Corrigendum to "On strategy-proofness and semilattice single-peakedness" [Games Econ. Behav. 124 (2020) 219-238]*

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As it was pointed out to us by Huaxia Zeng, Theorem 1 in Bonifacio and Massó (2020), henceforth BM20, is not correct. In this note we recall former Theorem 1, exhibit a counterexample of its statement, identify the mistake in its faulty proof, and state and prove the new version of Theorem 1. At the end we give an alternative proof of Lemma 9, whose former proof used incorrectly Lemma 5.

Notation and definitions are as in the section of Preliminaries in BM20.

1 Former Theorem 1

1.1 Wrong statement

Let $(A, \succeq)$ be a semilattice and let $\mathcal{SSP}(\succeq)$ be the set of semilattice single-peaked preferences on $(A, \succeq)$. The *supremum rule*, denoted as $\sup_{\succeq} : \mathcal{SSP}(\succeq)^n \to A$, is defined by setting, for each profile $R = (R_1, \ldots, R_n) \in \mathcal{SSP}(\succeq)^n$,

$$\sup_{\succeq}(R_1, \ldots, R_n) = \sup_{\succeq} t(R),$$

where $t(R) = \{t(R_i) \mid i \in N\}$.

Given $R \in \mathcal{SSP}(\succeq)^n$ and $x \in A$, define $N(R, x) = \{i \in N \mid t(R_i) = x\}$ as the set of agents whose top is x at R. Assume A has a supremum, denoted as $\alpha \equiv \sup_{\succeq} A$.[1] Let

$$A^\star(\succeq) = \{x \in A \mid \text{ for each } y \in A \setminus \{\alpha\}, x \not\succeq y \text{ and } y \not\succeq x\}$$

be the set of alternatives that, according to $\succeq$, are not related to any other alternative but α. Observe that $A^\star(\succeq)$ may be empty and $\alpha \notin A^\star(\succeq)$.

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[1]We abuse the notation a bit and use $\sup_{\succeq}$ to denote the supremum rule and $\sup_{\succeq} X$ to denote the supremum of a set $X \subseteq A$.

Definition 1 *Let $\succeq$ be a semilattice over A such that $\sup_{\succeq} A$ exists. The rule $f : \mathcal{SSP}(\succeq)^n \to A$ is a* quota-supremum *rule if there are $x \in A^\star(\succeq)$ and integer q^x with $1 \leq q^x < n$ such that, for every $R \in \mathcal{SSP}(\succeq)^n$,*

$$f(R) = \begin{cases} x & \text{if } |N(R,x)| \geq q^x \\ \sup_{\succeq} t(R) & \text{otherwise.} \end{cases}$$

The wrong statement in BM20 was as follows.

Theorem 1 *Let $\succeq$ be a semilattice over A. The rule $f : \mathcal{SSP}(\succeq)^n \to A$ is strategy-proof and simple if and only if $f = \sup_{\succeq}$ or f is a quota-supremum.*

1.2 Counterexample

A slight modification of Example 2 in BM20 provides a counterexample of Theorem 1 in BM20. Let $A = \{x, y, z\}$ be the set of alternatives and let $(A, \succeq)$ be the semilattice where $x \succ y$, $x \succ z$, $y \not\succ z$ and $z \not\succ y$. Observe that $A^\star(\succeq) = \{y, z\}$. Consider the linear order $\rhd$ over A where $y \rhd x \rhd z$ and let $\mathcal{SP}(\rhd)$ be the domain of single-peaked preferences (relative to $\rhd$). It is easy to check that $\mathcal{SP}(\rhd) = \mathcal{SSP}(\succeq)$.

Let $N = \{1, 2, 3, 4, 5\}$ be the set of agents. For y and z and integers $1 \leq q^y < 5$ and $1 \leq q^z < 5$, consider the quota-supremum rules $f^{q^y} : \mathcal{SSP}(\succeq)^5 \to A$ and $f^{q^z} : \mathcal{SSP}(\succeq)^5 \to A$ defined according to Definition 1.

By definition, for all $R \in \mathcal{SSP}(\succeq)^5$ with $t(R) = \{y, z\}$, $f^{q^y}(R) \in \{y, x\}$, $f^{q^z}(R) \in \{z, x\}$, and $\sup_{\succeq}(R) = x$. Hence, the three conditions below hold.

(C.1) There does not exist $R \in \mathcal{SSP}(\succeq)^5$ with $t(R) = \{y, z\}$ such that $f^{q^y}(R) = z$.

(C.2) There does not exist $R \in \mathcal{SSP}(\succeq)^5$ with $t(R) = \{y, z\}$ such that $f^{q^z}(R) = y$.

