\in N} p_i \geq 0$ and $x = 1$ implies $\sum_{i\in N} p_i \geq 1$. An allocation $(x, S, p) \in A$ satisfies the *no-waste* condition if $x = 1$ implies $S^f(\alpha) \neq \emptyset$. Let FA be the set of allocations which satisfy the no-deficit and no-waste conditions. Henceforth, we will focus on social choice functions that satisfy these conditions. Namely, for all $\alpha \in \mathbb{R}^N_+$, $x^f(\alpha) = 0$ if and only if $S^f(\alpha) = \emptyset$ and $\sum_{i \in N} p_i^f(\alpha) \geq 0$, and $x^f(\alpha) = 1$ if and only if $S^f(\alpha) \neq \emptyset$ and $\sum_{i \in N} p_i^f(\alpha) \geq 1$. This will allow us to omit the function $x^f : \mathbb{R}^N_+ \rightarrow \{0, 1\}$ in the description of the social choice function f because after specifying the set of users and the vector of individual contributions, we can infer whether or not the public good is provided.

Let Φ be the class of strategy-proof and individually rational social choice functions that satisfy the no-deficit and no-waste conditions.

3 Efficiency, Welfare Loss and Equal-cost Sharing

3.1 Purely Efficient Social Choice Functions

There is a natural notion of (first-best) *efficiency* for pure public goods. Remember that a public good is *pure* if once the public good is produced no agent can be excluded from its consumption ($x = 1$ implies $S = N$). Assume the public good is pure. Then, the following notion of *efficiency* is natural.

Definition 3 A social choice function $f : \mathbb{R}^N_+ \longrightarrow FA$ is *(purely) efficient* if for all $\alpha \in \mathbb{R}^N_+$:
(i) $\sum_{i \in N} \alpha_i \geq 1$ implies $S^f(\alpha) = N$.
(ii) $\sum_{i \in N} \alpha_i < 1$ implies $S^f(\alpha) = \emptyset$.

Observe that (purely) efficient social choice functions refer to pure public goods; as soon as there is exclusion (and a non-user has strictly positive valuation) efficiency is violated.

3.2 Minimizing the Maximal Welfare Loss

For any binary public good, pure or excludable, the first-best at profile $\alpha \in \mathbb{R}^N_+$ requires provision of the public good if $\sum_{i \in N} \alpha_i \geq 1$ and non-provision if $\sum_{i \in N} \alpha_i < 1$. Given $\alpha \in \mathbb{R}^N_+$, let $W(FB, \alpha) \equiv \max\{\sum_{i \in N} \alpha_i - 1, 0\}$ be the welfare of the first-best at profile α. We will show in Theorem 1 that, among the social choice functions in Φ, the equal-cost sharing social choice function minimizes the maximal welfare loss from the first-best.

Consider again the case of a public good with exclusion. Fix $f \in \Phi$ and consider $\alpha \in \mathbb{R}^N_+$.

The welfare of f at α is

$$W(f,\alpha) = \begin{cases} \sum_{i \in S^f(\alpha)} \alpha_i - \sum_{i \in N} p_i^f(\alpha) & \text{if } S^f(\alpha) \neq \emptyset \\ -\sum_{i \in N} p_i^f(\alpha) & \text{if } S^f(\alpha) = \emptyset. \end{cases}$$

Hence, the welfare loss from the first best of f at α is

$$WL(f,\alpha) = W(FB,\alpha) - W(f,\alpha) \tag{1}$$

$$= \begin{cases} \max\{\sum_{i \in N} \alpha_i - 1, 0\} - (\sum_{i \in S^f(\alpha)} \alpha_i - \sum_{i \in N} p_i^f(\alpha)) & \text{if } S^f(\alpha) \neq \emptyset \\ \max\{\sum_{i \in N} \alpha_i - 1, 0\} + \sum_{i \in N} p_i^f(\alpha) & \text{if } S^f(\alpha) = \emptyset \end{cases}$$

$$= \begin{cases} \max\{\sum_{i \notin S^f(\alpha)} \alpha_i - 1 + \sum_{i \in N} p_i^f(\alpha), -\sum_{i \in S^f(\alpha)} \alpha_i + \sum_{i \in N} p_i^f(\alpha)\} & \text{if } S^f(\alpha) \neq \emptyset \\ \max\{\sum_{i \in N} \alpha_i - 1, 0\} + \sum_{i \in N} p_i^f(\alpha) & \text{if } S^f(\alpha) = \emptyset. \end{cases}$$

Thus, the maximal welfare loss from the first best of f is

$$MWL(f) = \sup_{\alpha \in \mathbb{R}_+^N} WL(f,\alpha).$$

We want to minimize the maximal welfare loss on Φ by a choice of f, i.e. to find a social choice function $\hat{f} \in \Phi$ with the property that $MWL(\hat{f}) \leq MWL(f)$ for all $f \in \Phi$.

3.3 Equal-cost Sharing for Binary and Excludable Public Goods

The equal-cost sharing social choice function splits equally the cost of providing the binary and excludable public good among the maximal set of users for whom equal split is individually rational. Formally, for each $\alpha \in \mathbb{R}_+^N$, define the family of subsets of agents

$$U(\alpha) = \{S \in 2^N \mid \alpha_i \geq \frac{1}{\#S} \text{ for all } i \in S \text{ and } \alpha_j \leq \frac{1}{\#S+1} \text{ for all } j \notin S\}.$$

Given $\alpha \in \mathbb{R}_+^N$, $U(\alpha)$ is the family of sets of users that (i) satisfy individual rationality at α when the cost of the public good is uniformly distributed among the set of users and (ii) non-users do not strictly prefer to become a user by joining the group of users and pay the corresponding uniform contribution. Thus, every set in the family $U(\alpha)$ satisfies an internal and external stability property at α if the cost of providing the binary and excludable public good is equally shared among the set of users. Observe that for some profile $\alpha \in \mathbb{R}^N_+$, the family $U(\alpha)$ may contain only the empty set while for other profiles $\alpha' \in \mathbb{R}^N_+$ the family $U(\alpha')$ may contain more than one subset of agents. However, if $S, S' \in U(\alpha')$ then $S \cup S' \in U(\alpha')$. Therefore, for all $\alpha \in \mathbb{R}^N_+$, there exists a unique (set-inclusion) maximal set in $U(\alpha)$. Denote it by S_α.

Definition 4 The *equal-cost sharing social choice function* $f^{EC} : \mathbb{R}^N_+ \to FA$ is the social choice function that, for each $\alpha \in \mathbb{R}^N_+$,

$$S^{f^{EC}}(\alpha) = \begin{cases} S_\alpha & \text{if } U(\alpha) \neq \{\emptyset\} \\ \emptyset & \text{if } U(\alpha) = \{\emptyset\} \end{cases}$$

and, for all $i \in N$,

$$p_i^{f^{EC}}(\alpha) = \begin{cases} \frac{1}{\# S^{f^{EC}}(\alpha)} & \text{if } i \in S^{f^{EC}}(\alpha) \\ 0 & \text{if } i \notin S^{f^{EC}}(\alpha). \end{cases}$$

The following theorem is the main result of the paper.

