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The Uniqueness of Stable Matchings

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Abstract

This paper analyses conditions on agents' preferences for a unique stable matching in models of two-sided matching with non-transferable utility. The No Crossing Condition (NCC) is sufficient for uniqueness; it is based on the notion that a person's characteristics, for example their personal qualities or their productive capabilities, not only form the basis of their own attraction to the opposite sex but also determine their own preferences. The paper also shows that a weaker condition, alpha-reducibility, is both necessary and sufficient for a population and any of its sub-populations to have a unique stable matching. If preferences are based on utility functions with agents' characteristics as arguments, then the NCC may be easy to verify. The paper explores conditions on utility functions which imply that the NCC is satisfied whatever the distribution of characteristics. The usefulness of this approach is illustrated by two simple models of household formation.

KEYWORDS: uniqueness, matching, marriage, no crossing condition, supermodularity, Spence-Mirrlees single crossing condition

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1 Introduction

In 1962 Gale and Shapley posed and solved "the stable marriage problem", which asks whether it is possible to pair the members of one set (men) with members of another, disjoint, set (women), in such a way that no man and woman who are not paired with each other would both prefer to leave their partners and marry each other. They proved that at least one such equilibrium, called a stable matching, exists and they showed how to find it. Since Gale and Shapley's paper, there has arisen a considerable literature which has focused on the size and structure of the set of stable matchings. For example, Irving and Leather have shown that if there are n men and n women (so that there are $n!$ possible matchings) and if n is a power of 2 then there exist rankings of men by women and *vice versa* such that there are at least 2^{n-1} stable matchings (Irving and Leather, 1986).[1]

By contrast, little attention has been paid to exploring conditions under which there is a unique stable matching. This is remarkable given the importance generally attached to uniqueness, particularly its usefulness for prediction and comparative statics. In a matching context, uniqueness has a further advantage: if the actual pairings of men and women are a stable matching based upon *reported* preferences, then truth-telling by all agents is a Nash equilibrium if and only if there is a unique stable matching based upon *actual* preferences.[2] But even assuming agents cannot or do not conceal their true preferences, if there is more than one stable matching then although the lattice structure referred to in Footnote 1 may simplify the choice problem of a social planner or mechanism designer, it does not help identify the optimal stable matching unless the planner or designer is concerned only with the interests of one side of the market (e.g. men).

One route to uniqueness, identified by Gusfield and Irving (1989), is to assume one side, men for example, all have the same preferences, with the

[1] Furthermore, the set of stable matchings is a distributive lattice under the common interests of men dual to the common interests of women; see Roth and Sotomayor (1990), Section 3.

[2] From Theorem 4.6 in Roth and Sotormayor, (1990), non-uniqueness implies that truth-telling is not a Nash equilibrium of the revelation game induced by any stable mechanism (i.e. one whose outcome is a stable matching). Conversely, a special case of a theorem due to Demange, Gale and Sotormayor (1987) (Theorem 4.11 in Roth and Sotormayor, 1990), can be stated as follows: if only agent a lies about his/her preferences, then there is no reported stable matching (i.e. based upon reported preferences) which is preferred by a to all actual stable matchings (based upon actual preferences). Therefore if there is only one actual stable matching, so that it is the matching selected by any stable mechanism when all agents report their true preferences, then truth-telling is a Nash equilibrium.

actual matching determined by the women's preferences: the universally most preferred woman gets her most preferred man, the second most preferred woman gets her preferred man from those who remain, and so on. More recently, Eeckhout (2000) has shown that there is a unique stable matching if the sets of men and women can each be ordered so that any man and woman with the same rank prefer each other above any other partner with a lower rank; i.e. man i prefers woman i to women $i+1, i+2, ..., n$, and woman i prefers man i to men $i+1, i+2, ..., n$. I call this the Sequential Preference Condition (SPC). Then man 1 and woman 1 must be matched in any stable matching, as must man 2 and woman 2 since they most prefer each other once man 1 and woman 1 have been paired off; working through the rankings, man i must be matched with woman i, implying a unique stable matching.

The SPC is a great improvement on Gusfield and Irving's condition, and much less restrictive. However, it has two drawbacks: firstly, by itself it provides no clue as to why such orderings should exist, or how they might be constructed. For example, if men's rankings of a particular set of women result from applying one or more criteria that could be applied to any set of women (in effect by treating a woman's characteristics as arguments in a utility function) then how is the SPC related to the criteria used by men, and by women to rank men? If the same set of men were to be matched with a different set of women, then there is no reason to expect that even if the SPC were still satisfied the ordering of the set of men required by the SPC would remain the same. In this sense, the SPC orderings are endogenous, and may bear little relation to the underlying structure of agents' tastes or their characteristics.

Secondly, although a population may satisfy the SPC, a subset of it may not. This has, of course, the immediate consequence that when analysing such subpopulations the benefits of having a unique stable matching cannot be guaranteed.[3] More generally it implies that if we look for conditions on the set of all possible men and women, a "super-population" from which a particular population to be analysed is drawn, then knowing that the SPC is satisfied for the super-population does not imply that the particular population has a unique stable matching.

The starting point for this paper is to describe and analyse a condition that meets these objections. It is based on the notion that a person's characteristics, for example their physical appearance, their personal qualities, or

[3] This might be problematic, for example, if we sought to analyse a stratified population, in which agents could only match with others from the same stratum. That the SPC is satisfied by the larger population does not imply that it is satisfied by each of its constituent strata. For an analysis of matching in stratified populations, see Clark (2003).

their productive capabilities, both form the basis of their attraction to the opposite sex and determine to whom they themselves are attracted. The condition I propose is called the No Crossing Condition (NCC). It has two components: firstly, we order completely the set of men, M, and the set of women, W, the implication being that such orderings are based on one or more personal characteristics which make up their type; secondly, men further along the ordering of M tend to prefer women further along the ordering of W, and *vice versa.* The exact sense of "tend to prefer" is made clear in the next section, but the NCC encompasses two special cases: when all members of one sex agree on their preferences for the other sex, and when each person would prefer a partner who is similar to themselves.

It must be recognised at the outset that the NCC is a stronger condition than the SPC, and one aim of this paper is to provide a sufficient condition for the SPC that is intuitive and in many applications relatively easy to verify, in particular when preference orderings are based on given utility functions. However, the paper goes beyond the analysis of uniqueness in a given population P to identify a condition which is both necessary and sufficient for every subpopulation of P to have a unique stable matching. This condition, $\alpha-$*reducibility*, was developed originally by Alcalde (1995) in the context of roommate problems and is weaker than the NCC but stronger than the SPC. Taking P to be the "super-population" referred to above, then there is a unique stable matching whatever the subpopulation of P being analysed if and only P is $\alpha-$reducible.

If preferences are based on given utility functions whose arguments are agents' characteristics, then in many instances it may be straightforward to verify if the NCC is satisfied by direct examination of the utility functions. Ordering the sets M and W may be simple if preferences are based on a single characteristic for each sex: e.g. height, taste in music, wealth, or competence at a particular task. The NCC then says, for instance, that taller men tend to prefer taller woman and *vice versa;* or, to take an example based on the market for hospital interns (see Roth, 1984), that those medical graduates who are more interested in the academic rather than the pastoral aspects of medicine tend to prefer hospitals that are more oriented towards medical research, and *vice versa.* But if agents' types are a bundle of characteristics, e.g. height and wealth, or research orientation and locality, it may not be straightforward to order M and W to see if the NCC is satisfied. This issue is addressed in Section 4 of the paper, which seeks restrictions on agents' utility functions that ensure that regardless of the distribution of characteristics in the population the NCC is satisfied, and hence that there is a unique stable matching. This is achieved by combining a limited degree of separability of the utility functions with an assumption of modularity, a property which has hitherto been used to establish results on comparative statics rather than uniqueness (e.g. Milgrom and Shannon 1994). As Milgrom and Shannon establish, there is a close link between modularity and the Spence-Mirrlees Single Crossing Condition, and one special case of the general result established in Section 4 can be summarised as "Spence-Mirrlees plus quasi-linearity implies uniqueness". This connects the NCC to two commonly used assumptions about tastes; in particular it opens the way for a range of applications in which utility depends not only on the interaction of individual characteristics such as height, taste for jointly consumed goods, or interest in research (where it might be appropriate to assume modularity/single crossing) but also on money or wealth (where quasi-linearity might apply).

The next section of the paper sets up the formal matching framework, defines the NCC, and proves the central theorem of the paper. Section 3 compares the NCC with the SPC, and goes on to show how they are related to $\alpha-$ reducibility and Gusfield and Irving's condition. Section 4 considers issues that are raised when we treat agents' characteristics as arguments in utility functions, which then form the basis of preferences. The usefulness of looking at the properties of utility functions to check for uniqueness is illustrated with an application to household formation. Section 5 concludes. Unless otherwise stated, all proofs are in the Appendix.

