INTERNATIONAL ECONOMIC REVIEW
Vol. 59, No. 1, February 2018

ON STRATEGY-PROOFNESS AND THE SALIENCE OF SINGLE-PEAKEDNESS*

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We consider strategy-proof rules operating on a rich domain of preference profiles. We show that if the rule satisfies in addition tops-onlyness, anonymity, and unanimity, then the preferences in the domain have to satisfy a variant of single-peakedness (referred to as semilattice single-peakedness). We do so by deriving from the rule an endogenous partial order (a semilattice) from which the concept of a semilattice single-peaked preference can be defined. We also provide a converse of this main finding. Finally, we show how well-known restricted domains under which nontrivial strategy-proof rules are admissible are semilattice single-peaked domains.

1. INTRODUCTION

Strategy-proofness plays a central role in mechanism design. A social choice function (SCF) is strategy-proof if, for every preference profile, truth telling is a dominant strategy in its induced game form. Hence, the potentially complex strategic decision problems of agents involved in a strategy-proof SCF are extremely simple, indeed, for whether or not an agent's strategy is dominant depends only on the preferences of the agent and not on the other agents' preferences. Under strategy-proofness, the interlinked decisions become a collection of independent optimization problems. Thus, the use of a strategy-proof SCF does not require (as any other solution concept related to Nash equilibrium would) any informational hypothesis about the beliefs that each agent holds about the other agents' preferences, and the subsequent iteration of beliefs until the preference profile becomes common knowledge. However, the Gibbard–Satterthwaite theorem states that requiring truthful reporting of preferences in weakly dominant strategies implies dictatorship whenever preferences of agents are unrestricted. This fundamental result has directed subsequent research on social choice in the presence of private information toward suitably restricted domains of preferences that permit the design of anonymous, and hence nondictatorial, strategy-proof SCFs. Particularly prominent in this regard is the class of single-peaked preferences and its variants, and the strategy-proof SCFs characterized for such domains are extensions of the median voter scheme.[2] Single-peaked preferences are well known to have desirable properties in the context of aggregation theory. They also provide the underpinnings of many models in political and public economics.[3]

*Manuscript received January 2016; revised July 2016.

[1] We would like to thank Huaxia Zeng for helpful discussions and detailed comments, Salvador Barberà for insightful comments, and Masaki Aoyagi and three referees of the journal for very helpful and constructive comments. Chatterji would like to acknowledge research support from SMU grant number C244/MSS13E001 and thank KIER, Kyoto University for its hospitality. Massó would like to acknowledge financial support from the Spanish Ministry of Economy and Competitiveness, through Grant ECO2014-53051 and the Severo Ochoa Programme for Centers of Excellence in R&D (SEV-2015-0563), and from the Generalitat de Catalunya, through the research Grant SGR2014-515. Please address correspondence to: Jordi Massó, Departament d'Economia i d'Història Econòmica, Edifici B, UAB Campus, 08193 Cerdanyola del Vallès (Bellaterra), Spain. E-mail: *jordi.masso@uab.es*.

[2] Single-peakedness was initially proposed by Black (1948) and Inada (1964). The surveys of Barberà (2001, 2010) and Sprumont (1995) contain several axiomatic characterizations of the median voter scheme and its extensions.

[3] See Austen-Smith and Banks (1999, 2005).

Single-peaked preferences have been specified by postulating an underlying structure on the set of alternatives that allows one to state for every triple x, y, and z of alternatives, that y is between x and z, and so on, and the restriction imposed by single-peakedness is that if x is top-ranked for a particular preference ordering, then y, by virtue of being in between x and z, be ranked at least as high as z. This article formulates a more general concept of single-peakedness in terms of a partial order on the set of alternatives with the property that every pair of alternatives possesses a supremum under the postulated partial order.[4] Our concept of single-peakedness requires that for any triple x, y, and z of alternatives, a preference ordering that has x as its top-ranked alternative should rank the supremum of the pair (x, y) at least as high as the supremum of the pair (z, y).[5]

Our main finding is that this concept of single-peakedness is *implied* by the existence of a strategy-proof and anonymous SCF that is determined completely by the profile of the agents top-ranked alternatives (i.e., is tops-only) and satisfies additionally the innocuous requirement of unanimity whenever such a SCF can be defined for an even number of agents and the underlying domain satisfies a *richness* requirement.[6] Our approach reconstructs the partial order on alternatives in a natural way from the SCF with the four stated properties. Observe that the partial order depends on the particular SCF under consideration, and, hence, the derived concept of single-peakedness may differ across different SCFs. Our methodology applies to domains that allow the design of *well-behaved* SCFs for any even number of voters. Although this restriction to an even number of voters is somewhat awkward, we do not necessarily view it as a drawback of our approach, given that our intention is to reconstruct features of a domain of preferences that allows the design of well-behaved SCFs for all societies; indeed, while our methodology would not identify a domain that allows well-behaved SCFs to be designed only for societies with an odd number of agents, one might argue that such a domain would be too specific. The semilattice single-peaked condition identified by our methodology suffices for the design of well-behaved strategy-proof SCFs for all, in particular odd, numbers of agents.

Fix a tops-only and unanimous SCF. Assume the number of agents is two and let x and y be two alternatives. We say that $x \succeq y$ if and only if x is chosen at any profile of preferences where one agent has x as the top-ranked alternative and the other y. The assumed axioms of unanimity and anonymity imply that $\succeq$ is reflexive and antisymmetric, respectively. Our requirement that the domain of preferences be rich ensures that $\succeq$ is transitive and that the SCF must be of a particular form: At any profile of preferences, the social choice *is* the supremum of the pair of alternatives that are top-ranked by the two agents. Our definition of single-peakedness now obtains as a direct consequence of strategy-proofness. This methodology applies whenever the number of agents is even. A similar finding holds under an additional axiom of invariance when a SCF with the aforementioned properties can be defined only for an odd number of agents. As a converse to our main finding, we show that any domain of preferences (there is no richness requirement) that is semilattice single-peaked with respect to a partial order possessing the supremum property admits a strategy-proof, anonymous, and unanimous SCF that is completely determined by the profile of the agents top-ranked alternatives for any number of agents.

In the literature on social choice on restricted domains, there has been interest in formulating a sort of converse to the Gibbard–Satterthwaite theorem; a statement that would identify features of a domain that are implied by the design of a unanimous, strategy-proof SCF that is “nondictatorial.” It has been conjectured that domain restrictions of the single-peaked variety and SCFs of the median voter scheme form are salient in this regard.[7] We formalize a nondictatorial SCF using the axiom of anonymity and require additionally that the SCF satisfy the tops-only property. For the complete domain, strategy-proofness and unanimity imply the

[4] A partial order is a reflexive, antisymmetric, and transitive binary relation.

[5] Later, in the article, we explain this property and discuss why it may be seen as a weakening of single-peakedness.

[6] Most well-known SCFs identified in the restricted domain literature generate binary relations that allow interesting preference domains to satisfy our richness requirement.

[7] Conjectures of this nature have been attributed by Barberà (2010) to Faruk Gul and referred to as Gul’s conjecture.

tops-only property. Given that we work in a restricted domain setting with no structure on the set of alternatives, it does not appear feasible to derive the tops-only property as a consequence of strategy-proofness and unanimity; we accordingly impose it as an axiom. Our methods lead to a simple and fairly general version of a statement to the effect that a particular form of single-peakedness is implied by strategy-proofness in conjunction with anonymity and other natural axioms and that this form of single-peakedness suffices for the design of SCFs with these properties. In particular, the semilattice structure on alternatives arises endogenously as it is implied by the axioms on the SCF and a richness condition relative to the SCF and does not rely on any a priori structure on the set of alternatives or preferences (apart from the requirement that each preference has a unique top-ranked alternative).

