Game Theory

Choice under Uncertainty

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0.1.- Choice under Certainty

- Let X be a *choice* set.
- An agent's preference on X is a binary relation $\succsim$ on X (i.e., $\succsim \subset X \times X$).
- Given $x, y \in X$, we write $x \succsim y$ to denote that $(x, y) \in \succsim$.
- We assume that $\succsim$ is *complete* and *transitive*. Namely:
  - for all $x, y \in X$, we have $x \succsim y$ or $y \succsim x$ or both,
  - for all $x, y, z \in X$ such that $x \succsim y$ and $y \succsim z$, it holds that $x \succsim z$.
- We define *indifference* $x \sim y$ as $x \succsim y$ and $y \succsim x$.
- We define *strict preference* $x \succ y$ as $x \succsim y$ and $x \nsim y$.

0.1.- Choice under Certainty

- A function $u : X \to \mathbb{R}$ *represents* $\succsim$ on X if for all $x, y \in X$,

$$x \succ y \text{ if and only if } u(x) > u(y) \text{ and}$$

$$x \sim y \text{ if and only if } u(x) = u(y).$$

- **Result 1:** *Let X be a choice set and $\succsim$ be a preference on X. Assume there exists a utility function $u : X \to \mathbb{R}$ that represents $\succsim$. Then, $\succsim$ is complete and transitive.*
- **Result 2:** *Assume X is finite and $\succsim$ is complete and transitive. Then, there exists a utility function $u : X \to \mathbb{R}$ representing $\succsim$.*
- Not unique.
  - A function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if $x < y$ implies $f(x) < f(y)$.
  - Assume $u : X \to \mathbb{R}$ represents $\succsim$ and let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Then, $f \circ u : X \to \mathbb{R}$ also represents $\succsim$, where for all $x \in X$, $f \circ u(x) = f(u(x))$.

0.1.- Choice under Certainty

- Let X be a metric space. A preference $\succsim$ on X is *continuos* if for all $x \in X$, the sets $\{y \in X \mid y \succsim x\}$ and $\{y \in X \mid x \succsim y\}$ are closed.
- **Result 3:** *Let X be a compact metric space and $\succsim$ be a continuous, complete and transitive preference on X. Then, there exists a continuous utility function $u : X \to \mathbb{R}$ that represents $\succsim$.*
- Let T be a set such that for every $t \in T$, $F(t)$ is a subset of X.
- **Result 4:** *Fix $t \in T$. Assume $F(t) \subseteq X$ is compact and $\succsim$ is a continuous, complete and transitive preference on X. Then, there exists $x^* \in F(t)$ such that $x^* \succeq y$ for all $y \in F(t)$.*
- This means that the problem

$$\max u(x)$$
$$\text{s.t. } x \in F(t),$$

where u is a (continuous) utility function that represents $\succsim$, has a solution.

0.2.- Basic Lotteries

- To model uncertainty we need to enlarge the set X to the set of *lotteries* on X, denoted by $\mathcal{L}(X)$.
- Fix the set X of outcomes (or *sure things*). Assume that $\#X = N$.
- We assume that the agent faces risky alternatives.
- Given $x, y \in X$ and $p \in [0, 1]$, a *basic lottery* is a risky alternative where the agent receives the prize x with probability p and the prize y with probability $1 - p$. We represent a basic lottery by

$$\ell(p; x, y) = p \odot x \oplus (1 - p) \odot y.$$

0.3.- Composed Lotteries

- Given $x, y, z, w \in X$ and $p, q, m \in [0, 1]$ we define the *composed lottery* as the risky alternative where the agent receives the basic lottery $\ell(q; x, y)$ with probability p and the basic lottery $\ell(m; z, w)$ with probability $1 - p$. We represent this lottery $\ell(p; \ell(q; x, y), \ell(m; z, w))$ by

$$p \odot [q \odot x \oplus (1 - q) \odot y] \oplus (1 - p) \odot [m \odot z \oplus (1 - m) \odot w].$$

- Let $\mathcal{L}(X)$ be the set of all composed lotteries. The set $\mathcal{L}(X)$ is closed; *i.e.*, for all $x, y, z, w \in X$ and $p, q, m \in [0, 1]$

$$\ell(q; x, y), \ell(m; z, w) \in \mathcal{L}(X) \Longrightarrow \ell(p; \ell(q; x, y), \ell(m; z, w)) \in \mathcal{L}(X).$$

0.3.- Composed Lotteries: Basic Assumptions

- We make the following (natural) assumptions on how the agent perceives basic and composed lotteries.
  - Alternatively, we could make (equivalent) assumptions directly on agent's preferences on lotteries.
- For all $x, y \in X$ and $p, q \in [0, 1]$ the following holds:
  - (P.1) $p \odot x \oplus (1-p) \odot x = x$.
  - (P.2) $p \odot x \oplus (1-p) \odot y = (1-p) \odot y \oplus p \odot x$.
  - (P.3) $q \odot [p \odot x \oplus (1-p) \odot y] \oplus (1-q) \odot y = p \cdot q \odot x \oplus (1 - p \cdot q) \odot y$.

- We assume that the agent has complete and transitive preferences $\succsim$ on $\mathcal{L}(X)$.
- It would be very useful to have a utility representation of a preference $\succsim$ on $\mathcal{L}(X)$.
- But observe that $\mathcal{L}(X)$ is not finite even when X is finite. So, we can not apply Result 2.
- We could proceed as we did in Result 3: to identify sufficient conditions under which $\succsim$ on $\mathcal{L}(X)$ is representable by a utility function. Namely,
    - Assume X satisfies (BLA 1) and $\succsim$ satisfies (BLA 2). Then, there exists $U : \mathcal{L}(X) \to \mathbb{R}$ that represents $\succsim$; *i.e.*, for all $\ell = \ell(p; x, y) \in \mathcal{L}(X)$ and $\ell' = \ell(q; z, w) \in \mathcal{L}(X)$,

$$\ell \succ \ell' \Longleftrightarrow U(p \odot x \oplus (1-p) \odot y) > U(q \odot z \oplus (1-p) \odot w)$$

and

$$\ell \sim \ell' \Longleftrightarrow U(p \odot x \oplus (1-p) \odot y) = U(q \odot z \oplus (1-p) \odot w).$$

- But this approach would be insufficient. We need that this representation has an additional property: the expected utility property.

