Games and Economic Behavior 67 (2009) 745–749

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Note

Group incentive compatibility for matching with contracts

John William Hatfield[a], Fuhito Kojima[b,*]

[a] Graduate School of Business, Stanford University, Stanford, CA 94305, USA
[b] Cowles Foundation, Yale University, New Haven, CT 06510, USA

A R T I C L E   I N F O

Article history:
Received 15 May 2008
Available online 30 January 2009

JEL classification:
C78
D44

Keywords:
Matching
Matching with contracts
Substitutes
Law of aggregate demand
Stability
Strategy-proofness
Group strategy-proofness

A B S T R A C T

Hatfield and Milgrom [Hatfield, John William, Milgrom, Paul R., 2005. Matching with contracts. Amer. Econ. Rev. 95, 913–935] present a unified model of matching with contracts, which includes the standard two-sided matching and some package auction models as special cases. They show that the doctor-optimal stable mechanism is strategy-proof for doctors if hospitals' preferences satisfy substitutes and the law of aggregate demand. We show that the doctor-optimal stable mechanism is group strategy-proof for doctors under these same conditions. That is, no group of doctors can make each of its members strictly better off by jointly misreporting their preferences. We derive as a corollary of this result that no individually rational allocation is preferred by all the doctors to the doctor-optimal stable allocation.

1. Introduction

The theory of two-sided matching markets has interested researchers for its theoretical appeal and its relevance to the design of real-world institutions. The medical residency matching mechanism in the U.S. (National Resident Matching Program) and student assignment systems in New York City and Boston (among others) are examples of mechanisms designed by economists using the theory.[1] Hatfield and Milgrom (2005) present a unified framework of matching with contracts, which includes the two-sided matching models and package auction models as special cases. They introduce the substitutes condition of contracts, which generalizes the substitutability condition in the matching literature (Roth and Sotomayor, 1990). They also introduce a new condition they call the law of aggregate demand, which states that hospitals choose more contracts as the set of contracts to choose from expands. They show that the doctor-optimal stable mechanism is strategy-proof for doctors if hospitals' preferences satisfy substitutes and the law of aggregate demand. That is, no single doctor is made strictly better off by misreporting her preferences.

We show that this same mechanism is in fact group strategy-proof for doctors. That is, no group of doctors can make each of its members strictly better off by jointly misreporting their preferences. Under the substitutes condition, the law of aggregate demand is essentially the weakest possible sufficient condition for group strategy-proofness (or even strategy-

* Corresponding author.
E-mail addresses: hatfield_john@gsb.stanford.edu (J.W. Hatfield), fuhitokojima1979@gmail.com (F. Kojima).

[1] The theory was pioneered by Gale and Shapley (1962). For applications to labor markets, see Roth (1984) and Roth and Peranson (1999). For applications to student assignment, see for example Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu et al. (2005a, 2005b).

0899-8256/$ – see front matter
doi:10.1016/j.geb.2009.01.007

proofness).[2] The many previous results on strategy-proofness and group strategy-proofness (Dubins and Freedman, 1981; Roth, 1982; Martinez et al., 2004; Abdulkadiroğlu, 2005; Hatfield and Milgrom, 2005) are special cases of this result. Furthermore, it is well known that for the doctor-optimal stable mechanism a group of doctors can deviate in such a way to make some members strictly better off and none worse off (Dubins and Freedman, 1981) and that the mechanism is not even strategy-proof for hospitals (Roth, 1982).

As an application of our main result, we show a welfare result known as "weak Pareto optimality." More specifically, if preferences of every hospital satisfy substitutes and the law of aggregate demand, then there exists no individually rational allocation that every doctor strictly prefers to the doctor-optimal stable allocation. This result has been shown by a number of studies under various assumptions (Roth, 1982; Martinez et al., 2004; Kojima, 2007).