2 Uniqueness of Stable Matchings

2.1 The Matching Framework

The standard matching framework considers two finite and disjoint sets, both with n elements: a set of men M and a set of women W. We refer to $P = M \cup W$ as the population. Each man has complete, reflexive, and transitive preferences over the set W. We assume that these preferences are strict (so that no man is indifferent between two women), and are such that each man would rather be married to any woman than remain single. The preferences of a man $x \in M$ can thus be described by a complete ordering Ω_x of the set W.[4] The statement $y \succ_x y'$ denotes that x prefers y to y'. Similar assumptions are made for women's preferences, *mutatis mutandis*, with the preferences of

[4] Let $\succeq_x$ be the binary relation on W such that $y \succeq_x y'$ denotes that x does not prefer y' to y. Since (i) $y \succeq_x y$;(ii) $y \succeq_x y'$ and $y' \succeq_x y''$ implies $y \succeq_x y''$;(iii) $y \succeq_x y'$ and $y' \succeq_x y$ implies $y = y'$ (because preferences are strict); and (iv) either $y \succeq_x y'$ or $y' \succeq_x y$ or both, then $\succeq_x$is a complete ordering, hereinafter labelled Ω_x.

a woman $y \in W$ described by a complete ordering Ω_y of the set M; $x \succ_y x'$ denotes that y prefers x to x'. Let $\Omega = \{\Omega_i, i \in P\}$ be the preference profile (or set of preference orderings) of the population P. The pair (P, Ω) constitutes a *marriage market*.

A *matching* μ of P is a one-to-one function from P onto itself such that (i) $x = \mu(y)$ if and only if $y = \mu(x)$; (ii) if $x \in M$ then $\mu(x) \in W$ and if $y \in W$ then $\mu(y) \in M$. A matching μ can be *blocked* by a pair $(x, y) \in M \times W$ for whom $x \neq \mu(y)$ if $y \succ_x \mu(x)$ and $x \succ_y \mu(y)$. A matching μ is *stable* if it cannot be blocked by any pair. Then as shown by Gale and Shapley in 1962, we have:

Theorem 1 *A stable matching exists for every marriage market.*

2.2 The No Crossing Condition

To define the No Crossing Condition, it is convenient to consider M and W when ordered as vectors, with different orderings represented by different vectors. For any positive integer q let $I_q = \{1, 2, ..., q\}$. Then the vector $m = (m_i) \in M^n$ is an ordering of M if for all $x \in M, x = m_i$ for some $i \in I_n$; similarly the vector $w = (w_k) \in W^n$ is an ordering of W if for all $y \in W, y = w_k$ for some $k \in I_n$. The No Crossing Condition (NCC) may now be stated quite simply:

Definition 1 *A population P with preference profile Ω satisfies the No Crossing Condition if there exists an ordering m of M and an ordering w of W such that if $i < j$ and $k < l$ then*
(i) $m_l \succ_{w_i} m_k \implies m_l \succ_{w_j} m_k$;
and
(ii) $w_j \succ_{m_k} w_i \implies w_j \succ_{m_l} w_i$.

The NCC is thus a condition on the set of ordinal preference relations Ω, and it is sometimes convenient to refer to the orderings m and w themselves as satisfying the NCC. Part (i) of the definition may be interpreted as saying that it cannot be the case that the woman further back in the female ordering, w_i, prefers the man further forward in the male ordering, m_l, and at the same time the woman further forward in the female ordering, w_j, prefers the man further back in the male ordering, m_k. Diagrammatically this rules out the preferences depicted in Figure 1, where the sexes are ordered along the two horizontal lines, and the arrow from each woman points to the man she prefers out of the two shown. Likewise for part (ii) of the definition which applies to

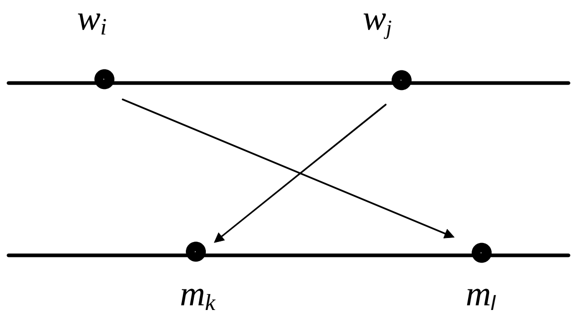

Figure 1: Preferences ruled out by the No Crossing Condition

men's preferences. If the NCC is satisfied there exist orderings m and w such that for any pair of women and any pair of men the two arrows representing the women's preferences do not cross, nor do the two arrows representing the men's preferences.

The condition does not rule out the possibility that both women prefer the same man; indeed, it allows for all members of one sex to have the same preferences. A simple instance of the NCC when all agents have different tastes arises if everyone would most prefer a partner of the same height as themselves, with an agent's utility a decreasing function of the absolute difference between his (or her) height and his (or her) partner's.[5] In this case, height provides a natural way to order M and W, and a graph of any agent's utility against his or her partner's height would display a single symmetric peak at the agent's own height. But note that the NCC does not imply, nor it implied by, 'single-peakedness' of preferences.[6] However, the No Crossing Condition does naturally bring to mind the Single Crossing Condition, much used in information economics; here there is a connection, the analysis of which requires us first to consider in more detail the dimensions over which tastes are defined, an issue which is explored in Section 4.

[5] A similar instance arises if paired agents care about the difference in their ages, as in Bergstrom and Lam (1989a, 1989b).

[6] For example if $m_1 \succ_{w_1} m_2 \succ_{w_1} m_3$, $m_1 \succ_{w_2} m_3 \succ_{w_2} m_2$, $m_3 \succ_{w_3} m_2 \succ_{w_3} m_1$, and $w_1 \succ_{m_i} w_2 \succ_{m_i} w_3$ for $i = 1, 2, 3$, then m and w satisfy the NCC but there is no ordering of M such that each woman's preferences display a single peak. Conversely, we have single-peakedness if $m_2 \succ_{w_i} m_1 \succ_{w_i} m_3$ $(i = 1, 3)$, $m_1 \succ_{w_3} m_2 \succ_{w_3} m_3$, $w_1 \succ_{m_i} w_2 \succ_{m_i} w_3$ $(i = 1, 3), w_2 \succ_{m_2} w_1 \succ_{m_2} w_3$; but the NCC is not met, as m_1 prefers w_1 who prefers m_2 who prefers w_2 who prefers m_1; *any* orderings of M and W must therefore produce a crossing of the type shown in Figure 1. This second example has two stable matchings, so single peakedness does not by itself imply uniqueness.

A very important property of the No Crossing Condition is that if it holds for the population P then it also holds for any sub-population of P. Formally, $P' = M' \cup W'$ is a *subpopulation* of $P = M \cup W$ if $M' \subseteq M$, $W' \subseteq W$, and $\#(M') = \#(W')$. Then we have:

Lemma 1 *If the population P satisfies the No Crossing Condition, then every subpopulation of P satisfies the No Crossing Condition.*

2.2.1 No Crossing and the Existence of Fixed Pairs

We now develop two lemmas that lie at the heart of the main theorem on uniqueness. If we can find any couple who prefer each other then each such a couple, called a *fixed pair,* must be matched in a stable matching. The main theorem then uses the No Crossing Condition to identify a sequence of n fixed pairs, thus generating a unique stable matching.

Formally, a couple $(x, y) \in M \times W$ *is a fixed pair of the population P* if $y \succ_x y'$ for all $y' \in W \backslash \{y\}$ and $x \succ_y x'$ for all $x' \in M \backslash \{x\}$. Suppose P has p fixed pairs, and denote by F the set constituted by these p men and p women;[7] then:

Lemma 2 *Let μ be a stable matching of the population P. Then (i) for any fixed pair (x, y), $\mu(x) = y$; (ii) the function μ' defined by $\mu'(z) = \mu(z)$ for all $z \in P \backslash F$ is a stable matching of $P \backslash F$.*

Thus the problem of finding a unique stable matching of the population P can thus be broken down into finding the fixed pairs of P, and then finding a unique stable matching of $P \backslash F$. But this requires that P does indeed have at least one fixed pair. The No Crossing Condition now assumes a central role:

Lemma 3 *If a population satisfies the No Crossing Condition, then it has a fixed pair.*

The proof considers the function that for each man x gives x's *rival*, the preferred man of x's preferred woman. Given orderings m and w satisfying the NCC, this function is non-decreasing in the sense that if x is further along the ordering m than x', then x's rival can be no further back in the ordering than x''s rival. The existence of a fixed point, and hence of a fixed pair, is

[7] I.e. if (x, y) is a fixed pair of P, then $x \in F$ and $y \in F$; and if $x \in F$ then there exists $y \in F$ such that either (x, y) or (y, x) is a fixed pair of P.