1.1. *Related Literature.* An early formulation of a partial converse statement to the Gibbard–Satterthwaite theorem is Bogomolnaia (1998). In a model with finitely many alternatives and two agents, she identifies the features of any anonymous and tops-only SCF under which the finite set of alternatives can be embedded into a finite-dimensional Euclidean space with a grid structure with the property that the SCF takes the form of a (multidimensional) median voter scheme. This embedding depends crucially on the set of alternatives being finite. These features of tops-only and anonymous SCFs are stated in terms of the same binary relation induced by a two-agent tops-only and anonymous SCF that we use in our study and are the following: (i) the binary relation is transitive and a semilattice and (ii) the SCF is the supremum of the pair of alternatives that are the top-ranked alternatives of the two agents. These findings are extended to the three-agent case under similar, but somewhat more demanding, hypotheses, and she derives additionally that the domain of preferences must be multidimensional single-peaked on the set of alternatives. Our work extends this methodology in the following sense: We postulate a richness condition on the domain in terms of the binary relation on alternatives induced by a two-agent SCF satisfying our axioms and *derive* that the binary relation is transitive and that the SCF has the supremum property. This is used to establish the salience of the supremum rule and a version of single-peaked preferences in a general setting with an arbitrary number of agents without requiring the set of alternatives to be finite. In particular, under our richness condition, the set of alternatives need not turn out to be embedded in a finite-dimensional Euclidean space with a grid structure as in Bogomolnaia (1998), but the identification of the SCF as a supremum rule on our version of a single-peaked domain remains valid.[8]

More recently, work by Nehring and Puppe (2007a, 2007b, 2010) and Chatterji et al. (2013) provides formulations of such a converse statement. Our article complements these approaches and is closely related to the approach of these papers in that our axioms on the SCF are similar. However, there are important differences in the scope of our model and our methodology. The richness condition in these papers is specified independently of the SCF whose existence is postulated, whereas in our article, the richness condition is specified in relation to the SCF. But more importantly, the methodology in these papers strongly relies also on the finiteness of the set of alternatives and on strict preferences. The approach of Nehring and Puppe (2007a, 2007b, 2010) assumes a specific structure on the finite set of alternatives by means of a given property space from which a betweenness relation can be derived. Again, their approach assumes that the set of alternatives is finite and it is endowed with an a priori structure, the property space. The richness condition in Chatterji et al. (2013) is specified in terms of alternatives that appear as the first- and second-ranked alternatives in different preference orderings that make it specific to a model with finitely many alternatives with strict preferences and also exclude the consideration of preferences commonly employed in the study of multidimensional models. Our formulation is more permissive in that we impose no finiteness requirement on the set of alternatives and, provided the top-ranked alternative is unique, we admit indifferences. As a consequence, our

[8] Observe that neither Bogomolnaia (1998) nor we pretend to characterize a subclass of strategy-proof SCFs on a given restricted domain of preferences. Rather, the objective is to identify the key property of any domain that admits a well-behaved and strategy-proof SCF.

methodology is of necessity different from and somewhat more direct than that of those papers. Many prominent restricted domains of preferences studied in the literature appear as special cases of our formulation.

The article is organized as follows: Section 2 introduces basic definitions and notation, whereas Section 3 contains the main results for the case of an even number of agents. In Section 4, we relate our results to the large literature on domain restrictions for nontrivial strategy-proof SCFs. Section 5 elaborates on our methodology and axioms and gathers some final remarks. An appendix contains an analysis of the case of an odd number of agents, the proofs of two results omitted in the main text, and the case of a finite set of alternatives.

2. BASIC DEFINITIONS AND NOTATION

Let $N = \{1, \ldots, n\}$ be the finite set of *agents*, with $n \geq 2$, and A be any set of *alternatives*. We do not assume any a priori structure on the set of alternatives. Each agent $i \in N$ has a preference (relation) $R_i \in \mathcal{D}$ over A, where $\mathcal{D}$ is a subset of complete, reflexive, and transitive binary relations on A.[9] The set $\mathcal{D}$ is referred to as the *domain* of preferences. For any $x, y \in A$, xR_iy means that agent i considers alternative x to be at least as good as alternative y. Let P_i and I_i denote the strict and indifference relations induced by R_i over A, respectively; namely, for any $x, y \in A$, xP_iy if and only if xR_iy and $\neg yR_ix$, and xI_iy if and only if xR_iy and yR_ix. We assume that for each $R_i \in \mathcal{D}$, there exists $t(R_i) \in A$, the *top* of R_i, such that $t(R_i)P_iy$ for all $y \in A\backslash\{t(R_i)\}$. For $x \in A$, let R_i^x denote any preference in $\mathcal{D}$ with $t(R_i^x) = x$. Moreover, we assume that for each $x \in A$, the domain $\mathcal{D}$ contains at least one preference R_i^x. A *profile* $R = (R_1, \ldots, R_n) \in \mathcal{D}^n$ is an n-tuple of preferences, one for each agent. To emphasize the role of agent i, we will often write the profile R as (R_i, R_{-i}).

An *SCF* is a mapping $f : \mathcal{D}^n \to A$ that assigns to every profile $R \in \mathcal{D}^n$ an alternative $f(R) \in A$.

An SCF $f : \mathcal{D}^n \to A$ is *tops-only* if for all $R, R' \in \mathcal{D}^n$ such that $t(R_i) = t(R_i')$ for all $i \in N$, $f(R) = f(R')$. Hence, a tops-only SCF $f : \mathcal{D}^n \to A$ can be written as $f : A^n \to A$. Accordingly, we will on occasion use the notation $f(t(R_1), \ldots, t(R_n))$ interchangeably with $f(R_1, \ldots, R_n)$.

An SCF $f : \mathcal{D}^n \to A$ is *unanimous* if for all $R \in \mathcal{D}^n$ and $x \in A$ such that $t(R_i) = x$ for all $i \in N$, $f(R) = x$.

To define an anonymous SCF on $\mathcal{D}^n$, for every profile $R \in \mathcal{D}^n$ and every one-to-one mapping $\sigma : N \to N$, define the profile $R^\sigma = (R_{\sigma(1)}, \ldots, R_{\sigma(n)})$ as the $\sigma-$permutation of R, where for all $i \in N$, $R_{\sigma(i)}$ is the preference that agent $\sigma(i)$ had in the profile R. Observe that the domain $\mathcal{D}^n$ is closed under permutations, since it is the Cartesian product of the same set $\mathcal{D}$. An SCF $f : \mathcal{D}^n \to A$ is *anonymous* if for all one-to-one mappings $\sigma : N \to N$ and all $R \in \mathcal{D}^n, f(R^\sigma) = f(R)$.

An SCF $f : \mathcal{D}^n \to A$ is *strategy-proof* if for all $i \in N$, all $R \in \mathcal{D}^n$, and all $R_i' \in \mathcal{D}$,

$$f(R)R_if(R_i', R_{-i}).$$

An SCF f is strategy-proof if for every agent at every preference profile R, truth telling is a weakly dominant strategy in the direct revelation game induced by f at R.

In this article, in addition to strategy-proofness, we will require the SCF to satisfy anonymity. This is a key assumption in our analysis and is in some ways an opposite of dictatorship, as the identity of no particular agent matters in determining the social outcome. The appeal of this axiom is obvious. In addition, we will impose that the SCF also satisfy the tops-only requirement. This axiom simplifies considerably the specification of the SCF, as well as the act of reporting preferences and checking that there are no gainful manipulations and is pervasive in the literature on the characterization of strategy-proof SCFs on restricted domains. This axiom has some normative appeal, and it is of interest to study what sort of preference domains permit the design of a strategy-proof SCF that is also tops-only. We discuss the role of

[9] A binary relation $\succeq$ over A is *complete* if for all $x, y \in A$ either $x \succeq y$ or $y \succeq x$, it is *reflexive* if for all $x \in A$, $x \succeq x$, and it is *transitive* if for all $x, y, z \in A$, $[x \succeq y$ and $y \succeq z] \Rightarrow [x \succeq z]$.

this axiom further in Section 5. The axiom of unanimity is natural to impose and is mild, as it follows as a consequence of strategy-proofness whenever the SCF is required to be onto the set of alternatives.

3. RESULTS

3.1. *Obtaining the Induced Binary Relation.* In this subsection, we assume that $n = 2$ and indicate how to obtain a binary relation $\succeq$ from a tops-only SCF $f : \mathcal{D}^2 \to A$ and show that if the SCF satisfies in addition unanimity and anonymity, then $\succeq$ is reflexive and antisymmetric.[10] In doing so, we follow a procedure introduced by Bogomolnaia (1998).