In addition to its theoretical appeal, our results have interesting implications for the various applications of matching models, especially those regarding school choice (Abdulkadiroğlu and Sönmez, 2003). Focusing on strategic behavior and welfare of students is natural in the school choice setting, since school priorities are often set by the law and manipulations by schools are impossible, and schools are regarded as objects whose priorities are not taken into account in welfare consideration. The generality of our preferences is important, since school priorities often violate assumptions under which the incentive and welfare conclusions have been shown in the literature. Some of the public schools in New York City are, for example, required to admit a balanced mix of students from high, middle and low test score populations (Abdulkadiroğlu et al., 2005b). Such requirements can easily be seen to violate the previous conditions, but still satisfy substitutes and the law of aggregate demand. Thus our analysis gives some additional justification for the use of the student-optimal stable mechanism originally advocated by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003).

2. Model

There are finite sets D and H of doctors and hospitals. X is the set of contracts. Each contract $x \in X$ is bilateral, so that it is associated with a doctor $x_D \in D$ and a hospital $x_H \in H$. We assume that each doctor can sign at most one contract. The null contract, meaning that the doctor has no contract, is denoted by $\emptyset$. For each $d \in D$, P_d is a strict preference relation on $\{x \in X \mid x_D = d\} \cup \{\emptyset\}$. A contract is *acceptable* if it is preferred to the null contract and *unacceptable* if it is not preferred to the null contract. For each $d \in D$ and $X' \subset X$, we define the *chosen set* C_d by

$$C_d(X') = \begin{cases} \emptyset & \text{if all } x \in X' \text{ such that } x_D = d \text{ are unacceptable to } d, \\ \max_{P_d}\{x \in X' \mid x_D = d\} & \text{otherwise,} \end{cases}$$

for any $X' \subset X$. Let $C_D(X') = \bigcup_{d \in D} C_d(X')$ be the set of contracts chosen from X' by some doctor.

We allow each hospital to sign multiple contracts, and assume that each hospital $h \in H$ has a preference relation P_h on subsets of contracts involving it. For any $X' \subset X$, define $C_h(X')$ by

$$C_h(X') = \max_{P_h}\{X'' \subset X' \mid x \in X'' \;\Rightarrow\; x_H = h \text{ and } x, x' \in X'',\; x \neq x' \;\Rightarrow\; x_D \neq x'_D\}.$$

Let $C_H(X') = \bigcup_{h \in H} C_h(X')$ be the set of contracts chosen from X' by some hospital.

We write $P = (P_d)_{d \in D}$ to denote a preference profile of doctors. We also write P_{-d} to denote $(P_{d'})_{d' \neq d}$ for $d \in D$, and $P_{D'}$ to denote $(P_d)_{d \in D'}$ and $P_{-D'}$ to denote $(P_d)_{d \notin D'}$ for $D' \subset D$.

A set of contracts $X' \subset X$ is an *allocation* if $x, x' \in X'$ and $x \neq x'$ imply $x_D \neq x'_D$. In words, a set of contracts is an allocation if each doctor signs at most one contract.

Definition 1. A set of contracts $X' \subset X$ is a *stable allocation* (or a *stable set of contracts*) if

1. $C_D(X') = C_H(X') = X'$, and
2. there exists no hospital h and set of contracts $X'' \neq C_h(X')$ such that $X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'')$.

When condition (2) is violated by some X'', we say that X'' is a blocking set for X'. For any $X' \subset X$ and $h \in H$, define the *rejected set* by $R_h(X') = X' - C_h(X')$. $R_h(X')$ is the set of contracts in X' which h is willing to reject.

Definition 2. Contracts are *substitutes* for h if we have $R_h(X') \subset R_h(X'')$ for all $X' \subset X'' \subset X$.

Hatfield and Milgrom (2005) show that there exists a stable set of contracts when contracts are substitutes for every hospital.