Theorem 1 *The equal-cost sharing social choice function minimizes the maximal welfare loss among the set of all strategy-proof and individually rational social choice functions.*

4 Proof of Theorem 1

The structure of the proof of Theorem 1 is as follows. We restrict the class of strategy-proof and individually rational social choice functions by successively imposing some additional properties; in particular, the properties of being non-subsidizing, budged balanced, demand monotonic, cross-monotonic, and satisfying consumer sovereignty. We show that imposing the additional conditions is without loss of generality because if a social choice function f does not satisfy one of these additional properties we can always find a social choice function $\tilde{f}$ satisfying them with an equal or smaller maximal welfare loss. Finally, we show that the equal-cost sharing social choice function minimizes the maximal welfare loss among all strategy-proof, individually rational, non-subsidizing, budget balanced, demand monotonic, and cross monotonic social choice functions that satisfy consumer sovereignty. This suffices to prove the result.

4.1 Non-subsidizingness and Budget Balancedness

Some social choice functions may require that at some profile an agent is subsidized (i.e., pays a negative price). Social choice functions that exclude this possibility are called non-subsidizing.

Definition 5 A social choice function $f : \mathbb{R}^N_+ \to FA$ is *non-subsidizing at* $\alpha \in \mathbb{R}^N_+$ if for all $i \in N$, $p^f_i(\alpha) \geq 0$. A social choice function $f : \mathbb{R}^N_+ \to FA$ is *non-subsidizing* if it is non-subsidizing at all profiles.

Another basic property of social choice functions is that the sum of the prices be equal to the cost of providing the public good. For many applications we want to consider social choice functions that balance the budget.

Definition 6 A social choice function $f : \mathbb{R}^N_+ \to FA$ is *budget balanced at* $\alpha \in \mathbb{R}^N_+$ if $S^f(\alpha) = \emptyset$ implies $\sum_{i \in N} p^f_i(\alpha) = 0$, and $S^f(\alpha) \neq \emptyset$ implies $\sum_{i \in N} p^f_i(\alpha) = 1$. A social choice function $f : \mathbb{R}^N_+ \to FA$ is *budget balanced* if it is budget balanced at all profiles.

Lemma 1 below says that for any strategy-proof and individually rational social choice function f, we can find another strategy-proof, individually rational, non-subsidizing and budget balanced social choice function $\tilde{f}$ with the property that the welfare loss of $\tilde{f}$ at every profile α is smaller or equal than the welfare loss of f at α.

Lemma 1 *Let $f : \mathbb{R}^N_+ \to FA$ be a strategy-proof and individually rational social choice function. Then, there exists a strategy-proof, individually rational, non-subsidizing, and budget balanced social choice function $\tilde{f} : \mathbb{R}^N_+ \to FA$ such that for all $\alpha \in \mathbb{R}^N_+$,*

$$WL(\tilde{f}, \alpha) \leq WL(f, \alpha).$$

Proof Let $f : \mathbb{R}^N_+ \to FA$ be a strategy-proof and individually rational social choice function. We construct a strategy-proof, individually rational, non-subsidizing, and budget balanced social choice function $\tilde{f} : \mathbb{R}^N_+ \to FA$ with the property that for all $\alpha \in \mathbb{R}^N_+$, $WL(\tilde{f}, \alpha) \leq WL(f, \alpha)$ by applying conditions (P1.a), (P1.b) and (P1.c) in Proposition 1 to the family of duples $(\phi^{\tilde{f}}_i(\alpha_{-i}), h^{\tilde{f}}_i(\alpha_{-i}))_{i \in N}$ defined below. Moreover, $\tilde{f}$ will coincide with f in solving the indifference condition in (P1.c).

First, for all $\alpha \in \mathbb{R}^N_+$, set $h_i^{\tilde{f}}(\alpha_{-i}) = 0$ for all $i \in N$.

Second, to define the family $(\phi_i^{\tilde{f}})_{i \in N}$, let $\alpha \in \mathbb{R}^N_+$ be arbitrary. We distinguish between two cases:

Case 1: $S^f(\alpha) = \emptyset$. Then, set $\phi_i^{\tilde{f}}(\alpha_{-i}) = \phi_i^f(\alpha_{-i})$ for all $i \in N$, and if $\alpha_i = \phi_i^f(\alpha_{-i})$ then $i \notin S^{\tilde{f}}(\alpha)$. Note that $S^{\tilde{f}}(\alpha) = S^f(\alpha) = \varnothing$ and $p_i^{\tilde{f}}(\alpha) = 0$ for all $i \in N$. Moreover, $\tilde{f}$ is individually rational and non-subsidizing at α. Since $S^f(\alpha) = \emptyset$ and $\sum_{i \in N} p_i^{\tilde{f}}(\alpha) = 0$, $\tilde{f}$ is budget balanced at α. Obviously, since f is individually rational $\sum_{i \in N} p_i^f(\alpha) = 0$ and hence $WL(f, \alpha) = \max\{\sum_{i \in N} \alpha_i - 1, 0\} = WL(\tilde{f}, \alpha)$.

Case 2: $S^f(\alpha) \neq \emptyset$. Then, for all $i \notin S^f(\alpha)$ set $\phi_i^{\tilde{f}}(\alpha_{-i}) = \phi_i^f(\alpha_{-i})$ and if $\alpha_i = \phi_i^f(\alpha_{-i})$ then $i \notin S^{\tilde{f}}(\alpha)$. For all $i \in S^f(\alpha)$ choose any $\phi_i^{\tilde{f}}(\alpha_{-i}) \leq \phi_i^f(\alpha_{-i})$ with the property that $\sum_{i \in S^f(\alpha)} \phi_i^{\tilde{f}}(\alpha_{-i}) = 1$ and if $\alpha_i = \phi_i^f(\alpha_{-i})$ then $i \in S^{\tilde{f}}(\alpha)$. Observe that this is possible since

$$1 \leq \sum_{i \in S^f(\alpha)} p_i^f(\alpha) = \sum_{i \in S^f(\alpha)} \phi_i^f(\alpha_{-i}) - \sum_{i \in S^f(\alpha)} h_i^f(\alpha_{-i}) \leq \sum_{i \in S^f(\alpha)} \phi_i^f(\alpha_{-i}),$$

where the first inequality follows from the no-deficit condition and the last one from Remark 1. Then, $S^{\tilde{f}}(\alpha) = S^f(\alpha)$. Assume that $\sum_{i \in N} \alpha_i \geq 1$, otherwise f violates individual rationality or the no-deficit condition. Thus,

$$\sum_{i \notin S^f(\alpha)} \alpha_i \geq 1 - \sum_{i \in S^f(\alpha)} \alpha_i.$$

Hence, by (1), $WL(f, \alpha) = \sum_{i \notin S^f(\alpha)} \alpha_i + \sum_{i \in N} p_i^f(\alpha) - 1$. Again, by individual rationality and the no-deficit condition of f, $\sum_{i \in S^f(\alpha)} \alpha_i \geq 1$. Then, $S^{\tilde{f}}(\alpha) = S^f(\alpha)$. Since $p_i^{\tilde{f}}(\alpha) = \phi_i^{\tilde{f}}(\alpha_{-i}) \leq \alpha_i$ for all $i \in S^{\tilde{f}}(\alpha)$ and $p_i^{\tilde{f}}(\alpha) = 0$ for all $i \notin S^{\tilde{f}}(\alpha)$, $\tilde{f}$ is individually rational and non-subsidizing at α. Since $S^f(\alpha) \neq \emptyset$ and $\sum_{i \in N} p_i^{\tilde{f}}(\alpha) = \sum_{i \in S^{\tilde{f}}(\alpha)} \phi_i^{\tilde{f}}(\alpha_{-i}) = 1$, $\tilde{f}$ is budget balanced at α. Moreover, $WL(\tilde{f}, \alpha) = \sum_{i \notin S^{\tilde{f}}(\alpha)} \alpha_i \leq \sum_{i \notin S^f(\alpha)} \alpha_i + \sum_{i \in N} p_i^f(\alpha) - 1 = WL(f, \alpha)$.