Let $f : \mathcal{D}^2 \to A$ be a tops-only SCF. Define the binary relation $\succeq$ induced by f over A as follows: for all $x, y \in A$,

(1) $$x \succeq y \text{ if and only if } f(x, y) = x.$$

An SCF aggregates individual preferences and can be seen as a systematic procedure specifying how a society resolves its members' disagreements. Hence, the binary relation $\succeq$ induced by an SCF f over A may be interpreted as the outcome of this procedure applied to the family of basic situations in which there are only two agents; the relation $x \succeq y$ reflects the fact that in this scenario, the alternative x prevails over y.[11] We will show later that if the SCF f is strategy-proof, tops-only, and anonymous, then its induced binary relation $\succeq$ is transitive, provided that the domain of f satisfies a richness condition. Here, we note that the following result is immediate.

REMARK 1. Let $f : \mathcal{D}^2 \to A$ be a tops-only SCF and $\succeq$ be the binary relation induced by f over A. If f is unanimous, then $\succeq$ is reflexive. If f is anonymous, then $\succeq$ is antisymmetric.

Although this construction of the binary relation $\succeq$ induced by $f : \mathcal{D}^2 \to A$ over A might seem very specific to the two agent case, we can extend this methodology to the case where n is any positive even integer as follows.

Given a strategy-proof, tops-only, and anonymous SCF $g : \mathcal{D}^n \to A$ where n is a positive even integer, we start by stating the following fact that appears as proposition 2 in Chatterji et al. (2013).

FACT 1. *Let $\mathcal{D}$ be an arbitrary domain and let n be a positive even integer. Suppose there exists a strategy-proof, tops-only, and anonymous SCF $g : \mathcal{D}^n \to A$. Let $N_1 = \{1, \dots, \frac{n}{2}\}$ and $N_2 = \{\frac{n}{2} + 1, \dots, n\}$. Then, SCF $f : \mathcal{D}^2 \to A$, defined by setting, for all $(R_1, R_2) \in \mathcal{D}^2$, $f(R_1, R_2) = g(\bar{R})$ where $\bar{R} \in \mathcal{D}^n$ is such that $\bar{R}_j = R_1$ for all $j \in N_1$ and $\bar{R}_j = R_2$ for all $j \in N_2$, is strategy-proof, tops-only, and anonymous. Moreover, if g is unanimous, then so is f.*

In view of Fact 1, we say that a strategy-proof, tops-only, anonymous, and unanimous SCF $g : \mathcal{D}^n \to A$, where n is a positive even integer, *induces a binary relation $\succeq$ over A*, where it is understood that $\succeq$ is the binary relation induced by f over A where f is *induced* from g by "cloning" the first $\frac{n}{2}$ agents as agent 1 and the remaining as agent 2.

3.2. *An Illustration of the Main Result.* We use the prominent instance of a median voter rule defined on a domain of single-peaked preferences (originally proposed by Black, 1948, and studied by Moulin, 1980) to illustrate our main finding and summarize what the study tries to accomplish. Following Moulin (1980), assume that the set of alternatives is the unit interval in the real line endowed with the linear order $>$; that is, $A = [0, 1]$. A preference R_i is *single-peaked*

[10] A binary relation $\succeq$ over A is *antisymmetric* if for all $x, y \in A$, $[x \succeq y$ and $y \succeq x] \Rightarrow [x = y]$.

[11] Since the binary relation is not required to be complete, it may be the case that neither alternative prevails over the other and $f(x, y)$ is a third alternative z.

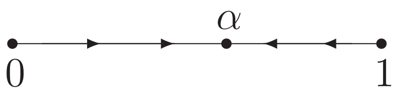

FIGURE 1

THE SEMILATTICE OF A MEDIAN VOTER RULE

on A if there exists a unique alternative $t(R_i) \in A$ such that, for all $x \in A \backslash \{t(R_i)\}$, $t(R_i)P_i x$, and for all $x, y \in A$, $xR_i y$ whenever either $t(R_i) \geq x > y$ or $y > x \geq t(R_i)$. Let SP be the set of all single-peaked preferences on A. An SCF $f : SP^2 \to A$ is a *median voter rule* if there exists a fixed ballot $\alpha \in A$ such that for all $(R_1, R_2) \in SP^2$,[12]

$$f(R_1, R_2) = med_{>}(t(R_1), t(R_2), \alpha).$$

A characterization result in Moulin (1980) implies that any strategy-proof, tops-only, anonymous, and unanimous SCF $f : SP^2 \to A$ is a median voter rule.

So, assume the SCF $f : SP^2 \to A$ is a median voter rule and let α be its associated fixed ballot. Since a median voter rule depends only on the top-ranked alternatives of the agents' preferences, it will be convenient to write $f(R_1, R_2)$ as $f(x, y)$ where $x = t(R_1)$ and $y = t(R_2)$. We now apply condition (1) to f to generate a binary relation on A by saying that for all $x, y \in A, x \succeq y$ if and only if $f(x, y) = x$. Since $f(x, x) = x$ (the median voter rule is unanimous), $\succeq$ is reflexive and since $f(x, y) = f(y, x)$ for all x, y, $\succeq$ is also antisymmetric. It is, however, not complete since if x and y lie on opposite sides of α, $f(x, y) = \alpha$, and so $x \not\succeq y$ and $y \not\succeq x$. Furthermore, the domain of single-peaked preferences satisfies our richness condition (this is formally defined in Subsection 3.3), and this will imply that $\succeq$ is transitive. As a consequence, $\succeq$ is a partial order and (we will prove that in general) for every pair $x, y \in A$, $\sup_{\succeq}(x, y)$ exists and is given by $f(x, y)$, so that $\succeq$ is a semilattice. Hence, $f(x, y) = med_{>}(x, y, \alpha) = \sup_{\succeq}(x, y)$. Figure 1 gives a geometric representation of this semilattice, where the arrows indicate the direction of the partial order $\succeq$ on the interval $[0, 1]$, so $y \succeq x$ whenever x and y are such that either $0 \leq x < y \leq \alpha$ or $\alpha \leq y < x \leq 1$.

Our main result will show that for the strategy-proofness of the median voter rule f, one does not actually need the domain of preferences to be single-peaked; it may be larger. To see this, suppose agent 2 has the top-ranked alternative y and agent 1's true preference puts x on top. Strategy-proofness requires that $f(x, y)R_i^x f(z, y)$ for all $z \in A$, which is equivalent to the requirement that $\sup_{\succeq}(x, y)R_i^x \sup_{\succeq}(z, y)$. This last condition is our concept of semilattice single-peakedness. Figure 2 illustrates a semilattice single-peaked preference R_i on $(A, \succeq)$ when $\sup_{\succeq} A = \alpha$. Observe four features of R_i. First, R_i is far from being single-peaked on A. Second, R_i is monotonically (not necessarily strictly) decreasing on the segment $[t(R_i), \alpha]$, and hence single-peaked on it, for should there exist $y, z \in (t(R_i), \alpha)$ such that $y < z$ and $zP_i y$, then $f(z, y) = zP_i y = f(t(R_i), y)$, a manipulation. Third, no condition is imposed between pairs on $[0, t(R_i))$. Fourth, $\alpha R_i x$ for each alternative $x \in (\alpha, 1]$ and no condition is imposed between pairs of alternatives on this segment. The reason underlying the last two conditions is that if $t(R_i) < \alpha$, then $f(t(R_i), y) \in [t(R_i), \alpha)]$, and hence how R_i orders pairs of alternatives that are either below $t(R_i)$ or above α is irrelevant for the manipulability of f.

We establish the following general version of the configuration presented above: If a preference domain admits a strategy-proof, tops-only, anonymous, and unanimous SCF for an even number of agents, then the preferences must be semilattice single-peaked. We also establish that given a semilattice single-peaked domain of preferences, we can define a strategy-proof, tops-only, anonymous, and unanimous SCF for any number of voters.

[12] Given a list of K real numbers $(x_1, \ldots, x_K)$, where K is a positive odd integer, define $med_{>}(x_1, \ldots, x_K) = y$, where $y \in \mathbb{R}$ is such that $\#\{t \in \{1, \ldots, K\} \mid x_t \geq y\} = \#\{t \in \{1, \ldots, K\} \mid x_t \leq y\} = \frac{K+1}{2}$.

