Game Theory

Cooperative Games

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8.1.- Preliminaries

- von Neumann and Morgenstern (1944) studied non-cooperative games with two players.
- They realized that to study the general case they had to consider the role of coalitions of players.
- For instance, what does happen if two players agree to behave against a third player?
- Since this type of considerations make the analysis too complex, they proposed and studied a reduced-form model: cooperative games with transferable utility.
- Two main hypothesis:
  - Any subset of players has the instruments to reach enforceable agreements among them.
  - Side-payments are feasible; namely, there is a good (utility or money) that can go from one player to another player. Thus, we can refer to the total amount (of utility) that a coalition can guarantee by itself.

8.1.- Preliminaries

- A *cooperative game with transferable utility* (a TU-game or a game with side-payments) is a pair (N, v) (or just v if the set N is clear from the context) where
  - $N = \{1, ..., n\}$ is the set of *players*,
  - $v : 2^N \longrightarrow \mathbb{R}$ is the *characteristic function* of the game that associates to every coalition $S \in 2^N$ a real number $v(S)$, which is interpreted as the total utility that coalition S can obtain by their own means. By convention, we set $v(\{\varnothing\}) = 0$.
  - There are several alternative non-cooperative justifications of this function v.

8.1.- Preliminaries

Definitions We say that the cooperative game (N, v) is

- *Essential* if $v(N) \geq \sum_{i=1}^{n} v(\{i\})$.
  - Cooperation only makes sense in essential games.
- *Monotonic* if for every $S \subset T \subseteq N$, $v(S) \leq v(T)$.
  - Adding players to a coalition can not decrease its value.
- *Superadditive* if for all $S, T \subseteq N$ such that $S \cap T = \varnothing$, $v(S \cup T) \geq v(S) + v(T)$.
  - Cooperation is always better (increasing or constant returns to cooperation).
  - Except when we say otherwise, we will assume that the game is superadditive and hence, it is also essential.

8.1.- Preliminaries

- *Convex* if for all $S, T \subseteq N$,

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

  - Every convex game is superadditive.
  - Equivalently, a game v is *convex* if for all $i \in N$ and all $S \subseteq T \subseteq N \backslash \{i\}$,

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

  - In a convex game the marginal contribution of a player becomes larger, the larger is the coalition he contributes to.
- *Simple* if for all $S \subseteq N$, $v(S) \in \{0, 1\}$.
- Let G be the class of all (superadditive cooperative) games.
  - $v \in G$ (or $(N, v) \in G$) is a game.

8.1.- Preliminaries

Example Consider the game (N, v) where $N = \{1, 2, 3\}$ and

$$\begin{aligned} v(\{1\}) &= 0 \\ v(\{2\}) &= 0 \\ v(\{3\}) &= 0 \\ v(\{1,2\}) &= 10 \\ v(\{1,3\}) &= 10 \\ v(\{2,3\}) &= 12 \\ v(N) &= 15. \end{aligned}$$

- The game v is essential, monotonic and superadditive. Hence, $v \in G$.
- But v is not convex since for $i = 1$, $S = \{2\}$ and $T = \{2, 3\}$,

  $$v(\{1,2,3\}) - v(\{2,3\}) = 15 - 12 = 3 < 10 = v(\{1,2\}) - v(\{2\}).$$

8.1.- Preliminaries

- In superadditive games, and given the possibility that players can reach binding agreements, players will finally end up cooperating. Thus, they will be splitting efficiently $v(N)$ among them.
- Question: How are they going to do it?
- Positive question: what is the set of stable divisions of $v(N)$?

8.1.- Preliminaries

- Let $v \in G$ be a game.
- The set of *pre-imputations* of $v \in G$, denoted by $I^*(v)$, is defined by

$$I^*(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N) \right\}.$$

- Given a family of games B, a *solution* on B is a correspondence $\Phi : B \twoheadrightarrow \mathbb{R}^n$ such that for every $v \in B$, Φ selects a subset of $I^*(v)$; namely, $\Phi(v) \subseteq I^*(v)$ for all $v \in B$.
- In many situations it is reasonable to require that the solution satisfies individual rationality.
- The set of *imputations of* $v \in G$, denoted by $I(v)$, is defined by

$$I(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N \right\}.$$

8.1.- Preliminaries

Definition A solution $\Phi : B \twoheadrightarrow \mathbb{R}^n$ is *individually rational* on B if for all $v \in B$, $\Phi(v) \subseteq I(v)$.

- Given a game $v \in G$, let $x, y \in I(v)$.
- Suppose players are confronted to choose between x and y. A solution may provide criteria to choose one rather than the other.
- Unless $x = y$, and since $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = v(N)$, some players prefer x and some other players prefer y.
- But v may contain information about the power (the strength) of a coalition to enforce the choice of x over y, or of y over x.

