Group Incentive Compatibility for Matching with Contracts

John William Hatfield and Fuhito Kojima*

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Abstract

Hatfield and Milgrom (2005) present a unified model of matching with contracts, which includes the standard two-sided matching and some package auction models as special cases. They show that the doctor-optimal stable mechanism is strategy-proof for doctors if hospitals' preferences satisfy substitutes and the law of aggregate demand. We show that the doctor-optimal stable mechanism is group strategy-proof for doctors under these same conditions. That is, no group of doctors can make each of its members strictly better off by jointly misreporting their preferences. We derive as a corollary of this result that no individually rational matching is preferred by all the doctors to the doctor-optimal stable match. *Journal of Economic Literature Classification Numbers*: C78, D44.

Key Words: matching, matching with contracts, substitutes, law of aggregate demand, stability, strategy-proofness, group strategy-proofness.

1 Introduction

The theory of two-sided matching markets has interested researchers for its theoretical appeal and its relevance to the design of real-world institutions. The medical residency matching mechanism in the U.S. (National Residency Matching Program) and student assignment systems in New York City and

*Hatfield: Graduate School of Business, Stanford University, Stanford, CA 94305, e-mail: hatfield_john@gsb.stanford.edu. Kojima: Department of Economics, Harvard University, Cambridge, MA 02138, e-mail: kojima@fas.harvard.edu. We are grateful to Paul Milgrom and Al Roth for comments.

Boston (among others) are examples of mechanisms designed by economists using the theory.[1] Hatfield and Milgrom (2005) present a unified framework of matching with contracts, which includes the two-sided matching models and package auction models as special cases. They introduce the substitutes condition of contracts, which generalizes the substitutability condition in the matching literature (Roth and Sotomayor, 1990). They also introduce a new condition they call the law of aggregate demand, which states that hospitals choose more contracts as the set of contracts to choose from expands. They show that the doctor-optimal stable mechanism is strategy-proof for doctors if hospitals' preferences satisfy substitutes and the law of aggregate demand. That is, no single doctor is made strictly better off by misreporting her preferences.

We show that this same mechanism is in fact group strategy-proof for doctors. That is, no group of doctors can make each of its members strictly better off by jointly misreporting their preferences. The class of preferences with substitutes and the law of aggregate demand is a maximal domain for such a conclusion, that is, it can be shown that no weaker condition on hospital preferences can guarantee group strategy-proofness (or even strategy-proofness). The many previous results on strategy-proofness and group strategy-proofness (Dubins and Freedman, 1981; Roth, 1982; Martinez et al., 2004; Abdulkadiroğlu, 2005; Hatfield and Milgrom, 2005) are special cases of this result. Furhtermore, it is well-known that a group of doctors can deviate in such a way to make some members strictly better off and none worse off (Dubins and Freedman, 1981) and that the mechanism is not even strategy-proof for hospitals (Roth, 1982).

As an application of our main result, we show a welfare result known as "weak Pareto optimality." More specifically, if preferences of every hospital satisfy substitutes and the law of aggregate demand, then there exists no individually rational allocation that every doctor strictly prefers to the doctor-optimal stable allocation. This result has been shown by a number of studies under various assumptions (Roth, 1982; Martinez et al., 2004; Kojima, 2007).

In addition to its theoretical appeal, our results have interesting implications for the various applications of matching models, especially those regarding school choice (Abdulkadiroğlu and Sönmez, 2003). Focusing on strategic behavior and welfare of students is natural in the school choice setting, since school priorities are often set by the law and manipulations by schools are im-

[1]The theory was pioneered by Gale and Shapley (1962). For applications to labor markets, see Roth (1984) and Roth and Peranson (1999). For applications to student assignment, see for example Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b).

possible, and schools are regarded as objects whose priorities are not taken into account in welfare consideration. The generality of our preferences is important, since school priorities often violate assumptions under which the incentive and welfare conclusions have been shown in the literature. Some of the public schools in the New York City are, for example, required to admit a balanced mix of students from high, middle and low test score populations. Such requirements violate the previous conditions, but still satisfy substitutes and the law of aggregate demand. Thus our analysis gives some additional justification for the use of the student-optimal stable mechanism originally advocated by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003).