Thus, we have defined a social choice function $\tilde{f}$ that is budget balanced, non-subsidizing, individually rational and, by Proposition 1, strategy-proof. Moreover, since for all $\alpha \in \mathbb{R}^N_+$, $WL(\tilde{f}, \alpha) \leq WL(f, \alpha)$, we have that $MWL(\tilde{f}) \leq MWL(f)$. ■

Remark 2 Given a strategy-proof social choice function f we can identify, by Proposition 1, a unique family of duples $(\phi_i^f, h_i^f)_{i \in N}$. On the other hand, given a family of duples $(\phi_i, h_i)_{i \in N}$ there are many social choice functions that generate it. This multiplicity is due to the alternative ways of solving the indifferences in condition (P1.c): either making the indifferent agent a user or a non-user. However, the social choice function that solves all indifferences by adding the indifferent agent to the set of users has larger welfare than any other social choice function (with the same family of duples) that solves some indifferences by making the indifferent agent a non-user. From now on, when we refer to the social choice function obtained from a family of duples $(\phi_i^f, h_i^f)_{i\in N}$ we will mean the particular social choice function that solves all indifferences by making the indifferent agent a user.

Let $\Phi^{NS\cap BB} \subsetneq \Phi$ be the set of strategy-proof, individually rational, non-subsidizing, and budget balanced social choice functions. Let $f \in \Phi^{NS\cap BB}$ and $\alpha \in \mathbb{R}_+^N$. Remark 3 below shows that the welfare loss of f at α can be written in a useful way.

Remark 3 Assume $S^f(\alpha) \neq \emptyset$. Since f is individually rational, $\alpha_i - p_i^f(\alpha) \geq 0$ for all $i \in S^f(\alpha)$. Summing up, $\sum_{i\in S^f(\alpha)} \alpha_i - \sum_{i\in S^f(\alpha)} p_i^f(\alpha) \geq 0$. By individual rationality, non-subsidizingness, and budget balancedness, $\sum_{i\in S^f(\alpha)} \alpha_i - 1 \geq 0$. Thus, $WL(f,\alpha)$ can be written as

$$WL(f,\alpha) = \begin{cases} \sum\limits_{i\notin S^f(\alpha)} \alpha_i & \text{if } S^f(\alpha) \neq \emptyset \\ \max\{\sum\limits_{i\in N} \alpha_i - 1, 0\} & \text{if } S^f(\alpha) = \emptyset. \end{cases}$$

The following result will be useful in the sequel.

Lemma 2 Let $\tilde{f}, f \in \Phi^{NS\cap BB}$ *and* $\alpha \in \mathbb{R}_+^N$ *be such that* $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$. *Then,* $WL(\tilde{f},\alpha) \leq WL(f,\alpha)$.

Proof First, note that if $\alpha \in \mathbb{R}_+^N$ is such that $x^f(\alpha) = x^{\tilde{f}}(\alpha)$, $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$ implies that $WL(\tilde{f},\alpha) \leq WL(f,\alpha)$. Suppose that $\alpha \in \mathbb{R}_+^N$ is such that $S^{\tilde{f}}(\alpha) \neq \emptyset$ and $S^f(\alpha) = \emptyset$ (the opposite cannot occur since $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$). Then, $WL(\tilde{f},\alpha) = \sum_{i\notin S^{\tilde{f}}(\alpha)} \alpha_i$ and $WL(f,\alpha) = \max\{\sum_{i\in N} \alpha_i - 1, 0\}$. Since $\tilde{f}$ satisfies the no-deficit condition and is individually rational, $\sum_{i\in S^{\tilde{f}}(\alpha)} \alpha_i \geq 1$ and therefore $WL(f,\alpha) = \sum_{i\in N} \alpha_i - 1 = \sum_{i\notin S^{\tilde{f}}(\alpha)} \alpha_i + \sum_{i\in S^{\tilde{f}}(\alpha)} \alpha_i - 1 \geq \sum_{i\notin S^{\tilde{f}}(\alpha)} \alpha_i = WL(\tilde{f},\alpha)$. ■

As an immediate consequence of Lemma 2 we have that if $\tilde{f}, f \in \Phi^{NS\cap BB}$ are such that $S^{\tilde{f}}(\alpha) \supseteq S^f(\alpha)$ for all $\alpha \in \mathbb{R}_+^N$ then, $MWL(\tilde{f}) \leq MWL(f)$.

4.2 Demand Monotonicity

We state now the property of demand monotonicity, introduced by Ohseto (2000), that will be very useful in the proof of Theorem 1. Demand monotonicity can be interpreted as a weak efficiency requirement. Its violation implies that at some profile the social choice function excludes an agent who is willing to join the group of users.

Definition 7 A social choice function $f : \mathbb{R}^N_+ \to FA$ is *demand monotonic* if for all $\alpha, \alpha' \in \mathbb{R}^N_+$ the following two conditions hold:
(DM.1) *if* $\alpha'_i \geq \alpha_i$ *for all* $i \in N$ *then* $S^f(\alpha') \supseteq S^f(\alpha)$*; and*
(DM.2) *if* $\alpha'_i \geq \alpha_i$ *for all* $i \in S^f(\alpha)$ *and* $\alpha'_j \leq \alpha_j$ *for all* $j \notin S^f(\alpha)$ *then* $S^f(\alpha') = S^f(\alpha)$.

Remark 4 Strategy-proofness, individual rationality, non-subsidizingness, and budget balancedness do not imply demand monotonicity (see Example 3 in Ohseto (2000)).

Definition 8 A social choice function $f : \mathbb{R}^N_+ \to FA$ is a *semiconstant cost sharing social choice function* if for all $\alpha, \alpha' \in \mathbb{R}^N_+$, $S^f(\alpha) = S^f(\alpha')$ implies $f(\alpha) = f(\alpha')$.

This is a simple class of social choice functions where agents' contributions only depend on the set of users. Ohseto (2000) shows that the following remark holds.

Remark 5 Any strategy-proof, individually rational, non-subsidizing, budget-balanced, and demand monotonic social choice function is a semiconstant cost sharing social choice function.