Definition Assume that $U : \mathcal{L}(X) \to \mathbb{R}$ represents the preferences $\succsim$ on $\mathcal{L}(X)$. We say that U satisfies the *expected utility property* if for all $\ell(p; x, y) \in \mathcal{L}(X)$,

$$U(p \odot x \oplus (1-p) \odot y) = p \cdot U(x) + (1-p) \cdot U(y).$$

- Namely, the utility of a lottery coincides with its expected utility.
- Observe that this is much more demanding than just to require that $\succsim$ has a utility representation.
- **Question:** Under which conditions a preference $\succsim$ on $\mathcal{L}(X)$ has a utility representation with the expected utility property?

0.5.- Expected Utility Property: Continuity Axiom

(A.1) A preference $\succsim$ on $\mathcal{L}(X)$ is *continuous* if for all $x, y, z \in \mathcal{L}(X)$ with $x \succ y \succ z$, there exists $p \in (0, 1)$ such that

$$[p \odot x \oplus (1 - p) \odot z] \sim y$$

and for all $q, r \in [0, 1]$ such that $q > p > r$, then

$$[q \odot x \oplus (1 - q) \odot z] \succ y \succ [r \odot x \oplus (1 - r) \odot z].$$

0.5.- Expected Utility Property: Independence Axiom

(A.2) A preference $\succsim$ on $\mathcal{L}(X)$ satisfies the *independence* axiom if for all $x, y, z \in \mathcal{L}(X)$ and all $p \in [0, 1]$,

$$x \succ y \text{ if and only if } [p \odot x \oplus (1-p) \odot z] \succ [p \odot y \oplus (1-p) \odot z]$$

and

$$x \sim y \text{ if and only if } [p \odot x \oplus (1-p) \odot z] \sim [p \odot y \oplus (1-p) \odot z].$$

0.5.- Expected Utility Theorem

Theorem (von Neumann and Morgenstern, 1944).

A preference $\succsim$ *on* $\mathcal{L}(X)$ *satisfies axioms* (A.1) *and* (A.2) *if and only if there exists a utility function* $U : \mathcal{L}(X) \to \mathbb{R}$ *representing* $\succsim$ *such that* U *satisfies the expected utility property.*

Moreover, U *is unique up to positive affine transformations.*

- Let Y be a set. The function $g : Y \to \mathbb{R}$ is a *positive affine transformation* of the function $f : Y \to \mathbb{R}$ if there exist $\alpha \in \mathbb{R}$ and $\beta > 0$ such that for all $y \in Y$,

$$g(y) = \alpha + \beta \cdot f(y).$$

0.5.- Expected Utility Theorem: Proof

($\Longleftarrow$)

- Assume there exists a utility function $U : \mathcal{L}(X) \rightarrow \mathbb{R}$ representing $\succsim$ such that U satisfies the expected utility property.
- To show that $\succsim$ is continuous, let $x, y, z \in \mathcal{L}(X)$ be such that $x \succ y \succ z$.
  - Since U represents $\succsim$, $U(x) > U(y) > U(z)$. The set of real numbers is convex, hence there exists $p \in (0, 1)$ such that $p \cdot U(x) + (1 - p) \cdot U(z) = U(y)$.
  - By the Expected Utility Property, $U(p \odot x \oplus (1 - p) \odot z) = p \cdot U(x) + (1 - p) \cdot U(z) = U(y)$.
  - Since U represents $\succeq$, $p \odot x \oplus (1 - p) \odot z \sim y$.

$(\Longleftarrow)$

- Let $q, r \in [0,1]$ be such that $q > p > r$. Then,

$$\begin{aligned} q \cdot U(x) + (1-q) \cdot U(z) &> p \cdot U(x) + (1-p) \cdot U(z) \\ &> r \cdot U(x) + (1-r) \cdot U(z). \end{aligned}$$

- By the Expected Utility Property, and the hypothesis that U represents $\succeq$,

$$q \odot x \oplus (1-q) \odot z \succ y \succ r \odot x \oplus (1-r) \odot z.$$

0.5.- Expected Utility Theorem: Proof

$(\Longleftarrow)$

- To show that $\succsim$ satisfies independence, let $x, y, z \in \mathcal{L}(X)$ and $p \in [0, 1]$.
- Since U represents $\succsim$, $x \succ y \Leftrightarrow U(x) > U(y)$.
  - Hence,
    $x \succ y \Leftrightarrow p \cdot U(x) + (1-p) \cdot U(z) > p \cdot U(y) + (1-p) \cdot U(z)$.
  - Since U satisfies the Expected Utility Property,
    $x \succ y \Leftrightarrow [p \odot x \oplus (1-p) \odot z] \succ [p \odot y \oplus (1-p) \odot z]$.
- Since U represents $\succsim$, $x \sim y \Leftrightarrow U(x) = U(y)$.
  - Hence,
    $x \sim y \Leftrightarrow p \cdot U(x) + (1-p) \cdot U(z) = p \cdot U(y) + (1-p) \cdot U(z)$.
  - Since U satisfies the Expected Utility Property,
    $x \sim y \Leftrightarrow [p \odot x \oplus (1-p) \odot z] \sim [p \odot y \oplus (1-p) \odot z]$.

$(\Longrightarrow)$

(a) Existence:

- For convenience, assume that there exist a best and a worst lottery; namely, there exist $b, w \in \mathcal{L}(X)$ such that for all $\ell \in \mathcal{L}(X)$, $b \succsim \ell \succsim w$ and that $b \succ w$. The case $b \sim w$ is trivial.
- Define $U(b) = 1$, $U(w) = 0$ and $U(\ell) = p_\ell$ for all $\ell \in \mathcal{L}(X)$, where $p_\ell \in [0, 1]$ satisfies

$$[p_\ell \odot b \oplus (1 - p_\ell) \odot w] \sim \ell. \tag{1}$$

- First, by (A.1), there exists such p_ℓ. Second, p_ℓ is unique.
- Assume otherwise; *i.e.*, (1) holds for p_ℓ and p'_ℓ, and $p_\ell > p'_\ell$.
- Hence, $[p_\ell \odot b \oplus (1 - p_\ell) \odot w] \sim \ell \sim [p'_\ell \odot b \oplus (1 - p'_\ell) \odot w]$.
- Since $b \succ w$, either $p_\ell \neq 1$ or $p'_\ell \neq 0$. Assume $1 > p_\ell > p'_\ell$. By (A.1),

$$[1 \odot b \oplus 0 \odot w] \succ \ell \succ [p'_\ell \odot b \oplus (1 - p'_\ell) \odot w],$$

a contradiction with $\ell \sim [p'_\ell \odot b \oplus (1 - p'_\ell) \odot w]$.