(C.3) There does not exist $R \in \mathcal{SSP}(\succeq)^5$ with $t(R) = \{y, z\}$ such that $\sup_{\succeq}(R) = y$ or $\sup_{\succeq}(R) = z$.

Consider now the median voter rule $f : \mathcal{SSP}(\succeq)^5 \to A$ where, for all $R \in \mathcal{SSP}(\succeq)^5$,

$$f(R) = med_{\rhd}(t(R_1), t(R_2), t(R_3), t(R_4), t(R_5), y, y, y, z). \tag{1}$$

By Moulin (1980), f is strategy-proof and simple on $\mathcal{SP}(\rhd) = \mathcal{SSP}(\succeq)$. Three facts hold.

First, $f(R^1) = z$ if $(t(R^1_1), t(R^1_2), t(R^1_3), t(R^1_4), t(R^1_5)) = (y, z, z, z, z)$. By (C.1), $f \neq f^{q^y}$.

Second, $f(R^2) = y$ if $(t(R^2_1), t(R^2_2), t(R^2_3), t(R^2_4), t(R^2_5)) = (y, y, z, z, z)$. By (C.2), $f \neq f^{q^z}$.

Third, $f(R) \in \{y, z\}$ for all $R \in \mathcal{SSP}(\succeq)^5$ with $t(R) = \{y, z\}$. By (C.3), $f \neq \sup_{\succeq}$.

Therefore, the rule $f : \mathcal{SSP}(\succeq)^5 \to A$ defined in (1) is strategy-proof and simple but it is neither the supremum rule nor a quota-supremum rule. Thus, Theorem 1 is not correct.

1.3 Error in the proof

The error in the proof of former Theorem 1 was that it used Lemma 7, which is not correct and whose statement was the following.

Lemma 7 *Let $\succeq$ be a semilattice over A, let $f : SSP(\succeq)^n \to A$ be a strategy-proof and simple rule and let k be such that $1 \leq k < n$. Assume $x, y \in A$ are such that $x \succ y$ and there is $R \in \mathcal{SSP}(\succeq)^n$ such that $t(R) = \{x, y\}$, $|N(R, y)| = k$ and $f(R) = y$. Then, $f(\widetilde{R}) \in \{y, \sup_{\succeq} t(\widetilde{R})\}$ for all $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ such that $|N(\widetilde{R}, y)| < k$.*

To see that Lemma 7 does not hold, consider again the semilattice $(A, \succeq)$ and the rule f of the counterexample. Let $R \in \mathcal{SSP}(\succeq)^5$ be such that $t(R) = \{x, y\}$ and $|N(R, y)| = 2$. Then, $f(R) = y \prec x = \sup_{\succeq} t(R)$. Let $\widetilde{R} \in \mathcal{SSP}(\succeq)^5$ be such that $t(\widetilde{R}) = \{y, z\}$, and $|N(\widetilde{R}, y)| = 1$. Then, $\sup_{\succeq} t(\widetilde{R}) = x$ and $f(\widetilde{R}) = z \notin \{y, x\}$, so Lemma 7 does not hold. After stating the new and correct Theorem 1 in the next section, we will comeback to this example to check that this rule f is indeed covered by the correct characterization.

The error in the proof of Lemma 7 was that we mistakenly assumed that the profile $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ used in the proof had the property that $x \in t(\widetilde{R})$ (see the end of the first line in the proof, where we concluded that $t(\widetilde{R}) = \{x\}$), but as we just saw this does not have to be necessarily the case.

2 (New) Theorem 1

2.1 Generalized quota-supremum rules

Let $(A, \succeq)$ be a semilattice such that $A^\star(\succeq) \neq \emptyset$. A *quota system* $q = \{q^x\}_{x \in A^\star(\succeq)}$ assigns to each $x \in A^\star(\succeq)$ a quota $1 \leq q^x \leq n$ satisfying the following two properties.

(QS.1) There is $x \in A^\star(\succeq)$ such that $1 \leq q^x < n$.

(QS.2) For any two distinct alternatives $x, y \in A^\star(\succeq)$, $q^x + q^y > n$.