8.1.- Preliminaries

Definition Given $v \in G$, $x, y \in I(v)$ and $S \subseteq N$ we say that

- *x dominates y through S*, written $xd_S y$, if
  - $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leq v(S)$.
    - All members of S strictly prefer x to y and they are able to obtain what x gives to them.
    - We say that S *blocks* $y \in I(v)$ if there exists $x \in I(v)$ such that $xd_S y$.
- *x dominates y*, written xdy, if there exists $S \subseteq N$ such that $xd_S y$.
- **Remarks**
  - Given S, d_S is an incomplete, irreflexive and transitive binary relation on $I(v)$.
  - d is an incomplete, irreflexive and non-transitive binary relation on $I(v)$.

8.2.- Core

Definition The *Core* of a game $v \in G$, denoted by $C(v)$, is the set of all undominated imputations of v. That is,

$$C(v) = \{x \in I(v) \mid \nexists y \in I(v) \text{ such that } ydx\}.$$

- Gillies, D. B. *Some Theorems on n−Person Games*. Ph.D. Thesis, Princeton University Press, 1953.

Theorem

(Gillies, 1953) For all $v \in G$,

$$\begin{aligned} C(v) &= \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\right\} \\ &\equiv A(v). \end{aligned}$$

8.2.- Core

Proof

- $A(v) \subseteq C(v)$.
  - First notice that if $x \in A(v)$ then $\sum_{i=1}^{n} x_i = v(N)$, and by letting $S = \{i\}$, $x_i \geq v(\{i\})$ for all $i \in N$. Hence, $x \in I(v)$.
  - Let $x \in A(v)$ and assume $x \notin C(v)$.
  - Then, there exist $y \in I(v)$ and $S \subset N$ such that $y d_S x$.
  - That is, $y_i > x_i$ for all $i \in S$.
  - Thus, $\sum_{i \in S} y_i > \sum_{i \in S} x_i \geq v(S)$, where the weak inequality follows from the fact that $x \in A(v)$.
  - But $\sum_{i \in S} y_i > v(S)$ contradicts that $y d_S x$.

8.2.- Core

- $C(v) \subseteq A(v)$.
  - Let $y \notin A(v)$. We show that $y \notin C(v)$.
  - If $\sum\limits_{i=1}^{n} y_i \neq v(N)$ then $y \notin I(v)$ and hence, $y \notin C(v)$.
  - Thus, assume $\sum\limits_{i=1}^{n} y_i = v(N)$.
  - Since $y \notin A(v)$, there exists $S (\subsetneq N)$ such that $\sum\limits_{i \in S} y_i < v(S)$.
  - Let $\varepsilon > 0$ be such that $\sum\limits_{i \in S} y_i = v(S) - \varepsilon$, and let $\alpha = v(N) - v(S) - \sum\limits_{i \in N \setminus S} v(\{i\})$. By superadditivity, $\alpha \geq 0$.
  - Define for all $i \in N$,

$$z_i = \begin{cases} y_i + \frac{\varepsilon}{\#S} & \text{if } i \in S \\ v(\{i\}) + \frac{\alpha}{n - \#S} & \text{if } i \notin S. \end{cases}$$

8.2.- Core

- $C(v) \subseteq A(v)$ (Continuation).
  - Note that $z = (z_i)_{i\in N}$ is an imputation since

$$\begin{aligned}\sum_{i=1}^{n} z_i &= \sum_{i\in S} y_i + \varepsilon + \sum_{i\in N\backslash S} v(\{i\}) + \alpha \\ &= \sum_{i\in S} y_i + \varepsilon + \sum_{i\in N\backslash S} v(\{i\}) + v(N) - v(S) - \sum_{i\in N\backslash\{S\}} v(\{i\}) \\ &= v(N),\end{aligned}$$

  - if $i \notin S$, $z_i = v(\{i\}) + \frac{\alpha}{n-\#S} \geq v(\{i\})$,
  - and if $i \in S$, $z_i = y_i + \frac{\varepsilon}{\#S} > y_i \geq v(\{i\})$ (the weak inequality follows because otherwise, $y \notin I(v)$ would imply that $y \notin C(v)$ and then, we would be done).
  - Hence, $z \in I(v)$ and there exists $S \subset N$ such that $z d_S y$ (remember that $\sum_{i\in S} z_i = \sum_{i\in S} y_i + \varepsilon = v(S)$).
  - Thus, $y \notin C(v)$. ■

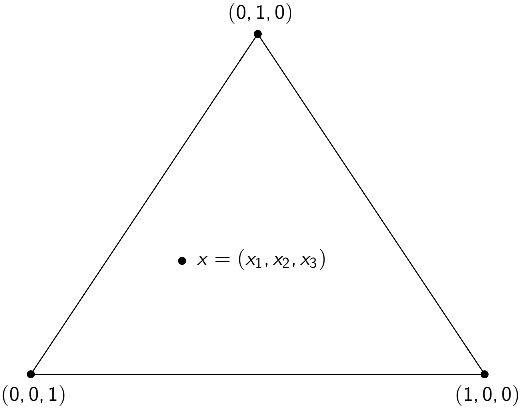
8.2.- Core

Remark $C(v)$ is a convex and compact subset of $\mathbb{R}^n$: it is a *polytope* (a geometric object with flat sides).