2 Model and Results

There are finite sets D and H of doctors and hospitals. X is the set of contracts. Each contract $x \in X$ is bilateral, so that it is associated with a doctor $x_D \in D$ and a hospital $x_H \in H$. We assume that each doctor can sign at most one contract. The null contract, meaning that the doctor has no contract, is denoted by $\emptyset$. For each $d \in D$, P_d is a strict preference relation on $\{x \in X | x_D = d\} \cup \{\emptyset\}$. A contract is **acceptable** if it is preferred to the null contract and **unacceptable** if it is not preferred to the null contract. For each $d \in D$ and $X' \subset X$, we define the **chosen set** C_d by

$$C_d(X') = \max_{P_d} \{x \in X' | x_D = d\} \cup \{\emptyset\}$$

for any $X' \subset X$. Let $C_D(X') = \bigcup_{d \in D} C_d(X')$ be the set of contracts chosen from X' by some doctor.

We allow each hospital to sign multiple contracts, and assume that each hospital $h \in H$ has a preference relation P_h on subsets of contracts involving it. For any $X' \subset X$, define $C_h(X')$ by

$$C_h(X') = \max_{P_h} \{X'' \subset X' | x \in X'' \Rightarrow x_H = h \text{ and } x, x' \in X'', x \neq x' \Rightarrow x_D \neq x'_D\}.$$

Let $C_H(X') = \bigcup_{h \in H} C_h(X')$ be the set of contracts chosen from X' by some hospital.

We write $P = (P_d)_{d \in D}$ to denote a preference profile of doctors. We also write P_{-d} to denote $(P'_d)_{d' \neq d}$ for $d \in D$, and $P_{D'}$ to denote $(P_d)_{d \in D'}$ and $P_{-D'}$ to denote $(P_d)_{d \notin D'}$ for $D' \subset D$.

A set of contracts $X' \subset X$ is an **allocation** if $x, x' \in X'$ and $x \neq x'$ imply $x_D \neq x'_D$. In words, a set of contracts is an allocation if each doctor signs at most one contract.

Definition 1. *A set of contracts $X' \subset X$ is a **stable allocation** (or a **stable set of contracts**) if*

1. *$C_D(X') = C_H(X') = X'$, and*
2. *there exists no hospital h and set of contracts $X'' \neq C_h(X')$ such that $X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'')$.*

When condition (2) is violated by some X'', we say that X'' is a blocking set for X'. For any $X' \subset X$ and $h \in H$, define the **rejected set** by $R_h(X') = X' \setminus C_h(X')$. $R_h(X')$ is the set of contracts in X' which h is willing to reject.

Definition 2. *Contracts are **substitutes** for h if we have $R_h(X') \subset R_h(X'')$ for all $X' \subset X'' \subset X$.*

Hatfield and Milgrom (2005) show that there exists a stable set of contracts when contracts are substitutes for every hospital.

Result 1 (Hatfield and Milgrom 2005). *Suppose that contracts are substitutes for every hospital. Then there exists a stable set of contracts. Moreover, there exists a stable allocation which every doctor weakly prefers to any other stable allocation.*

A stable allocation which every doctor weakly prefers to any other stable allocation is called the **doctor-optimal stable allocation**. The **doctor-optimal stable mechanism** is a mechanism which, for any reported preference profile P, produces the doctor-optimal stable allocation under P.

Definition 3. *The preferences of hospital $h \in H$ satisfy the **law of aggregate demand** if for all $X' \subset X'' \subset X$, $|C_h(X')| \leq |C_h(X'')|$.*

A mechanism is **strategy-proof** if, for any preference profile P, there is no doctor d and preferences P'_d such that d strictly prefers y_d to x_d according to P_d, where x_d and y_d are the doctor-optimal stable allocations for d under P and (P'_d, P_{-d}), respectively.