Let $\Phi^{NS \cap BB \cap DM} \subsetneq \Phi$ be the class of strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic social choice functions. Lemma 3 below shows that we can restrict our search of social choice functions minimizing the maximal welfare loss to the class $\Phi^{NS \cap BB \cap DM}$ without loss of generality.

Lemma 3 Let $f : \mathbb{R}^N_+ \to FA$ *be a strategy-proof, individually rational, non-subsidizing, and budget balanced social choice function and assume that* f *is not demand monotonic. Then, there exists a strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic social choice function* $\tilde{f} : \mathbb{R}^N_+ \to FA$ *that has a lower or equal maximal welfare loss than* f.

Proof Assume $f \in \Phi^{NS \cap BB}$ and f does not satisfy demand monotonicity. Then, there exist $\alpha'', \alpha' \in \mathbb{R}^N_+$ such that either

(a) $\alpha_i'' \geq \alpha_i'$ for all $i \in N$ and $S^f(\alpha') \nsubseteq S^f(\alpha'')$ or

(b) $\alpha_i'' \geq \alpha_i'$ for all $i \in S^f(\alpha')$ and $\alpha_i'' \leq \alpha_i'$ for all $i \notin S^f(\alpha')$ and $S^f(\alpha') \neq S^f(\alpha'')$.

The proof consists of constructing another social choice function that satisfies all the properties of the original social choice function and demand monotonicity additionally. Moreover, it has a maximal welfare loss no higher than that associated with the original social choice function. We proceed in three steps.

Step 1: Suppose that (a) occurs; otherwise, set $\bar{f} = f$ and go to Step 2. We informally describe the construction of a new social choice function $\bar{f}$. Let $Inf^f_{(1)}$ denote the set of profiles α' such that (i) there exists a profile α'' where all agents have higher valuations but the set of users at α'' does not contain the set of users at α' and (ii) there does not exist a profile α''' where all agents have lower valuations than α' but the set of users at α' does not contain the set of users at α'''. Let $NoDM^f_{(1)}(\alpha')$ denote the set of profiles α'' such that there exists $\alpha' \in Inf^f_{(1)}$ with respect to which (a) occurs. We construct $\bar{f}$ as follows. For all α not in $NoDM^f_{(1)}(\alpha')$ for any $\alpha' \in Inf^f_{(1)}$, we let $\bar{f}(\alpha) = f(\alpha)$. For all other α there exists at least one $\alpha' \in Inf^f_{(1)}$ with respect to which (a) occurs. Pick an arbitrary α'. Let $S^{\bar{f}}(\alpha) = N$ and $p^{\bar{f}}(\alpha) = p^f(\alpha')$. Thus, users at α' continue to pay the same amount while non users at α' pay zero. Formally, let

$$\begin{aligned} Inf^f_{(1)} = \quad & \{\alpha' \in \mathbb{R}^N_+ \mid \text{there exists } \alpha'' \in \mathbb{R}^N_+ \text{ such that } \alpha_i'' \geq \alpha_i' \text{ for all } i \in N,\ S^f(\alpha') \nsubseteq S^f(\alpha'') \\ & \text{and for all } \alpha''' \in \mathbb{R}^N_+ \backslash \{\alpha'\} \text{ such that } \alpha_i' \geq \alpha_i''' \text{ for all } i \in N,\ S^f(\alpha') \supseteq S^f(\alpha''')\}. \end{aligned}$$

For each $\alpha' \in Inf^f_{(1)}$, let

$$NoDM^f_{(1)}(\alpha') = \{\alpha'' \in \mathbb{R}^N_+ \mid \alpha_i'' \geq \alpha_i' \text{ for all } i \in N \text{ and } S^f(\alpha') \nsubseteq S^f(\alpha'')\}.$$

Let $\{\phi^f_i\}_{i \in N}$ be the family of functions associated to f identified in Proposition 1. By non-subsidizingness, the corresponding family $\{h^f_i\}_{i \in N}$ has the property that for all $i \in N$, $h^f_i(\alpha_{-i}) = 0$ for all $\alpha_{-i} \in \mathbb{R}^{N \backslash \{i\}}_+$. From $\{\phi^f_i\}_{i \in N}$, define the social choice function $\bar{f}$ by describing its associated family of functions $\{\phi^{\bar{f}}_i\}_{i \in N}$ as follows.

- For each $\alpha \notin \bigcup_{\alpha' \in Inf^f_{(1)}} NoDM^f_{(1)}(\alpha')$, set for all $i \in N$, $\phi^{\bar{f}}_i(\alpha_{-i}) = \phi^f_i(\alpha_{-i})$.
- For each $\alpha \in \bigcup_{\bar{\alpha} \in Inf^f_{(1)}} NoDM^f_{(1)}(\bar{\alpha})$, let $\alpha' \in Inf^f_{(1)}$ be a particular profile for which $\alpha \in NoDM^f_{(1)}(\alpha')$ (since this union is not necessarily disjoint, the specific defined

social choice function $\bar{f}$ may depend on this choice). Then, set for all $i \in N$,

$$\phi_i^{\bar{f}}(\alpha_{-i}) = \begin{cases} \phi_i^{f}(\alpha'_{-i}) & \text{if } i \in S^f(\alpha') \\ 0 & \text{otherwise.} \end{cases}$$

Note that $S^{\bar{f}}(\alpha) = N$ for all $\alpha \in \bigcup_{\bar{\alpha} \in Inf^f_{(1)}} NoDM^f_{(1)}(\bar{\alpha})$. Now all violations of type (a) have been accounted for. We now eliminate all violations of type (b).

Step 2: Consider $\bar{f}$, the outcome of Step 1, and assume (b) occurs for $\bar{f}$; otherwise, set $\tilde{f} = \bar{f}$ and go to Step 3. Let $Inf^f_{(2)}$ denote the set of profiles α' such that (i) there exists a profile α'' where all agents belonging to $S^{\bar{f}}(\alpha')$ have higher valuations than α', all agents not belonging to $S^{\bar{f}}(\alpha')$ have lower valuations than α' but $S^{\bar{f}}(\alpha') \neq S^{\bar{f}}(\alpha'')$ and (ii) there does not exist a profile α''' where all agents belonging to $S^{\bar{f}}(\alpha''')$ have lower valuations than α', all agents not belonging to $S^{\bar{f}}(\alpha''')$ have higher valuations than α', but $S^{\bar{f}}(\alpha''') \neq S^{\bar{f}}(\alpha')$. Let $NoDM^f_{(2)}(\alpha')$ denote the set of profiles α'' such that there exists $\alpha' \in Inf^{\bar{f}}_{(2)}$ with respect to which (b) occurs. We construct $\tilde{f}$ as follows. For all $\alpha \notin NoDM^{\bar{f}}_{(2)}(\alpha')$ for any $\alpha' \in Inf^{\bar{f}}_{(2)}$, we let $\tilde{f}(\alpha) = \bar{f}(\alpha)$. For each $\alpha' \in Inf^{\bar{f}}_{(2)}$, we let $S^{\tilde{f}}(\alpha') = N$ and $p^{\tilde{f}}(\alpha') = p^{\bar{f}}(\alpha')$. Thus, users at α' continue to pay the same amount while non users at α' pay zero. For all other α there exists at least one $\alpha' \in Inf^{\bar{f}}_{(2)}$ with respect to which (b) occurs. Pick an arbitrary α' and let $\tilde{f}(\alpha) = \tilde{f}(\alpha')$. Formally, let