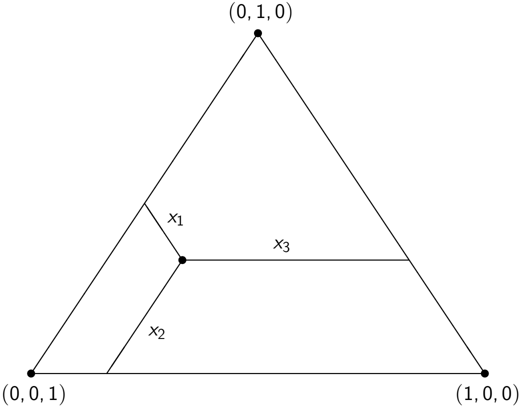
Example

- Consider any (0-1)$-$normalized game $(N, v) \in G$, where $v(N) = 1$ and $v(\{i\}) = 0$ for all $i \in N$. In particular assume that $N = \{1, 2, 3\}$.
- The set of imputations $I(v)$ are all vectors $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ such that $x_1 + x_2 + x_3 = 1$.
    - They can be represented graphically by the following triangle.

8.2.- Core



8.2.- Core



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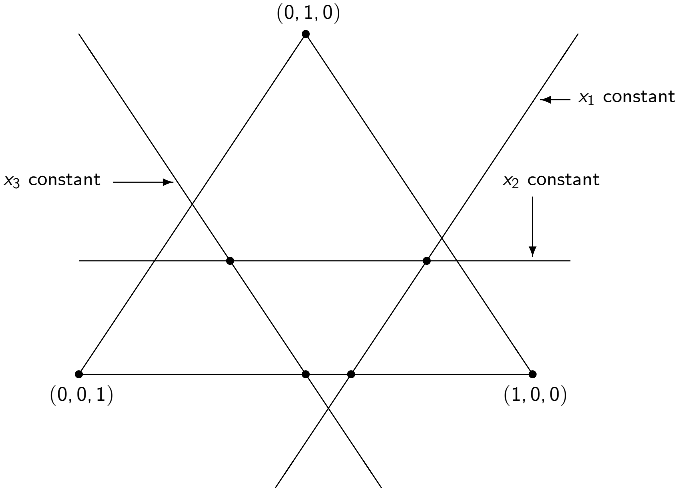
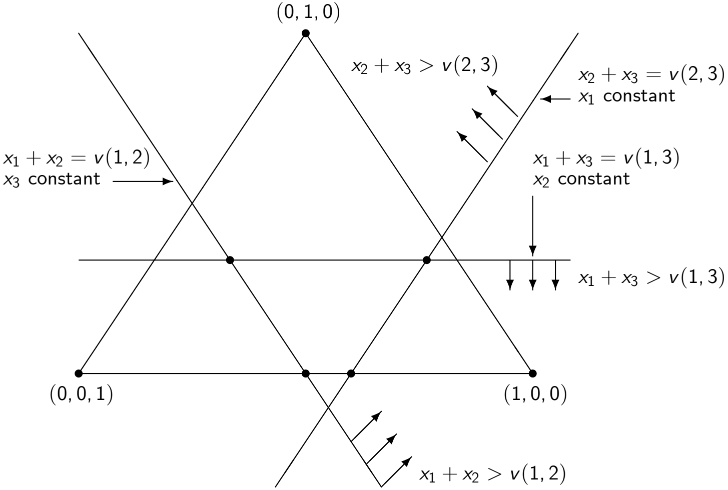
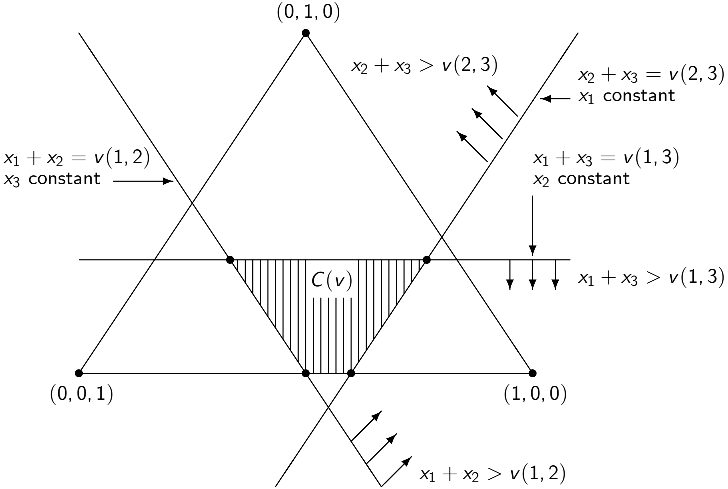


