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MS. MACHIAVELLI AND THE STABLE MATCHING PROBLEM

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This paper is a sequel to a paper, "Machiavelli and the Gale-Shapley Algorithm," written by Dubins and Freedman [**1**] in 1981. That paper was, in turn, a sequel to "College Admissions and the Stability of Marriage," by Gale and Shapley [**2**], 1962. For the benefit of readers who may have missed the previous installments, a brief recapitulation is given in the following section.

1. The Story So Far. Paper [2] above was concerned with a situation in which there are two sets of "agents", such as students and universities or workers and employees, or women and men. For the sake of concreteness, we will describe the problem in terms of this last group. It is assumed that each man and woman has prepared a list (possibly empty) containing the names of

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those people of the opposite sex that he or she would accept as a marriage partner and that the names appear on the list in order of preference. The lists are then submitted to a matchmaker who has the job of finding a suitable pairing of the participants. If we think of an unmatched person as being matched with him or herself, then the only requirement for the matching is that it should be free of *instability*; that is, there should not be any man m and woman w who are not matched with each other but prefer each other to their mates. The matchings are thus, "divorce-proof". The main result of [**2**] is an existence theorem which shows that for any sets M and W of men and women and any preference pattern, there is always at least one *stable matching*. The proof is constructive, giving an algorithm for finding the desired matching.

Since we know that stable matchings exist, it is an instructive and rather easy exercise to show that the set of stable matchings forms a lattice in the following sense: if μ and μ' are matchings, we write $\mu \underset{M}{\succsim} \mu'$ if every man likes his μ-mate at least as well as his μ'-mate. It turns out that this partial ordering of stable matchings is a lattice, as we will show shortly. Further, it is not hard to show that if $\mu \underset{M}{\succsim} \mu'$, then $\mu' \underset{W}{\succsim} \mu$, meaning that if all men prefer μ to μ', then all women prefer μ' to μ (these observations are due to J. H. Conway and are contained in [**4**]). As a corollary, since every finite lattice has a largest and a smallest element, it follows that among all stable matchings there is one which is preferred to all others by the men and another by the women. These will be called the *$M(W)$-optimal matchings*. The algorithms of [**2**] always arrive at one of these extreme matchings.

Some twenty years later, Dubins and Freedman considered the following question: suppose it is known that the matchmaker will impose the M-optimal stable matching and suppose some participants have full knowledge of the preference of all of the others. Will it ever be possible for an individual or group of individuals to obtain a preferred mate by falsifying preference lists? It was shown in [**1**] that there are cases in which a woman can do better by falsifying. On the other hand, the main result showed that no man or coalition of men can ever be made better off by falsifying their true preferences. The conclusion is that Machiavellian behavior on the part of the men is not profitable and they can do no better than to list their true preferences, regardless of what the women do. The purpose of the present paper is to give a detailed analysis of the strategic possibilities for the women. It will turn out that it will almost always pay for some of the women to be "Machiavellian", and under certain plausible assumptions we will describe their best competitive behavior.

2. Some History. The scenario described above involving marriages is, of course, rather fanciful, but there are genuine real world situations in which the matching problem comes up regularly. The problem of assigning students to colleges (which was the original motivation for these investigations) is an example. The difficulty here is that, as is so often the case, there are complicating factors such as the possibility of financial aid of various kinds. However, there is at least one situation where the methods of [**2**] are not only applicable but are actually being applied and have been for more than thirty years. The "National Residents Matching Program", headquarters in Evanston, Illinois, has the task each year of assigning graduates of medical schools throughout the country to hospital programs in which they are to fulfill a residency requirement.* It turns out that the procedure used by NRMP is precisely the one described in [**2**], except that their procedure leads to the hospital-optimal rather than the student-optimal matching of [**2**]. The Dubins-Freedman Theorem shows that when the student optimal procedure is used the students can do no better than to list the hospitals they are interested in in their true order of preference. Under the NRMP system, on the other hand, the results presented here show that in general there will be some students who can get into a preferred hospital by suitably falsifying their preferences.

It should be pointed out that the student-college problem is more general than the marriage problem in that the colleges are "polygamous" and may admit any number of students up to some fixed quota q. Many of the properties of the marriage problem carry over to the polygamous case, including those to be presented here. However, not all results carry over to the more general case.