[2] Theorem 12 of Hatfield and Milgrom (2005) presents a sense in which the law of aggregate demand is "essentially the weakest" sufficient condition for strategy-proofness of the doctor-optimal stable mechanism. Explaining the theorem, they write "to the extent that the law of aggregate demand for hospital preferences has observable consequences for the progress of the doctor-offering algorithm, it is an indispensable condition to ensure the dominant strategy property for doctors." For a more precise statement of their result, see footnote 3 below.

Result 1. *(See Hatfield and Milgrom, 2005.) Suppose that contracts are substitutes for every hospital. Then there exists a stable set of contracts. Moreover, there exists a stable allocation which every doctor weakly prefers to any other stable allocation.*

A stable allocation which every doctor weakly prefers to any other stable allocation is called the *doctor-optimal stable allocation*. A *mechanism* is a mapping F from the set of preference profiles to the set of allocations. The *doctor-optimal stable mechanism* is a mechanism which, for any reported preference profile P, produces the doctor-optimal stable allocation under P.

Hatfield and Milgrom (2005) also introduce the law of aggregate demand, which is the key condition for the doctor-optimal stable mechanism to be strategy-proof.

Definition 3. The preferences of hospital $h \in H$ satisfy the *law of aggregate demand* if for all $X' \subset X'' \subset X$, $|C_h(X')| \leqslant |C_h(X'')|$.

A mechanism F is *strategy-proof* if, for any preference profile P, there is no doctor d and preferences P'_d such that d strictly prefers y_d to x_d according to P_d, where x_d and y_d are the (possibly null) contracts for d in $F(P)$ and $F((P'_d, P_{-d}))$, respectively.

Result 2. *(See Hatfield and Milgrom, 2005.) Suppose that preferences of every hospital satisfy substitutes and the law of aggregate demand. Then the doctor-optimal stable mechanism is strategy-proof.*

A mechanism F is *group strategy-proof* if, for any preference profile P, there is no group of doctors $D' \subset D$ and a preference profile $P'_{D'} = (P'_d)_{d \in D'}$ such that every $d \in D'$ strictly prefers y_d to x_d according to P_d, where x_d and y_d are the (possibly null) contracts for d in $F(P)$ and $F((P'_{D'}, P_{-D'}))$, respectively.

3. Result

We build on and extend Result 2 by Hatfield and Milgrom (2005) to obtain the following main result. To do this, we first show the following lemma:

Lemma 1. *Let Z be the doctor-optimal stable allocation under preference profile P. Let d obtain $z \in Z$ and suppose that y is preferred to z under P_d. Let P'_d be the preference ordering where y is ranked first by d and is otherwise unchanged from P_d. Then the doctor-optimal stable allocation for the preference profile (P'_d, P_{-d}) is Z.*

Proof. Z is stable under (P'_d, P_{-d}) as if any blocking set Y' existed, Y' would block Z under P, as the set of contracts preferred by the doctors to their contract in Z is unchanged.

Now suppose that another allocation, Y, is stable under (P'_d, P_{-d}). If $y \notin Y$, then the set Y' that blocks Y under P also blocks Y under (P'_d, P_{-d}). If $y \in Y$, then the doctor-optimal stable mechanism would not be strategy-proof, violating Result 2. Hence Z is the doctor-optimal stable allocation under (P'_d, P_{-d}). $\square$

We now state and prove the main result.

Theorem 1. *Suppose that preferences of every hospital satisfy substitutes and the law of aggregate demand. Then the doctor-optimal stable mechanism is group strategy-proof.*

Proof. Suppose by way of contradiction that doctors' true preferences are given by the preference profile P, and suppose there exists a coalition D' that can profitably misreport its members' preferences. Let $\{z_d\}_{d \in D'}$ be the set of contracts doctors in D' obtain when they report their true preferences $P_{D'}$ and $\{y_d\}_{d \in D'}$ those they obtain when they report false preferences $\hat{P}_{D'}$ such that every $d \in D'$ strictly prefers y_d to z_d under P_d. Finally, let Y be the outcome of the doctor-optimal stable mechanism for the report $(\hat{P}_{D'}, P_{-D'})$, and Z the outcome of the doctor-optimal stable mechanism for the report P.