8.2.- Core


$(0,1,0)$
$x_2 + x_3 > v(2,3)$
$x_2 + x_3 = v(2,3)$
x_1 constant
$x_1 + x_2 = v(1,2)$
x_3 constant
$x_1 + x_3 = v(1,3)$
x_2 constant
$x_1 + x_3 > v(1,3)$
$(0,0,1)$
$(1,0,0)$
$x_1 + x_2 > v(1,2)$

8.2.- Core


$(0,1,0)$
$x_2 + x_3 > v(2,3)$
$x_2 + x_3 = v(2,3)$
x_1 constant
$x_1 + x_2 = v(1,2)$
x_3 constant
$x_1 + x_3 = v(1,3)$
x_2 constant
$C(v)$
$x_1 + x_3 > v(1,3)$
$(0,0,1)$
$(1,0,0)$
$x_1 + x_2 > v(1,2)$

8.2.- Core

- Gillies's Theorem does not say anything about the possibility that $C(v) = \varnothing$.
- **Example** Majority voting with three voters.
  - $N = \{1, 2, 3\}$.
  - $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 1$.
  - Suppose that $x \in C(v)$. Then

$$\begin{array}{c} x_1 + x_2 \geq 1 \\ x_1 + x_2 \geq 1 \\ x_1 + x_2 \geq 1 \\ \hline 2x_1 + 2x_2 + 2x_3 \geq 3. \end{array}$$

  - Hence, $x_1 + x_2 + x_3 \geq \frac{3}{2} > 1 = v(N)$, contradicting that $x \in I(v)$.
  - Thus, $C(v) = \varnothing$

8.2.- Core

- **Example** (Continuation)
  - Why $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \notin C(v)$?
  - Coalition $\{1,2\}$ (or any other coalition with two players) can block the egalitarian imputation with $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, but then, $\{2,3\}$ could block it with $\left(0, \frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon\right)$, and then, $\{1,3\}$ could block it with ...
  - Suggestion: Introduce forward looking behavior; *i.e.*, when blocking an imputation players also foresee the consequences of their blocking (Bargaining Set).
- There is a characterization of the class of games with non-empty Core (balanced games).
  - Bondareva, O.N. "Certain Applications of the Methods of Linear Programming to the Theory of Cooperative Games," *Problemy Kibernet* 10, 1963.
  - Shapley, L. "On Balanced Sets and Cores," *Naval Research Logistics Quarterly* 14, 1967.

8.2.- Core

- Given a set of players $N = \{1, ..., n\}$ we say that $\{\delta_S\}_{S \in 2^N}$ is a *balanced family of weights* if for all $S \in 2^N$, $\delta_S \in [0, 1]$ and for all $i \in N$,

$$\sum_{\substack{S \in 2^N \\ i \in S}} \delta_S = 1.$$

- **Interpretation** Every player i has a fixed amount of time (1 unit) and if $i \in S$, δ_S is the proportion of time that player i cooperates with coalition S.

- **Examples** $N = \{1, 2, 3\}$.
  - $\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = \delta_{\{1,2,3\}} = 0$ and $\delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = \frac{1}{2}$.
  - $\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = 1$ and $\delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = \delta_{\{1,2,3\}} = 0$.

8.2.- Core

Theorem

(Bondareva, 1963; Shapley, 1967) Let $v \in G$. *Then,* $C(v) \neq \emptyset$ *if and only if for all balanced family of weights* $\{\delta_S\}_{S \in 2^N}$,

$$\sum_{S \subsetneq N} \delta_S \cdot v(S) \leq v(N).$$

- **Example** $n = 3$ and majority voting; *i.e.*, $v(S) = 1$ if and only if $\#S \geq 2$. Consider the balanced family of weights $\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = \delta_{\{1,2,3\}} = 0$ and $\delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = \frac{1}{2}$. Then,

$$\delta_1 v(1) + \delta_2 v(2) + \delta_3 v(3) + \delta_{12} v(12) + \delta_{13} v(13) + \delta_{23} v(23) \overset{?}{\leq} v(123).$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \overset{?}{\leq} 1$$

  - NO. Hence, $C(v) = \emptyset$.