In particular, Roth has recently shown [**6**] by an example that even when the college optimal matching is used, as in the case of NRMP, it may still be possible for a college to get a better class by appropriate misrepresentation of preferences.

We now return to our main subject.

3. Preliminaries. In this section we present two propositions which will be needed later and which are of some interest in themselves. These are special cases of some results of [**3**]. The first is rather surprising. Recall that in general there may be many stable matchings for a given set of men and women. Our result asserts, however, that the people who are unmatched or, as we shall say, self-matched, are the same for all of these matchings. This is particularly striking in the context of the student-hospital situation. Suppose NRMP were to change its policy and impose the student rather than hospital-optimal matching. Of course this would make all the students at least as well off, but those students who were not accepted by any hospital would still not be accepted and, on the other hand, each hospital would end up admitting the same number of students, though in general not the same set, under the student-optimal as under the hospital-optimal scheme. This rather non-obvious fact turns out to be quite easy to prove. Our second proposition, by contrast, seems extremely plausible. It asserts that if additional women are added to the "pool", this can never make any of the men worse off. The proof of this, however, is not so straightforward.

We will need a little terminology. A *preference pattern* will consist of a triple $(M, W; P)$ where M and W are the men and women and P represents their preferences, that is, the ordered lists of the members of $M \cup W$. Because of our convention that an unmatched person is considered to be self-matched, we may define a matching μ as a bijection of $M \cup W$ onto itself which is of order 2, that is, $\mu \cdot \mu =$ identity, and such that if m or w is not self-matched, then $\mu(m) \in W$ and $\mu(w) \in M$. We call $\mu(m)$ ($\mu(w)$) the *mate* of m (w) under μ. We say that a pair (m, w) *blocks* the matching μ if m prefers w to $\mu(m)$ and w prefers m to $\mu(w)$. The matching μ is then *stable* if it is not blocked by any pair. If μ and μ' are matchings, we say that m (w) prefers μ' to μ if he (she) prefers $\mu'(m)$ to $\mu(m)$ ($\mu'(w)$ to $\mu(w)$). We write $\mu \succsim_M \mu'$ if every man is at least as well off under μ as under μ'. A key result is now the following.

LEMMA 1. *Suppose $W \subset W'$ and μ is a stable matching for $(M, W; P)$ and μ' for $(M, W'; P')$ where P' agrees with P on W. Let M_μ be all men who prefer μ to μ' and let $W_{\mu'}$ be all women who prefer μ' to μ. Then μ and μ' are bijections between M_μ and $W_{\mu'}$.*

This is a sort of decomposition lemma represented schematically by the diagram below.

$$M_\mu \underset{\mu'}{\overset{\mu}{\leftrightarrows}} W_{\mu'}$$

$$M - M_\mu \underset{\mu'}{\overset{\mu}{\leftrightarrows}} W' - W_{\mu'}$$

It will suffice to show that $\mu(M_\mu) \subset W_{\mu'}$ and $\mu'(W_{\mu'}) \subset M_\mu$, for since M_μ and $W_{\mu'}$ are finite and μ and μ' are injective, they must be surjective as well. For m in M_μ we know $\mu(m) \neq m$ since m prefers $\mu(m)$ to $\mu'(m)$; so let $w = \mu(m)$. Then $w \in W$ and $w \neq \mu'(m)$. Further, w prefers $\mu'(w)$ to m for if not (m, w) would block μ', so $w \in W_{\mu'}$. A symmetric argument shows that $\mu'(W_{\mu'}) \subset M_\mu$. □

Our first result is an immediate consequence of the lemma.

PROPOSITION 1. *If μ and μ' are stable matchings for (M, W), then the people who are self-matched are the same for both.*

□ Suppose $\mu'(m) = m$ but $\mu(m) = w$. Then $m \in M_\mu$ but then, by the Lemma for the case $W = W'$, $m \in \mu'(W_{\mu'})$, contradicting $\mu'(m) = m$. □

LEMMA 2. *Let μ and μ' be as in Lemma 1 and define μ'' to agree with μ on $M_\mu \cup W_{\mu'}$ and with μ' on $(M - M_\mu) \cup (W' - W_{\mu'})$. Then μ'' is stable for (M, W').*

□ Clearly μ'' restricted to $M_\mu \cup W_{\mu'}$ or $(M - M_\mu) \cup (W' - W_{\mu'})$ is stable, so if there is a blocking pair, we must have either $m \in M_\mu$, $w \notin W_{\mu'}$ or $m \notin M_\mu$ and $w \in W_{\mu'}$. Suppose w prefers m to $\mu''(w)$.