$$\begin{aligned} Inf^{\bar{f}}_{(2)} = \;& \{\alpha' \in \mathbb{R}^N_+ \mid \text{there exists } \alpha'' \in \mathbb{R}^N_+ \text{ such that } \alpha''_i \geq \alpha'_i \text{ for all } i \in S^{\bar{f}}(\alpha'), \\ & \alpha''_i \leq \alpha'_i \text{ for all } i \notin S^{\bar{f}}(\alpha'),\ S^{\bar{f}}(\alpha') \neq S^{\bar{f}}(\alpha'') \text{ and for all } \alpha''' \in \mathbb{R}^N_+ \backslash \{\alpha'\} \\ & \text{such that } \alpha'''_i \leq \alpha'_i \text{ for all } i \in S^{\bar{f}}(\alpha''') \text{ and } \alpha'''_i \geq \alpha'_i \text{ for all } i \notin S^{\bar{f}}(\alpha'''), \\ & S^{\bar{f}}(\alpha''') = S^{\bar{f}}(\alpha')\}. \end{aligned}$$

For each $\alpha' \in Inf^{\bar{f}}_{(2)}$, define

$$\begin{aligned} NoDM^{\bar{f}}_{(2)}(\alpha') = \;& \{\alpha'' \in \mathbb{R}^N_+ \mid \alpha''_i \geq \alpha'_i \text{ for all } i \in S^{\bar{f}}(\alpha'), \alpha''_i \leq \alpha'_i \text{ for all } i \notin S^{\bar{f}}(\alpha'), \\ & \text{and } S^{\bar{f}}(\alpha') \neq S^{\bar{f}}(\alpha'')\}. \end{aligned}$$

Let $\{\phi_i^{\bar{f}}\}_{i \in N}$ be the family of functions associated to $\bar{f}$ identified in Proposition 1. From $\{\phi_i^{\bar{f}}\}_{i \in N}$, define the social choice function $\tilde{f}$ by describing its associated family of functions $\{\phi_i^{\tilde{f}}\}_{i \in N}$ as follows.

- For each $\alpha \notin Inf^{\bar{f}}_{(2)} \cup (\bigcup_{\alpha' \in Inf^{\bar{f}}_{(2)}} NoDM^{\bar{f}}_{(2)}(\alpha'))$, set for all $i \in N$, $\phi^{\tilde{f}}_i(\alpha_{-i}) = \phi^{\bar{f}}_i(\alpha_{-i})$.
- For each $\alpha' \in Inf^{\bar{f}}_{(2)}$, set for all $i \in N$,

$$\phi^{\tilde{f}}_i(\alpha'_{-i}) = \begin{cases} \phi^{\bar{f}}_i(\alpha'_{-i}) & \text{if } i \in S^{\bar{f}}(\alpha') \\ 0 & \text{otherwise.} \end{cases}$$

- For each $\alpha \in \bigcup_{\bar{\alpha} \in Inf^{\bar{f}}_{(2)}} NoDM^{\bar{f}}_{(2)}(\bar{\alpha}) \backslash Inf^{\bar{f}}_{(2)}$, let $\alpha' \in Inf^{f}_{(2)}$ be a particular profile for which $\alpha \in NoDM^{f}_{(2)}(\alpha')$. Then, set for all $i \in N$, $\phi^{\tilde{f}}_i(\alpha_{-i}) = \phi^{\tilde{f}}_i(\alpha'_{-i})$, where $\alpha' \in Inf^{\bar{f}}_2$ is such that $\alpha \in NoDM^{\bar{f}}_{(2)}(\alpha')$.

Step 3: Consider $\tilde{f}$, the outcome of Step 2. It is easy to see that $\tilde{f}$ and $\{\phi^{\tilde{f}}_i\}_{i\in N}$ satisfy properties (P1.a), (P1.b), and (P1.c) of Proposition 1. Hence, $\tilde{f}$ is strategy-proof. Moreover, $\tilde{f}$ is non-subsidizing and individually rational. For all $\alpha \in \mathbb{R}^N_+$ there exists $\alpha' \in \mathbb{R}^N_+$ such that $\sum_{i\in N} p^{\tilde{f}}(\alpha) = \sum_{i\in N} p^{f}(\alpha')$ and hence $\tilde{f}$ is budget balanced, since f is budget balanced. Furthermore, by construction of $\tilde{f}$, there do not exist $\alpha, \alpha' \in \mathbb{R}^N_+$ for which either property (a) or (b) of the negation of demand monotonicity holds; thus, by Lemma 2, $\tilde{f}$ has a lower or equal maximal welfare loss than f since, for all $\alpha \in \mathbb{R}^N_+$, $S^{\tilde{f}}(\alpha) \supseteq S^{f}(\alpha)$ holds. ■

From now on we restrict our search to the class $\Phi^{NS\cap BB\cap DM}$ without loss of generality. The following three lemmata will be very useful because they will allow us to pay attention only to profiles where the set of users is empty and define the maximal welfare loss of a social choice function as its aggregate loss (the maximal sum of non-users' valuations).

Lemma 4 *For all* $f \in \Phi^{NS\cap BB\cap DM}$ *and all* $\alpha \in R^N_+$ *such that* $\#S^f(\alpha) \geq 2$ *there exists* $\alpha' \in R^N_+$ *such that* (i) $S^f(\alpha') \subsetneq S^f(\alpha)$ *and* (ii) $WL(f,\alpha) < WL(f,\alpha')$.

Proof Let $f \in \Phi^{NS\cap BB\cap DM}$ and assume that $\alpha \in \mathbb{R}^N_+$ is such that $\#S^f(\alpha) \geq 2$. Since f is individually rational there exists at least one user in $S^f(\alpha)$ who pays a strictly positive price. Let $i \in S^f(\alpha)$ be one of such users. Then, by Proposition 1, $\alpha_i \geq \phi^f_i(\alpha_{-i}) > 0$. Since $S^f(\alpha) \neq \varnothing$, by Remark 3, $WL(f,\alpha) = \sum_{k\notin S^f(\alpha)} \alpha_k$. Let $j \in S^f(\alpha)\backslash\{i\}$ and consider any $\alpha'_j > \max\{1, \alpha_j\}$. Since f is demand monotonic, (DM.2) implies $S^f(\alpha) = S^f(\alpha'_j, \alpha_{-j})$. By Remark 5, $f(\alpha) = f(\alpha'_j, \alpha_{-j})$. Hence, $p^f_i(\alpha) = p^f_i(\alpha'_j, \alpha_{-j})$. Thus, by individual rationality of f and Proposition 1, $\phi^f_i(\alpha_{-i}) = \phi^f_i(\alpha'_j, \alpha_{-\{i,j\}})$. Moreover, $WL(f,\alpha) =$