8.2.- Core

- **Definition** We say that a game $v \in G$ is *balanced* if $C(v) \neq \varnothing$.
- **Remark** Every convex game $v \in G$ is balanced.

8.3.- Stable Sets

Definition (vNM, 1944) Let $v \in G$ be a game. A set $V \subseteq I(v)$ is a *stable set* (solution) for v if

(INTERNAL STABILITY) for all $x, y \in V$, neither xdy nor ydx,

(EXTERNAL STABILITY) for all $x \notin V$ there exists $y \in V$ such that ydx.

Remarks

1. A game may have no stable set.
   - Lucas (1969) has an example of a game with $n = 10$ without any stable set (for $n < 9$??).
   - Characterization of the class of games with at least one stable set?
2. A game may have many stable sets.
3. The Core is a subset of every stable set.
4. No stable set is a proper subset of any other stable set.
5. If the Core is a stable set then it is the unique stable set.

8.3.- Stable Sets

Example Consider the majority voting game with $n = 3$; *i.e.*, $v(S) = 1$ if and only if $\#S \geq 2$.

- What are the reasonable solutions of this game?
- It seems that any coalition with two players, splitting equally 1 unit of utility.
- Indeed,

$$V = \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0 \right), \left(\frac{1}{2}, 0, \frac{1}{2} \right), \left(0, \frac{1}{2}, \frac{1}{2} \right) \right\}.$$

- Homework.

8.4.- Bargaining Sets

- **Idea:** To introduce forward-looking behavior when players block an imputation.
  - If an imputation x is not in the core is because it is blocked by a coalition S (with the imputation y). But, is y stable? Is there another coalition T that blocks y (with z)? ...
  - Assume that players are forward looking and understand the consequences of their blocks.
- The Bargaining Sets capture this idea.
- It constitutes an enlargement of the notion of Core (good direction: to be applied to non-balanced games).
- Many forward-looking notions: many bargaining sets.

8.4.- Bargaining Sets

- Aumann, R.J. and M. Maschler. "The Bargaining Set for Cooperative Games," in *Advances in Game Theory*, M. Dresher, L. Shapley, and A. Tuckers, Eds. Princeton, NJ: Princeton University Press (1964).
- Mas-Colell, A. "An Equivalence Theorem for a Bargaining Set," *Journal of Mathematical Economics* 18, 1989.
- Zhou, L. "A New Bargaining Set of an *N*-person Game and Endogenous Coalition Formation," *Games and Economic Behavior* 6, 1994.

8.4.- Bargaining Sets

Zhou's Bargaining Set

- Let (N, v) be a cooperative game.
- Let $x \in I(v)$. An *objection* from S to x is a pair (S, y) such that $y d_S x$.
  - Namely, $y_i > x_i$ for all $i \in S$ and $\sum\limits_{i \in S} y_i \leq v(S)$.
- Let (S, y) be an objection from S to x. A *counter-objection* from T against (S, y) is a pair (T, z) such that:
  1. $\sum\limits_{i \in T} z_i \leq v(T)$.
  2. $z_k \geq x_k$ for all $k \in T \backslash S$ and $z_j \geq y_j$ for all $j \in S \cap T$.
  3. $T \backslash S \neq \varnothing$, $S \backslash T \neq \varnothing$ and $S \cap T \neq \varnothing$.

8.4.- Bargaining Sets

Zhou's Bargaining Set

- An objection (S, y) against x is *justified* if there is no counter-objection from any other coalition T against (S, y).
- The *Zhou's Bargaining Set*, denoted by $BS^Z(v)$, is the set of imputations against which no coalition has justified objections. Namely,

$$BS^Z(v) = \{x \in I(v) \mid \text{every objection } (S, y) \text{ to } x \text{ is not justified}\}.$$

- **Remark** For every game v, $C(v) \subseteq BS^Z(v)$.

Theorem

(Zhou, 1994) The Zhou's Bargaining Set is non-empty for every superadditive game; namely, for all $v \in G$, $BS^Z(v) \neq \varnothing$.

8.4.- Bargaining Sets

Mas-Colell's Bargaining Set

- Let (N, v) be a cooperative game.
- Let $x \in I(v)$. An *objection* from S to x is a pair (S, y) such that $yd_S x$.
    - Namely, $y_i > x_i$ for all $i \in S$ and $\sum\limits_{i \in S} y_i \leq v(S)$.
- Let (S, y) be an objection from S to x. A *counter-objection** from T against (S, y) is a pair (T, z) such that:
    1. $\sum\limits_{i \in T} z_i \leq v(T)$.
    2. $z_k \geq x_k$ for all $k \in T \backslash S$, $z_j \geq y_j$ for all $j \in S \cap T$ and one of these inequalities holds strictly.