(b) U **represents** $\succeq$:

- First, assume $\ell \succ \ell'$. Then,

$$U(\ell) = p_\ell \text{ and } U(\ell') = p'_\ell,$$

where p_ℓ and p'_ℓ are the unique numbers that satisfy

$$[p_\ell \odot b \oplus (1 - p_\ell) \odot w] \sim \ell \succ \ell' \sim [p'_\ell \odot b \oplus (1 - p'_\ell) \odot w]. \quad (2)$$

- To obtain a contradiction, assume $U(\ell) \leq U(\ell')$. By (2), $U(\ell) < U(\ell')$.

- Either $U(\ell') \neq 1$ or $U(\ell) \neq 0$. Assume, $U(\ell) < U(\ell') < 1$. By (A.1),

$$[1 \odot b \oplus 0 \odot w] \succ \ell' \succ [p_\ell \odot b \oplus (1 - p_\ell) \odot w],$$

a contradiction with (2).

(b) U **represents** $\succeq$:

- Second, assume $\ell \sim \ell'$. Then,

$$U(\ell) = p_\ell,$$

where p_ℓ is the unique number such that

$$[p_\ell \odot b \oplus (1 - p_\ell) \odot w] \sim \ell \sim \ell'$$

so that, by definition of U, $U(\ell') = p_\ell$.

- Hence, $U(\ell) = p_\ell = U(\ell')$.

(c) U satisfies the expected utility property:

- Let $x, y \in \mathcal{L}(X)$ and $p \in [0,1]$ be arbitrary. Then, $U(x) = p_x$ and $U(y) = p_y$, where

$$x \sim [p_x \odot b \oplus (1 - p_x) \odot w] \quad (3)$$

and

$$y \sim [p_y \odot b \oplus (1 - p_y) \odot w]. \quad (4)$$

- By (A.2), (3), (4), and (P.3),
$[p \odot x \oplus (1-p) \odot y]$
$\sim [p \odot [p_x \odot b \oplus (1-p_x) \odot w] \oplus (1-p) \odot [p_y \odot b \oplus (1-p_y) \odot w]$
$\sim [(pp_x + (1-p)p_y) \odot b \oplus (p(1-p_x) + (1-p)(1-p_y)) \odot w]$
$\sim [(pU(x) + (1-p)U(y)) \odot b \oplus (p(1-U(x)) + (1-p)(1-$
$U(y)) \odot w]$.

- Hence, by definition of U,

$$U(p \odot x \oplus (1-p) \odot y) = pU(x) + (1-p)U(y).$$

(d) U **is unique up to positive affine transformations:**
We have to show two things.

- First, assume $U : \mathcal{L}(X) \to \mathbb{R}$ represents the preferences $\succsim$ and has the expected utility property and let $V : \mathcal{L}(X) \to \mathbb{R}$ be a positive affine transformation of U. We have to show that V represents $\succsim$ and has the expected utility property.
- Since V is a positive affine transformation of U, there exist $\alpha \in \mathbb{R}$ and $\beta > 0$ such that for all $\ell \in \mathcal{L}(X)$,

$$V(\ell) = \alpha + \beta U(\ell).$$

(d) U is unique up to positive affine transformations:

- Then, for any $\ell, \ell' \in \mathcal{L}(X)$

$$\begin{aligned} \ell \quad \succ \quad & \ell' \\ \iff \quad & U(\ell) > U(\ell') \\ \iff \quad & \alpha + \beta U(\ell) > \alpha + \beta U(\ell') \\ \iff \quad & V(\ell) > V(\ell'). \end{aligned}$$

and

$$\begin{aligned} \ell \quad \sim \quad & \ell' \\ \iff \quad & U(\ell) = U(\ell') \\ \iff \quad & \alpha + \beta U(\ell) = \alpha + \beta U(\ell') \\ \iff \quad & V(\ell) = V(\ell'). \end{aligned}$$

- Hence, V represents $\succsim$.

(d) U is unique up to positive affine transformations:

- We want to show now that V has the expected utility property.
- Let $x, y \in \mathcal{L}(X)$ and $p \in [0,1]$ be arbitrary. Then,

$$
\begin{aligned}
& V(p \odot x \oplus (1-p) \odot y) \\
= \;& \alpha + \beta U(p \odot x \oplus (1-p) \odot y) \\
= \;& \alpha p + \alpha(1-p) + \beta(pU(x) + (1-p)U(y)) \\
= \;& p(\alpha + \beta U(x)) + (1-p)(\alpha + \beta U(y)) \\
= \;& pV(x) + (1-p)V(y).
\end{aligned}
$$

- Hence, V has the expected utility property.

(d) U is unique up to positive affine transformations:

- Second, assume that U and V represent $\succsim$ and satisfy the expected utility property. We want to show that V is a positive affine transformation of U; namely, there exist $\alpha \in \mathbb{R}$ and $\beta > 0$ such for all $\ell \in \mathcal{L}(X)$,
$$V(\ell) = \alpha + \beta U(\ell).$$
- Since U and V represent $\succsim$ there exists a strictly increasing transformation of U, $f : \mathbb{R} \to \mathbb{R}$ such that $V = f \circ U$ represents the same preference as U.
- Since U satisfies the Expected Utility Property, for all $x, y \in \mathcal{L}(X)$ and all $p \in [0, 1]$,
$$\begin{aligned} & V(p \odot x \oplus (1-p) \odot y) \\ = \ & f(U(p \odot x \oplus (1-p) \odot y)) \\ = \ & f(pU(x) + (1-p)U(y)). \end{aligned}$$

(d) U is unique up to positive affine transformations:

- Since V satisfies the expected utility property, for all $x, y \in \mathcal{L}(X)$ and all $p \in [0,1]$,

$$\begin{aligned} & V(p \odot x \oplus (1-p) \odot y) \\ = \ & pV(x) + (1-p)V(y) \\ = \ & pf(U(x)) + (1-p)f(U(y)). \end{aligned}$$

- Hence,

$$f(pU(x) + (1-p)U(y)) = pf(U(x)) + (1-p)f(U(y)).$$

- Thus, f is simultaneously concave and convex. Namely, f is affine. ■

- A positive affine transformation represents a change of the 0 (performed by α) and a change of units (performed by β).

- **Example:** Suppose that x is the temperature measured in Celsius degrees. The temperature measured in Farenheit degrees is a positive affine transformation and represents a change of the 0 and of the units: namely, $g(x) = 32 + \frac{5}{9}x$ is the temperature in Farenheit degrees of the temperature x in Celsius.