$WL(f,(\alpha'_j,\alpha_{-j})) = \sum_{k\notin S^f(\alpha)} \alpha_k$. Consider now any profile $\alpha' = (\alpha'_i, \alpha'_j, \alpha_{-\{i,j\}})$ such that $0 < \alpha'_i < \phi_i^f(\alpha'_{-i}) = \phi_i^f(\alpha'_j, \alpha_{-\{i,j\}}) = \phi_i^f(\alpha_{-i})$. By Proposition 1, $i \notin S^f(\alpha')$. By (DM.1) in demand monotonicity, $S^f(\alpha') \subsetneq S^f(\alpha'_j, \alpha_{-j}) = S^f(\alpha)$. This proves that (i) holds. If $x^f(\alpha') = 1$ then $WL(f,\alpha') = \sum_{k\notin S^f(\alpha')} \alpha'_k > \sum_{k\notin S^f(\alpha)} \alpha_k = WL(f,\alpha)$. If $S^f(\alpha') = \emptyset$, then $WL(f,\alpha') = \sum_{k\in N} \alpha'_k - 1 = \sum_{k\notin S^f(\alpha)} \alpha'_k + \sum_{k\in S^f(\alpha)} \alpha'_k - 1 > \sum_{k\notin S^f(\alpha)} \alpha'_k = \sum_{k\notin S^f(\alpha)} \alpha_k$, where the strict inequality follows since $\alpha'_j > 1$ and $j \in S^f(\alpha)$ and the last equality holds by definition of α'. This proves that (ii) holds. ■

Let $f \in \Phi^{NS\cap BB\cap DM}$ and $\alpha \in \mathbb{R}^N_+$ be such that $S^f(\alpha) = \{i\}$. Define

$$A_i^\alpha(f) = \{\alpha' \in \mathbb{R}^N_+ \mid \alpha'_j = \alpha_j \text{ for all } j \neq i \text{ and } \alpha'_i < 1\}.$$

Lemma 5 *For all $f \in \Phi^{NS\cap BB\cap DM}$ and all $\alpha \in \mathbb{R}^N_+$ such that $S^f(\alpha) = \{i\}$,*

$$WL(f,\alpha) = \sup_{\alpha' \in A_i^\alpha(f)} WL(f,\alpha').$$

Proof Let $f \in \Phi^{NS\cap BB\cap DM}$ and assume that $\alpha \in \mathbb{R}^N_+$ is such that $S^f(\alpha) = \{i\}$. By individual rationality of f and Proposition 1, $\phi_i^f(\alpha_{-i}) = 1 \leq \alpha_i$. By Remark 3, and since $S^f(\alpha) \neq \emptyset$, $WL(f,\alpha) = \sum_{j\neq i} \alpha_j$. Now, consider any $\alpha' \in A_i^\alpha(f)$. By (DM.1) in demand monotonicity and Proposition 1, $S^f(\alpha') = \varnothing$. Thus, by Remark 2, $WL(f,\alpha') = \max\{\sum_{j\neq i} \alpha'_j + \alpha'_i - 1, 0\}$. Hence,

$$\sup_{\alpha' \in A_i^\alpha(f)} WL(f,\alpha') = \sup_{\alpha' \in A_i^\alpha(f)} \max\{\textstyle\sum_{j\neq i} \alpha'_j + \alpha'_i - 1, 0\} = \sum_{j\neq i} \alpha'_j = \sum_{j\neq i} \alpha_j = WL(f,\alpha).$$

This completes the proof of Lemma 5. ■

Given $f \in \Phi^{NS\cap BB\cap DM}$, define the set of profiles where the set of users is empty as $A^\varnothing(f) \equiv \{\alpha \in \mathbb{R}^N_+ \mid S^f(\alpha) = \varnothing\}$. By Proposition 1 and Remark 3,

$$A^\varnothing(f) = \{\alpha \in \mathbb{R}^N_+ \mid \alpha_i < \phi_i^f(\alpha_{-i}) \text{ for all } i \in N\}. \tag{2}$$

Observe that for any $\alpha \in \mathbb{R}^N_+$ such that $S^f(\alpha) = \{i\}$,

$$A_i^\alpha(f) \subseteq A^\varnothing(f). \tag{3}$$

Lemma 6 *For all* $f \in \Phi^{NS \cap BB \cap DM}$, $\sup_{\alpha \in \mathbb{R}_+^N} WL(f, \alpha) = \sup_{\alpha \in A^{\varnothing}(f)} \sum_{i \in N} \phi_i^f(\alpha_{-i}) - 1.$

Proof Let $f \in \Phi^{NS \cap BB \cap DM}$. Then,

$$
\begin{aligned}
\sup_{\alpha \in \mathbb{R}_+^N} WL(f, \alpha) &= \sup_{\alpha \in \{\alpha' \in \mathbb{R}_+^N | \# S^f(\alpha') \geq 2\} \cup \{\alpha' \in \mathbb{R}_+^N | \# S^f(\alpha') = 1\} \cup A^{\varnothing}(f)} WL(f, \alpha) \\
&= \sup_{\alpha \in \{\alpha' \in \mathbb{R}_+^N | \# S^f(\alpha') = 1\} \cup A^{\varnothing}(f)} WL(f, \alpha) \\
&= \sup_{\alpha \in A^{\varnothing}(f)} WL(f, \alpha) \\
&= \sup_{\alpha \in A^{\varnothing}(f)} \max\{\sum_{i \in N} \alpha_i - 1, 0\} \\
&= \sup_{\alpha \in A^{\varnothing}(f)} \sum_{i \in N} \phi_i^f(\alpha_{-i}) - 1.
\end{aligned}
$$

The second equality follows from Lemma 4, the third one from Lemma 5 and condition (3), the fourth equality follows from the definition of $WL(f, \alpha)$, and the fifth equality follows from Proposition 1. ■

From Lemma 6 it follows immediately that the maximal welfare loss of the equal-cost sharing social choice function is equal to

$$
MWL(f^{EC}) = \frac{1}{2} + ... + \frac{1}{n} = \sum_{i=2}^{n} \frac{1}{i}. \tag{4}
$$

This result is well-known in the literature; see for instance Dobzinski et al. (2008) and Roughgarden and Sundararajan (2006).

4.3 Consumer sovereignty

The next property requires social choice functions to respond to agents' valuations: each agent has a valuation that guarantees that the public good is provided and that she is a user.

Definition 9 A social choice function $f : \mathbb{R}_+^N \to FA$ satisfies *consumer sovereignty* if for all $i \in N$ there exists $\overline{\alpha}_i \in \mathbb{R}_+$ such that for all $\alpha_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$, $i \in S^f(\overline{\alpha}_i, \alpha_{-i})$.

We state in Lemma 7 below the observation that a social choice function that minimizes the maximal welfare loss has to satisfy *consumer sovereignty*.