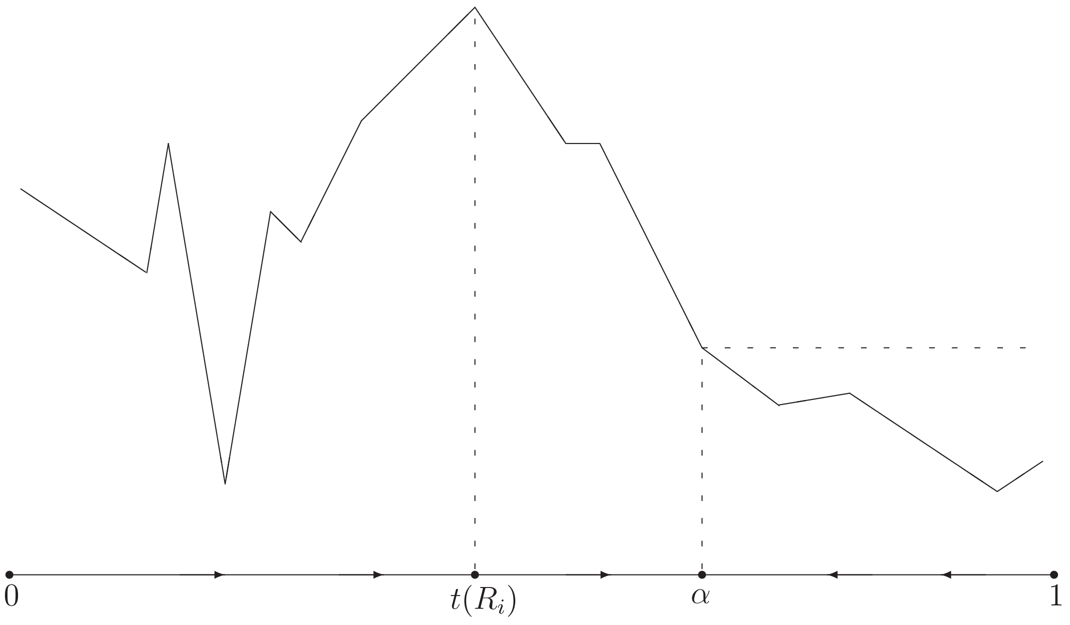

FIGURE 2

A SEMILATTICE SINGLE-PEAKED PREFERENCE

3.3. *Rich Domains and Semilattice Single-Peaked Preferences.* We now turn to a description of the domain of preferences that we characterize in this article. First, we present the concept of a rich domain on a set of alternatives endowed with a binary relation. Fix a binary relation $\succeq$ over A. Given two alternatives $x, y \in A$ with $y \succeq x$, define the set $[x, y]$ as

$$[x, y] = \{x, y\} \cup \{z \in A \mid y \succeq z \text{ and } z \succeq x\}.$$

If x and y are distinct alternatives and related by $\succeq$ as $y \succeq x$, then the set $[x, y]$ is obtained by adding to the set $\{x, y\}$ all alternatives in A that "lie between" x and y according to $\succeq$. For $y \not\succeq x$, define $[x, y] = \emptyset$.

DEFINITION 1. Let $\succeq$ be a binary relation over A. The domain $\mathcal{D}$ is *rich* on $(A, \succeq)$ if for all $x, y \in A$ with $[x, y] \neq \emptyset$ and $z \notin [x, y]$, there exist $R_i^x, R_i^y \in \mathcal{D}$ such that $yP_i^x z$ and $xP_i^y z$.

Richness says that for any pair of distinct alternatives x and y related by $\succeq$ and any alternative z not lying between x and y, a rich domain has to contain two preference relations with the properties that, for one of the preferences, x is the top-ranked alternative and y is strictly preferred to z and for the other preference, y is the top-ranked alternative and x is strictly preferred to z. Our concept of a rich domain is relative to the binary relation induced by the SCF that is applied to it. Thus, whether or not a domain $\mathcal{D}$ is rich depends on the particular SCF $f : \mathcal{D}^2 \to A$ operating on it. Below, we will illustrate the concept of rich domain by means of an example.

We now exhibit conditions under which $\succeq$ is transitive.

LEMMA 1. *Let $(\mathcal{D}, A, \succeq)$ be such that $\mathcal{D}$ is rich on $(A, \succeq)$. If there exists an SCF $f : \mathcal{D}^2 \to A$ that induces $\succeq$ over A and is strategy-proof, tops-only, and anonymous, then $\succeq$ is transitive.*

PROOF. Assume the three distinct alternatives $x, y, z \in A$ are such that $x \succeq y$ and $y \succeq z$. We show that $x \succeq z$; namely, $f(x, z) = x$. First, suppose $f(x, z) = w \notin \{x, y\}$. By strategy-proofness,

$f(x, w) = w$. Hence, $w \succeq x \succeq y$ and $w \notin [y, x] \neq \emptyset$. Since $\mathcal{D}$ is rich on $(A, \succeq)$, there exists $R_1^x \in \mathcal{D}$ such that $yP_1^x w$. But then,

$$f(y, z) = yP_1^x w = f(x, z),$$

a contradiction with strategy-proofness of f. Thus, $f(x, z) \in \{x, y\}$. Assume $f(x, z) = y$. But then, by strategy-proofness, $f(x, y) = y$, a contradiction with $x \succeq y$. Hence, $f(x, z) = x$ and $x \succeq z$. Thus, $\succeq$ is transitive. ■

A *partial order* $\succeq$ over A is a reflexive, antisymmetric, and transitive binary relation over A. A partial order $\succeq$ over A is a *(join-)semilattice* if for all $(x, y) \in A \times A$, $\sup_{\succeq}(x, y)$ exists.[13] In Lemma 2 below, we will establish that the binary relation $\succeq$ induced by f on A is a (join-) semilattice, provided that f is strategy-proof, tops-only, anonymous, and unanimous and $\mathcal{D}$ is rich on $(A, \succeq)$. We now turn to our concept of a single-peaked preference in this setting.

DEFINITION 2. Let $\succeq$ be a semilattice over A. The preference $R_i^x \in \mathcal{D}$ is *semilattice single-peaked on* $(A, \succeq)$ if for all $y, z \in A$, $\sup_{\succeq}(x, y) R_i^x \sup_{\succeq}(z, y)$.

We say that a domain $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq)$ if it is a subset of all semilattice single-peaked preferences on $(A, \succeq)$.

Single-peaked preferences embody the idea that an alternative y that is "closer" to the top x of a preference ordering R_i^x than is an alternative z should be ranked at least as high as z. We now argue that semilattice single-peakedness embodies in some measure this idea in its treatment of those pairs of alternatives that arise as suprema under the semilattice $\succeq$. Given a triple of alternatives x, y, z, we say that y is "closer" to x than is z according to the semilattice $\succeq$ if $x \preceq y$ holds and $x \preceq z \preceq y$ (equivalently, $z \in [x, y]$) does not hold. Now consider any preference $R_i^x \in \mathcal{D}$ and consider any pair of alternatives y, z. Assume first that $\sup_{\succeq}(z, y) \succeq x$. Then, we have $x \preceq \sup_{\succeq}(x, y) \preceq \sup_{\succeq}(z, y)$ holds, so that $\sup_{\succeq}(x, y)$ is closer to the top x of R_i^x than is $\sup_{\succeq}(z, y)$. Even when the condition $\sup_{\succeq}(z, y) \succeq x$ does not hold, we have at any rate that $\sup_{\succeq}(z, y) \notin [x, \sup_{\succeq}(x, y)]$, and here too $\sup_{\succeq}(x, y)$ is closer to the top x of R_i^x than is $\sup_{\succeq}(z, y)$. Indeed, the condition of semilattice single-peakedness requires that in this situation, $\sup_{\succeq}(x, y)$ being closer to the top x, should be ranked by R_i^x at least as high as $\sup_{\succeq}(z, y)$.

To better understand the concepts of richness and semilattice single-peakedness on $(A, \succeq)$, it is convenient to look at the semilattice $(A, \succeq)$ as a partially directed graph. To make the argument more transparent, assume A is finite and that $\sup_{\succeq} A$ exists and is denoted by α. Figure 3 represents an example of such a semilattice $(A, \succeq)$ as a partially directed graph, where $A = \{x, y, z, \alpha, x_1, \ldots, x_{13}\}$, and the direction of an arrow on the edge linking two alternatives indicates how they are related according to the partial order $\succeq$; for example, $x \longrightarrow y$ means that $y \succeq x$ (arrows that can be obtained from the transitivity of $\succeq$ are omitted).