- We have two comparability notions:
    - Ordinal: an object is hotter than another one (for instance, 22 degrees is hotter than 10 degrees the numerical representation does not play any role; 30 degrees is also hotter than 10 degrees).
    - Cardinal: 22 degrees is 2.2 times hotter than 10 degrees (the difference of temperature is 12 degrees).

- An strictly increasing transformation of U, $f : \mathbb{R} \to \mathbb{R}$ where $f' > 0$, only maintains the sign of the first derivative but not of the second (which is related with the concavity or convexity of f). $V = f \circ U$, we have $signV' = signf' \cdot signU' = signU'$ since $f' > 0$. However, $signV''$ may be different than $signU''$ (if $f'' < 0$).
- A positive affine transformation of U, $V = \alpha + \beta U$, maintains the sign of the first and second derivatives since $signV' = \beta signU' = signU'$ and $signV'' = \beta signU'' = signU''$.
- Hence, under uncertainty the concavity or convexity of the utility function representing $\succsim$ will be relevant and, as we will see, this will be meaningful and crucial.

- (P.3) says that the risky alternative where the agent receives x with probability $\frac{1}{3}$, y with probability $\frac{1}{3}$ and z with probability $\frac{1}{3}$ can be described as the composed lottery

$$\tfrac{2}{3} \odot \left[\tfrac{1}{2} \odot x \oplus \tfrac{1}{2} \odot y\right] \oplus \tfrac{1}{3} \odot z.$$

- Fix the finite set of possible outcomes (prizes) $X = \{x_1, ..., x_N\}$. Denote the $(N-1)-$dimensional simplex by

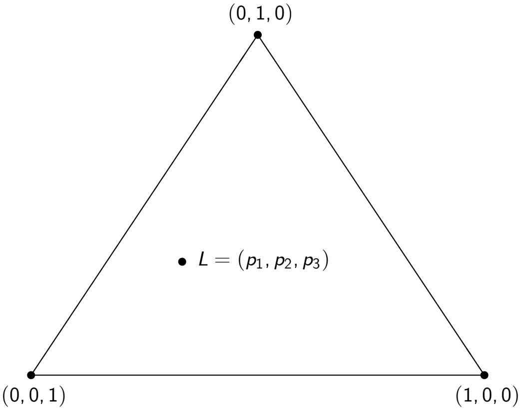
$$\Delta^N = \left\{ (y_1, ..., y_N) \in \mathbb{R}^N \mid y_i \geq 0 \text{ for all } i = 1, ..., N \text{ and } \sum_{i=1}^{N} y_i = 1 \right\}.$$

Definition: A *simple lottery* L is a vector $L = (p_1, ..., p_N) \in \Delta^N$, where p_i is interpreted as the probability of outcome x_i occurring.

- **Remark:** The set of simple lotteries Δ^N is a compact and convex subset of $\mathbb{R}^N$.

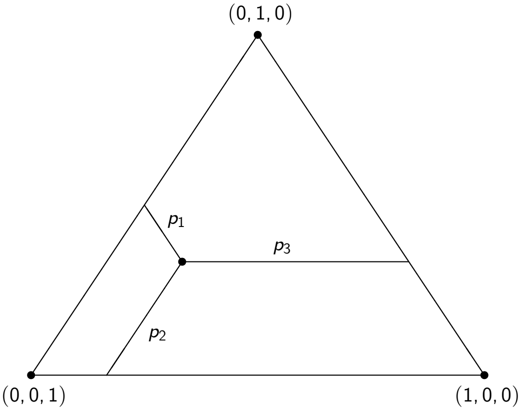
0.6.- Discussion

Lotteries and the Expected Utility Property: A Geometric Interpretation



0.6.- Discussion

Lotteries and the Expected Utility Property: A Geometric Interpretation



- The prizes of a simple lottery are certain outcomes. The prizes of a compound lottery may themselves be simple lotteries.

Definition: Given K simple lotteries $L_k = (p_1^k, ..., p_N^k)$, $k = 1, ..., K$, and probabilities $(q_1, ..., q_K) \in \Delta^K$, the *compound lottery* $(L_1, ..., L_K; q_1, ...q_K)$ is the risky alternative where the agent receives each prize L_k with probability q_k.

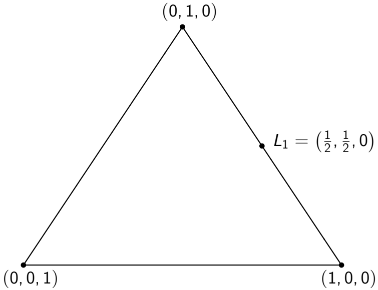
- Given a compound lottery $(L_1, ..., L_K; q_1, ...q_K)$ we can compute the simple lottery $L = (p_1, ..., p_N)$ that generates the same probability distribution over certain prizes as follows. For each $i = 1, ..., N$,

$$p_i = \sum_{k=1}^{K} q_k \cdot p_i^k.$$

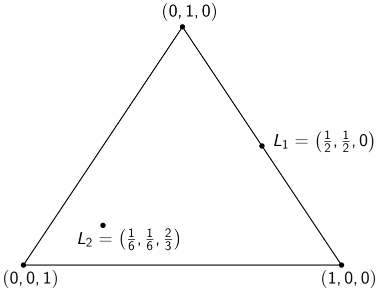
- Therefore, the simple lottery L of any compound lottery $(L_1, ..., L_K; q_1, ...q_K)$ can be obtained as the convex combination

$$L = \sum_{k=1}^{K} q_k \cdot L_k \in \Delta^N.$$

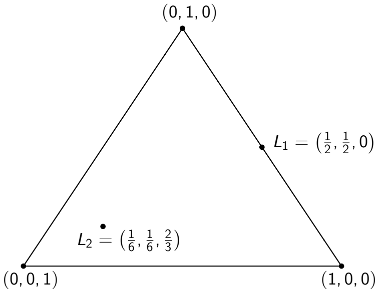
Example: Consider $L_1 = (\frac{1}{2}, \frac{1}{2}, 0)$,



Example: Consider $L_1 = (\frac{1}{2}, \frac{1}{2}, 0)$, $L_2 = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$

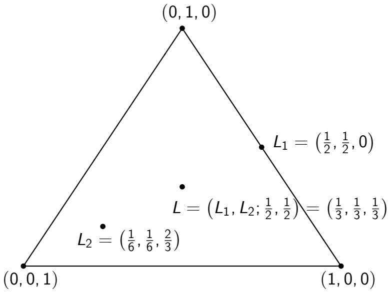


Example: Consider $L_1 = (\frac{1}{2}, \frac{1}{2}, 0)$, $L_2 = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ and $q_1 = q_2 = \frac{1}{2}$.