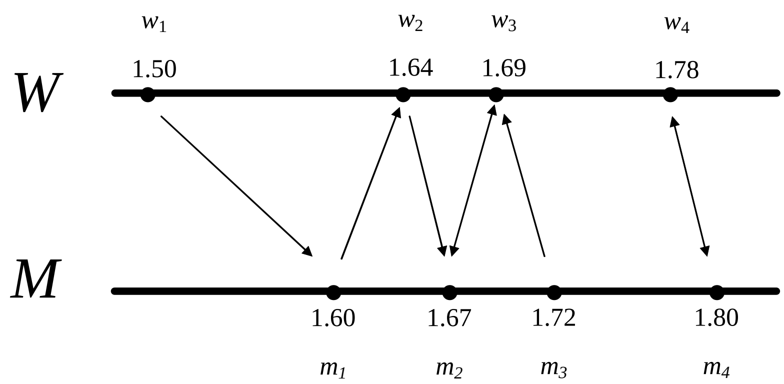

Figure 2: Preferences to illustrate Theorem 2

almost immediate. But there is no reason to suppose that if m_i and w_k are a fixed pair then $i = k$. For example, the shortest man and the tallest woman form a fixed pair if everyone prefers a partner of the same height as themselves and all men are taller than all women. Of course, in general the fixed point is not necessarily unique: a population may have more than one fixed pair.

2.3 The Main Theorem

Bringing together Lemmas 1, 2, and 3, we now have the main uniqueness result:

Theorem 2 *If a population satisfies the No Crossing Condition, then there exists a unique stable matching.*

The proof shows how to construct the unique stable matching: first match the fixed pairs of P; take the remaining population P_2, match the fixed pairs of P_2; and so on, until the whole population is matched. To illustrate how the successive identification of fixed pairs leads to a unique stable matching, consider the following example of a population of women with heights 1.50, 1.64, 1.69, 1.78, and men with heights 1.60, 1.67, 1.72, 1.80. Each individual would prefer to be matched with someone as near to their own height as possible i.e. someone of height h_1 matched with someone of height h_2 has utility that is a negative function of $|h_1 - h_2|$. Such preferences satisfy the No Crossing Condition, and are illustrated in Figure 2 (the arrows from each person point to her/his most preferred partner).

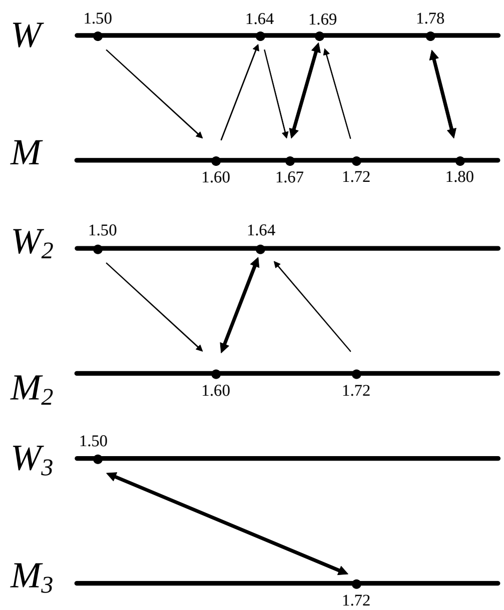

Figure 3: The emergence of fixed pairs (shown in bold arrows)

The fixed pairs of this population are (w_3, m_2) and (w_4, m_4), so they are matched in any stable matching.; the remaining population, P_2, equals $\{w_1, w_2, m_{1,} m_3\}$ with heights 1.50, 1.64, 1.60, and 1.72 respectively; P_2 also satisfies the No Crossing Condition and the couple (w_2, m_1) are a fixed pair; note that w_2 would have preferred m_2 but he is already matched with w_3; finally, w_1 and m_3 are left and must be matched. This process by which new fixed pairs emerge as others are taken out of the population is illustrated in Figure 3, where the bold double arrows denote a fixed pair of the population or sub-population under consideration.

The example in Figures 2 and 3 also illustrates how the NCC can be used to establish interesting and counter-intuitive results in matching markets where preferences are based on "nearness". Although like may be attracted to like, like is not necessarily matched to like: in the example the shortest woman is most attracted to the shortest man but ends up with the second tallest man.

3 No Crossing, Sequential Preferences, and Uniqueness in all Subpopulations.

Let m and w be orderings of M and W respectively and suppose that, for $i < n$, m_i prefers w_i to all the women from w_{i+1} to w_n, and w_i prefers m_i to all the men from m_{i+1} to m_n. This is Eeckhout's Sequential Preference Condition (SPC), and I refer to m and w as satisfying the SPC. Then there is a unique stable matching in which m_i is matched with w_i, for all i (Theorem 1 in Eeckhout, 2000). This can be constructed by a sequential process: m_1 and w_1 prefer each other above all others and so must be paired in any stable matching; m_2 and w_2 prefer each other to anyone else in $W \backslash \{w_1\}$ and $M \backslash \{m_1\}$ respectively and hence must also be paired (since they could block any matching in which they were not paired but m_1 and w_1 were paired); and so on, until we are left with m_n and w_n, who would rather marry each other than remain single.

What is the relationship between the NCC and the SPC? Suppose the population P satisfies the NCC. Then orderings m and w satisfying the SPC can be derived from the order in which the fixed pairs of P and its subpopulations are generated in constructing the unique stable matching of P. Briefly, let the k^{th} elements of m and w be the k^{th} fixed pair in a sequence $\{\sigma_s\}, s = 1, ..., n$ such that σ_s is a fixed pair of the population P_s and $P_{s+1} = P_s \backslash \{\sigma_s\}$, with $P_1 = P$.[8] Then the vectors m and w satisfy the SPC. The NCC therefore implies the SPC for a given population P. The reverse is not true, as shown by the example of Table 1, which gives the preferences of a population of three men and three women:

agent	1st preference	2nd preference	3rd preference
m_1	w_1	w_2	w_3
m_2	w_2	w_3	w_1
m_3	w_1	w_2	w_3
w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_2	m_1

Table 1

The SPC is satisfied and m_i is matched with w_i, $i = 1, 2, 3$. However, w_1 prefers m_2 to m_3 and w_3 prefers m_3 to m_2, whereas m_2 prefers w_3 to w_1 and

[8] Because a population may have more than one fixed pair, the sequence $\{\sigma_s\}$ is not unique, and nor, therefore, are the orderings satisfying the SPC.

m_3 prefers w_1 to w_3. Consequently, it is not possible to order $\{m_2, m_3\}$ and $\{w_1, w_3\}$ (and therefore M and W) and avoid a crossing such as in Figure 1. Thus the NCC implies the SPC but not *vice versa*. .

However, the NCC has several strengths not shared by the SPC. The SPC may be seen as a statement that there exists a sequence of n fixed pairs, each from a successively smaller subpopulation of P. But the order in which these fixed pairs emerge, which immediately gives the orderings that satisfy the SPC, need bear no obvious relation to the preferences or characteristics of the population. For example, in Figure 3, the NCC orderings based on height are $w = (w_1, w_2, w_3, w_4)$ and $m = (m_1, m_2, m_3, m_4)$; the SPC is satisfied not by w and m but by (w_3, w_4, w_2, w_1) and (m_2, m_4, m_1, m_3) or (w_4, w_3, w_2, w_1) and (m_4, m_2, m_1, m_3). If we remove w_4 and m_2 from this population, then the NCC orderings of the remaining sub-population based on height remain the same, (w_1, w_2, w_3) and (m_1, m_3, m_4) but the SPC is now satisfied by (w_2, w_3, w_1) and (m_1, m_3, m_4) or (w_3, w_2, w_1) and (m_3, m_1, m_4). The SPC orderings seem almost arbitrary.[9] The NCC orderings, on the other hand, aim to capture a property of the preference profile of the population: men further on in the male ordering tend to prefer women further on in the female ordering and *vice versa*. As a result, it may well be straightforward to test whether the NCC is satisfied, particularly, as we see in Section 4, if preferences are based on known utility functions. The SPC, on the other hand, gives little clue about the underlying structure of tastes that might result in the condition being satisfied.

3.1 Uniqueness in all Subpopulations

The merits of the NCC may be viewed from a different perspective. We know from Lemma 1 that if the NCC holds for a population P it also holds for any subpopulation of P. This is not true of the SPC, as the example in Table 1 shows: the subpopulation $\{m_2, m_3, w_1, w_3\}$ has no fixed pairs and both possible matchings are stable. Consequently, whether the SPC holds or not may depend critically on the exact membership of the subpopulation being considered.[10] Suppose we regard M and W as the sets of *all possible* men and

[9] They may contain some interesting information, however. For example, from the SPC orderings for the example in Figure 3 we can derive what the stable matching implies for the rank correlation between agents' heights and their partners'.