Lemma 7 Let $f : \mathbb{R}^N_+ \to FA$ *be a social choice function that minimizes the maximal welfare loss. Then,* f *satisfies consumer sovereignty.*

Proof By Proposition 1, the welfare loss of any social choice function f that does not satisfy consumer sovereignty is unboundedly large. ■

4.4 Cross monotonicity

The final additional property on social choice functions that we consider is *cross monotonicity.* It imposes conditions on the price vector chosen by the social choice function at two profiles for which the set of users at one profile is contained in the set of users at the other profile: the price paid by the users can not increase if more agents become users at the new profile.[7]

Definition 10 A social choice function $f : \mathbb{R}^N_+ \to FA$ satisfies *cross monotonicity* if for all $\alpha, \alpha' \in \mathbb{R}^N_+$ such that $S^f(\alpha) \subset S^f(\alpha')$, $p_i^f(\alpha) \geq p_i^f(\alpha')$ for all $i \in S^f(\alpha)$.

Lemma 8 below says that cross monotonicity is satisfied by all strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic social choice functions for which consumer sovereignty holds.[8]

Lemma 8 Let $f : \mathbb{R}^N_+ \to FA$ *be a strategy-proof, individually rational, non-subsidizing, budget balanced, and demand monotonic social choice function that satisfies consumer sovereignty. Then,* f *is cross monotonic.*

Proof Let f be a social choice function satisfying the hypothesis of Lemma 8 and let $\alpha, \alpha' \in \mathbb{R}^N_+$ be such that $S^f(\alpha) \subset S^f(\alpha')$. We want to show that $p_i^f(\alpha) \geq p_i^f(\alpha')$ for all $i \in S^f(\alpha)$. If $S^f(\alpha) = N$ then, by Remark 5, f is cross monotonic. Let $\widetilde{\alpha} \in \mathbb{R}^N_+$ be such that

$$\widetilde{\alpha}_j = \begin{cases} \alpha_j & \text{if } j \in S^f(\alpha) \\ 0 & \text{if } j \notin S^f(\alpha). \end{cases}$$

[7] This property has already been used in a more general public good setting by Moulin and Shenker (2001). Similar notions have also been used in different settings under the name of monotonicity with respect to changes in the number of agents or population monotonicity (see for instance Thomson (1983a, 1983b) and Sprumont (1990)).

[8] Variants of this result have been proved in Ohseto (2000) and Juarez (2008c).

By (DM.2), $S^f(\alpha) = S^f(\widetilde{\alpha})$. By Remark 5, $p^f(\alpha) = p^f(\widetilde{\alpha})$. By Proposition 1, for all $j \in S^f(\widetilde{\alpha})$, $p^f_j(\widetilde{\alpha}) = \phi^f_j(\widetilde{\alpha}_{-j})$ and by individual rationality $\widetilde{\alpha}_j \geq \phi^f_j(\widetilde{\alpha}_{-j})$. Take any $j^1 \in S^f(\widetilde{\alpha})$ and let $\widetilde{\alpha}^1 = (\phi^f_{j^1}(\widetilde{\alpha}_{-j^1}), \widetilde{\alpha}_{-j^1})$. By Proposition 1, $j^1 \in S^f(\widetilde{\alpha}^1)$. By (DM.2), $S^f(\widetilde{\alpha}^1) = S^f(\widetilde{\alpha})$. By Remark 5, $p^f(\widetilde{\alpha}^1) = p^f(\widetilde{\alpha})$. Iterating this argument for all $j \in S^f(\widetilde{\alpha})$ we obtain a profile $\widehat{\alpha}$, where

$$\widehat{\alpha}_j = \begin{cases} \phi^f_j(\widetilde{\alpha}_{-j}) & \text{if } j \in S^f(\widetilde{\alpha}) \\ 0 & \text{if } j \notin S^f(\widetilde{\alpha}) \end{cases}$$

such that $S^f(\widehat{\alpha}) = S^f(\alpha)$ and $p^f(\widehat{\alpha}) = p^f(\alpha)$. By consumer sovereignty, for each $j \in S^f(\alpha') \backslash S^f(\alpha) \equiv T$, there exists $\overline{\alpha}_j$ such that $\overline{\alpha}$ defined by

$$\overline{\alpha}_j = \begin{cases} \phi^f_j(\overline{\alpha}_{-j}) & \text{if } j \in S^f(\widetilde{\alpha}) \\ \overline{\alpha}_j & \text{if } j \in T \\ 0 & \text{if } j \notin S^f(\widetilde{\alpha}) \cup T \end{cases}$$

is such that, by (DM.1) in demand monotonicity, $S^f(\overline{\alpha}) \supseteq S^f(\alpha') = S^f(\alpha) \cup T$. We want to show that $S^f(\overline{\alpha}) = S^f(\alpha')$. Define $\alpha'' \in \mathbb{R}^N_+$ by

$$\alpha''_j = \begin{cases} \alpha'_j & \text{if } j \in S^f(\alpha') \\ 0 & \text{if } j \notin S^f(\alpha'). \end{cases}$$

By (DM.2) in demand monotonicity, $S^f(\alpha'') = S^f(\alpha')$. Consider any profile α^* such that $\alpha^*_j > \max\{\alpha''_j, \overline{\alpha}_j\}$ for all $j \in S^f(\alpha')$ and $\alpha^*_j = 0$ for all $j \notin S^f(\alpha')$. By (DM.2) in demand monotonicity, $S^f(\alpha^*) = S^f(\alpha'') = S^f(\alpha')$. Again, by (DM.2), $S^f(\alpha^*) = S^f(\overline{\alpha})$. Thus, $S^f(\overline{\alpha}) = S^f(\alpha')$. By Remark 5, $p^f(\overline{\alpha}) = p^f(\alpha')$. Since for all $i \in S^f(\alpha)$, $\overline{\alpha}_i = p^f(\alpha)$ and, by individual rationality $\overline{\alpha}_i \geq p^f(\overline{\alpha})$, we have that $p^f_i(\alpha') = p^f_i(\overline{\alpha}) \leq \overline{\alpha}_i = p^f_i(\alpha)$ ■

Let Φ^* be the class of all strategy-proof, individually rational, non-subsidizing, budget balanced, demand and cross monotonic social choice functions that satisfy consumer sovereignty.