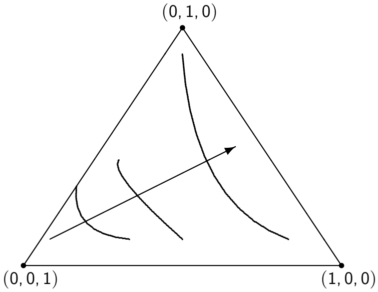
$$L = \left(L_1, L_2; \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right), \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right), \frac{1}{2}\left(0 + \frac{2}{3}\right)\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Example: Consider $L_1 = (\frac{1}{2}, \frac{1}{2}, 0)$, $L_2 = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ and $q_1 = q_2 = \frac{1}{2}$.



$$L = (L_1, L_2; \tfrac{1}{2}, \tfrac{1}{2}) = \left(\tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{6}\right), \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{6}\right), \tfrac{1}{2}\left(0 + \tfrac{2}{3}\right)\right) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

- The set of all lotteries $\mathcal{L}(X)$ on X can be identified with the set of simple lotteries Δ^N.
- A preference $\succsim$ on Δ^N can be geometrically represented by an indifference map.



Definition: The preference relation $\succsim$ on Δ^N is *continuous* if for all $L, L', L'' \in \Delta^N$ the sets

$$\{p \in [0,1] \mid pL + (1-p)L' \succsim L''\} \subseteq [0,1]$$

and

$$\{p \in [0,1] \mid pL + (1-p)L' \precsim L''\} \subseteq [0,1]$$

are closed.

Definition: The preference relation $\succsim$ on Δ^N satisfies the *independence axiom* if for all $L, L', L'' \in \Delta^N$ and all $p \in (0,1)$

$$L \succsim L' \text{ if and only if } pL + (1-p)L'' \succsim pL' + (1-p)L''.$$

Definition Assume that $U : \Delta^N \to \mathbb{R}$ represents the preferences $\succsim$ on Δ^N. We say that U satisfies the *expected utility property* if there exists a vector $u = (u_1, ..., u_N) \in \mathbb{R}^N$ such that for all $p = (p_1, ..., p_N) \in \Delta^N$,

$$U(p) = \sum_{i=1}^{N} p_i \cdot u_i.$$

- Namely, U is linear on Δ^N.

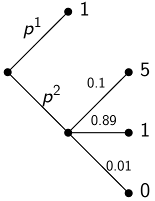
Theorem (von Neumann and Morgenstern, 1944).

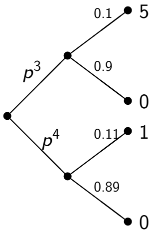
A preference $\succsim$ on Δ^N is continuous and satisfies the independence axiom if and only if there exists $U : \Delta^N \to \mathbb{R}$ representing $\succsim$ such that U satisfies the expected utility property.

Moreover, U is unique up to positive affine transformations.

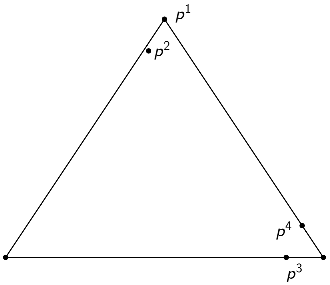
- There are three possible monetary prizes (in millions of Euros): $x_1 = 0$, $x_2 = 1$ and $x_3 = 5$. Consider the four simple lotteries in Δ^3 :

$$
\begin{aligned}
p^1 &= (0, 1, 0), \\
p^2 &= (0.01, 0.89, 0.1) \\
p^3 &= (0.9, 0, 0.1) \\
p^4 &= (0.89, 0.11, 0).
\end{aligned}
$$





- Express preferences between p^1 and p^2 and between p^3 and p^4.
- Often: $p^1 \succ p^2$ and $p^3 \succ p^4$.



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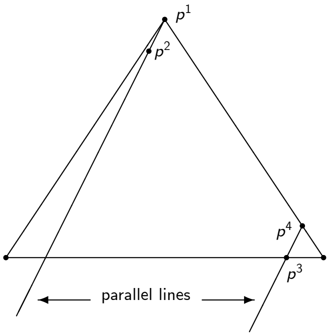


- This is inconsistent with the assumptions of the expected utility theorem.
- Let (u_1, u_2, u_3) be the utility values of the three monetary outcomes.
- Then, $p^1 \succ p^2$ implies

$$u_2 > 0.01 \cdot u_1 + 0.89 \cdot u_2 + 0.1 \cdot u_3.$$

- Adding $0.89 \cdot u_1 - 0.89 \cdot u_2$ to both sides, we obtain

$$0.89 \cdot u_1 + 0.11 \cdot u_2 > 0.90 \cdot u_1 + 0.1 \cdot u_3,$$

hence, any agent with v.N-M utility function must have $p^4 \succ p^3$.

Four reactions:

- Theory should help us to correct mistaken choices if they are proven to be inconsistent with principles ("to rectify is a sign of smartness").
- The choices presented in the Allais paradox are somehow artificial and without much significance for Economics.
- We should enlarge the theory to accommodate the fact that the agent not only values what he receives but also what he receives compared with what he might have received by choosing differently: *regret theory*.
- Stay with the model of lotteries but weaken the independence axiom.

0.7.- Monetary Lotteries

- There are many evidences showing that agents don't like risk.
- **Saint Petersburg Paradox:**
  - Consider the following lottery L. A coin is tossed repeatedly over time. Let n be the period in which Tails comes up for the first time. Then, the agent receives 2^n Euros.
  - How much would you pay to participate in such lottery?
  - Since all tosses are independent, the probability p_n of the event "Tails comes up for the first time at period n" is equal to $\frac{1}{2^n}$.
  - Suppose the agent's utility of x Euros is $u(x) = x$.
  - Then, the expected utility (and value) of this lottery is

$$U(L) = \sum_{n=1}^{\infty} p_n \cdot u(2^n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot 2^n = \infty.$$

  - However, if $\hat{u}(x) = \ln x$ then,
    $2 < \hat{U}(L) = \sum_{n=1}^{\infty} p_n \cdot \hat{u}(2^n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \ln 2^n < 3.$