Case 1. $m \in M_\mu$, $w \in W'_\mu - W_{\mu'}$. Then m prefers $\mu(m)$ to $\mu'(m)$ whom he likes at least as well as w, so (m, w) does not block μ''.

Case 2. $m \notin M_\mu$, $w \in W_{\mu'}$. Then m likes $\mu'(m)$ at least as well as $\mu(m)$ whom he prefers to w, since μ is stable so, again, (m, w) does not block μ''. □

The lattice property is a consequence of Lemma 2, for note that μ'' assigns each man his favorite of $\mu(m)$ and $\mu'(m)$, so in usual lattice notation we write $\mu'' = \mu \vee \mu'$. As stated in the introduction, it now follows that for any $(M, W; P)$ there exist matchings μ_M and μ_W which are M-optimal and W-optimal, respectively. We now have

PROPOSITION 2. *With the hypothesis of Lemma 1 let μ_M be the M-optimal matching for $(M, W; P)$ and let μ'_M be the M-optimal matching for $(M, W'; P')$. Then every man is at least as well off under μ'_M as under μ_M.*

□ From Lemma 2, $\mu_M \vee \mu'_M$ is stable for $(M, W'; P')$ and

$$\mu \underset{M}{\gtrsim} \mu' \vee \mu' \underset{W}{\gtrsim} \mu \quad . \square$$

4. The Matching Game. Since the Dubins-Freedman Theorem shows that it is never advantageous for the men to falsify their preferences, we will assume from here on that the men always submit their true preference list. For the special case where there is only one stable matching, the Dubins-Freedman Theorem implies that honesty is also the best policy for the women. In all other cases, however, this is not so.

THEOREM 1. *If there is more than one stable matching, then there is at least one woman who will be better off by falsifying, assuming the others tell the truth.*

□ By hypothesis $\mu_M \neq \mu_W$, so let w be any woman such that $\mu_W(w) \neq \mu_M(w)$. Now let w falsify by removing from her preference list all men who rank below $\mu_W(w)$. Clearly the matching μ_W will still be stable under these preferences (there are now fewer possible blocking pairs). Letting μ'_M be the M-optimal matching for these new preferences, it follows from Proposition 1 that w is not self-matched by μ'_M and hence she is matched with someone she likes at least as well as $\mu_W(w)$, since all other men have been removed from her list, and she prefers $\mu_W(w)$ to $\mu_M(w)$ so she prefers μ'_M to μ_M. □

Theorem 1 shows that the policy of honest revelation of preference is "unstable" for the women in a different sense of instability, namely, if each woman expects the others to be honest, it will in general be possible for at least one woman to improve her position by lying. This leads one to ask the following question: if honesty is not the best policy, is there any set of policies or "strategies" which have the property that once they are adopted by the women there will be no advantage to any one woman in changing her strategy? It will be useful here to introduce some standard game theoretic-terminology and we will henceforth refer to our model as the *matching game*. We are dealing here with perhaps the most important concept in game theory, that of an *equilibrium point*. An abstract game consists of a set of players each of whom has a certain set of *strategies*. In a play of the game each player selects one of her strategies and the "rules of the game" then assign her some *payoff* which may be a number ("score") or, as in our case, a mate. A set of strategies, one for each player, forms an *equilibrium point* if no player can, by changing her strategy, achieve a better payoff, assuming the other players do not change theirs. The strategies form a *strong equilibrium point* if no subset of players by changing its strategies can achieve a better payoff for all of its members. Are there any equilibrium point strategies for the women in the matching game? The next two theorems answer this question.

THEOREM 2. *Let μ be any stable matching for (M, W, P) and suppose each woman in $\mu(M)$ chooses the strategy of listing only $\mu(m)$ on her preference list. This is an equilibrium point.*

□ It is clear that μ is stable under these falsified preferences which we will denote by P'. Further, μ is the only stable matching for $(M, W; P')$, for any other matching would leave some w in $\mu(M)$ unmatched, which is not possible by Proposition 1. Hence μ is the M-optimal matching for $(M, W; P')$.