4.5 Proof of Theorem 1

By Lemmata 1, 3, 7, and 8 we can restrict our search for a social choice function in Φ (all strategy-proof and individually rational social choice functions) minimizing the maximal welfare loss to the class of social choice functions in Φ^*. Thus, consider any $f \in \Phi^*$ such that $f \neq f^{EC}$. By Remark 5, f is a semiconstant cost sharing social choice function. Hence, for all $\alpha, \alpha' \in \mathbb{R}_+^N$ such that $S^f(\alpha) = S^f(\alpha')$, $f(\alpha) = f(\alpha')$. Pick $\bar{\alpha} \in \mathbb{R}_+^N$ with the property that for each $i \in N$, $i \in S^f(\overline{\alpha}_i, \alpha_{-i})$ for all $\alpha_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$. Observe that by consumer sovereignty such an $\bar{\alpha}$ exists. Then $S^f(\bar{\alpha}) = N$. Let $i_1 \in N$ be an agent such that $p_{i_1}^f(\bar{\alpha}) = \max_{j \in N} p_j^f(\bar{\alpha})$. Observe that, by Remark 5, $p^f(\alpha) = p^f(\bar{\alpha})$ for all α such that $S^f(\alpha) = N$. By budget balancedness, $\sum_{j \in N} p_j^f(\bar{\alpha}) = 1$; therefore $p_{i_1}^f(\bar{\alpha}) \geq \frac{1}{n}$. By individual rationality and Proposition 1, $\phi_{i_1}^f(\bar{\alpha}_{-i_1}) = p_{i_1}^f(\bar{\alpha}) > 0$. Now pick $\bar{\alpha}^1 = (0, \bar{\alpha}_{-i_1})$. Observe that by consumer sovereignty and Proposition 1, $S^f(\bar{\alpha}^1) = N \backslash \{i_1\}$. Let $i_2 \in N$ be an agent such that $p_{i_2}^f(\bar{\alpha}^1) = \max_{j \in S^f(\bar{\alpha}^1)} p_j^f(\bar{\alpha}^1)$. Then $p_{i_2}(\bar{\alpha}^1) \geq \frac{1}{n-1}$. Proceeding in this manner we get a profile $(p_{i_1}^f(\bar{\alpha}), ..., p_{i_n}^f(\bar{\alpha}^{n-1}))$ with the property that $p_{i_j}^f(\bar{\alpha}^{j-1}) \geq \frac{1}{n-j+1}$ holds for all $j = 1, ..., n$. Note that by cross monotonicity the lowest price that agent i_1 pays for participating is $p_{i_1}^f(\bar{\alpha})$ since all other agents are users too at $\bar{\alpha}$. Similarly the lowest price that agent i_2 pays for participating as an user when i_1 does not participate is $p_{i_2}^f(\bar{\alpha}^1)$. In general, the lowest price that agent i_j pays for participating when agents from i_1 to i_{j-1} are not participating is $p_{i_j}^f(\bar{\alpha}^{j-1})$. Therefore, $S^f(p_{i_1}^f(\bar{\alpha}) - \frac{\varepsilon}{n}, ..., p_{i_n}^f(\bar{\alpha}^{n-1}) - \frac{\varepsilon}{n}) = \varnothing$ for any $\varepsilon > 0$. Hence, $MWL(f) \equiv \sup_{\alpha \in \mathbb{R}_+^N} \sum_{i \notin S^f(\alpha)} \alpha_i - 1 \geq \sum_{j=1}^n p_{i_j}^f(\bar{\alpha}^{j-1}) - 1 \geq \sum_{i=2}^n \frac{1}{i} = MWL(f^{EC})$. This establishes Theorem 1. ■

4.6 Discussion

In this subsection we illustrate the welfare loss associated with the equal-cost sharing social choice function in the case of two agents. This enables us to highlight the reason why it minimizes the maximal welfare loss with respect to other social choice functions in the class Φ^*. By our earlier arguments any social choice function f belonging to Φ^* in the two-agent case can be represented in the following way:

$$\varphi_1^f(\alpha_2) = \begin{cases} x & \text{if } \alpha_2 \geq y \\ 1 & \text{if } \alpha_2 < y \end{cases} \qquad \text{and} \qquad \varphi_2^f(\alpha_1) = \begin{cases} y & \text{if } \alpha_1 \geq x \\ 1 & \text{if } \alpha_1 < x \end{cases},$$

where $x, y \geq 0$ and $x + y = 1$. Hence, every f can be identified with a point (x, y) on the line $\alpha_1 + \alpha_2 = 1$. The social choice function f^{EC} corresponds to the point $(\frac{1}{2}, \frac{1}{2})$ represented in Diagram 1.

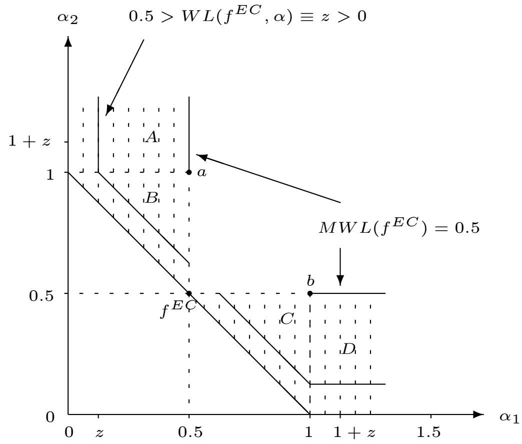

Diagram 1

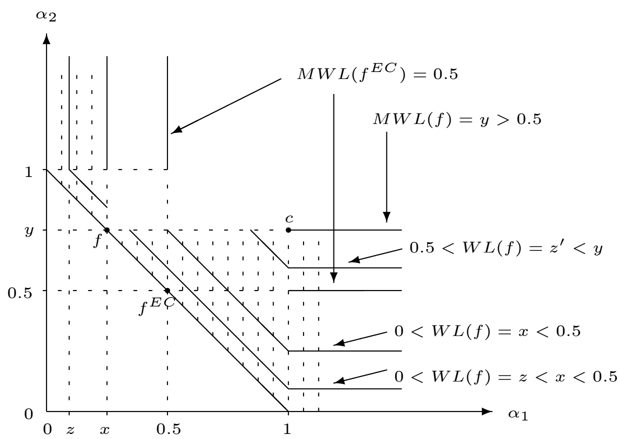

Diagram 2

The hatched area represents profiles where f^{EC} is suboptimal with respect to the first-best. In area A (resp. D) the public good is provided but only agent 2 (resp. 1) is a user, while in areas B and C it is not provided. In the first best the public good is provided to all agents in these regions. We also exhibit an "indifference curve" joining profiles where the welfare loss is z. Note that the maximal welfare loss occurs at points such as a and b corresponding to profiles where one agent has valuation 1 and the other 0.5, and is equal to 0.5. In Diagram 2 we show another rule f represented by (x, y) with $y > 0.5$. The hatched area represents now profiles where f is suboptimal with respect to the first best. Note that the maximal welfare loss now occurs at points such as c and is equal to $y > 0.5$.

5 Final Remarks

The equal-cost sharing social choice function is not the unique social choice function minimizing the maximal welfare loss in the class of strategy-proof and individually rational social choice functions. It is easy to construct a social choice function that differs from the equal-cost sharing social choice function at profiles where the welfare loss is not maximal but has the same maximal welfare loss as the equal-cost sharing social choice function.

We have considered a much broader set of social choice functions than those existing in the literature. Our result is therefore not a consequence of any existing result.[9]

[9] A working paper version available on the websites of the authors contains an example showing that the

Several open questions remain. Our Theorem 1 answers a question posed by Dobzinski et al. (2008) for the case of a binary public good with fixed cost. In particular, we show that the equal-cost sharing rule minimizes the maximal welfare loss, without imposing any equal treatment of equals or anonymity axiom. An important open question is whether our result extends to the non binary public good setting and/or for more general cost function specifications. An interesting case is when the public good is binary and the cost of provision is increasing in the cardinality of the set of users.

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set of social choice functions we consider is strictly larger than the set of social choice functions considered in Moulin and Shenker (2001).

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