0.7.- Monetary Lotteries

- To study such phenomenon, we are going to consider the special case of the previous model where the lottery prizes are money (in Euros), and we will model money as a continuous variable.
- Hence, the set of prizes (or outcomes) will be $X \subseteq \mathbb{R}$, not necessarily finite.
- Extension of the previous model to infinite settings.
  - Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is the sample space or the set of all possible outcomes, $\mathcal{F}$ is a $\sigma-$algebra of subsets of Ω (or the set of events) and P is a probability measure on Ω assigning to each event in $\mathcal{F}$ a number between 0 and 1.
  - Consider the pair $(X, \mathcal{B})$, where $\mathcal{B}$ is the family of all Borel subsets of $X \subseteq \mathbb{R}$.
  - Let $\tilde{x} : \Omega \to \mathbb{R}$ be a random variable; *i.e.*, a measurable function in the sense that for all $B \in \mathcal{B}$,

$$\tilde{x}^{-1}(B) = \{w \in \Omega \mid \tilde{x}(w) \in B\} \in \mathcal{F}.$$

0.7.- Monetary Lotteries

- A *monetary lottery* (*i.e.*, given $(\Omega, \mathcal{F}, P)$ the random variable $\tilde{x} : \Omega \to \mathbb{R}$ describes the monetary prize at each state of nature) will be described by a *cumulative distribution function* (cdf) $F : X \subseteq \mathbb{R} \to [0, 1]$, where for all $x \in X$,

$$F(x) = P\{w \in \Omega \mid \tilde{x}(w) \leq x\}.$$

- If f is the density function of F then, for all $x \in X$,

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

- Now, a monetary lottery can be identified with a cdf F; namely, with its associated distribution of money.

- The distribution of money of the monetary lottery L (or its cdf F) associated to the compound lottery $(L_1, ..., L_K; q_1, ..., q_K)$ (with their cdfs $F_1, ..., F_K$) is the convex combination of the cdfs; namely, for all $x \in X$,
$$F(x) = \sum_{k=1}^{K} q_k \cdot F_k(x)$$
- Observe that cdfs preserve the linear structure of lotteries.
- The set of all monetary lotteries $\mathcal{L}$ is the set of all cdfs over an interval $X = [a, +\infty)$.

- Fix X and let $\succsim$ be a preference on $\mathcal{L}$. The extension of the v.N-M Theorem says that if $\succsim$ is continuos and satisfies the independence axiom then, there exists $u : X \to \mathbb{R}$ such that for all $F, \hat{F} \in \mathcal{L}$,

$$F \succsim \hat{F} \text{ if and only if } U(F) = \int u(x)dF(x) \geq \int u(x)d\hat{F}(x) = U(\hat{F}).$$

- The following terminology, after Mas-Colell, Whinston and Green, is becoming standard.
  - $U : \mathcal{L} \to \mathbb{R}$ is the v.N-M expected utility function.
  - $u : X \to \mathbb{R}$ is the Bernoulli utility function.
- **Assumptions:**
  - u is increasing (for all $x, y \in X$ such that $x < y$, $u(x) \leq u(y)$).
  - u is continuous.

0.7.- Monetary Lotteries: Finite Setting

- Suppose $X = \{x_1, ..., x_N\} \subset \mathbb{R}$ is finite and for each $1 \leq k < N$, $x_k < x_{k+1}$.
- Then, we can identify a cdf F on X with the probability distribution (lottery) $p^F = \left(p_1^F, ..., p_N^F\right) \in \Delta^N$, where

$$F(x_1) = p_1^F$$

and for each $1 < k \leq N$,

$$F(x_k) = \sum_{j=1}^{k} p_j^F.$$

- In this finite case, for each cdf $F \in \mathcal{L}$,

$$\int x dF(x) = \sum_{k=1}^{N} x_k \cdot p_k^F = E_F.$$

and given the Bernoulli utility function $u : X \to \mathbb{R}$,

$$\int u(x) dF(x) = \sum_{k=1}^{N} u(x_k) \cdot p_k^F = U(F).$$

0.8.- Risk Aversion

- For $x \in X$ denote by δ_x the degenerate lottery that gives x for certain.
  - Namely, δ_x is the lottery $F \in \mathcal{L}$ where

$$F(y) = \begin{cases} 0 & \text{if } y < x \\ 1 & \text{if } y \geq x. \end{cases}$$

Definition: An agent is (*strictly*) *risk averse* if for any non-degenerate lottery $F \in \mathcal{L}$ with expected value $E_F = \int x dF(x)$ the agent (strictly) prefers δ_{E_F} to F.

Definition: An agent is *risk neutral* if for any lottery $F \in \mathcal{L}$ with expected value $E_F = \int x dF(x)$ the agent is indifferent between δ_{E_F} and F.

Definition: An agent is (*strictly*) *risk lover* if for any non-degenerate lottery $F \in \mathcal{L}$ with expected value $E_F = \int x dF(x)$ the agent (strictly) prefers F to δ_{E_F}.

0.8.- Risk Aversion

- Let X be an interval of $\mathbb{R}$. Hence, X is a convex set.

Definition: A function $f : X \to \mathbb{R}$ is (*strictly*) *concave* if for all $x, y \in X$, $(x \neq y)$, and all $\alpha \in [0, 1]$,

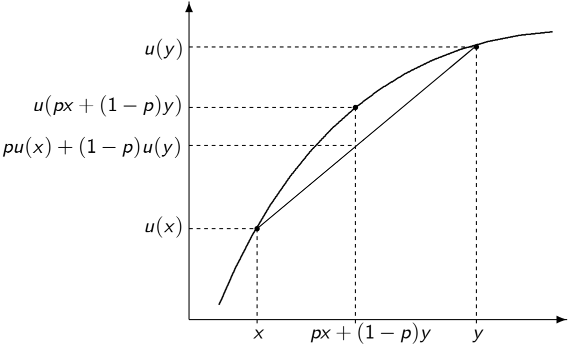
$$f\left(\alpha x + (1-\alpha)y\right)(>) \geq \alpha f(x) + (1-\alpha)f(y).$$

Proposition: A function $u : X \to \mathbb{R}$ is (*strictly*) *concave* if and only if for all $F \in \mathcal{L}$,

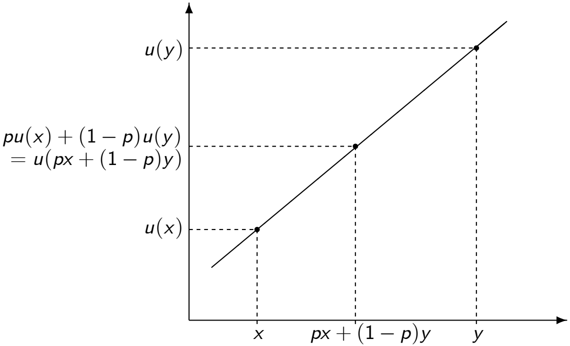
$$u\left(\int x dF(x)\right)(>) \geq \int u(x) dF(x).$$

- This inequality is called Jensen's inequality.
- An agent is (strictly) risk averse if and only if his Bernoulli utility function is (strictly) concave.
- An agent is risk neutral if and only if his Bernoulli utility function is linear.