Definition 2 *Let $(A, \succeq)$ be a semilattice such that $\sup_{\succeq} A$ exists. The rule $f : \mathcal{SSP}(\succeq)^n \to A$ is a* generalized quota-supremum *rule if there exists a quota system $q = \{q^x\}_{x \in A^\star(\succeq)}$ such that, for every $R \in \mathcal{SSP}(\succeq)^n$,*

$$f(R) = \begin{cases} x & \text{if } x \in A^\star(\succeq) \text{ and } |N(R, x)| \geq q^x \\ \sup_{\succeq} t(R) & \text{otherwise.} \end{cases}$$

A quota-supremum rule f^{q^x} is defined by preselecting an alternative $x \in A^\star(\succeq)$ and an integer $1 \leq q^x < n$. Then, at each profile R, f^{q^x} chooses x if it receives the support of at least q^x agents, and $\sup_{\succeq} t(R)$ otherwise. In contrast, a generalized quota-supremum rule f^q has to specify, for each $x \in A^\star(\succeq)$, an integer $1 \leq q^x \leq n$ in such a way that at least one is strictly smaller than n and no two distinct alternatives $x, y \in A^\star(\succeq)$ can receive simultaneously the support of at least q^x and q^y agents, respectively. Then, at each profile R, f^q chooses the alternative $x \in A^\star(\succeq)$ that receives the support of at least q^x agents, and

$\sup_{\succeq} t(R)$ if no such alternative exists. Of course, generalized quota-supremum rules are more flexible because the quota used at each profile is not preselected, but rather it depends on the profile. Moreover, each quota-supremum rule f^{q^x} can be written as a generalized quota-supremum rule f^q, where the quota for alternative x is q^x and it is equal to n for all other alternatives in $A^\star(\succeq)$. In Subsection 2.4 we show how the median voter rule f defined in (1) can be represented as a generalized quota-supremum rule.

2.2 Correct statement

Theorem 1 *Let $\succeq$ be a semilattice over A. The rule $f : \mathcal{SSP}(\succeq)^n \to A$ is strategy-proof and simple if and only if $f = \sup_{\succeq}$ or f is a generalized quota-supremum.*

2.3 Proof of Theorem 1

PROOF ($\Longrightarrow$) Assume $f : \mathcal{SSP}(\succeq)^n \to A$ is *strategy-proof* and *simple*. Suppose $\sup_{\succeq} A$ does not exist. By Lemma 9, $f(R) = \sup_{\succeq} t(R)$ for each $R \in \mathcal{SSP}(\succeq)^n$. Hence, $f = \sup_{\succeq}$. Suppose $f \neq \sup_{\succeq}$. Then, $\sup_{\succeq} A$ does exist. Let $\alpha \equiv \sup_{\succeq} A$. To show that f is a generalized quota-supremum rule, define

$$\overline{A} = \{x \in A \setminus \{\alpha\} \mid \text{ there is } R \in \mathcal{SSP}(\succeq)^n \text{ with } \alpha \in t(R) \text{ and } f(R) = x\}.$$

By Lemma 8, $\overline{A} \neq \emptyset$. For each $x \in \overline{A}$, define

$$k(x) = \min_{1 \leq k < n} \{k = |N(R, x)| \mid R \in \mathcal{SSP}(\succeq)^n \text{ with } t(R) = \{x, \alpha\} \text{ and } f(R) = x\}.$$

Step 1: $\overline{A} \subseteq A^\star(\succeq)$. We need to prove that, for each $x \in \overline{A}$ and each $y \in A \setminus \{\alpha, x\}$, $x \not\succ y$ and $y \not\succ x$.

To obtain a contradiction, first suppose $y \succ x$. Let $R \in \mathcal{SSP}(\succeq)^n$ be such that $|N(R, x)| = k(x)$, $t(R) = \{x, \alpha\}$ and $f(R) = x$. Let $i \in N(R, x)$ and consider any $R_i^y \in \mathcal{SSP}(\succeq)$. Since $\alpha \succ y \succ x$, by Lemma 4, $f(R_i^y, R_{-i}) \in [x, \alpha]$. There are three cases to consider. First, $f(R_i^y, R_{-i}) = x$. By Lemma 5, since $x \notin [y, \alpha]$, $f(R_i^\alpha, R_{-i}) = f(R_i^y, R_{-i}) = x$ for any $R_i^\alpha \in \mathcal{SSP}(\succeq)$. This contradicts the definition of $k(x)$. Second, $f(R_i^y, R_{-i}) = y$. Let $j \in N(R, \alpha)$. Since $\alpha \succ x$, by Remark 4 (ii) there is $\widetilde{R}_j^\alpha \in \mathcal{SSP}(\succeq)$ such that $x \widetilde{P}_j^\alpha y$. By *tops-onlyness,* $f(R_i^y, \widetilde{R}_j^\alpha, R_{-\{i,j\}}) = f(R_i^y, R_{-i}) = y$. Now consider $R_j^x \in \mathcal{SSP}(\succeq)$. Since $\left|N((R_i^y, R_j^x, R_{-\{i,j\}}), x)\right| = k(x)$, by Lemma 3, $f(R_i^y, R_j^x, R_{-\{i,j\}}) = x$. Therefore,

$$f(R_i^y, R_j^x, R_{-\{i,j\}}) = x \widetilde{P}_j^\alpha y = f(R_i^y, \widetilde{R}_j^\alpha, R_{-\{i,j\}}),$$

contradicting *strategy-proofness.* Third, $f(R_i^y, R_{-i}) \in [x, \alpha] \setminus \{y, x\}$. Since $y \succ x$, by Remark 4 (ii) and *tops-onlyness* we can assume that R_i^y is such that $x P_i^y f(R_i^y, R_{-i})$. As $f(R) = x$, we get $f(R) P_i^y f(R_i^y, R_{-i})$, contradicting *strategy-proofness.* Thus, $y \not\succ x$.