First, consider the pair of alternatives α, x. Since $\alpha \succeq x$, the set $[x, \alpha]$ is nonempty and equals $\{x, y, z, x_2, x_3, x_4, \alpha\}$. Richness would require that there exist for the set $[x, \alpha]$, two preferences $R_i^x, R_i^\alpha \in \mathcal{D}$ such that $\alpha P_i^x v$ and $x P_i^\alpha v$ only for alternatives $v \in \{x_1, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$.

We next illustrate the restrictions implied by semilattice single-peakedness on a preference ordering where the alternative x is top-ranked. The definition of semilattice single-peakedness imposes two sorts of restrictions on a preference relation R_i^x (in addition to $xP_i^x y$ for all $y \neq x$). The first of these applies to alternatives that appear along any $\succeq$-path emanating from x. There are two such paths from x to α (emphasized with bold-type links); namely, $x \preceq y \preceq x_3 \preceq z \preceq \alpha$ and $x \preceq x_2 \preceq x_4 \preceq z \preceq \alpha$. Along such paths, we have classical single-peakedness. Thus, since the

[13] Given $x, y \in A$, $\sup_{\succeq}(x, y) = z$ if and only if z is the least element in A that is greater than or equal (according to $\succeq$) to x and y; namely, $z \in \{w \in A \mid w \succeq x \text{ and } w \succeq y\}$ and, for all $z' \in \{w \in A \mid w \succeq x \text{ and } w \succeq y\}$, $z' \succeq z$. Since $\succeq$ is antisymmetric, if the supremum exists it is unique.

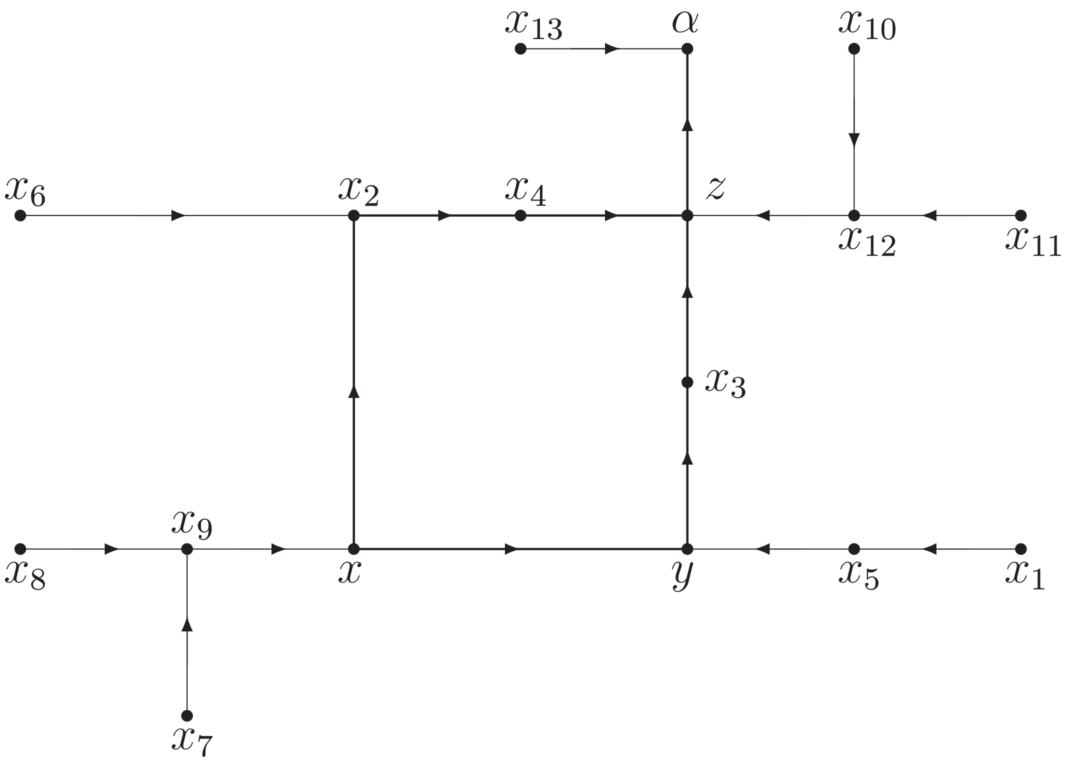

FIGURE 3

A SEMILATTICE AS A PARTIALLY DIRECTED GRAPH

pairs y, z belong to the first path, we have $yR_i^x z$. Observe that $\sup_{\succeq}(x, y) = y$ and $\sup_{\succeq}(z, y) = z$. However, note that since the alternatives x_3 and x_4 belong to different paths, there is no restriction on the relative ranking of these two alternatives in R_i^x; indeed, if one were to apply Definition 2 with x_3, x_4 playing the role of y, z, respectively, one only obtains $\sup_{\succeq}(x, x_3) = x_3 R_i^x z = \sup_{\succeq}(x_4, x_3)$.

The second restriction applies to alternatives that are not in a $\succeq$-path from x to α. Such alternatives are dispreferred to the "closest" alternative in the path; namely, if w and r are such that $x \preceq w \preceq \alpha$, $r \notin [x, \alpha]$, and $\sup_{\succeq}(x, r) = w$, then $wR_i^x r$ (observe that $\sup_{\succeq}(r, r) = r$). For instance, in Figure 3, $yR_i^x x_5$ and $yR_i^x x_1$ but no condition is imposed on the preference between x_5 and x_1; moreover, take any $z', z'' \in \{x_{10}, x_{11}, x_{12}\}$ such that $z' \neq z''$ and observe that $\sup_{\succeq}(x, z') = z$, $\sup_{\succeq}(z'', z') = x_{12}$, and $\sup_{\succeq}(z', z') = z'$. Then, $zR_i^x x_{12}$ and $zR_i^x z'$.

Finally, we enumerate below the restrictions implied on a preference R_i^x over A. By definition, we know that $xP_i^x y'$ for all $y' \notin A\backslash\{x\}$. Semilattice single-peakedness imposes the following relations among pairs of alternatives (observe that in Figure 3, z is the supremum of $A\backslash\{\alpha, x_{13}\}$):[14]

- $yR_i^x x_3 R_i^x z$ since $\sup_{\succeq}(x, y) = yR_i^x x_3 = \sup_{\succeq}(x_3, y)$ and $\sup_{\succeq}(x, x_3) = x_3 R_i^x z = \sup_{\succeq}(z, x_3)$.
- $x_2 R_i^x x_4 R_i^x z$ since $\sup_{\succeq}(x, x_2) = x_2 R_i^x x_4 = \sup_{\succeq}(x_4, x_2)$ and $\sup_{\succeq}(x, x_4) = x_4 R_i^x z = \sup_{\succeq}(z, x_4)$.
- $yR_i^x x_k$ for $k = 1, 5$ since $\sup_{\succeq}(x, x_k) = yR_i^x x_k = \sup_{\succeq}(x_k, x_k)$ (i.e., x_k plays simultaneously the role of y and z in Definition 2).
- $x_2 R_i^x x_6$ since $\sup_{\succeq}(x, x_6) = x_2 R_i^x x_6 = \sup_{\succeq}(x_6, x_6)$ (i.e., x_6 plays simultaneously the role of y and z in Definition 2).

[14] In addition to the relations derived from the transitivity of R_i^x, these are the *only* relations imposed on R_i^x by semilattice single-peakedness.

- $zR_i^x x_k$ for $k = 10, 11, 12$ since $\sup_{\succeq}(x, x_k) = zR_i^x x_k = \sup_{\succeq}(x_k, x_k)$.
- $\alpha R_i^x x_{13}$ since $\sup_{\succeq}(x, x_{13}) = \alpha R_i^x x_{13} = \sup_{\succeq}(x_{13}, x_{13})$ (i.e., x_{13} plays simultaneously the role of y and z in Definition 2).

Observe that semilattice single-peakedness leaves freedom to R_i^x on how it orders many pairs of alternatives. For instance, we have already noted that the relative ranking of the pair x_3, x_4 is not fixed. Consider next the path $x_7 \longrightarrow x_9 \longrightarrow x$. Here too, letting x_7, x_9 play the role of y, z in Definition 2 does not lead to any restriction on the relative rankings of x_7 and x_9 in R_i^x since $\sup_{\succeq}(x, x_9) = xR_i^x x_9 = \sup_{\succeq}(x_7, x_9)$.