8.4.- Bargaining Sets

Mas-Colell's Bargaining Set

- An objection (S, y) against x is *justified** if there is no counter-objection* from any other coalition T against (S, y).
- The *Mas-Colell's Bargaining Set*, denoted by $BS^{M-C}(v)$, is the set of imputations against which no coalition has justified* objections. Namely,

$$BS^{M-C}(v) = \{x \in I(v) \mid \text{every objection } (S, y) \text{ to } x \text{ is not justified}^*\}.$$

- In general, $BS^{Z}(v)$ and $BS^{M-C}(v)$ are unrelated (non-empty sets versus strict inequality). However, $BS^{Z}(\cdot)$ is upper-hemicontinuos (with a natural topology) but $BS^{M-C}(\cdot)$ is not.

- A game v is *weakly superadditive* if for all $S \subseteq N$,

$$v(N) \geq v(S) + \sum_{i \in N \setminus S} v(\{i\}).$$

- **Remark** Every superadditive game $v \in G$ is weakly superadditive.

Theorem

(Vohra, 1991) The Mas-Colell's Bargaining Set is non-empty for every weakly superadditive game.

- Vohra, R. "An Existence Theorem for a Bargaining Set," *Journal of Mathematical Economics* 20, 1991.

8.5.- The Shapley Value

- **Idea-objective:** To summarize in a single number the "value" for every player of playing the game.
- A normative approach: What should player i get in game $v \in G$?
- Let G be the set of all superadditive games. A *value* is a function ϕ on G such that for all $(N, v) \in G$, $\phi(N, v) \in \mathbb{R}^n$.
- The interpretation is that given the game $(N, v) \in G$, $\phi_i(N, v)$ is the value or expected worth of playing the game (N, v) for $i \in N$.
- Shapley (1953) approached the problem axiomatically by proposing reasonable properties that any value should satisfy and showed that there exists a unique value satisfying them: The Shapley value.

8.5.- The Shapley Value

- Shapley, L. "A Value for $n-$Person Games". In *Contributions to the Theory of Games II*, editors: H. Kuhn and A. Tucker. Princeton University Press, 1953.

EFFICIENCY (EFF) A value ϕ is *efficient* if for every $(N, v) \in G$,

$$\sum_{i\in N} \phi_i(N, v) = v(N).$$

8.5.- The Shapley Value

- Players $i, j \in N$ are *symmetric* in a game $(N, v) \in G$ if for all $S \in 2^N$ such that $\{i, j\} \cap S = \varnothing$ we have $v(S \cup \{i\}) = v(S \cup \{j\})$.

SYMMETRY (SYM) A value ϕ is *symmetric* if for all symmetric players $i, j \in N$ in game $(N, v) \in G$,

$$\phi_i(N, v) = \phi_j(N, v).$$

8.5.- The Shapley Value

A player $i \in N$ is a dummy in (N, v) if for all $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) - v(S) = v(\{i\})$.

DUMMY PLAYER (DUM) A value ϕ satisfy the dummy player property if for all player $i \in N$ dummy in (N, v), $\phi_i(N, v) = v(\{i\})$.

8.5.- The Shapley Value

Given $u, v \in G$ define a new game $(u + v) \in G$ as follows: for all $S \subseteq N$, $(u + v)(S) = u(S) + v(S)$.

Given $v \in G$ and $\beta > 0$ define a new game $(\beta v) \in G$ as follows: for all $S \subseteq N$, $(\beta v)(S) = \beta \cdot v(S)$.

LINEARITY (LIN) A value ϕ satisfies *linearity* if for all $u, v \in G$, all $\beta > 0$, and all $i \in N$,

$$\phi_i(u + (\beta v)) = \phi_i(u) + \beta \cdot \phi_i(v).$$

Interpretation Suppose the games u and v are going to be played with probabilities p and $1 - p$, respectively (with $p \in [0, 1]$). Then, $pu + (1 - p)v$ can be seen as this composed game. Then, assuming that the Expected Utility Property holds,

$$\phi_i((pu) + ((1 - p)v)) = p \cdot \phi_i(u) + (1 - p) \cdot \phi_i(v).$$

8.5.- The Shapley Value

Theorem

(Shapley, 1953) There exists a unique value ϕ on G satisfying $(\textsc{Eff})$, $(\textsc{Sym})$, $(\textsc{Dum})$, *and* $(\textsc{Lin})$. *This function is the Shapley value, denoted by Sh, and it is defined as follows: for all $v \in G$ and all $i \in N$,*

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(\#S)!(\#N - \#S - 1)!}{\#N!} \left[v(S \cup \{i\}) - v(S) \right].$$

Interpretation

- $[v(S \cup \{i\}) - v(S)]$ is the *marginal contribution* of player i to coalition S.
- Every player receives his "expected marginal contribution" in the following sense.