To show that $x \not\succ y$ holds as well, suppose $x \succ y$. If $y \in \overline{A}$, the proof follows a similar argument than the one used when we supposed $y \succ x$, interchanging the roles of x and y. Assume $y \notin \overline{A}$. Namely, for all $R \in \mathcal{SSP}(\succeq)^n$ such that $\alpha \in t(R)$, $f(R) \neq y$. Consider any profile R and agent i such that $N(R,\alpha) = \{i\}$ and $|N(R,y)| = n-1$. By hypothesis and Lemma 2, $f(R) = \alpha$. Let $j \in N(R,y)$ and $R_j^x \in \mathcal{SSP}(\succeq)$. Then, $t(R_j^x, R_{-j})$ is equal to $\{x,\alpha\}$ if $n=2$ and equal to $\{y,x,\alpha\}$ if $n>2$. Since $\alpha \succ x \succ y$ and $\{x,\alpha\} \subseteq t(R_j^x, R_{-j}) \subseteq \{y,x,\alpha\}$, by Lemma 4, $f(R_j^x, R_{-j}) \in [y,\alpha]$. There are two cases to consider. First, $f(R_j^x, R_{-j}) \neq \alpha$. Then, by semilattice single-peakedness and *tops-onlyness*, $f(R_j^x, R_{-j}) P_j \alpha = f(R)$, contradicting *strategy-proofness*. Second, $f(R_j^x, R_{-j}) = \alpha$. If $n=2$, $x \in \overline{A}$ and *tops-onlyness* imply that $f(R_i, \overline{R}_j) = x$ for some $\overline{R}_j$. By *strategy-proofness*, $f(R_i, R_j^x) = x$, a contradiction with $f(R_j^x, R_{-j}) = \alpha$. Hence, assume $n > 2$ and consider $j_1 \in N(R,y) \setminus \{j\}$ and $R_{j_1}^x \in \mathcal{SSP}(\succeq)$. By Lemma 4, $f(R_j^x, R_{j_1}^x, R_{-\{j,j_1\}}) \in [y,\alpha]$. Again, by semilattice single-peakedness and *strategy-proofness* $f(R_j^x, R_{j_1}^x, R_{-\{j,j_1\}}) = \alpha$. Using iteratively the same argument, we can identify a subprofile $(R_j^x, R_{j_1}^x, \ldots R_{j_{n-2}}^x) \in \mathcal{SSP}(\succeq)^{n-1}$ such that $f(R_j^x, R_{j_1}^x, \ldots R_{j_{n-2}}^x, R_i) = \alpha$. But since $|\{j, j_1, \ldots, j_{n-2}\}| = n-1 \geq k(x)$, together with *tops-onlyness*, we obtain a contradiction with $x \in \overline{A}$ and the definition of $k(x)$. Thus, $x \not\succ y$ and accordingly, $x \in A^\star(\succeq)$.

Now, define $q = \{q^x\}_{x \in A^\star(\succeq)}$ as follows: for each $x \in A^\star(\succeq)$,

$$q^x = \begin{cases} k(x) & \text{if } x \in \overline{A} \\ n & \text{otherwise.} \end{cases} \tag{2}$$

Step 2: q is a quota system. We need to show that conditions (QS.1) and (QS.2) in the definition of a quota system are satisfied. Since $\overline{A} \neq \emptyset$, there is at least one $x \in \overline{A}$ such that $q^x = k(x)$. By definition of $k(x)$, $1 \leq q^x < n$. By Step 1 in this proof, $x \in A^\star(\succeq)$. Hence, (QS.1) holds. To show that (QS.2) is satisfied, let $x \neq y$ and $x, y \in A^\star(\succeq)$ be arbitrary. Assume $x, y \in A^\star(\succeq) \diagdown \overline{A}$. Then, by (2), $q^x = q^y = n$ and, accordingly, $q^x + q^y > n$. Assume $x \in \overline{A}$ and $y \in A^\star(\succeq) \diagdown \overline{A}$. Then, by (2), $q^x = k(x)$ and $q^y = n$. By definition of $k(x)$, $1 \leq q^x < n$ and, accordingly, $q^x + q^y > n$. Finally, assume $x, y \in \overline{A}$ and to obtain a contradiction, suppose that $q^x + q^y \leq n$. Consider a profile $R \in \mathcal{SSP}(\succeq)^n$ with $|N(R,x)| = q^x$, $|N(R,y)| = q^y$, and (in case $q^x + q^y < n$) $|N(R,\alpha)| = n - (q^x + q^y)$. By Lemma 4, $f(R) \in \cup_{i \in N}[t(R_i), \sup_{\succeq} t(R)]$. As $x, y \in A^\star(\succeq)$, $\cup_{i \in N}[t(R_i), \sup_{\succeq} t(R)] = \{x, y, \alpha\}$. Then,