Proposition 1 below shows that the two restrictions used in the example characterize indeed semilattice single-peakedness. After stating the main result of the article in Proposition 3, we will be in a better position to comment on why semilattice single-peakedness emerges as an implication of strategy-proofness (and other desirable properties) and can be seen as a weakening of the classical concept of single-peakedness.

Proposition 1. *Let $\succeq$ be a semilattice over A. Then, the preference R_i^x is semilattice single-peaked on $(A, \succeq)$ if and only if the following two properties hold:*

(i) for all $y, z \in A$ such that $x \preceq y \preceq z$, $yR_i^x z$;
(ii) for all $w \in A$ such that $x \npreceq w$, $\sup_{\succeq}(x, w)R_i^x w$.

Proof. Assume that R_i^x is semilattice single-peaked on $(A, \succeq)$ and let $y, z \in A$ be such that $x \preceq y \preceq z$. Then, (i) follows because $\sup_{\succeq}(x, y) = yR_i^x z = \sup_{\succeq}(z, y)$. Let $w \in A$ be such that $x \npreceq w$. If $x \succeq w$, then (ii) follows because $x = \sup_{\succeq}(x, w)$ and $xP_i^x w$. If $x \nsucceq w$, then (ii) follows since $w = \sup_{\succeq}(w, w)$ and, by semilattice single-peakedness, $\sup_{\succeq}(x, w)R_i^x \sup_{\succeq}(w, w) = w$.

Assume that (i) and (ii) hold for R_i^x and let $z, y \in A$ be arbitrary. We show that

$$\sup_{\succeq}(x, y)R_i^x \sup_{\succeq}(z, y) \tag{2}$$

holds by distinguishing among three cases.

Case 1: $\sup_{\succeq}(x, y) = x$. Then (2) holds trivially since $x = t(R_i^x)$.

Case 2: $\sup_{\succeq}(x, y) = y \neq x$. The case $\sup_{\succeq}(z, y) = y$ is trivial. If $\sup_{\succeq}(z, y) = w \neq y$, then $x \preceq y \preceq w$ and, by (i), $y = \sup_{\succeq}(x, y)R_i^x \sup_{\succeq}(z, y)$. Thus, (2) holds.

Case 3: $\sup_{\succeq}(x, y) = w \notin \{x, y\}$. First, assume that $\sup_{\succeq}(z, y) = y$. Since $x \npreceq y$, by (ii), $wR_i^x y$. Hence, $\sup_{\succeq}(x, y) = wR_i^x y = \sup_{\succeq}(z, y)$. Thus, (2) holds. Assume now that $s = \sup_{\succeq}(z, y) \neq y$. If $s \succeq x$, then s is an upper bound of x and y. Hence, $x \preceq w \preceq s$. By (i), $wR_i^x s$. Hence, $\sup_{\succeq}(x, y) = wR_i^x s = \sup_{\succeq}(z, y)$. Thus, (2) holds. Assume finally that $s \nsucceq x$. By (ii), $r \equiv \sup_{\succeq}(x, s)R_i^x s$. Note that r is an upper bound of x and y. Hence, $x \preceq w \preceq r$. By (i), $wR_i^x r$. Hence, $\sup_{\succeq}(x, y) = wR_i^x rR_i^x s = \sup_{\succeq}(z, y)$. Thus, (2) holds. ■

3.4. *Semilattice Single-Peaked Domains Admit an SCF with the Desirable Properties.* Before presenting the main result of the article in Proposition 3 below, we show that a semilattice single-peaked domain admits a strategy-proof, tops-only, anonymous, and unanimous SCF for an arbitrary number of agents. This generalizes in a very simple way the converse of the main result in Chatterji et al. (2013) to settings where A is not necessarily finite and the underlying structure is not necessarily a tree.

Proposition 2. *Let $\mathcal{D}$ be a semilattice single-peaked domain on the semilattice $(A, \succeq)$. Then, there exists a strategy-proof, tops-only, anonymous, and unanimous SCF $f : \mathcal{D}^n \to A$ for all n. Moreover, if n is even, then $\succeq$ is induced by f over A.*

Proof. We first establish the following induction step: Suppose for $k \geq 2$, $\sup_{\succeq}(x_1, \ldots, x_k)$ exists for every set $\{x_1, \ldots, x_k\}$ of k distinct alternatives. Then, for any alternative $x_{k+1} \notin \{x_1, \ldots, x_k\}$, $\sup_{\succeq}(x_1, \ldots, x_{k+1})$ exists and is given by $\sup_{\succeq}(\sup_{\succeq}(x_1, \ldots, x_k), x_{k+1})$.

To verify this step, let $y = \sup_{\succeq}(x_1, \ldots, x_k)$. By the induction hypothesis, $\sup_{\succeq}(y, x_{k+1})$ exists and is denoted by w. Since $\succeq$ is transitive, w is an upper bound for $(x_1, \ldots, x_{k+1})$. Suppose there exists $v \in A\backslash\{w\}$ such that v is an upper bound for $(x_1, \ldots, x_{k+1})$. Then, it must be that $v \succeq y$ since $y = \sup_{\succeq}(x_1, \ldots, x_k)$. We also have $v \succeq x_{k+1}$. These imply that v is an upper bound for (y, x_{k+1}). But since $\sup_{\succeq}(y, x_{k+1})$ exists and is w, we must have $v \succeq w$ and so $w = \sup_{\succeq}(x_1, \ldots, x_{k+1})$.

Given a preference profile $R \in \mathcal{D}^n$, let $G(R) = \{x_1, \ldots, x_k\}$, $k \leq n$, be the set of distinct alternatives such that for each $r = 1, \ldots, k$, $x_r = t(R_i)$ for some $i \in N$.

For every $R \in \mathcal{D}^n$, define

$$f(R) = \sup_{\succeq} G(R). \tag{3}$$

Since $\succeq$ is a semilattice, the induction step verified earlier implies that f is well defined. By construction, f is tops-only, anonymous, and unanimous. We next show that f is strategy-proof. Given $R \in \mathcal{D}^n$ and $i \in N$, let $G(R_{-i}) = G(R)\backslash\{t(R_i)\}$ and observe that $f(R_i, R_{-i}) = \sup_{\succeq}(t(R_i), \sup_{\succeq} G(R_{-i}))$. To show that f is strategy-proof, we wish to show for arbitrary $R_i^x \in \mathcal{D}$ and $R_i^z \in \mathcal{D}$, $z \in A\backslash\{x\}$,

$$f(R_i^x, R_{-i}) = \sup_{\succeq}(x, \sup_{\succeq} G(R_{-i})) R_i^x \sup_{\succeq}(z, \sup_{\succeq} G(R_{-i})) = f(R_i^z, R_{-i}). \tag{4}$$

By the definition of f in (3) and the definition of semilattice single-peakedness, (4) holds.

It is straightforward to verify that when n is an even positive integer, $\succeq$ is induced by f as defined following Fact 1. ■

3.5. *Results for the Case of n Even.* We now proceed by first showing that any strategy-proof, tops-only, and anonymous SCF $f : \mathcal{D}^2 \to A$ can be seen as the supremum of the binary relation $\succeq$ induced by f over A, provided that the domain of f is rich on $(A, \succeq)$.[15]

LEMMA 2. *Let $(\mathcal{D}, A, \succeq)$ be such that $\mathcal{D}$ is rich on $(A, \succeq)$. If there exists an SCF $f : \mathcal{D}^2 \to A$ that induces $\succeq$ over A and is strategy-proof, tops-only, and anonymous, then for all $x, y \in A$, $f(x, y) = \sup_{\succeq}(x, y)$.*