[10] Put differently, the NCC may be seen as a global condition, involving a comparison of the preferences of all pairs of men over all pairs of women, whereas the SPC is more local in chararacter: for each man m_i it is his comparisons of w_i and w_j for $j > i$ that are relevant; (and similarly for women's preferences). A local condition may suffice for a

women. In any particular instance, we are therefore dealing with subsets M' and W'. From Lemma 1, it is sufficient, when analysing the subpopulation $P' = M' \cup W'$, to show or assume that $P = M \cup W$ satisfies the NCC. Moreover, when considering the orderings m' and w' of M' and W' that satisfy the NCC, the order in which any two men or two women appear in m' or w' is independent of the other elements in those vectors. In effect, the orderings m and w of M and W that satisfy the NCC may be treated as "master orderings". But if we show or assume that P satisfies the SPC, we cannot be sure that P' also satisfies the SPC. Even if P' does so, consider the emergence of fixed pairs as the stable matching is constructed (as in Figure 3, for example). The individuals who constitute the first fixed pair, the second fixed pair, and so on, will typically vary with the precise membership of M' and W'. But it is the order in which they emerge as fixed pairs that gives the orderings that satisfy the SPC. Both theoretically and from an applied perspective, it seems an advantage that if P satisfies the NCC then when analysing P' neither the question of whether the NCC is satisfied nor the orderings satisfying the NCC depends on the particular groups M' and W' being considered.

Lemma 1 and Theorem 2 imply that if a population P satisfies the NCC, then any subpopulation of P has a unique stable matching. It is useful to break down this result into two propositions using the notion of $\alpha-$*reducibility*, developed originally by Alcalde (1995) in the context of roommate problems: a marriage market (P, Ω) is $\alpha-$*reducible* if every subpopulation of P has a fixed pair.[11] Then Lemmas 1 and 3 imply

Theorem 3 *(i) If a population P with preference profile Ω satisfies the NCC, then (P, Ω) is $\alpha-$reducible.*

And Lemma 2 forms the main basis of the proof of

Theorem 4 *If (P, Ω) is $\alpha-$reducible then every subpopulation of P has a unique stable matching.*

But we can also show the following:[12]

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given population, but the extra information contained in the NCC ensures uniqueness in all subpopulations.

[11] Banerjee, Konishi, and Somnez (2001) extend $\alpha-$reducibility to more general coalition games in the form of the *top-coalition property.*

[12] Alcalde (1995) shows that certain types of preferences imply a roommate market's $\alpha-$reducibility and hence that it has a unique equilibrium; however he does not state or prove the roommate equivalent of Theorem 5.

Theorem 5 *If every subpopulation of P has a unique stable matching then (P, Ω) is $\alpha-$reducible.*

If (P, Ω) is not $\alpha-$reducible, then it is possible to find a subpopulation P' of P with at least two men and two women and no fixed pairs such that each man in P' is most preferred by some woman in P',and each woman in P' is most preferred by some man in P'.[13] Then the matching in which each man is paired with his preferred woman is stable, as is the matching in which each woman is matched with her preferred man. As there are no fixed pairs of P', these two stable matchings of P' are different. Thus $\alpha-$reducibility is a sufficient and necessary condition for every subpopulation of P, including of course P itself, to have a unique stable matching.[14]

The relationships between the various conditions for uniqueness, summarised as a Venn diagram in Figure 4 , are as follows: if P satisfies the NCC then (P, Ω) is $\alpha-$reducible; $\alpha-$reducibility is a necessary and sufficient condition for P and each of its subpopulations to have a unique stable matching; $\alpha-$reducibility also implies the SPC, which in turn implies that P has at least one fixed pair; the SPC is a sufficient but not necessary condition for P to have a unique stable matching; and having at least one fixed pair is neither necessary nor sufficient for P to have a unique stable matching. We can also fit into this framework Gusfield and Irving's (1989) condition for uniqueness; if all men have the same preferences, then (P, Ω) is $\alpha-$ reducible (in any subpopulation, the most preferred woman and her preferred man are a fixed pair), but the NCC is not necessarily satisfied; similarly if all women have the same preferences. But if all men and all women have the same preferences, then any orderings of M and W satisfy the NCC: a crossing as in Figure 1 can only occur if two men or two women disagree. [15]

[13] Or equivalently, that no man (resp. woman) prefers the same woman (resp. man) as any other man (woman).

[14] As with the SPC, $\alpha-$reducibility gives little clue as to when or why it should be satisfied; indeed, almost by definition (P, Ω) is $\alpha-$reducible if and only the SPC holds for P and each of its subpopulations.

[15] Conditions such as the NCC, $\alpha-$ reducibilty, and the SPC concern the preferences of a population. An alternative route to uniqueness is to restrict what pairs may form (beyond the bar on same sex partnerships). In recent work on general coalition formation games Papai (2004) has shown that if, and only if, the set of permitted coalitions satisfies a *single lapping* property, then there is a unique stable coalition structure for any preference profile of the population. But in the context of two-sided matching, single lapping is extremely demanding; for example, at least one agent has only one permitted partner.

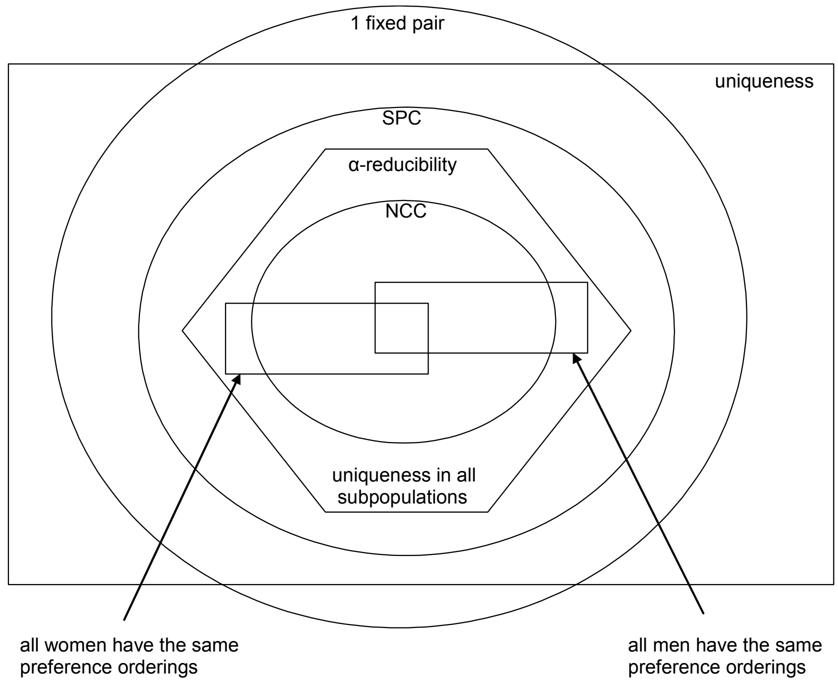

Figure 4: Conditions on the preference profile Ω for a unique stable matching. Note that the set of profiles for which all subpopulations have a unique stable matching coincides with the set for which (P, Ω) is $\alpha-$ reducible.

4 No Crossing and Utility Functions

The idea underlying the No Crossing Condition is that an individual's characteristics both form the basis of their attraction to the opposite sex and determine to whom they are attracted. In this section, we treat agents' characteristics as arguments in utility functions, which then form the basis of the preference profile Ω. We shall see that in many instances it is relatively easy to verify if the NCC is satisfied by looking at the utility functions directly. This approach may be seen as an application of the property of the NCC, established in the previous section, that if it satisfied by a population P, then it is also satisfied by any subpopulation of P. In effect we take P to be a "super-population" of all possible agents, defined as those with given utility functions. If the NCC is satisfied by P, then in any particular instance knowing that all agents have these utility functions implies that they collectively constitute a subpopulation of P, which then satisfies the NCC.

The NCC requires the sets M and W to be ordered. This may be straightforward if preferences are based on a single characteristic for each sex: men are arranged along one dimension (height, wealth, political views or taste in music), as are women. The NCC then says, for example, that more left-wing men tend to prefer women with a stronger taste for jazz *and* that women with a stronger taste for jazz tend to prefer more left-wing men. The issues raised by going beyond one dimension for men and one for women are illustrated in the following. We describe each individual by the same two characteristics, political views and taste for music. An individual i has political views v_i and taste for music t_i, where v_i ranges from 1 (left-wing/liberal) to 4 (right-wing/conservative) and t_i ranges from 1 (preference for pop music) to 4 (preference for classical music). Let $n = 2$, where individuals 1 and 2 are men, and 3 and 4 are women. Suppose that if a male i is matched with a female k then he gets utility $-2(v_i - v_k)^2 - (t_i - t_k)^2$ and she gets utility $-(v_i - v_k)^2 - 2(t_i - t_k)^2$, so men care mostly about politics and women mostly about music. Assume: $(v_1, t_1) = (1, 1), (v_2, t_2) = (4, 4), (v_3, t_3) = (2, 3), (v_4, t_4) = (3, 2)$. Then, of the members of the opposite sex, individual 1 prefers 3; 3 prefers 2; 2 prefers 4; 4 prefers 1. Even though preferences satisfy the NCC if we take one dimension at a time (e.g. a more conservative individual prefers a more conservative partner, *ceteris paribus),* the NCC fails overall: we cannot order M and W and avoid a crossing as in Figure 1. Uniqueness cannot be guaranteed and in this example both of the two possible matchings are stable.