Result 2 (Hatfield and Milgrom 2005). *Suppose that preferences of every hospital satisfy substitutes and the law of aggregate demand. Then the doctor-optimal stable mechanism is strategy-proof.*

A mechanism is **group strategy-proof** if, for any preference profile P, there is no group of doctors $D' \subset D$ and a preference profile $P'_{D'} = (P'_d)_{d \in D'}$ such that every $d \in D'$ strictly prefers y_d to x_d according to P_d, where x_d and y_d are the doctor-optimal stable allocations for d under P and $(P'_{D'}, P_{-D'})$, respectively.

We build on and extend Result 2 by Hatfield and Milgrom (2005) to obtain the following main result.

Theorem 1. *Suppose that preferences of every hospital satisfy substitutes and the law of aggregate demand. Then the doctor-optimal stable mechanism is group strategy-proof.*

Proof. We prove the result by induction on the size (cardinality) of the coalition whose members misreport their preferences. No coalition of size 1 can profitably misreport preferences by Result 2. Suppose that no coalition of size at most $n-1$ can profitably misreport preferences, and suppose by way of contradiction that there exists a preference profile P and a coalition D' of size n that can profitably misreport its members' preferences. Let $\{x_d\}_{d\in D'}$ be the set of contracts doctors in D' obtain when they report their true preferences and $\{y_d\}_{d\in D'}$ those they obtain when they misreport their preferences such that every $d \in D'$ strictly prefers y_d to x_d under P_d. Let P'_d be preferences of d in which y_d is the most preferred contract to d and the ranking of every other contract is identical to P_d. It is easy to see each doctor in D' obtains y_d under the doctor-optimal stable mechanism when $(P'_{D'}, P_{-D'})$ is reported. Now consider a preference profile (P'_d, P_{-d}) with a fixed $d \in D'$. The set of stable allocations under (P'_d, P_{-d}) coincides with the set of stable allocations under P, since:

1. **Any stable set Z' of contracts under P is stable under (P'_d, P_{-d}).** Doctor d prefers y_d to x_d under P_d and so if a set Y including a contract with d blocks Z' under P'_d, then the set Y blocks Z' under P_d, contradicting the stability of Z' under P. Moreover, no blocking set of Z' that does not involve d is possible under (P'_d, P_{-d}), since such a blocking set would be a blocking set under P, again contradicting stability of Z' under P.

2. **Any set Z' of contracts that is unstable under P is unstable under (P'_d, P_{-d}).** By construction of P'_d, no allocation that does not assign y_d to d and is unstable under P is stable under (P'_d, P_{-d}). Suppose there exists an allocation that assigns y_d to d and is stable under (P'_d, P_{-d}). Then, since the doctor-optimal stable mechanism assigns the most preferred contract to d that can be assigned in some stable allocation, d is assigned y_d by the doctor-optimal stable mechanism under (P'_d, P_{-d}). This implies that d could profitably misreport her preferences by submitting P'_d when P is the true preference profile, a contradiction to Result 2.

In particular, we conclude from the above that the doctor-optimal stable allocations under P and (P'_d, P_{-d}) coincide. Therefore every doctor $d' \in D' \setminus \{d\}$ obtains $x_{d'}$ under (P'_d, P_{-d}). But this implies that the group of doctors

$D' \setminus \{d\}$ can profitably misreport preferences by declaring $P'_{D'\setminus\{d\}}$ instead of $P_{D'\setminus\{d\}}$ when the true preferences are (P'_d, P_{-d}), since by assumption every doctor $d' \in D' \setminus \{d\}$ obtains $y_{d'}$ under $(P'_{D'}, P_{-D'})$. This contradicts the inductive assumption that no group of size at most $n-1$ can profitably misreport its members' preferences. $\square$

Many incentive compatibility results in the literature are special cases of Theorem 1. Strategy-proofness was first obtained by Dubins and Freedman (1981) and Roth (1982) in one-to-one matching markets, and extended by Abdulkadiroğlu (2005) to more general preferences and then by Hatfield and Milgrom (2005) to the matching markets with contracts under the substitutes and the law of aggregate demand conditions. Group strategy-proofness was first shown by Dubins and Freedman (1981) in one-to-one matching. Martinez et al. (2004) obtained group strategy-proofness in many-to-one matching when hospitals' preferences satisfy substitutes and what they call q-separability, a condition stronger than the law of aggregate demand. All these results are special cases of Theorem 1.