$$f(R) \in \{x, y, \alpha\}. \tag{3}$$

We now show that (3) leads to a contradiction. First, assume $f(R) = x$. Let $i \in N(R,x)$ and consider $R_i^\alpha \in \mathcal{SSP}(\succeq)$ such that $x P_i^\alpha y$ (such preference exists by Remark 4 (ii)). By Lemma 4, $f(R_i^\alpha, R_{-i}) \in \{x, y, \alpha\}$. If $f(R_i^\alpha, R_{-i}) = y$, it follows that

$$f(R) = x P_i^\alpha y = f(R_i^\alpha, R_{-i}),$$

contradicting *strategy-proofness.* If $f(R_i^\alpha, R_{-i}) = \alpha$, by a repeated use of *strategy-proofness* it follows that $f(\widetilde{R}) = \alpha$ for $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ such that $t(\widetilde{R}) = \{y, \alpha\}$, $N(\widetilde{R}, y) = N(R, y)$ and $\left|N(\widetilde{R}, y)\right| = q^y = k(y)$, contradicting the definition of $k(y)$. Therefore, $f(R_i^\alpha, R_{-i}) = x$. If $n = 2$, $t(R_i^\alpha, R_{-i}) = \{y, \alpha\}$ and, by Lemma 2, $f(R_i^\alpha, R_{-i}) \in \{y, \alpha\}$, a contradiction. Therefore, assume $n > 2$. By the same reasoning we can replace the preferences of all agents with top equal to x, one by one except the preference R_j of an agent $j \in N(R, x)$, with preferences with top equal to α, and the alternative selected by f would not change. Namely, let $\widehat{R} \in \mathcal{SSP}(\succeq)^n$ be a profile with $t(\widehat{R}) = \{x, y, \alpha\}$, $N(\widehat{R}, x) = \{j\} \subseteq N(R, x)$, and $N(\widehat{R}, y) = N(R, y)$ such that

$$f(\widehat{R}) = x. \tag{4}$$

Now, let $\overline{R} \in \mathcal{SSP}(\succeq)^n$ be such that $t(\overline{R}) = \{y, \alpha\}$, $N(\overline{R}, y) = N(R, y)$ and $t(\overline{R}_j) = \alpha$. By $\left|N(\overline{R}, y)\right| = q^y = k(y)$ and the definition of $k(y)$,

$$f(\overline{R}) = y. \tag{5}$$

Without loss of generality, assume $\overline{R}_j^\alpha$ is such that $x\overline{P}_j^\alpha y$ (this is possible by Remark 4 (ii)). Hence, by (4) and (5) it follows that

$$f(\widehat{R}) = x\overline{P}_j^\alpha y = f(\overline{R}),$$

contradicting *strategy-proofness.* This implies that $f(R) \neq x$. The proof of the fact that $f(R) \neq y$ follows a similar argument, after replacing the roles of x and y, and it is omitted. Finally, assume $f(R) = \alpha$. By a repeated use of *strategy-proofness* and *tops-onlyness*, $f(R') = \alpha$ for each $R' \in \mathcal{SSP}(\succeq)^n$ with $t(R') = \{y, \alpha\}$ and $N(R', y) = N(R, y)$ with $|N(R', y)| = q^y = k(y)$, contradicting the definition of $k(y)$. Hence, (QS.2) holds.

Step 3: $\boldsymbol{f}$ is a generalized quota-supremum with respect to the quota system $\boldsymbol{q}$. Let $R \in \mathcal{SSP}(\succeq)^n$. There are two cases to consider.

3.1. **There is $\boldsymbol{x \in A^\star(\succeq)}$ such that $\boldsymbol{|N(R, x)| \geq q^x}$.** If $x \in \overline{A}$, then $f(R) = x$ by Lemma 3. If $x \in A^\star(\succeq) \setminus \overline{A}$, then $q^x = n$ and, by *unanimity*, $f(R) = x$.

3.2. **For all $\boldsymbol{x \in A^\star(\succeq)}$, $\boldsymbol{|N(R, x)| < q^x}$.** We want show that $f(R) = \sup_{\succeq} t(R)$. There are two cases to consider.