8.5.- The Shapley Value

- Players arrive randomly, one after the other.
- Every order of arrival is equally likely: $\frac{1}{\#N!}$ probability of each of the $\#N!$ orderings.
- For each $S \subset N$ and $i \notin S$, $(\#S)!(\#N - \#S - 1)!$ is the number of orderings in which

$$\underbrace{S}\quad i \quad \underbrace{N \backslash (S \cup \{i\})}.$$

- Thus,

$$\frac{(\#S)!(\#N - \#S - 1)!}{\#N!}$$

  is the probability that exactly the members of S (in any ordering) are ahead of i, and i arrives just after all members of S (*i.e.*, just before $N \backslash (S \cup \{i\})$).

8.5.- The Shapley Value

- There are games $(N, v) \in G$ for which $Sh(N, v) \notin C(v) \neq \varnothing$ (remember the gloves game of the Introduction).

Theorem

(Shapley, 1969) Let $(N, v) \in G$ be a convex game. Then, $Sh(N, v) \in C(v)$.

8.5.- The Shapley Value

- Let $(N, v) \in G$, $S \subseteq N$ and $i \notin S$. Denote $D_i(v, S) = v(S \cup \{i\}) - v(S)$.

$\textsc{Strong Monotonicity}$ (SM) A value ϕ is *strong monotonic* if for all $(N, v), (N, u) \in G$ such that for all $S \subseteq N$ and all $i \notin S$, $D_i(v, S) \geq D_i(u, S)$ then, $\phi_i(v) \geq \phi_i(u)$.

Theorem

(Young, 1985) *Let $v \in G$. Then, a value ϕ satisfies* (SM), ($\textsc{Sym}$), *and* ($\textsc{Eff}$) *if and only if ϕ is the Shapley value.*

- Young, H. P. "Monotonic Solutions of Cooperative Games," *International Journal of Game Theory* 14, 1985.

8.5.- The Shapley Value

- Fix N and let $S \subseteq N$. Denote by G_S all superadditive TU-games with player set S. From now on, let $G = \bigcup\limits_{S \subseteq N} G_S$; *i.e.*, G is the class of all superadditive games whose player sets are the subsets of N.
- Let ϕ be a value on G; *i.e.*, for all $(S, v) \in G$, with $S \subseteq N$, $\phi(S, v) \in \mathbb{R}^{\#S}$. Given $v \in G_N$ and $S \subseteq N$, (S, v) means the TU-game v restricted to S; *i.e.*, v is seen as $v : 2^S \longrightarrow \mathbb{R}$.
- Let $v \in G_N$, $S \subseteq N$ and ϕ be a value on G. The *Hart and Mas-Colell reduced game* (S, v_S^ϕ) is defined as follows: for all $T \in 2^S$,

$$v_S^\phi(T) = v(T \cup S^c) - \sum_{i \in S^c} \phi_i(T \cup S^c, v).$$

Definition A value ϕ on G is *Hart and Mas-Colell consistent* if for all $(N, v) \in G$, all $S \subseteq N$, and all $i \in S$,

$$\phi_i(N, v) = \phi_i(S, v_S^\phi).$$

8.5.- The Shapley Value

Definition A value ϕ on G is *Standard for two* if for all two-player games $(\{i, j\}, v) \in G$,

$$\begin{aligned}\phi_i(\{i,j\},v) &= v(\{i\}) + \frac{1}{2}\left[v(\{i,j\}) - v(\{i\}) - v(\{j\})\right].\\ \phi_j(\{i,j\},v) &= v(\{j\}) + \frac{1}{2}\left[v(\{i,j\}) - v(\{i\}) - v(\{j\})\right].\end{aligned}$$

Theorem

(Hart and Mas-Colell, 1989) Let ϕ be a value on G. Then, ϕ is Hart and Mas-Colell consistent and Standard for two if and only if ϕ is the Shapley value.

Theorem

(Hart and Mas-Colell, 1989) Let ϕ be a value on G. Then, ϕ is symmetric, efficient and Hart and Mas-Colell consistent if and only if ϕ is the Shapley value.

8.6.- The Nucleolus

- Schmeidler, D. "The Nucleolus of a Characteristic Function Game," *SIAM Journal on Applied Mathematics* 17, 1969.
- **Objective:** To define a value on G with the property that whenever the Core is non-empty, it always belongs to the Core.
- Let (N, v) be an essential game. For every $x \in \mathbb{R}^n$ and every $S \subseteq N$ define the *excess* of S at x as

$$e_S(x) = \sum_{i \in S} x_i - v(S).$$

- Interpretation: $e_S(x)$ is a measure of the aggregate welfare of coalition S if the proposed vector is x.
- We call $e(x)$ as the vector of all excesses at x; *i.e.*, $e(x) = (e_S(x))_{S \in 2^N \setminus \{\varnothing\}} \in \mathbb{R}^{2^n - 1}$.