To see that P' is an equilibrium point, suppose some w now changes her preference list leading to a new M-optimal matching μ' which gives her a mate $m' = \mu'(w)$ whom she prefers to $\mu(w)$ under true preference. Then m' must have been matched by μ to some w', for if not (m', w) would have blocked μ in $(M, W; P)$. But then w' is self-matched under μ' since m' was the only man on her P'-list. This means m' prefers w to w', but if this were so, again (m', w) would have blocked μ, a contradiction. □

Theorem 2 says that the women can force any matching μ which is stable under the true preferences by equilibrium point strategies. We now present a sort of converse.

THEOREM 3. *Suppose the women choose any set of strategies P'_W (preference lists) that form an equilibrium point for the matching game. Then the corresponding M-optimal matching for $(M, W; P')$ is one of the stable matchings of $(M, W; P)$.*

This theorem has been proved by Roth [**5**] making use of the properties of the matching algorithm of [**2**]. We present here a direct proof.

□ Suppose μ' is the M-optimal matching for $(M, W; P')$ but (m, w) blocks μ' under w's true preference. We will show that P' is not an equilibrium point. Namely, let w refalsify by listing only m on her preference list. Then she will get him, for let μ'' be the M-optimal matching for the new preference P''. If w does not get m, then she is self-matched by μ'' so by stability of μ'', m prefers $\mu''(m)$ to w and by assumption m prefers w to $\mu'(m)$, so m prefers $\mu''(m)$ to $\mu'(m)$. But clearly the matching μ'' would be stable for $(M, W - w; P')$, where P' is restricted to $W - w$. Thus m' is worse off under the M-optimal matching for $(M, W; P')$ than he is under μ'' for $(M, W - w, P')$, which directly contradicts Proposition 2 of the previous section. □

To summarize this section, we see that by falsifying appropriately the women can achieve by equilibrium point strategies any stable matching, thus, in particular, the W-optimal matching. On the other hand, the women cannot get too greedy for if any set of strategies gives some woman w a mate whom she likes better than $\mu_W(w)$, this will not be an equilibrium point, by Theorem 3, so some other woman can change the matching to her advantage by choosing a different strategy.

5. Strong Equilibrium Points. By Theorem 2, the women can achieve any stable matching μ by equilibrium point strategies. However, unless $\mu = \mu_W$ the equilibrium point will not be strong. To see this note that if $\mu \neq \mu_W$, then $\mu(w) \neq \mu_W(w)$ for at least two women, for say, $\mu(w_1) \neq \mu_W(w_1) = m_1$. By Proposition 1, m_1 is not self-matched by μ and hence $\mu(m_1) = w_2$ and $\mu(w_2) = m_1 \neq \mu_W(w_2)$. To show μ is not the matching of a strong equilibrium point we let $\tilde{W}$ be all w such that $\mu(w) \neq \mu_W(w)$. Let all w's in $\tilde{W}$ refalsify by pretending $\mu_W(w)$ is the only man on their list. Then μ_W is stable for this new preference and hence since all $\tilde{W}$ are matched by μ_W, they are matched by the M-optimal matching for the new preference.

Do there exist strong equilibrium points? Yes.

THEOREM 4. *Let each woman w submit a preference list in true order of preference but removing all men who are ranked below $\mu_W(w)$. These preferences P' are a strong equilibrium point.*

□ We claim that μ_W is the only stable matching for $(M, W; P')$, for clearly μ_W is stable and any other stable matching μ' must have $\mu'(w) \neq \mu_W(w)$ for some w; hence if w is not self-matched, then $\mu'(w)$ is preferred by w to $\mu_W(w)$. Since μ_W is the W-optimal stable matching, this means μ' is unstable under true preferences, hence is blocked by some pair (m, w). But by construction of P', this means (m, w) blocks μ' under P' preferences, contradicting P'-stability of μ'.