3.2.1. **$\boldsymbol{\sup_{\succeq} t(R) = \alpha}$.** Let $y \equiv f(R)$ and assume $y \neq \alpha$. Then, $y \prec \alpha$. By Lemma 6, we can assume that $\alpha \in t(R)$. By definition of $\overline{A}$, $y \in \overline{A}$ and, by Step 1 in this proof, $y \in A^\star(\succeq)$. First, notice that $y \notin t(R)$. Suppose otherwise. Then, $y \in t(R) \cap A^\star(\succeq)$. Assume first that $t(R) = \{y, \alpha\}$. Then, by the hypothesis of 3.2 and the definition of q^y, $|N(R, y)| < q^y = k(y)$ which together with $f(R) = y$ contradict the definition of $k(y)$. Assume now that $t(R_i) \in t(R) \setminus \{y, \alpha\}$ for some i. Then, since $\alpha \succ t(R_i)$, $\alpha \succ f(R)$ and $f(R) \notin [t(R_i), \alpha]$, by Lemma 5 we have that $f(\widetilde{R}_i^{\alpha}, R_{-i}) = f(R)$ for any $\widetilde{R}_i^{\alpha}$. By the repeated use of Lemma 5, applied to the remaining agents whose top at R does not belong to $\{y, \alpha\}$, we obtain that there exists a profile $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ with $t(\widetilde{R}) = \{y, \alpha\}$ and $N(\widetilde{R}, y) = N(R, y)$ such that $\mid N(\widetilde{R}, y) \mid < q^y = k(y)$ and $f(\widetilde{R}) = y$, contradicting the definition of $k(y)$. Hence, $y \notin t(R)$. By Lemma 4, $y \in \cup_{i \in N}[t(R_i), \alpha]$, so there is $i \in N$ such that $t(R_i) \prec y \prec \alpha$, contradicting the fact that $y \in A^{\star}(\succeq)$. This proves that $f(R) = \alpha = \sup_{\succeq} t(R)$.

3.2.2. $\boldsymbol{\sup_{\succeq} t(R) \neq \alpha}$. Notice that $t(R_i) \neq \alpha$ for all $i \in N$. Let $s = \sup_{\succeq} t(R)$ and assume $f(R) \neq s$. By Lemma 4, $f(R) \prec s$. As $s \prec \alpha$, by Lemma 6 it is without loss of generality to assume that $s \in t(R)$. Let $i \in N(R, s)$. By richness, there is $R_i^s \in \mathcal{SSP}(\succeq)$ such that $\alpha P_i^s f(R)$. By *tops-onlyness*, $f(R_i^s, R_{-i}) = f(R)$. Let $R_i^{\alpha} \in \mathcal{SSP}(\succeq)$. By the previous case 3.2.1, $f(R_i^{\alpha}, R_{-i}) = \alpha$. Therefore,

$$f(R_i^{\alpha}, R_{-i}) = \alpha P_i^s f(R_i^s, R_{-i}),$$

contradicting *strategy-proofness.* This proves that $f(R) = \sup_{\succeq} t(R)$.

($\Longleftarrow$) That $\sup_{\succeq}$ is *strategy-proof* and *simple* is shown in Section 3 in BM20. Let f^q be a generalized quota-supremum rule with respect to the quota system $q = \{q^x\}_{x \in A^{\star}(\succeq)}$. By definition, f^q is *unanimous, anonymous* and *tops-only*, and therefore *simple.* We now show that f^q is also *strategy-proof.* Let $R \in \mathcal{SSP}(\succeq)^n$, $i \in N$ and $R_i' \in \mathcal{SSP}(\succeq)$ be arbitrary, and assume $f^q(R) \neq t(R_i)$. There are two cases to consider.

1. **There is $\boldsymbol{x \in A^{\star}(\succeq)}$ such that $\boldsymbol{|N(R, x)| \geq q^x}$.** Hence, $f^q(R) = x$. As $t(R_i) \neq x$, $f^q(R_i', R_{-i}) = x$ because $|N((R_i', R_{-i}), x)| \geq |N(R, x)|$. Thus, agent i can not manipulate f at R.