Let P'_d be the preferences of d in which y_d is the most preferred contract and the ranking of every other contract is identical to P_d. Since y_d is d's first choice under P'_d for each $d \in D'$, Y is stable under $(P'_{D'}, P_{-D'})$ as it was stable under $(\hat{P}_{D'}, P_{-D'})$ and there are now fewer possible blocking coalitions of Y. Hence, because the doctor-optimal stable allocation is weakly most preferred by every doctor amongst all stable allocations, each $d \in D'$ must obtain y_d as part of the doctor-optimal stable allocation under $(P'_{D'}, P_{-D'})$.

Now consider the outcome of the doctor-optimal stable mechanism for (P'_d, P_{-d}) for some $d \in D'$. By Lemma 1, the doctor-optimal stable allocation for these preferences must be Z. Iterating this result for each agent in D', the doctor-optimal stable allocation for $(P'_{D'}, P_{-D'})$ must be Z. But this contradicts our result in the previous paragraph that the doctor-optimal stable mechanism under $(P'_{D'}, P_{-D'})$ must provide y_d for each $d \in D'$. $\square$

Many incentive compatibility results in the literature are special cases of Theorem 1. Strategy-proofness was first obtained by Dubins and Freedman (1981) and Roth (1982) in one-to-one matching markets without contracts, and extended by

Abdulkadiroğlu (2005) to more general preferences and then by Hatfield and Milgrom (2005) to the matching markets with contracts under the substitutes and the law of aggregate demand conditions. Group strategy-proofness was first shown by Dubins and Freedman (1981) in one-to-one matching without contracts. Martinez et al. (2004) obtained group strategy-proofness in many-to-one matching without contracts when hospitals' preferences satisfy substitutes and what they call q-separability, a condition stronger than the law of aggregate demand. All these results are special cases of Theorem 1.

The proof presented is also more straightforward than the previous ones in the literature. For example, the proof in Martinez et al. (2004) refers to a technical result (the Blocking Lemma of Gale and Sotomayor, 1985, adapted to their environment), which is proved in a separate full-length paper (Martinez et al., 2006). Here we simply build upon the previous strategy-proofness result. The key insight is that if a deviation by a group can make all members of that group strictly better off, then there exists a specification of preferences such that one agent can make herself strictly better off by a false report.

A natural question is whether Theorem 1 can be further generalized. Within preferences satisfying the substitutes condition, Hatfield and Milgrom (2005) show that the law of aggregate demand is essentially the weakest possible condition to guarantee strategy-proofness.[3] Since group strategy-proofness is a stronger property than strategy-proofness, their result implies the same conclusion for group strategy-proofness.[4]

Another possible extension of this result is the claim that no coalition of doctors can misreport preferences and make some of its members strictly better off without making any of its members strictly worse off. Unfortunately, such a result does not hold even in one-to-one matching markets (Dubins and Freedman, 1981).[5] This fact suggests that, while outside the model, it may be possible for a coalition of doctors to manipulate the doctor-optimal stable mechanism if even small transfers are allowed among doctors: for the application of our results to problems in, say, auction design, this is a substantial limitation. See Section 4.3.1. of Roth and Sotomayor (1990) for exposition and discussion of this point.

Furthermore, Theorem 1 cannot be further generalized to include coalitions including hospitals. In one-to-one matching markets, it is well known that a hospital can improve its outcome by not reporting truthfully under the doctor-optimal stable mechanism (Roth, 1982). Further, in many-to-one matching markets, even a single hospital can sometimes manipulate the hospital-optimal stable mechanism (Roth, 1985).