PROOF. Let $x, y \in A$ and assume first that $x \neq y$. If $f(x, y) = x$, then $x \succeq y$. By strategy-proofness, $f(x, x) = x$ and hence, $x \succeq x$. Thus, $x = \sup_{\succeq}(x, y)$. Similarly if $f(x, y) = y$. Assume $f(x, y) = z \notin \{x, y\}$. By strategy-proofness, $f(z, y) = f(x, z) = z$. Hence, $z \succeq x$ and $z \succeq y$. Thus, z is an upper bound of (x, y). Assume $z \neq \sup_{\succeq}(x, y)$; namely, there exists $\bar{z} \in A$, $\bar{z} \neq z$, such that $\bar{z} \succeq x$ and $\bar{z} \succeq y$ and either $z \succeq \bar{z}$ or z is not comparable to $\bar{z}$. In either case, we have $\bar{z} \not\succeq z$, and hence, $z \notin [x, \bar{z}] \neq \emptyset$. Furthermore, we have $f(\bar{z}, y) = \bar{z}$. Since $\mathcal{D}$ is rich on $(A, \succeq)$, there exists $R_1^x \in \mathcal{D}$ such that $\bar{z} P_1^x z$. But then,

$$f(\bar{z}, y) = \bar{z} P_1^x z = f(x, y),$$

a contradiction with strategy-proofness of f. Assume now that $x = y$ and $f(x, x) = z$. We want to show that $\sup_{\succeq}(x, x) = z$. Suppose not; i.e., there exists $w \in A$ such that $w \succeq x$ and either $z \succeq w$ or z is not comparable to w. In either case, we have $w \not\succeq z$ and so $z \notin [x, w] \neq \emptyset$. Since $\mathcal{D}$ is rich on $(A, \succeq)$, there exists $R_1^x \in \mathcal{D}$ such that $w P_1^x z$. But then,

$$f(w, x) = w P_1^x z = f(x, x),$$

a contradiction with strategy-proofness of f. ■

[15] Subsection 5.2 contains an example of a set A and a strategy-proof, tops-only, anonymous, and unanimous SCF f on a domain that is not rich on $(A, \succeq)$ with the property that $\succeq$ is not a semilattice and f does not take the supremum form.

Lemmas 1 and 2 do not require that the SCF should be unanimous. If the SCF f in Lemmas 1 and 2 is unanimous, then the binary relation $\succeq$ induced by f over A is reflexive. Similarly, if the SCF g in Fact 1 is unanimous, then so is f. From now on we will be interested only in unanimous SCFs.

Recall that (in view of Fact 1) when we say that a strategy-proof, tops-only, anonymous, and unanimous SCF $g : \mathcal{D}^n \to A$, where n is a positive even integer, *induces a binary relation* $\succeq$ *over* A, it is understood that $\succeq$ is the binary relation induced by f over A where f is *induced* from g by "cloning" the first $\frac{n}{2}$ agents as agent 1 and the remaining as agent 2. We now state our principal finding.

PROPOSITION 3. *Let* $(\mathcal{D}, A, \succeq)$ *be such that* $\mathcal{D}$ *is rich on* $(A, \succeq)$*. If there exists an SCF* $g : \mathcal{D}^n \to A$*, where n is a positive even integer that induces* $\succeq$ *over A and is strategy-proof, tops-only, anonymous, and unanimous, then (i)* $\succeq$ *is a semilattice over A, (ii) for all* $x, y \in A$*,* $f(x, y) = \sup_{\succeq}(x, y)$*, where f is induced from g, and (iii)* $\mathcal{D}$ *is semilattice single-peaked on* $(A, \succeq)$*.*

PROOF. The proofs of (i) and (ii) follow from Lemmas 1 and 2, respectively. To show that the condition specified in Definition 2 holds, observe that by Lemma 2 and strategy-proofness, $f(x, y) = \sup_{\succeq}(x, y) R_1^x \sup_{\succeq}(z, y) = f(z, y)$. ■

In light of Proposition 3, we are now ready to comment on why semilattice single-peakedness becomes necessary and constitutes a weakening of single-peakedness. Condition (i) in Proposition 1 inherits (only partially) the general concept of single-peakedness: The preference is decreasing when alternatives are farther from the top, according *only* to the increasing direction of the partial order (i.e., $x \preceq y \preceq z$ implies $y R_i^x z$). To see why no condition is imposed on the preference in the decreasing direction of the partial order (i.e., $z \preceq y \preceq x$), fix $x \in A$, R_i^x and an SCF f with the desirable properties (and its induced semilattice $\succeq$ over A). Define the set of options left open by x at f as the set of alternatives that may be selected by f for some top alternative of the other agent; namely, $o^f(x) = \{z \in A \mid z = f(x, y) \text{ for some } y \in A\}$. We know from the maximal domain literature that,[16] to guarantee that f be strategy-proof, R_i^x has to be single-peaked only on $o^f(x)$. But observe that according to (ii) in Proposition 3, $o^f(x) = \{z \in A \mid z \succeq x\}$. Hence, $y, z \notin o^f(x)$ whenever $z \preceq y \preceq x$. This explains the form of condition (i) in Proposition 1. Moreover, if $x \not\preceq z$, then, by (ii) in Proposition 3, strategy-proofness, and unanimity, $\sup_{\succeq}(x, z) = f(x, z) R_i^x f(z, z) = z$, which implies that condition (ii) in Proposition 1 holds.

4. RELATED LITERATURE

In this section, we relate our results to the large literature on restricted domains. The starting point of this approach is to assume that the set of alternatives A has a particular structure (e.g., A is a linearly ordered set). Using this structure, one can define a meaningful domain restriction on preferences over A (e.g., single-peakedness) under which nontrivial strategy-proof SCFs can be defined (e.g., the median voter rule). Our Proposition 2 (and its proof) follows partially this approach. We start by hypothesizing that the set A, together with the binary relation $\succeq$, is a semilattice from which we define the domain $\mathcal{D}$ of semilattice single-peaked preferences on $(A, \succeq)$. We then show that there exists a strategy-proof, tops-only, anonymous, and unanimous SCF f on the domain $\mathcal{D}$ that, when n is a positive even integer, is such that $\succeq$ is induced by f over A. We want to emphasize, however, that our main contribution is Proposition 3, which follows a very different approach. Without assuming any structure on the set of alternatives A, we suppose that there is a strategy-proof, tops-only, anonymous, and unanimous SCF g on a given domain $\mathcal{D}$ of preferences over A. Following Bogomolnaia (1998), we show how to identify using

[16] See, for instance, Barberà et al. (1999), Berga and Serizawa (2000), Ching and Serizawa (1998), Hatsumi et al. (2014), and Serizawa (1995).

condition (1) a binary relation $\succeq$ over A. Then, provided that the domain $\mathcal{D}$ is rich on $(A, \succeq)$, we prove that $(A, \succeq)$ is a semilattice, the domain $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq)$, and g can be obtained as the supremum rule of a two-agent SCF f induced from g. Hence, the semilattice structure on A follows from the existence of an SCF satisfying the desirable properties without imposing any condition on A whatsoever. We now relate with more detail our results to some representative results of the restricted domains literature. In particular, we consider well-studied SCFs in this literature and uncover the associated semilattice that is implicit in each formulation.

4.1. *Single-Peaked Preferences on a Line.* We return to the median voter rule $f : \mathcal{D}^2 \to A$ (where $A = [0, 1]$), represented by a fixed ballot $\alpha \in A$ and already presented in Subsection 3.2. Let $\succeq$ be the binary relation induced by f on A. As we have already seen, the following facts hold:

(1) The median voter rule f is strategy-proof, tops-only, anonymous, and unanimous on $\mathcal{D}$.
(2) The binary relation $\succeq$ induced from f using (1) is as follows: If either $y < x \leq \alpha$ or $\alpha \leq x < y$, then $x \succeq y$ and if $x > \alpha > y$, then $x \not\succeq y$ and $y \not\succeq x$.
(3) The domain of single-peaked preferences SP is rich on $(A, \succeq)$ and is a strict subset of the set of all semilattice single-peaked preferences on $(A, \succeq)$, already represented in Figure 2.
(4) For all $x, y \in A$, $f(x, y) = \sup_{\succeq}(x, y) = med_{>}\{x, y, \alpha\}$.

Moreover, the following fact can be verified:

(1) The domain of single-peaked preferences coincides with the intersection of all semilattice single-peaked preferences, where each of these sets is associated to each of all possible values α in A. In other words, the set of single-peaked preferences is the largest preference domain that is semilattice single-peaked relative to *all* binary relations induced by all median voter rules (i.e., all strategy-proof, tops-only, anonymous, and unanimous SCFs).