Now since μ_W is the only stable matching for $(M, W; P')$ it is the W-optimal matching for P'. If some subset $\tilde{W}$ could get a better payoff by falsifying, then it would get a better payoff than from the W-optimal matching of $(M, W; P')$, but by the Dubins-Freedman Theorem this is impossible. □

It seems reasonable to consider the falsification of P' of Theorem 4 as the best method of play for the women. It is (a) a strong equilibrium point so no woman or set of women would be tempted to deviate from P', and, (b) among all equilibrium point strategies it gives the women the highest possible payoff. It would have been nice to be able to assert that the matching μ_W is the only matching obtainable from a strong equilibrium point. Unfortunately this is not the case, as shown by the following example. The true preferences are given by Table 1.

TABLE 1

	w_1	w_2	w_3	w_4
m_1	2,2	1,3	3,x	x,x
m_2	1,3	3,2	2,3	4,x
m_3	x,x	x,x	1,2	2,x
m_4	3,1	4,1	2,1	1,1

TABLE 2

	w_1	w_2	w_3	w_4
m_1	x	2	x	x
m_2	2	x	2	x
m_3	x	x	3	x
m_4	1	1	1	1

The first entry in box i, j is the ranking of w_j on the list of m_i. If it is x, this means that w_j is not in the list of m_i. Thus, m_2 ranks w_2 in third place and w_1 is not in the list of m_3. The second entry is the ranking of m_i by w_j. If it is x, this signifies that m_i is not in the list of w_j. So, w_1 ranks m_1 in second place and m_3 is not in the list of w_4. For these preferences the matching μ_W is given by (m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4). Now suppose all women use the system of preferences, P', given by Table 2 above. The entry in box i, j is the ranking of m_i by w_j. The M-optimal matching for these new preferences is given by (m_2, w_1), (m_1, w_2), (m_3, w_3), (m_4, w_4). We assert that P' is a strong equilibrium point. In fact, no subset $W' \subseteq W$ which contains w_4, by falsifying, can improve the situation of all its members, since the mate of w_4 is the best possible; if W' contains w_3, it cannot improve the situation of w_3, for if so (m_4, w_4) would block the new matching. If W' contains w_2, it cannot get a better mate for w_2 by falsifying, for if so (m_4, w_4) or (m_2, w_3) would block the new matching. If W' contains w_1, it is not possible to improve the payoff of w_1, for if so (m_4, w_4) or (m_1, w_2) would block the new matching.

6. Dominated Strategies. There is one more important game-theoretic concept which is illustrated by the Matching Game. Let σ and σ' be two strategies for some player w in a general n-person game. We say that σ *dominates* σ' if the payoff to w when she plays σ is at least as high as when she plays σ' no matter what strategies the other players play, and σ *strictly dominates* σ' if the payoff to w when she plays σ is higher than when she plays σ' for at least one set of strategies for the other players. A strategy is called *dominant* if it dominates all other strategies.

As an illustration of these concepts, the Dubins-Freedman Theorem shows that for each man revealing true preferences is a dominant strategy. For the women there are no dominant strategies except for the special case where $|W| = 1$. If there is only one woman w, then revealing true preferences strictly dominates any other strategy, for if she submits her true preferences, she will get the highest man on her list who has listed her. If she falsifies by, for example, listing $m' \succ m$ instead of $m \succ m'$, she will be worse off in the case where m and m' are the only men who are willing to marry her. From now on it will be assumed that there are at least two women in W. We will show that the strategy of listing only one man, as in Theorem 2, is dominated unless that man happens to be the woman's true first choice. In fact, we can essentially characterize the dominated strategies of the women by the following theorems, where $a \underset{x}{\succ} b$ means that x prefers a to b under the true preference P_x.

THEOREM 5. *Any strategy P'_w in which w does not list her true first choice at the head of her list is strictly dominated.*

□ This has been proved by Roth [5] using the matching algorithm. We present here a direct proof.