4.1 Modularity and Separability

We now seek restrictions on agents' utility functions that are sufficient to ensure uniqueness, whatever the values of agents' characteristics. For each sex, we isolate a single characteristic, which we refer to as a for men and b for women. If x is male, the value of a is denoted by a_x (a scalar) and his remaining characteristics by $\boldsymbol{\alpha}_x$ (possibly a vector); if y is female, the value of b is denoted by b_y and her remaining characteristics by $\boldsymbol{\beta}_y$. The analysis proceeds by imposing sufficient separability on utility functions to confine any interactions to a and b and then by looking for orderings of M and W based on the characteristics a and b that satisfy the No Crossing Condition.

Assumption 1: *For any matched pair $(x, y) \in M \times W$, the utility of x can be written as*

$$f(a_x, b_y) + \tau(\boldsymbol{\alpha}_x) + \phi(\boldsymbol{\beta}_y)$$

and the utility of y *can be written as*

$$g(a_x, b_y) + \chi(\boldsymbol{\alpha}_x) + \psi(\boldsymbol{\beta}_y)$$

The effect of this is that when a man is comparing two women the function τ is not important, and when we compare two men's comparisons of the same two women (in order to see if the NCC is satisfied) the function ϕ is not important; similarly for χ and ψ. By differencing twice we take out the fixed effects embodied in the functions τ, ϕ, χ, and ψ. However, as we show below, these functions still have a potentially important role in determining the stable matching.

The separability assumption focuses attention on the form of f and g. The main result of this section relies on the notions of supermodularity and submodularity:[16]

Theorem 6 *If Assumption 1 is satisfied, and if the functions* f *and* g *are either both supermodular or both submodular, then there is a unique stable matching.*

This result prompts a number of remarks. Firstly, it is not sufficient that f and g each be either supermodular or submodular.[17] The theorem requires that f and g must either both be supermodular or both be submodular, a condition we label the *comodularity* of f and g. Is comodularity necessary for uniqueness? No, not if we take as given the values of agents' characteristics and the functions τ, ϕ, χ, and ψ.[18] But if there exist $a_1 < a_2$ and $b_3 < b_4$ such that $f(a_2, b_4) - f(a_2, b_3) > f(a_1, b_4) - f(a_1, b_3)$ and $g(a_2, b_4) - g(a_2, b_3) < g(a_1, b_4) - g(a_1, b_3)$ (or such that both inequalities are reversed) then for these values of a and b we can always find values of agents' other characteristics, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and functions ϕ and χ to construct a marriage market where there is

[16] A function $f : R^2 \to R$ is supermodular if for $a_1 < a_2$ and $b_1 < b_2$, $f(a_2, b_2) - f(a_2, b_1) > f(a_1, b_2) - f(a_1, b_1)$; it is submodular if for $a_1 < a_2$ and $b_1 < b_2$, $f(a_2, b_2) - f(a_2, b_1) < f(a_1, b_2) - f(a_1, b_1)$. For a comprehensive analysis of modularity, see Topkis (1998).

[17] To take an example where f is supermodular and g is submodular, suppose $f(a, b) = (a + b)^2$,$\tau(\boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha}$, $\phi(\boldsymbol{\beta}) = \boldsymbol{\beta}$, $g(a, b) = (a + b)^{0.5}$, $\chi(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, and $\psi(\boldsymbol{\beta}) = 0$ for all $\boldsymbol{\beta}$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are scalars. Let $n = 2$, where individuals 1 and 2 are men, and 3 and 4 are women. If $(a_1, \boldsymbol{\alpha}_1) = (1, 0.3), (a_2, \boldsymbol{\alpha}_2) = (2, 0), (b_3, \boldsymbol{\beta}_3) = (1, 6)$, and $(b_4, \boldsymbol{\beta}_4) = (2, 0)$, then individual 1 prefers 3 to 4, 3 prefers 2 to 1, 2 prefers 4 to 3, and 4 prefers 1 to 2. Thus both of the two possible matchings are stable.

[18] To see this amend the example of the previous footnote so that $\chi(\boldsymbol{\alpha}) = \phi(\boldsymbol{\beta}) = \mathbf{0}$ for all $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$; then both individuals 1 and 2 prefer 4 to 3, and both 3 and 4 prefer 2 to 1. Thus 2 are 4 are a fixed pair and must be matched, as then must 1 and 3.

more than one stable matching.[19]

Secondly, modularity is typically invoked in matching theory when utility is transferable: supermodularity (resp. submodularity) then implies that maximisation of total utility requires positive (resp. negative) sorting; for example, Becker (1981) or Legros and Newman (2002). The theorem above has nothing to do with efficiency or sorting. In this paper utility is non-transferable and any stable matching is Pareto efficient, whether or not the conditions of the theorem are satisfied.[20] Furthermore, even with the separability and modularity assumed by the theorem, the unique stable matching may display positive or negative sorting. For example, suppose the characteristics a and b are both height and $f(a,b) = g(a,b) = -(a-b)^2$; thus f and g are both supermodular. Let τ, ϕ, χ, ψ be constant functions, so only height matters. If the distribution of height is the same amongst men and women, then everyone can find a perfect partner and sorting is positive. But if all men are taller than all women (or *vice versa*), then in the stable matching the i^{th} shortest man matches with the i^{th} tallest woman, even though preferences are such that like attracts like.[21]

Thirdly, it is essential to Theorem 6 that there are interactions only between the characteristics a and b; otherwise there may be more than one stable matching, even if each utility function consists of the sum of two supermodular functions (as in the example above of political views and taste for music). However, confining interaction effects to the functions f and g does not imply that τ, ϕ, χ, and ψ have no effect on the stable matching. Suppose $f(a,b) = g(a,b) = -(a-b)^2$; for simplicity let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be scalars such that $\tau(\boldsymbol{\alpha}) = \chi(\boldsymbol{\alpha}) = \lambda\boldsymbol{\alpha}$, and $\phi(\boldsymbol{\beta}) = \psi(\boldsymbol{\beta}) = \lambda\boldsymbol{\beta}$. To fix ideas, interpret both a and b as political views (1 = liberal, 4 = conservative) and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as money. Then individuals are attracted to partners who are politically like-minded and/or rich, with λ determining the relative importance of these two characteristics. The functions f and g are supermodular, so for any distribution of

[19] For $n = 2$ we can choose $\chi(\boldsymbol{\alpha}_1)$, $\chi(\boldsymbol{\alpha}_2)$ $\phi(\boldsymbol{\beta}_3)$ and $\phi(\boldsymbol{\beta}_4)$ so that $f(a_2,b_4) - f(a_2,b_3) + \phi(\boldsymbol{\beta}_4) - \phi(\boldsymbol{\beta}_3) > 0 > f(a_1,b_4) - f(a_1,b_3) + \phi(\boldsymbol{\beta}_4) - \phi(\boldsymbol{\beta}_3)$ and $g(a_2,b_4) - g(a_1,b_4) + \chi(\boldsymbol{\alpha}_2) - \chi(\boldsymbol{\alpha}_1) < 0 < g(a_2,b_3) - g(a_1,b_3) + \chi(\boldsymbol{\alpha}_2) - \chi(\boldsymbol{\alpha}_1)$. Then individual 1 prefers 3 to 4, 3 prefers 2, 2 prefers 4, and 4 prefers 1, so both possible matchings are stable. This example can easily be embedded in a larger marriage market to generate non-uniqueness when $n > 2$.

[20] Suppose man x and woman y are not matched by μ but are matched by some alternative matching μ' (if $\mu \neq \mu'$, there must exist some such couple). Since μ cannot be blocked, either x prefers $\mu(x)$ to y or y prefers $\mu(y)$ to x or both. As preferences are strict, either x or y (or both) is worse off when matched by μ'.