- Concavity of u means that the marginal utility of money is decreasing.



0.8.- Risk Neutrality: A geometric interpretation



0.8.- Risk Aversion: Two definitions

Definition: Let u be a Bernoulli utility function of an agent.

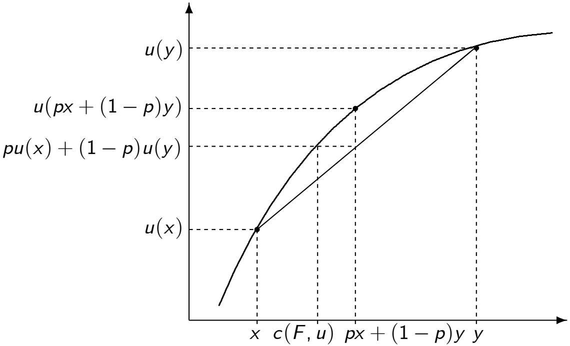
- The *certainty equivalent* of $F \in \mathcal{L}$, denoted by $c(F, u)$, is the amount of money for which the agent is indifferent between the lottery F and the certain amount $c(F, u)$. Namely,

$$u(c(F, u)) = \int u(x) dF(x)$$

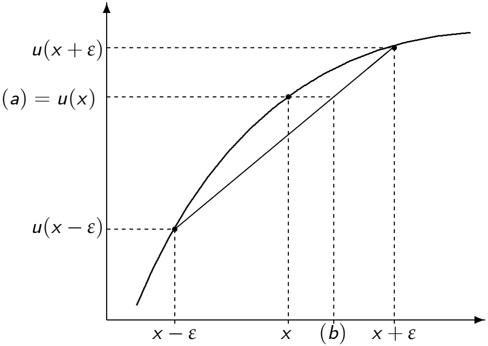
- For any fixed amount of money x and $\varepsilon > 0$, the *probability premium*, denoted by $\pi(x, \varepsilon, u)$, is the excess in winning probability over fair odds that makes the agent indifferent between the amount x with certainty and a lottery between the two outcomes $x - \varepsilon$ and $x + \varepsilon$. Namely,

$$u(x) = \left(\tfrac{1}{2} + \pi(x, \varepsilon, u)\right) \cdot u(x + \varepsilon) + \left(\tfrac{1}{2} - \pi(x, \varepsilon, u)\right) \cdot u(x - \varepsilon).$$

0.8.- Risk Aversion: Certainty equivalent



0.8.- Risk Aversion: Probability premium



$$(b) = [\tfrac{1}{2} + \pi(x,\varepsilon,u)](x+\varepsilon) + [\tfrac{1}{2} - \pi(x,\varepsilon,u)](x-\varepsilon) = x + 2\varepsilon\pi(x,\varepsilon,u)$$

$$(a) = [\tfrac{1}{2} + \pi(x,\varepsilon,u)]u(x+\varepsilon)) + [\tfrac{1}{2} - \pi(x,\varepsilon,u)]u(x-\varepsilon) = u(x)$$

0.8.- Risk Aversion: Characterizations

Proposition: *Let u be a Bernoulli utility function of an agent. Then, the following statements are equivalent:*

1. *The agent is risk averse.*
2. *u is concave.*
3. $c(F, u) \leq \int x dF(x)$ *for all* $F \in \mathcal{L}$.
4. $\pi(x, \varepsilon, u) \geq 0$ *for all* $x \in X$ *and* $\varepsilon > 0$.

- We have already saw that 1 is equivalent to 2.
- To see that 1 and 3 are equivalent, let $F \in \mathcal{L}$ be arbitrary. Then,

$$c(F,u) \leq \int x dF(x)$$

$$\Leftrightarrow u\left(c(F,u)\right) \leq u\left(\int x dF(x)\right) \qquad \text{since } u \text{ is increasing}$$

$$\Leftrightarrow \int u(x) dF(x) \leq u\left(\int x dF(x)\right) \qquad \text{by definition of } c(F,u),$$

but the last inequality is the definition of risk aversion.

- To show that 2 is equivalent to 4, see Problem Set #0.

0.8.- Risk Aversion: Its measurement

- Recall that if u is twice-differentiable then, u is concave if and only if $u''(x) \leq 0$ for all $x \in X$.

Definition: Given a twice differentiable Bernoulli utility function u the *Arrow-Pratt coefficient of absolute risk aversion at* x is defined as

$$r_A(x, u) = -\frac{u''(x)}{u'(x)}.$$

- A natural measure of risk aversion is the curvature of u at x and hence $u''(x)$ seems natural. However, the second derivative is not invariant with respect to positive affine transformations because if $v(x) = a + bu(x)$, for $b > 0$, then $v'(x) = bu'(x)$ and $v''(x) = bu''(x)$; hence v and u would have different risk aversions at x since $b > 0$. However,

$$r_A(x, v) = -\frac{v''(x)}{v'(x)} = -\frac{bu''(x)}{bu'(x)} = -\frac{u''(x)}{u'(x)} = r_A(x, u).$$

- Consider the family of Bernoulli utility functions

$$u(x) = \beta - \alpha e^{-ax}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$.

- Then, $u'(x) = a\alpha e^{-ax}$ and $u''(x) = -a^2\alpha e^{-ax}$.

- Hence,

$$r_A(x, u) = -\frac{u''(x)}{u'(x)} = -\frac{-a^2\alpha e^{-ax}}{a\alpha e^{-ax}} = a.$$

- A Bernoulli utility function in this family is know as CARA (Constant Absolute Risk Averse).

- **Objective:** To compare agents in terms of its risk aversion. Is it meaningful to say that agent 1 with u_1 is more risk averse than agent 2 with u_2?