Let P'_w be a strategy described above. We will show that P'_w is strictly dominated by P''_w which lists m_1 (w's favorite man) in first place and leaves the rest of the list unchanged. Let μ' and μ'' be the corresponding M-optimal matchings (the strategies of all other players are assumed unchanged). If $\mu''(w) = m_1$, there is nothing to prove so we suppose that $\mu''(w) \neq m_1$. Then

(1) $\mu''(m_1)$ is preferred by m_1 to w or w is not on m_1's list,

for if not (m_1, w) would block μ''. Hence μ'' is stable under preference P''_w. So,

$$\mu'(m_1) \underset{m_1}{\succ} \mu''(m_1), \tag{2}$$

from the M-optimality of μ'. Furthermore, from (1) and (2) it follows that $\mu'(m_1) \underset{m_1}{\succ} w$, and since all the other elements different from m_1 are ranked in the same ordering in both lists P'_w and P''_w, it follows that μ' is stable under P''_w, and hence $\mu''(w) = \mu'(w)$; so w is no worse off using P''_w than using P'_w.

To show that P''_w strictly dominates P'_w, let m' be the first element in P'_w. Consider the following preference pattern: m': $\{w\}$, $m_1 = \{w\}$ and no other man lists w. Then we can see that $m_1 = \mu''(w) \underset{w}{\succ} \mu'(w) = m'$, which concludes the proof. □

Our final result states that Theorem 5 describes essentially all the dominated strategies.

THEOREM 6. *Let P'_w be any strategy for w in which* (a) *m_1 (w's favorite man) is listed first,* (b) *P'_w contains only men m who are on w's true preference list P_w. Then P'_w is not dominated.*

□ We will show that for any other strategy P''_w there exists strategies $\breve{P}$ for the other players such that $\mu'(w) \underset{w}{\succ} \mu''(w)$, where μ' and μ'' are the M-optimal matchings for $(M, W; P'_w \cup \breve{P})$ and $(M, W, P''_w \cup \breve{P})$, respectively. There are three cases. We first suppose P''_w also satisfies (a) above.

Case 1. P'_w contains some m not on P''_w. Then for $\breve{P}$ we suppose that m lists w as first choice and no other man lists w. Then $\mu'(w) = m$ while w is self-matched under μ'' (here we used that $m \in P_w$).

Case 2. P''_w contains some m not on P'_w. Then for $\breve{P}$ suppose m has preference list $\{w \succ w'\}$ for some w', m_1 has preference list $\{w' \succ w\}$, and w' has preference list $\{m \succ m_1\}$ and no other man lists w or w'. Then one verifies that $\mu'(w) = m_1 \underset{w}{\succ} \mu''(w) = m$.

Case 3. Let lists P'_w and P''_w be the same but w prefers m to m' in P'_w and m' to m in P''_w. Then suppose preference list for m_1 is $\{w' \succ w\}$, for m is $\{w \succ w'\}$, for m' is $\{w \succ w'\}$, and for w' is $\{m' \succ m_1 \succ m\}$, and no other men list either w or w'. It is an instructive exercise to verify that $\mu'(w) = m_1$ while $\mu''(w) = m'$.

We have seen that if P_w' satisfies (a) and (b) and P_w'' satisfies (a), then for some $\tilde{P}$, $\mu'(w) \underset{w}{\succ} \mu''(w)$. If P_w'' does not satisfy (a), then by Theorem 5 there is some P_w''' which dominates P_w'' and P_w''' satisfies (a) so we construct $\tilde{P}$ so that $\mu'(w) \underset{w}{\succ} \mu'''(w)$ but $\mu'''(w) \underset{w}{\succsim} \mu''(w)$ by dominance. The proof is now complete. □

REMARK 1. Condition (b) above is needed to avoid cases in which $P_w' = \{m_1 \succ m\}$ where m is not in P_w. It is clear that this is strictly dominated by $P_w'' = \{m_1\}$.

REMARK 2. Note that the counterexample of Section 4 uses only undominated strategies for the women. Thus we do not have a unique strong equilibrium point for W even when restricting to the use of undominated strategies.

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144. MISCELLANEA

Inflation

The growth of mathematics is producing discontent:
The inflation rate per annum is pushing 10 per cent.
Faced with so much information, it's not easy to succeed
In locating any theorem that you appear to need.
Our plethora of indices can leave you in the lurch,
For it takes less time to prove it than to go and make a search.
I can offer a solution, but it's totally upsetting:
We need to introduce a way of constantly forgetting
The results that won't be needed for another 20 years,
At the end of which they'll surface with appropriate loud cheers,
While the ones that won't be needed till forever and a day,
Once their authors get their tenure will be firmly thrown away.

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