8.6.- The Nucleolus

- Given $x \in \mathbb{R}^K$, define $x^{\leq} = (x_1^{\leq}, ..., x_K^{\leq}) \in \mathbb{R}^K$, where $x_1^{\leq} \leq ... \leq x_K^{\leq}$ and there exists a one to one mapping $\pi : \{1, ..., K\} \longrightarrow \{1, ..., K\}$ such that for each $j = 1, ..., K$, $x_j^{\leq} = x_{\pi(j)}$.
  - Namely, $x^{\leq}$ is obtained from x after rearranging the components of x by an increasing order.
- Given $x, y \in \mathbb{R}^K$ we say that x is *leximin superior* to y, written $x \geq^{Lex} y$, if there exists $i = 1, ..., K$ such that $x_i^{\leq} > y_i^{\leq}$ and for all $j = 1, ..., i-1$ (if any), $x_j^{\leq} = y_j^{\leq}$.
- **Examples**
  - $x = (2, 1, 3, 4)$, $y = (0, 5, 4, 6)$. Then, $x^{\leq} = (1, 2, 3, 4)$ and $y^{\leq} = (0, 4, 5, 6)$. Then $x \geq^{Lex} y$.
  - $x = (0, 3, 4, 1)$, $y = (0, 3, 3, 1)$. Then, $x^{\leq} = (0, 1, 3, 4)$ and $y^{\leq} = (0, 1, 3, 3)$. Then $x \geq^{Lex} y$.

8.6.- The Nucleolus

Definition Let (N, v) be an essential game. The *Nucleolus* of v, denoted by $Nu(v)$, is the set imputations $x \in I(v)$ such that there does not exist $y \in I(v)$ with the property that $e(y)$ is leximin superior to $e(x)$; namely,

$$Nu(v) = \left\{ x \in I(v) \mid \nexists y \in I(v) \text{ s.t. } e(y) \geq^{Lex} e(x) \right\}.$$

Theorem

(Schmeidler, 1969) Let v be an essential game. Then, $\#Nu(v) = 1$.

- Therefore, the Nucleolus can be seen as a value.

Theorem

(Schmeidler, 1969) Let v be a balanced game. Then, $Nu(v) \in C(v)$.

8.6.- The Nucleolus

- Given $v, u \in G$ we say that u is a *positive affine transformation* of v if there exists $\alpha \in \mathbb{R}$ and $\beta > 0$ such that for all $S \subseteq N$,

$$u(S) = \alpha + \beta v(S).$$

COVARIANCE A value on G satisfies *covariance* if for all $u, v \in G$ such that u is a positive affine transformation of v then, for all $i \in N$,

$$\phi_i(u) = \alpha + \beta\phi_i(v).$$

8.6.- The Nucleolus

- Let $v \in G_N$, $S \subseteq N$ and $x \in \mathbb{R}^n$ be given. The *Davis and Maschler reduced game* $(S, \tilde{v}_S^x)$ is defined as follows:

  $$\tilde{v}_S^x(S) = v(N) - \sum_{i \notin S} x_i$$

  and for all $T \in 2^S \backslash \{\varnothing, S\}$,

  $$\tilde{v}_S^x(T) = \max_{Q \subseteq N \backslash S} \left\{ v(T \cup Q) - \sum_{i \in Q} x_i \right\}.$$

Definition A value ϕ on G is *consistent in the sense of Davis and Maschler* if for all $(N, v) \in G$, all $S \subseteq N$, and all $i \in S$,

$$\phi_i(N, v) = \phi_i(S, \tilde{v}_S^{\phi(N,v)}).$$

8.6.- The Nucleolus

Theorem

(Sobolev, 1975) Let ϕ be a value on G. Then, ϕ satisfies symmetry, covariance and consistency in the sense of Davis and Maschler if and only if ϕ is the Nucleolus.

Theorem

(Maschler, 1992) Let ϕ be a value on G. Then, ϕ is standard for two and Davis and Maschler consistent if and only if ϕ is the Nucleolus.

- Sobolev, A.I. "The characterization of optimality principles in cooperative games by functional equations," *Mathematical Methods in the Social Sciences* 6, 1975.
- Maschler, M. "The Bargaining Set, Kernel, and Nucleolus". In *Handbook of Game Theory with Economic Applications I*, editors: R. Aumann and S. Hart. North-Holland, 1992.