Proposition: *The following five statements are equivalent.*

1. $r_A(x, u_1) \geq r_A(x, u_2)$ *for all* $x \in X$.
2. *There exists an increasing concave function* $g : \mathbb{R} \rightarrow \mathbb{R}$ *such that for all* $x \in X$, $u_1(x) = g(u_2(x))$.
3. $c(F, u_1) \leq c(F, u_2)$ *for all* $F \in \mathcal{F}$.
4. $\pi(x, \varepsilon, u_1) \geq \pi(x, \varepsilon, u_2)$ for all $x \in X$ and $\varepsilon > 0$.
5. *Whenever agent 1 fins a lottery* F *at least as good as a riskless outcome* $\bar{x}$, *then agent 2 also finds* F *at least as good as* $\bar{x}$. *Namely, for all* $F \in \mathcal{F}$ and all $\bar{x} \in X$, $\int u_1(x)dF(x) \geq u_1(\bar{x})$ *implies* $\int u_2(x)dF(x) \geq u_2(\bar{x})$.

0.8.- Risk Aversion: Comparisons across wealth levels

- **Objective:** To compare the risk aversion of an agent at different wealth levels.
- Consider two wealth levels $x < y$.
- Let u be the Bernoulli utility function of the agent.
- We want to compare how the agent evaluates risk at x and at y.
- Denote by z the variation of wealth (around x and around y).
- Then, the agent evaluates risk at x and at y by, respectively, the induced Bernoulli utility functions $u_x(z) = u(x + z)$ and $u_y(z) = u(y + z)$. That is, it is like if we had two different individuals (with their u_x and u_y) evaluating his wealth variations.
- But the previous Proposition gave an answer to this comparison.

0.8.- Risk Aversion: Comparisons across wealth levels

Definition: The Bernoulli utility function u exhibits *decreasing absolute risk aversion* if $r_A(x, u)$ is a decreasing function of x.

Proposition: *The following five statements are equivalent.*

1. *The Bernoulli utility function u exhibits decreasing absolute risk aversion.*
2. *Whenever $x < y$, $u_x(z) = u(x + z)$ is a concave transformation of $u_y(z) = u(y + z)$.*
3. *For any F, fix $x \in X$ and define c_x to be $u(c_x) = \int u(x + z)dF(z)$ (the certainty equivalent of the lottery formed by adding z to x). The expression $(x - c_x)$ is decreasing in x (the higher is x, the less is the agent willing to pay to get rid of the risk).*
4. *The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.*
5. *For any F, if $x < y$ and $\int u(x + z)dF(z) \geq u(x)$, then $\int u(y + z)dF(z) \geq u(y)$.*

0.8.- Risk Aversion: Comparisons across wealth levels

Definition: Given the Bernoulli utility function u, the *coefficient of relative risk aversion at* x is

$$r_R(x, u) = -x \cdot \frac{u''(x)}{u'(x)};$$

namely, $r_R(x, u) = x \cdot r_A(x, u)$.

Proposition: *The following three statements are equivalent.*

1. $r_R(x, u)$ *is decreasing in* x.
2. *Whenever* $x < y$, $\widehat{u}_x(t) = u(tx)$ *is a concave transformation of* $\widehat{u}_y(t) = u(ty)$.
3. *Given* $X = \mathbb{R}_{++}$ *the certainty equivalent* $\hat{c}_x$ *defined by* $u(\hat{c}_x) = \int u(tx)dF(t)$ *is such that* $\frac{x}{\hat{c}_x}$ *is decreasing in* x.

0.8.- Risk Aversion: Examples

- CARA: $u(x) = \beta - \alpha e^{-ax}$.
- $r_A(x, u) = a$ (CARA).
- $r_R(x, u) = a \cdot x$ (IRRA).

0.8.- Risk Aversion: Examples

- IARA: $u(x) = x - \alpha x^2$, where $\alpha > 0$.
  - $u'(x) = 1 - 2\alpha x$.
  - $u''(x) = -2\alpha$.
- $r_A(x, u) = -\frac{-2\alpha}{1-2\alpha x} = \frac{2\alpha}{1-2\alpha x}$.
- Thus, $r'_A(x, u) = \frac{4\alpha^2}{(1-2\alpha x)^2} > 0$ (IARA).
- $r_R(x, u) = \frac{2\alpha x}{1-2\alpha x}$.
- Thus, $r'_R(x, u) = \frac{2\alpha(1-2\alpha x)-(-2\alpha)\cdot(2\alpha x)}{(1-2\alpha x)^2} = \frac{2\alpha}{(1-2\alpha x)^2} > 0$ (IRRA).

0.8.- Risk Aversion: Examples

- CRRA: $u(x) = \ln x$.
  - $u'(x) = \frac{1}{x}$.
  - $u''(x) = -\frac{1}{x^2}$.
- $r_A(x, u) = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$ (DARA).
- $r_R(x, u) = -x\frac{-\frac{1}{x^2}}{\frac{1}{x}} = 1$ (CRRA).

0.8.- Comparing Risk Prospects

- **Objectives:** Is it meaningful to say first that a distribution F gives unambiguously higher returns than G and second, F is unambiguously less risky than G?
- The answers to these questions are related to the notions of first-order stochastic dominance and second-order stochastic dominance.
- Assumption: from now on we consider only lotteries on $X = \mathbb{R}_+$ and with the property that $F(0) = 0$ and $F(x) = 1$ for some $x \in X$.

Definition: The distribution F first-order stochastically dominates distribution G if for every nondecresing function u we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition: *The distribution F first-order stochastically dominates G if and only if $F(x) \leq G(x)$ for every $x \in X$ $[\Leftrightarrow 1 - F(x) \geq 1 - G(x)]$.*

0.8.- Comparing Risk Prospects: Example

- Consider $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$.
-

	f	g	F	G
x_5	0.2	0.1	1.0	1.0
x_4	0.0	0.1	0.8	0.9
x_3	0.4	0.3	0.8	0.8
x_2	0.3	0.4	0.4	0.5
x_1	0.1	0.1	0.1	0.1

- For all $u(x_1) < u(x_2) < u(x_3) < u(x_4) < u(x_5)$,

$$\int u(x)dF(x) = \sum_{i=1}^{5} u(x_i)f(x_i) \geq \sum_{i=1}^{5} u(x_i)g(x_i) = \int u(x)dG(x)$$

Definition: We say that the distribution F *second-order stochastically dominates* (or *is less risky than*) distribution G if for every nondecreasing concave function u we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition

1. $F \succsim_{FOSD} G \Rightarrow E_F \geq E_G$.
2. $F \succsim_{FOSD} G \Rightarrow F \succsim_{SOSD} G$.
3. $F \succsim_{SOSD} G \Rightarrow E_F \geq E_G$ and $VarF \leq VarG$.