2. **For all $\boldsymbol{x \in A^{\star}(\succeq)}$, $\boldsymbol{|N(R, x)| < q^x}$.** Hence,

$$f^q(R) = \sup_{\succeq} t(R). \tag{6}$$

We want to show that

$$f^q(R) R_i f^q(R_i', R_{-i}). \tag{7}$$

Suppose $f^q(R_i', R_{-i}) = \sup_{\succeq} t(R_i', R_{-i})$. Then, by *semilattice single-peakedness* and associativity of the supremum,

$$f^q(R) = \sup_{\succeq}\{t(R_i), \sup_{\succeq} t(R_{-i})\}\ R_i\ \sup_{\succeq}\{t(R_i'), \sup_{\succeq} t(R_{-i})\} = f^q(R_i', R_{-i}),$$

so (7) holds. Suppose $f^q(R_i', R_{-i}) = x$ for some $x \in A^{\star}(\succeq)$ with $|N((R_i', R), x)| \geq q^x$. Then, $t(R_i) \neq x$. Otherwise, $t(R_i) = x$ and $|N(R, x)| \geq |N((R_i', R_{-i}), x)| \geq q^x$ implying that $f^q(R) = x \in A^{\star}(\succeq)$, contradicting our hypothesis. Therefore, $t(R_i) \not\preceq x$. By Remark 1 (ii),

$$\alpha = \sup_{\succeq}\{t(R_i), x\} R_i x. \tag{8}$$

Also, since $t(R_i) \preceq \sup_{\succeq} t(R) \preceq \alpha$, by Remark 1 (i),

$$\sup_{\succeq} t(R) R_i \alpha. \tag{9}$$

Therefore, by (6), (8) and (9), we obtain

$$f^q(R) = \sup_{\succeq} t(R) R_i x = f^q(R'_i, R_{-i}),$$

so (7) holds.

Hence, the generalized quota-supremum rule f^q is *strategy-proof.* ■

2.4 Counterexample again

We show that the rule

$$f(R) = med_{\triangleright}(t(R_1), t(R_2), t(R_3), t(R_4), t(R_5), y, y, y, z),$$

used in the counterexample in Subsection 1.2, can be written as a generalized quota-supremum f^q with $q^y = 2$ and $q^z = 4$. Hence, it is one of the strategy-proof and simple rules identified in Theorem 1. To see that, let $R \in \mathcal{SSP}(\succeq)^5$ be arbitrary. We distinguish among several cases depending on the supports received by the alternatives at profile R.

- $|N(R, y)| \geq 2$. Then, $f(R) = f^q(R) = y$.
- $|N(R, y)| = 1$.
  - $|N(R, x)| \geq 1$ (*i.e.*, $|N(R, z)| \leq 3$). Then, $f(R) = f^q(R) = x$.
  - $|N(R, x)| = 0$ (*i.e.*, $|N(R, z)| = 4$). Then, $f(R) = f^q(R) = z$.
- $|N(R, y)| = 0$.
  - $|N(R, z)| < 4$ (*i.e.*, $|N(R, x)| \geq 2$). Then, $f(R) = f^q(R) = x$.
  - $|N(R, z)| \geq 4$ (*i.e.*, $|N(R, x)| \leq 1$). Then, $f(R) = f^q(R) = z$.

Hence, f can be represented as a generalized quota-supremum rule.

3 Lemma 9

3.1 Statement of Lemma 9

Lemma 9 *Let* $\succeq$ *be a semilattice over* A *such that* $\sup_{\succeq} A$ *does not exist and let* $f : \mathcal{SSP}(\succeq)^n \to A$ *be a strategy-proof and simple rule. Then,* $f(R) = \sup_{\succeq} t(R)$ *for each* $R \in \mathcal{SSP}(\succeq)^n$.

The proof of Lemma 9 in BM20 uses incorrectly Lemma 5 in the second line of the proof of Claim 2. To be applied, the hypothesis of Lemma 5 would require that, in addition to $s \succ t(R_i)$ and $s \succ f(R)$, $f(R) = f(\widetilde{R}_{\widetilde{N}}, R_{-\widetilde{N}}) \notin [t(R_i), s]$ holds as well. But this is not necessarily true. Figure 1 shows an instance where this is the case, with the convention that an arrow from alternative z pointing to alternative y means that $z \prec y$.

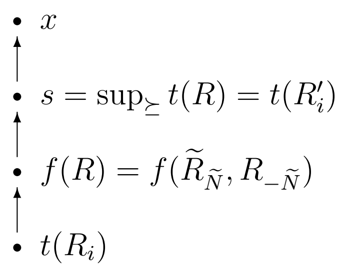

Figure 1: $f(R) = f(\widetilde{R}_{\widetilde{N}}, R_{-\widetilde{N}}) \in [t(R_i), s]$

However, the statement of Lemma 9 is correct, and we present below an alternative proof.