[21] Furthermore, if social welfare is the sum of individual utilities, then in this case, out of all possible matchings the equilibrium matching, because it has perfect negative sorting, minimises welfare.

characteristics the NCC is satisfied and there is a unique stable matching. Let $n = 2$, where individuals 1 and 2 are men, and 3 and 4 are women and suppose $(a_1, \boldsymbol{\alpha}_1) = (1,1), (a_2, \boldsymbol{\alpha}_2) = (2,2), (b_3, \boldsymbol{\beta}_3) = (2,2), (b_4, \boldsymbol{\beta}_4) = (3,3)$ i.e. richer people are more conservative. If $0 \leq \lambda < 1$, a fixed pair is formed by individuals 2 and 3; but if $\lambda > 1$ individual 2 prefers the moderately conservative but richer 4, who in turn prefers 2 to the liberal and poor 1.[22]

Finally, the NCC is concerned with preference orderings, whereas modularity is a cardinal concept. Theorem 6 looks only for a particular representation of agents' preferences. In the simple case where the functions τ, ϕ, χ, and ψ are all constant, even if f and g are not comodular there may nevertheless exist increasing transformations η and ξ of f and g respectively such that the compositions $\eta \circ f$ and $\xi \circ g$ are comodular. Since these compositions represent the original preferences, the NCC is then satisfied.[23]

4.2 The Spence-Mirrlees Single Crossing Condition

We can relate the modularity condition of Theorem 6 to the Spence-Mirrlees *Single Crossing Condition,* which states that if an agent with a characteristic c has preferences for goods a and b represented by the utility function $u(a, b, c)$ then the marginal rate of substitution $\frac{\partial u}{\partial a} / \frac{\partial u}{\partial b}$ is either always increasing or always decreasing in c.[24] Given the utility function $f(a,b)+\tau(\boldsymbol{\alpha})+\phi(\boldsymbol{\beta})$ (where for simplicity we treat $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as scalars), the Spence-Mirrlees condition for men means that the marginal rate of substitution between b and $\boldsymbol{\beta}$, $\frac{\partial f}{\partial b} / \frac{\partial \phi}{\partial \boldsymbol{\beta}}$, is always increasing or always decreasing in a (the characteristic $\boldsymbol{\alpha}$ having no effect on the marginal utilities of b and $\boldsymbol{\beta}$). Now, since $\frac{\partial \phi}{\partial \boldsymbol{\beta}}$ is not a function of a, then, assuming $\frac{\partial \phi}{\partial \boldsymbol{\beta}} > 0$, the Spence-Mirrlees condition for men requires that $\frac{\partial^2 f}{\partial a \partial b}$ does not change sign, i.e. that f is either supermodular or submodular. A similar line of argument holds for g and the Spence-Mirrlees condition for women. But it is important to recall that Theorem 6 requires that f and g must be both supermodular or both submodular; assuming $\frac{\partial \phi}{\partial \boldsymbol{\beta}} > 0$ and $\frac{\partial \chi}{\partial \boldsymbol{\alpha}} > 0$ this is equivalent to requiring that $\frac{\partial^2 f}{\partial a \partial b}$ and $\frac{\partial^2 g}{\partial a \partial b}$ have the same sign, not just

[22] Our matching framework requires strict preferences, so we rule out the possibility that $\lambda = 1$ as then individual 2 would be indifferent between 3 and 4.

[23] For example, if $f(a_x, b_y) = g(a_x, b_y) = -|a_x - b_y|^{0.5}$, f and g are neither submodular nor supermodular. But if $\eta(z) = \xi(z) = -z^4$ (which is increasing since its domain is the non-positive real numbers), then $(\zeta \circ f)(a_x, b_y) = (\xi \circ g)(a_x, b_y) = -(a_x - b_y)^2$. We are now back in familiar territory.

[24] The connections between supermodularity and single-crossing have been extensively analysed by Milgrom and Shannon (1994), but in the context of comparative statics, not uniqueness of equilibrium.

that each has a constant sign.

Assumption 1 is satisfied if both the male and female utility functions are quasi-linear in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$; hence the motto "Spence-Mirrlees plus quasi-linearity implies uniqueness". One example of this was given at the end of Section 3.1, in which the utility of all agents is quasi-linear in money. As the value of λ increases money become more important, and for the given distribution of political views and money the stable matching switches at $\lambda = 1$. But as long as preferences are strict, the stable matching is always unique whatever the distribution of characteristics. Thus there is no indeterminacy arising from multiple stable matchings and attention can be fully focused on the role of λ and the distribution of characteristics.

4.3 An Application to Family Formation.

Since the pioneering work of Becker (1965, 1981), matching theory has made major contributions to the economic analysis of marriage and the family.[25] We now briefly consider two simple models of household formation in which knowledge of agents' utility and cost functions make it straightforward to verify that the NCC is satisfied. We can then ask without fear of multiple equilibria whether the outcome of partnership formation leads to assortative mating, an issue that has been the subject of extensive empirical work.[26]

4.3.1 Partnerships based on comparative advantage

Becker (1965) studied the family as an economic institution that provides efficiency gains in production and consumption. Matched partners typically have different skills and productiveness, and so specialisation allows both to be better off than had they remained single. Thus we may think of a matched couple as a bilateral trading relationship: one partner "exports" a good or service to the other by specialising in it, producing more than he or she consumes, and in exchange "importing" another good. We model this using the tools of partial equilibrium trade theory, focusing on one good, X, with a second good, Y, in the background. Agent i has a demand or marginal benefit curve for X given by $a - bq_i^d$, and a supply or marginal cost curve $c_i + dq_i^s$, where q_i^d and q_i^s are the quantities consumed and produced by i. The values of the parameters a, b, and d are common across all agents, but c_i is specific to i, and is a (negative) measure of his or her productivity. If i remains single,

[25] For further references see Bergstom, (1997), Weiss, (1997), and Ermisch (2003).

[26] See for example, Dalmia and Lawrence (2001), Rose (2001), Jepsen and Jepsen (2002), and Ermisch, Francesoni, and Siedler, (2006).

$q_i^d = q_i^s$, with a shadow price $p_i = \frac{ad+bc_i}{b+d}$. If i and j form a partnership, they trade, with $q_i^d + q_j^d = q_i^s + q_j^s$. I assume that trade is efficient and that the gains from trade are split equally; then the benefits to both i and j from their partnership are[27]

$$(c_i - c_j)^2 \frac{b}{8d(b+d)}.$$

Thus an agent would most prefer a partner who is as different from him or her as possible, where "different" means having a different value of c.

Consider now a population of men and women, each of whom is characterised by a particular value of c. Each man can only "trade" with one woman, and vice versa. To find orderings that satisfy the NCC, we need only order M by increasing value of c and W by decreasing value of c, or M by decreasing value of c and W by increasing value of c; we thus have a unique stable matching. Alternatively, since $(c_i - c_j)^2$ is a submodular function of c_i and c_j, we can invoke Theorem 6.

Furthermore, it is straightforward to verify that the stable matching will display negative assortment; to see this, note that in any subpopulation, a fixed pair is formed either by the man with lowest c and the woman with the highest c, or by the woman with lowest c and the man with the highest c. This result is independent of the distribution of c in M and W, and forms a theoretical basis for those empirical studies that look for negative assortment with respect to wages and hours worked in the labour market.[28]

4.3.2 Partnerships based on joint consumption

Cohabitation offers substantial economies of scale and many goods are in effect jointly consumed. We now present a simple model, based on the analysis in Clark and Kanbur (2004), in which there are two household public goods and no private goods. Agents differ in their preferences, and a matched pair must decide on the allocation of their combined income between the two goods.

If a matched pair $(x, y) \in M \times W$ jointly consume quantities A and B of the two public goods, then they get utility $u_x = a_x \ln A + (1 - a_x) \ln B$ and $u_y = a_y \ln A + (1 - a_y) \ln B$ respectively, where $0 < a_x < 1$ and $0 < a_y < 1$. We assume that A and B are determined by maximising $u_x + u_y$, subject to

[27] The gains from trade can easily be calculated from a supply and demand diagram in which trade occurs at a price $p = (p_i + p_i)/2$.

[28] Such negative assortment arises if we interpret good X as a bundle of market commodities bought with wage income and good Y as household or domestically produced goods. Then an agent has a comparative advantage in X if they have a higher wage than their partner.

the constraint $A + B = 2R$, where R is each agent's income, the price of each good being taken by choice of units to be 1. Then the resultant utilities are

$$v_x = a_x \ln(a_x + a_y) + (1 - a_x)\ln(2 - a_x - a_y) + \ln R$$

and

$$v_y = g(a_x, a_y) = a_y \ln(a_x + a_y) + (1 - a_y)\ln(2 - a_x - a_y) + \ln R.$$

Thus each agent would prefer to be matched with someone with the same preferences.[29]

In such a population, where agents differ only in the value of a, the NCC is satisfied by ordering both M and W by increasing values of a (or by ordering both by decreasing values of a). Thus there is a unique stable matching. Alternatively, note that v_x and v_y are both supermodular functions of a_x and a_y, so Theorem 6 applies. In contrast to the comparative advantage model of the previous section, the pattern of assortment depends on the distribution of taste parameters in the population, so the model's empirical predictions are not so clear cut. Of course, tastes cannot be directly observed, but in applied work they are sometimes proxied by traits such as age, education, race, or class. There is a broad empirical consensus of positive sorting based on these characteristics (see for example, Rose, 2001, and Jepsen and Jepsen, 2002); as shown in Clark and Kanbur (2004), this suggests a significant overlap in the male and female distributions of traits.