Interestingly, the doctor-optimal stable mechanism is both group strategy-proof and, from the perspective of the doctors, bossy, that is, a doctor can change her preferences and change the allocation without changing her own outcome.[6] This observation may look contradictory to the well-known result (see Lemma 1 of Papai, 2000) that group strategy-proofness is equivalent to strategy-proofness and nonbossiness in models such as the house allocation problem (Shapley and Scarf, 1974; Hylland and Zeckhauser, 1979). This is not a contradiction, however, since nonbossiness and strategy-proofness are equivalent to a stronger claim that no group can misreport in such a way that some agent in the group is made strictly better off while no one in the group is made strictly worse off.

Our result can also be used to derive another result that has been proven before in a number of special contexts. An immediate corollary of Theorem 1 is the following welfare result.

Corollary 1. *Suppose that preferences of every hospital satisfy substitutes and the law of aggregate demand. Then there exists no individually rational allocation that every doctor strictly prefers to the doctor-optimal stable allocation.*

Proof. Suppose by way of contradiction that there is an individually rational allocation $\{y_d\}_{d\in D}$ that each doctor strictly prefers to the doctor-optimal stable allocation. For each $d \in D$, let P'_d be preferences that declare y_d as the unique acceptable contract. It is easy to show that $\{y_d\}_{d\in D}$ is the unique stable allocation when each d declares P'_d. This contradicts Theorem 1, completing the proof. $\square$

[3] As shown in Theorem 12 of Hatfield and Milgrom (2005), if there exists a violation of the law of aggregate demand such that there are three contracts x, y, z and set of contracts X' such that $y_D \neq z_D \neq x_D \neq y_D$ and $y, z \in R_h(X' \cup \{x\}) - R_h(X')$, and one other hospital, then there exist preferences for the doctors and singleton preferences for the other hospital such that the mechanism is not strategy-proof for the doctors.

[4] On the other hand, Hatfield and Kojima (2008) show that the conclusion of Theorem 1 holds under a weaker condition than the substitutes condition called unilateral substitutes.

[5] A simple example below shows that a stronger version of the group strategy-proofness as stated in the main text does not hold. Let each hospital find only singleton sets of contracts acceptable, and preferences of doctors d, d', d'' and hospitals h, h' be:

$P_d\colon\ (d,h'), (d,h)$ $\qquad P_h\colon\ (d,h), (d',h), (d'',h)$

$P_{d'}\colon\ (d',h)$ $\qquad P_{h'}\colon\ (d'',h'), (d,h')$

$P_{d''}\colon\ (d'',h), (d'',h')$

where the above notation is understood to mean that d prefers contract (d,h') best and (d,h) second and the null contract third, and so on. Then the doctor-optimal stable allocation is $\{(d,h), (d'',h')\}$. However, if d' were to report that no contract is acceptable and d and d'' report $P_{d'}$ and $P_{d''}$ respectively, then the doctor-optimal stable mechanism results in allocation $\{(d,h'), (d'',h)\}$, which every doctor weakly prefers and doctors d and d'' strictly prefer to $\{(d,h), (d'',h')\}$.

[6] An example in footnote 5 is an environment where nonbossiness is violated by the doctor-optimal stable mechanism.

The result is known as “weak Pareto optimality” in the literature and first presented by Roth (1982) for one-to-one matching. Martinez et al. (2004) obtain the result in many-to-one matching when hospitals' preferences satisfy substitutes and q-separability. Kojima (2007) gives a direct proof of Corollary 1 except that he does not consider matching with contracts and focuses on the simpler many-to-one setting without contracts.

4. Concluding remarks

The matching problem with contracts subsumes a large class of problems, such as the matching model with fixed terms of contract, the job matching model with adjustable wages of Kelso and Crawford (1982) and the package auction model of Ausubel and Milgrom (2002). The current paper shows that the substitutes condition and the law of aggregate demand are sufficient for group strategy-proofness and the weak Pareto optimality property.