3.2 Alternative proof of Lemma 9

Proof Let the hypothesis of the Lemma hold. If $|t(R)| = 1$, the result follows by *unanimity*. To obtain a contradiction, let $R \in \mathcal{SSP}(\succeq)^n$ be such that $|t(R)| > 1$ and $f(R) \neq \sup_{\succeq} t(R)$. Since $f(R) \in \cup_{i \in N}[t(R_i), \sup_{\succeq} t(R)]$ by Lemma 4, we have $\sup_{\succeq} t(R) \succ f(R)$. By Lemma 6, it is without loss of generality to assume that $\sup_{\succeq} t(R) \in t(R)$. Let $s \equiv \sup_{\succeq} t(R)$, $x \equiv f(R)$ and $S = N(R, s)$ (notice that $S \neq \emptyset$). By a repeated use of *strategy-proofness,*

$$f(R_S, \widehat{R}_{-S}) = x \tag{10}$$

for any $\widehat{R}_{-S} \in \mathcal{SSP}(\succeq)^{n-|S|}$ such that $t(\widehat{R}_i) = x$ for each $i \in N \setminus S$.

First, let $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ be such that $t(\widetilde{R}) = \{s, x\}$, $f(\widetilde{R}) = x$ and $|N(\widetilde{R}, x)|$ is minimal in the sense that $f(\overline{R}) = s$ for each $\overline{R} \in \mathcal{SSP}(\succeq)^n$ with $t(\overline{R}) = \{s, x\}$ and $|N(\overline{R}, x)| < |N(\widetilde{R}, x)|$. Notice that such $\widetilde{R} \in \mathcal{SSP}(\succeq)^n$ exists because of (10).

Second, we claim that there is $y \in A$ such that $y \succ s$. To see this, notice that since there is no $\sup_{\succeq} A$, there exists $z \in A$ such that $s \not\succeq z$. If $z \succ s$, take $y = z$; whereas if $z \not\succeq s$, take $y = \sup_{\succeq}\{s, z\}$.

Next, let $i \in N(\widetilde{R}, x)$ and consider $R_i^y \in \mathcal{SSP}(\succeq)$. Then,

$$f(R_i^y, \widetilde{R}_{-i}) \in \{s, y\}. \tag{11}$$

Assume otherwise; that is, $f(R_i^y, \widetilde{R}_{-i}) \notin \{s, y\}$. Since $y \succ s$, by Remark 4 (ii) and *tops-onlyness* we can assume that R_i^y is such that $sP_i^y f(R_i^y, \widetilde{R}_{-i})$. By the minimality of $|N(\widetilde{R}, x)|$ and Lemma 2, $f(R_i^s, \widetilde{R}_{-i}) = s$ for any $R_i^s \in \mathcal{SSP}(\succeq)$. Therefore, $f(R_i^s, \widetilde{R}_{-i})P_i^y f(R_i^y, \widetilde{R}_{-i})$, contradicting *strategy-proofness.* This shows that (11) holds, so there are two cases to consider.

1. $\boldsymbol{f(R_i^y, \widetilde{R}_{-i}) = s}$. As $s = \sup_{\succeq} t(R)$, $f(R) = x$ and $\sup_{\succeq} t(R) \succ f(R)$, we have that $s \succ x$. Together with $y \succ s$ and transitivity, we have $y \succ x$. By Remark 4 (ii) and *tops-onlyness*, we can assume that R_i^y is such that $xP_i^y s$. Therefore, $f(\widetilde{R}) = xP_i^y s = f(R_i^y, \widetilde{R}_{-i})$, contradicting *strategy-proofness.*

2. $\boldsymbol{f(R_i^y, \widetilde{R}_{-i}) = y}$. Let $j \in N(\widetilde{R}, s)$ and $\overline{R}_j^x \in \mathcal{SSP}(\succeq)$. Since $|N((R_i^y, \overline{R}_j^x, \widetilde{R}_{-\{i,j\}}), x)| = |N(\widetilde{R}, x)|$, by Lemma 3, $f(R_i^y, \overline{R}_j^x, \widetilde{R}_{-\{i,j\}}) = x$. As $y \succ s \succ x$, by Remark 4 (ii) and *tops-onlyness* we can assume that $\widetilde{R}_j$ is such that $x\widetilde{P}_j y$. Therefore, since $(R_i^y, \widetilde{R}_{-i}) = (R_i^y, \widetilde{R}_j, \widetilde{R}_{-\{i,j\}})$, $f(R_i^y, \overline{R}_j^x, \widetilde{R}_{-\{i,j\}}) = x\widetilde{P}_j y = f(R_i^y, \widetilde{R}_j, \widetilde{R}_{-\{i,j\}})$, contradicting *strategy-proofness.*

As in each case we reach a contradiction, we conclude that $f(R) = \sup_{\succeq} t(R)$ for each $R \in \mathcal{SSP}(\succeq)^n$. ■

References

[1] Bonifacio, A., Massó, J., 2020. On strategy-proofness and semilattice single-peakedness. Games Econ. Behav. 124, 219-238.