5 Conclusion

Uniqueness of equilibrium is a generally regarded as a desirable characteristic of an economic model. It helps to make prediction and comparative statics unambiguous. The main theoretical result of this paper is that the standard model of two-sided matching has a unique stable matching if agents' preferences satisfy the No Crossing Condition (NCC). Although the NCC is stronger than Eeckhout's Sequential Preference Condition (SPC), it is both intuitive and easy to interpret, being based on the notion that a person's characteristics not only form the basis of their attraction to the opposite sex but also determine their own preferences. It requires that men and women be ordered

[29] For example, for a given value of a_x, v_x attains its maximimum of $v_x^* = a_x \ln(a_x) + (1 - a_x)\ln(1 - a_x) + \ln 2R$ if $a_y = a_x$. Note also that to a second order approximation, $v_x \approx v_x^* - \frac{(a_x - a_y)^2}{2}$.

on the basis of their characteristics, and that men further along the male ordering tend to prefer women further along the female ordering and *vice versa*. The paper also establishes a necessary and sufficient condition, $\alpha-$reducibility, weaker than the NCC but stronger than the SPC, for all subpopulations of P to have a unique stable matching.

An advantage of the NCC is that if it is satisfied by a population P then it is satisfied by any subpopulation of P, which will therefore also have a unique stable matching. One use of this property is to base preferences on utility functions with agents' characteristics as arguments. In applications, it may well then be straightforward to establish that any population with the given utility functions satisfies the NCC. The fruitfulness of this approach is illustrated in the context of household formation and marital sorting.

Finally, the NCC is closely related to well-known concepts of modularity and single crossing. These have been used by other authors to analyse sorting in matching with transferable utility, and comparative statics. When utility is not transferable, modularity delivers uniqueness but takes us no further, leaving open the comparative statics of matching problems with non-transferable utility, an interesting area for further research.

APPENDIX

Proof of Lemma 1. Take the orderings m and w that satisfy the NCC for the population P, delete those elements corresponding to $M\backslash M'$ and $W\backslash W'$ to form the n' dimensional vectors m' and w'. Since m' and w' must continue to satisfy conditions (i) and (ii) in Definition 1 they are orderings that satisfy the NCC for the population P'. ■

Proof of Lemma 2. If (i) is not satisfied then μ can be blocked by one of the fixed pairs of P and hence cannot be stable, a contradiction. It follows that μ maps F onto F and P' onto P'. This implies that the function μ' is a matching of P'. If (i) is satisfied but μ' is not stable, then there exists a pair $(x', y') \in M' \times W'$ (where $M' = M \cap P'$ and $W' = W \cap P'$), with $x' \neq \mu'(y)$, who can block the matching μ' i.e. $y' \succ_{x'} \mu'(x')$ and $x' \succ_{y'} \mu'(y')$. The definition of μ' implies that $\mu'(x') = \mu(x')$ for all $x' \in M'$ and $\mu'(y') = \mu(y')$ for all $y' \in W'$, so that $y' \succ_{x'} \mu(x')$ and $x' \succ_{y'} \mu(y')$. This means that the pair (x', y') can block the matching μ, and hence μ cannot be stable, a contradiction. ■

Proof of Lemma 3. For any man $x \in M$, let $q(x) \in W$ denote his preferred woman in W; i.e. $q(x) \succ_x y$ for all $y \in W\backslash \{q(x)\}$. Since preferences are complete and strict and the set W is finite, $q(x)$ exists and is unique. Similarly, for any woman $y \in W$, let $r(y) \in M$ denote her preferred man in M; i.e. $r(y) \succ_y x$ for all $x \in W \backslash \{r(y)\}$. $r(y)$ also exists and is unique. Let m and w be orderings of M and W satisfying the NCC. For each element of m the function q specifies an element of w; this in turn defines a function $\theta : I_n \rightarrow I_n$ as follows: if $q(m_k) = w_i$ then $\theta(k) = i$, which may be read as "the k^{th} man prefers the i^{th} woman". Compare the preferences of m_k and m_l, where $k < l$. If $q(m_k) = q(m_l)$ then $\theta(k) = \theta(l)$. If $q(m_k) \neq q(m_l)$, then $\theta(k) \neq \theta(l)$ and $w_{\theta(k)} \succ_{m_k} w_{\theta(l)}$ and $w_{\theta(l)} \succ_{m_l} w_{\theta(k)}$; but if $\theta(l) < \theta(k)$ then since $k < l$ this would contradict part (ii) of Definition 1 of the NCC, with $i = \theta(l)$ and $j = \theta(k)$. Hence if $k < l$ then $\theta(k) \leq \theta(l)$ i.e. the function θ is non-decreasing. A similar argument applies to the function $\gamma : I_n \rightarrow I_n$, defined as follows: if $r(w_i) = m_k$ then $\gamma(i) = k$; if $i < j$ then $\gamma(i) \leq \gamma(j)$ i.e. the function γ is non-decreasing. Now, consider the composition of θ and γ, the function $\rho(k) = \gamma(\theta(k))$; this gives the position, in the ordering m, of the preferred man of the k^{th} man's preferred woman (the k^{th} man's rival); i.e. $m_{\rho(k)} = r(q(m_k))$. Since θ and γ both map I_n into I_n and are non-decreasing the function ρ also maps I_n into I_n and is non-decreasing. It therefore has a fixed point $k^* = \rho(k^*)$. Let $i^* = \theta(k^*)$; then $k^* = \gamma(i^*)$; thus $w_{i^*} = q(m_{k^*})$ and $m_{k^*} = r(w_{i^*})$; i.e. m_{k^*} and w_{i^*} are a fixed pair. ∎

Proof of Theorem 2. We consider a sequence of populations $\{P_s\}, s = 1, ..., S$ such that $P_{s+1} = P_s \backslash F_s$, with $P_1 = P$, where F_s is the set of individuals who constitute the fixed pairs of P_s; i.e. if (x, y) is a fixed pair of P_s then $\{x, y\} \subseteq F_s$. S is defined by the condition $P_S = F_S$. Since F_s is unique given P_s, the sequence $\{P_s\}$ is uniquely defined. By the repeated application of Lemmas 1 and 3, each element in the sequence $\{P_s\}$ satisfies the No Crossing Condition and has at least one fixed pair. Thus P_{s+1} is a proper subset of P_s, and since P is finite S exists and is finite.

Let μ be any stable matching of P, and for all $s \in I_S$, let μ_s be a matching of P_s defined by $\mu_s(z) = \mu(z)$ for all $z \in P_s$; i.e. μ_s is the matching μ as it applies to the population P_s. Lemma 2 , part (ii), says that if μ_s is a stable matching of P_s then μ_{s+1} is a stable matching of P_{s+1}. But since $\mu = \mu_1$ is a stable matching of $P = P_1$, this implies that for all $s \in I_S$, μ_s is stable,. Then by Lemma 2, part (i), for all $s \in I_S$, $x = \mu_s(y)$ for any fixed pair (x, y) of P_s; and hence, given the definition of μ_s, we have $x = \mu(y)$ for any fixed pair (x, y) of P_s and for all $s \in I_S$. But since $P = \cup_{s=1}^{S} F_s$ every individual in P is a member of some fixed pair of some population P_s and is therefore matched by μ with the other member of the fixed pair. Since the sequence $\{P_s\}$ and the associated sequence $\{F_s\}$ are independent of the choice of which stable matching μ of P we consider, the matching μ is uniquely determined. ∎

Proof of Theorem 4. Clearly if (P, Ω) is $\alpha-$reducible then so is (P', Ω'),where P' is a subpopulation of P and $\Omega' = \{\Omega_i, i \in P'\}$. Then to prove the theorem, it is sufficient to prove that if (P, Ω) is $\alpha-$reducible it has a unique stable matching. As in the proof of Theorem 2, we consider a sequence of populations $\{P_s\}$, $s = 1, ..., S$ such that $P_{s+1} = P_s \backslash F_s$, with $P_1 = P$. Since each element in the sequence $\{P_s\}$ is $\alpha-$reducible it has at least one fixed pair. Thus P_{s+1} is a proper subset of P_s, and since P is finite S exists and is finite.