The proof presented is also more straightforward than the previous ones in the literature. For example, the proof in Martinez et al. (2004) refers to a technical result (the Blocking Lemma of Gale and Sotomayor (1985) adapted to their environment), which is proved in a separate full-length paper (Martinez et al., 2006). Here we simply build upon the previous strategy-proofness result using induction. The key insight is that if a deviation by a group can make all members of that group strictly better off, but a deviation by one of them does not change the outcome, then there exists a specification of preferences such that a smaller group can make all of its members strictly better off by deviating.

A natural question is whether Theorem 1 can be further generalized. Hatfield and Milgrom (2005) show that the substitutes condition and the law of aggregate demand are the weakest conditions to guarantee strategy-proofness. Since group strategy-proofness is a stronger property than strategy-proofness, their result implies that it is impossible to obtain group strategy-proofness under weaker conditions on hospitals' preferences.

Another possible extension of this result is the claim that no coalition can misreport preferences and make some of its members strictly better off without making any of its members strictly worse off. Unfortunately, such a result does not hold even in one-to-one matching markets (Dubins and Freedman, 1981). This fact suggests that, while outside the model, it may be possible for a coalition of doctors to manipulate the doctor-optimal stable mechanism if even small transfers are allowed among doctors. See section 4.3.1. of Roth and Sotomayor (1990) for exposition and discussion of this point.

Furthermore, Theorem 1 can not be further generalized to include coalitions including hospitals. In one-to-one matching markets, it is well-known that a hospital can improve its outcome by not reporting truthfully under the doctor-optimal stable mechanism (Roth, 1982). Further, in many-to-one matching markets, even a single hospital can sometimes manipulate the hospital-optimal stable mechanism (Roth, 1985).

Our result can also be used to derive another result that has been proven before in a number of special contexts. An immediate corollary of Theorem 1 is the following welfare result.

Corollary 1. *Suppose that preferences of every hospital satisfy substitutes and the law of aggregate demand. Then there exists no individually rational allocation that every doctor strictly prefers to the doctor-optimal stable allocation.*

Proof. Suppose by way of contradiction that there is an individually rational allocation $\{y_d\}_{d \in D}$ that each doctor strictly prefers to the doctor-optimal stable allocation. For each $d \in D$, let P'_d be preferences that declare y_d as the unique acceptable contract. It is easy to show that $\{y_d\}_{d \in D}$ is the unique stable allocation when each d declares P'_d. This contradicts Theorem 1, completing the proof. $\square$

The result is known as "weak Pareto optimality" in the literature and first presented by Roth (1982) for one-to-one matching. Martinez et al. (2004) obtain the result in many-to-one matching when hospitals' preferences satisfy substitutes and q-separability. Kojima (2007) gives a direct proof of Corollary 1 except that he does not consider matching with contracts and focuses on simple many-to-one matching setting.

3 Concluding Remarks

The matching problem with contracts subsumes a large class of problems, such as the matching model with fixed terms of contract, the job matching model with adjustable wages of Kelso and Crawford (1982) and the package auction model of Ausubel and Milgrom (2002). The current paper shows that the substitutes condition and the law of aggregate demand are not only necessary and sufficient for the rural hospitals theorem and strategy-proofness of the doctor-optimal stable mechanism, but also are necessary and sufficent for group strategy-proofness and the weak Pareto property.

We note an interesting coincidence of maximal domains for strategy-proofness and group strategy-proofness. Hatfield and Milgrom (2005) show that substitutes and the law of aggregate demand cannot be weakened unless strategy-proofness is lost, and by implication the same is true for group strategy-proofness. It is more surprising, however, that the more demanding group strategy-proofness property holds under the same set of assumptions.

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