We note an interesting coincidence of maximal domains for strategy-proofness and group strategy-proofness. Hatfield and Milgrom (2005) show that, under the substitutes condition, the law of aggregate demand is essentially the weakest condition to ensure strategy-proofness, and by implication the same is true for group strategy-proofness. It is more surprising, however, that the more demanding group strategy-proofness property holds under the same set of assumptions.

Acknowledgments

We are grateful to Paul Milgrom and Al Roth as well as the editor and anonymous referees for comments.

References

Abdulkadiroğlu, Atila, 2005. College admission with affirmative action. Int. J. Game Theory 33, 535–549.
Abdulkadiroğlu, Atila, Sönmez, Tayfun, 2003. School choice: A mechanism design approach. Amer. Econ. Rev. 93, 729–747.
Abdulkadiroğlu, Atila, Pathak, Parag A., Roth, Alvin E., Sönmez, Tayfun, 2005a. The Boston Public School match. Amer. Econ. Rev. Pap. Proc. 95, 368–372.
Abdulkadiroğlu, Atila, Pathak, Parag A., Roth, Alvin E., 2005b. The New York City High School match. Amer. Econ. Rev. Pap. Proc. 95, 364–367.
Ausubel, Lawrence M., Milgrom, R. Paul, 2002. Ascending auctions with package bidding. Front. Theor. Econ. 1, 1.
Balinski, Michel, Sönmez, Tayfun, 1999. A tale of two mechanisms: Student placement. J. Econ. Theory 84, 73–94.
Dubins, L.E., Freedman, D.A., 1981. Machiavelli and the Gale–Shapley algorithm. Amer. Math. Monthly 88, 485–494.
Gale, David, Shapley, Lloyd S., 1962. College admissions and the stability of marriage. Amer. Math. Monthly 69, 9–15.
Gale, David, Sotomayor, Marilda A. Oliveira, 1985. Some remarks on the stable matching problem. Discrete Appl. Math. 11, 223–232.
Hatfield, John William, Kojima, Fuhito, 2008. Substitutes and stability for matching with contracts. Mimeo.
Hatfield, John William, Milgrom, Paul R., 2005. Matching with contracts. Amer. Econ. Rev. 95, 913–935.
Hylland, Aanund, Zeckhauser, Richard, 1979. The efficient allocation of individuals to positions. J. Polit. Economy 87, 293–314.
Kelso, Alexander, Crawford, Vincent P., 1982. Job matching, coalition formation, and gross substitutes. Econometrica 50, 1483–1504.
Kojima, Fuhito, 2007. The law of aggregate demand and welfare in the two-sided matching market. Econ. Lett. 99, 581–584.
Martinez, Ruth, Masso, Jordi, Neme, Alejandro, Oviedo, Jorge, 2004. On group strategy-proof mechanisms for a many-to-one matching model. Int. J. Game Theory 33, 115–128.
Martinez, Ruth, Masso, Jordi, Neme, Alejandro, Oviedo, Jorge, 2006. The Blocking Lemma for a many-to-one matching model. Mimeo.
Papai, Szilvia, 2000. Strategyproof assignment by hierarchical exchange. Econometrica 68, 1403–1433.
Roth, Alvin E., 1982. The economics of matching: Stability and incentives. Math. Oper. Res. 7, 617–628.
Roth, Alvin E., 1984. The evolution of the labor market for medical interns and residents: A case study in game theory. J. Polit. Economy 92, 991–1016.
Roth, Alvin E., 1985. The college admission problem is not equivalent to the marriage problem. J. Econ. Theory 36, 277–288.
Roth, Alvin E., Peranson, Elliott, 1999. The redesign of the matching market for American physicians: Some engineering aspects of economic design. Amer. Econ. Rev. 89, 748–780.
Roth, Alvin E., Sotomayor, A.O. Marilda, 1990. Two-Sided Matching: A Study in Game-theoretic Modeling and Analysis. Econom. Soc. Monogr., Cambridge.
Shapley, Lloyd, Scarf, Herbert, 1974. On cores and indivisibility. J. Math. Econ. 1, 23–37.