The proof now proceeds exactly as the second paragraph of the proof of Theorem 2. Since $P = \cup_{s=1}^{S} F_s$ every individual in P is a member of some fixed pair of some population P_s and, by Lemma 2, is matched by any stable matching μ of P with the other member of the fixed pair. Since the sequences $\{P_s\}$ and $\{F_s\}$ are independent of the choice of which stable matching μ of P we consider, the matching μ is uniquely determined. ■

Proof of Theorem 5. Suppose that (P, Ω) is not $\alpha-$reducible. Then there exists a subpopulation $\widetilde{P}$ of P that has no fixed pair. For any man x in $\widetilde{P}$ let $\phi(x)$ denote his most preferred woman in $\widetilde{P}$,and for any woman y in $\widetilde{P}$ let $\psi(y)$ denote her favourite man in $\widetilde{P}$;let $\zeta(x) = \psi(\varphi(x))$ and for $i \geq 0$ let $\zeta^{i+1}(x) = \zeta(\zeta^i(x))$, where $\zeta^0(x) = x$.Now, for any man x in $\widetilde{P}$ consider the following infinite sequence of men: $Q(x) = (\zeta^0(x), \zeta^1(x), \zeta^2(x), \zeta^3(x), ...)$.Since $\#(\widetilde{P})$ is finite, the sequence must at some point become periodic; i.e. there must exist finite numbers $n' \geq 0$ and $n'' \geq 1$ such that $\zeta^{n'}(x) = \zeta^{n'+n''}(x)$. Let $\widehat{n}(x) = \min\{n' | \zeta^{n'}(x) = \zeta^{n'+n''}(x)$ for some $n'' \geq 1\}$,(so elements of $Q(x)$ before $\zeta^{\widehat{n}(x)}(x)$ appear only once; the others are repeated), and let $\widetilde{n}(x) = \min\{n'' | \zeta^{\widehat{n}(x)}(x) = \zeta^{\widehat{n}(x)+n''}(x), n'' \geq 1\}$,(so $\widetilde{n}(x)$ is the eventual period of $Q(x)$). If $\widetilde{n}(x) = 1$ then $\zeta^{\widehat{n}(x)}(x) = \zeta^{\widehat{n}(x)+1}(x)$,which defines $\zeta^{\widehat{n}(x)}(x)$and $\phi(\zeta^{\widehat{n}(x)}(x))$ as a fixed pair of $\widetilde{P}$,a contradiction. Therefore $\widetilde{n}(x) \geq 2$.

Let $\widetilde{x}$ be some man in $\widetilde{P}$ such that $\widetilde{x} = \zeta^{\widehat{n}(\widetilde{x})}(x)$,so the set $\widetilde{M} = \{\widetilde{x}, \zeta(\widetilde{x}), \zeta^2(\widetilde{x}), ...$ $\zeta^{\widetilde{n}(x)-1}(\widetilde{x})\}$ consists of $\widetilde{n}$ different men, and the set of women $\widetilde{W} = \{\phi(\widetilde{x}), \phi(\zeta(\widetilde{x}))$, $\phi(\zeta^2(\widetilde{x})), ...\phi(\zeta^{\widetilde{n}-1}(\widetilde{x}))\}$ consists of $\widetilde{n}$ different women. By construction the subpopulation $P' = \widetilde{M} \cup \widetilde{W}$ has no fixed pairs, and at least two men and two women. Furthermore each man in $P'$is most preferred (out of all the men in P') by some woman in P',and each woman in $P'$is most preferred (out of all the women in P') by some man in P. Consider the following matching μ_M of P': each man is matched with his preferred woman ($\widetilde{x}$ is matched with $\phi(\widetilde{x})$, $\zeta(\widetilde{x})$with $\phi(\zeta(\widetilde{x}))$, so on). This is feasible, as no man prefers the same woman as any other man. It is also a stable matching, as in any pairing $(x', y') \in \widetilde{M} \times \widetilde{W}$, where $x' \neq \mu_M(y'), x'$ must be worse off as he is no longer matched with his preferred woman in $\widetilde{W}$. Thus (x', y') cannot block μ_M. Now consider the following matching μ_W of P': each woman is matched with her preferred man: $\phi(\widetilde{x})$is matched with $\zeta(\widetilde{x})$, $\phi(\zeta(\widetilde{x}))$with $\zeta^2(\widetilde{x})$and so on, and $\phi(\zeta^{\widetilde{n}-1}(\widetilde{x}))$ is matched with $\psi(\phi(\zeta^{\widetilde{n}-1}(\widetilde{x}))) = \zeta^{\widetilde{n}}(\widetilde{x}) = \widetilde{x}$. This is feasible, as no woman prefers the same man as any other man. It is also a stable matching, as in any pairing $(x', y') \in \widetilde{M} \times \widetilde{W}$, where $x' \neq \mu_W(y'), y'$ must be worse off as she is no longer matched with her preferred man in $\widetilde{M}$. Thus (x', y') cannot block μ_W.As $\widetilde{n}(x) \geq 2, \mu_M$ and μ_M are different, so P' does not have a unique stable matching. ■

Proof of Theorem 6. To prove the theorem we prove that if Assumption 1 is satisfied and f and g are either both supermodular or both submodular then we can find orderings of M and W that satisfy the NCC. We begin by ordering M and W by a and b respectively. Let m and w be orderings of M and W such that if $i < j$ and $k < l$ then $a_{m_i} \leq a_{m_j}$ and $b_{w_k} \leq b_{w_l}$; note that if some men (or women) have the same values of a (or b), then these orderings are not unique.

Let $i < j$ and $k < l$ and suppose that f is supermodular. Note that if $a_1 = a_2$ or $b_1 = b_2$ then $f(a_2, b_2) - f(a_2, b_1) = f(a_1, b_2) - f(a_1, b_1)$. Since $a_{m_i} \leq a_{m_j}$ and $b_{w_k} \leq b_{w_l}$, then

$$f(a_{m_j}, b_{w_l}) - f(a_{m_i}, b_{w_l}) \geq f(a_{m_j}, b_{w_k}) - f(a_{m_i}, b_{w_k}).$$

If $w_l \succ_{m_i} w_k$ then

$$f(a_{m_i}, b_{w_l}) + \phi(\boldsymbol{\beta}_{w_l}) > f(a_{m_i}, b_{w_k}) + \phi(\boldsymbol{\beta}_{w_k}).$$

Hence

$$f(a_{m_j}, b_{w_l}) + \phi(\boldsymbol{\beta}_{w_l}) > f(a_{m_j}, b_{w_k}) + \phi(\boldsymbol{\beta}_{w_k}).$$

so $w_l \succ_{m_j} w_k$. Thus supermodularity of f implies that it cannot be the case that both $w_l \succ_{m_i} w_k$ and $w_k \succ_{m_j} w_l$. By a similar argument, supermodularity of g implies that it cannot be the case that both $m_j \succ_{w_k} m_i$ and $m_i \succ_{w_l} m_j$. Thus if f and g are both supermodular, the orderings m and w satisfy the NCC.

Again let $i < j$ and $k < l$ but suppose now that f is submodular. Then

$$f(a_{m_j}, b_{w_l}) - f(a_{m_i}, b_{w_l}) \leq f(a_{m_j}, b_{w_k}) - f(a_{m_i}, b_{w_k}).$$

If $w_k \succ_{m_i} w_l$ then

$$f(a_{m_i}, b_{w_l}) + \phi(\boldsymbol{\beta}_{w_l}) < f(a_{m_i}, b_{w_k}) + \phi(\boldsymbol{\beta}_{w_k}).$$

Hence

$$f(a_{m_j}, b_{w_l}) + \phi(\boldsymbol{\beta}_{w_l}) < f(a_{m_j}, b_{w_k}) + \phi(\boldsymbol{\beta}_{w_k}).$$

Thus submodularity of f implies that it cannot be the case that both $w_k \succ_{m_i} w_l$ and $w_l \succ_{m_j} w_k$. Similarly, submodularity of g implies that it cannot be the case that both $m_i \succ_{w_k} m_j$ and $m_j \succ_{w_l} m_i$. Consider now the orderings m and w', where m is as defined above and w' orders W by decreasing value of b, so that if $i < j$ and $k < l$ then $a_{m_i} \leq a_{m_j}$ and $b_{w'_k} \geq b_{w'_l}$. By construction, $w_k = w'_{n+1-k}$

and $w_l = w'_{n+1-l}$, so the submodularity of f implies that it cannot be the case that both $w'_{n+1-k} \succ_{m_i} w'_{n+1-l}$ and $w'_{n+1-l} \succ_{m_j} w'_{n+1-k}$ and the submodularity of g implies that it cannot be the case that both $m_i \succ_{w'_{n+1-k}} m_j$ and $m_j \succ_{w'_{n+1-l}} m_i$. Therefore if f and g are both submodular the orderings m and w' satisfy the NCC. This completes the proof. ■

Remark: It is simple to show that if f and g are both submodular the orderings m' and w (where $m_i = m'_{n+1-i}$) also satisfy the NCC, as do m' and w' in the case where f and g are both supermodular.

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