4.2. *Semi-Single-Peaked Preferences.* In the previous subsection, the set of alternatives was assumed to be linearly ordered, and single-peakedness of preferences and the median was defined with respect to this linear order. We now turn to a more general formulation where the alternatives are arranged on a tree. In this case, one may move away from the top of a preference in more than two directions. Single-peakedness on a tree, introduced by Demange (1983) and studied further by Danilov (1994),[17] requires preferences to be monotonically nonincreasing along such directions. Subsequent work by Chatterji et al. (2013) shows that full single-peakedness in this sense is not required for the design of strategy-proof, tops-only, anonymous, and unanimous SCFs. They identify a weaker concept called *semi-single-peakedness*, which is an implication of strategy-proofness and the other properties and suffices for the design of such SCFs. The specification of semi-single-peakedness requires the selection of a particular alternative on the tree, called the threshold, whose projection on each path on the tree corresponds to a fixed ballot on that path, and the preference restriction of semi-single-peakedness on each path coincides with the one depicted in Figure 4. We proceed by first summarizing their findings and by showing that their formulation is a special case of semilattice single-peakedness.

Assume that the set of alternatives A is a finite tree; that is, for every pair of alternatives (nodes) $x, y \in A$, there is a *unique* path p linking them, denoted by $\langle x, y\rangle$. Two alternatives x, y are directly linked if $\langle x, y\rangle = \{x, y\}$.[18] Given alternatives $x, y, z \in A$, let $\pi(z, \langle x, y\rangle)$ denote

[17] Savaglio and Vannucci (2014) extend the analysis to graphs that are not necessarily trees. We further comment on this article in Subsection 5.3. Schummer and Vohra (2002) also study a model where the set of alternatives is possibly infinite and arranged as a graph. They consider separately the case where the graph is a tree and the case where the graph has cycles. They characterize strategy-proof and onto SCFs assuming preferences are Euclidean, which satisfy our richness condition.

[18] See Example 3 in Subsection 5.3 for a description of how, given a domain of preferences, to directly link two alternatives.

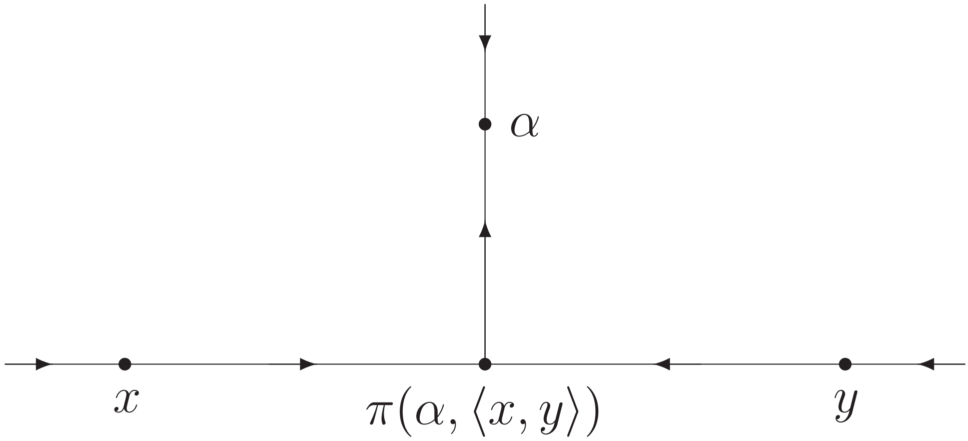

FIGURE 4

THE PROJECTION OF A THRESHOLD ON A PATH ON A TREE

the projection of z on the path $\langle x, y\rangle$ that is defined as the *unique* alternative $w \in A$ such that $\langle x, z\rangle \cap \langle y, z\rangle = \langle w, z\rangle$. A path p is "maximal" if it cannot be extended by adding more edges at either one of the two ends. Fix a particular alternative α on the tree A (call it *the threshold*), and use it to specify a threshold on *every* maximal path p, denoted by $\lambda(p)$, as $\lambda(p) = \pi(\alpha, p)$. Thus, for every maximal path p, if it contains the alternative α, set $\lambda(p) = \alpha$; otherwise, the threshold $\lambda(p)$ is the *unique* alternative that lies on every path from an alternative on the path p to alternative α.[19]

Given A with a tree structure and the threshold $\alpha \in A$, Chatterji et al. (2013) say that a preference R_i is *semi-single-peaked with respect to the threshold* α if for all $x, y \in A$:

(i) $[x, y \in p$ such that $x, y \in \langle t(R_i), \lambda(p)\rangle$ and $x \in \langle t(R_i), y\rangle] \Rightarrow [xR_iy]$ and
(ii) $[x \in p$ such that $\lambda(p) \in \langle t(R_i), x\rangle] \Rightarrow [\lambda(p)R_ix]$.

Moreover, they define the two-agent SCF f, where for all $x, y \in A$,

$$f(x, y) = \pi(\alpha, \langle x, y\rangle). \tag{5}$$

They show that the SCF f defined by (5) is strategy-proof, tops-only, anonymous, and unanimous on the domain of semi-single-peaked preferences on the tree A with respect to the threshold α.

The following facts can be verified:

(1) The binary relation $\succeq$ on A induced by f from (1) is such that for all $x, y \in A$,

$$x \succeq y \text{ if and only if } x = \pi(\alpha, \langle x, y\rangle).$$

(2) The domain of all semi-single-peaked preferences with respect to α is rich on $(A, \succeq)$, so that $(A, \succeq)$ is a semilattice and for all $x, y \in A$, $f(x, y) = \sup_{\succeq}(x, y)$. Figure 4 illustrates this construction.
(3) The set of strict semilattice single-peaked preferences on $(A, \succeq)$ coincides with the set of semi-single-peaked preferences with respect to the threshold α.

[19] The threshold seeks to identify an alternative on the path p that is "closest" to α. In the absence of a distance, the closest alternative is one that belongs to every path emanating from any alternative on p to α. On a tree, such an alternative is uniquely identified.

4.3. *Multidimensional Models.* In many social choice problems, alternatives are multidimensional. To describe an alternative, one has to specify the level reached in each of its attributes. Our setting also includes these cases. Border and Jordan (1983) and Barberà et al. (1993) are prototypical examples of this approach, and they can be seen as extensions of Moulin (1980). We first relate our results with the main ones contained in these two papers and, second, with the results in Barberà et al. (1991) on voting by quota, the case when each attribute can take only two possible values.

4.3.1. *Multidimensional single-peaked preferences.* We begin by postulating a multidimensional structure on the set of alternatives A, and specifying a restricted domain. A preference with top x is multidimensional single-peaked on A if an alternative z that lies on the shortest path from x to y is weakly preferred to y. General results in Border and Jordan (1983) and Barberà et al. (1993) imply that strategy-proof SCFs satisfying our properties are componentwise median voter rules. We first provide the details of this formulation and then identify it as a special instance of semilattice single-peakedness.

Assume the set of alternatives A is a Cartesian product of subsets of real numbers; i.e.,

$$A = \prod_{k=1}^{K} A_k,$$

where, for each $k = 1, \ldots, K$, $A_k \subseteq \mathbb{R}$ can be finite or infinite.[20] Define the L_1-norm in A as follows: For every $x \in A$,

$$\|x\| = \sum_{k=1}^{K} |x_k|.$$

Given $x, y \in A$, let

$$MB(x, y) = \{z \in A \mid \|x - y\| = \|x - z\| + \|z - y\|\}$$

be the minimal box containing x and y.

A preference $R_i \in \mathcal{D}$ is *multidimensional single-peaked on* A if, for all $y \in MB(x, t(R_i))$, yR_ix holds (namely, alternatives that lie on a L_1-path going from x to $t(R_i)$ should be ranked at least as high as x).[21] Let MSP be the set of all such preferences on A.

Let $f : MSP^2 \to A$ be an anonymous and unanimous SCF. Then, f is strategy-proof if and only if there exists a vector of fixed ballots $\alpha = (\alpha_1, \ldots, \alpha_K) \in A$ such that for all $(R_1, R_2) \in MSP^2$ and $k = 1, \ldots, K$,

$$f_k(R_1, R_2) = med_{>}(t_k(R_1), t_k(R_2), \alpha_k).$$

Consider a tops-only SCF $f : MSP^2 \to A$ and let $\succeq$ be the semilattice obtained from f using (1). Furthermore, assume f is strategy-proof, anonymous, and unanimous and let $\alpha \in A$ be its associated vector of fixed ballots. The following facts can be verified:

(1) For all $x, y \in A$,

$$x \succeq y \text{ if and only if } x \in MB(y, \alpha).$$

[20] Border and Jordan (1983) study the infinite case, whereas Barberà et al. (1993) study the finite case.

[21] It is possible to show that the set of star-shaped and separable preferences on A (defined in Border and Jordan, 1983) coincides with the set of multidimensional single-peaked preferences on A.

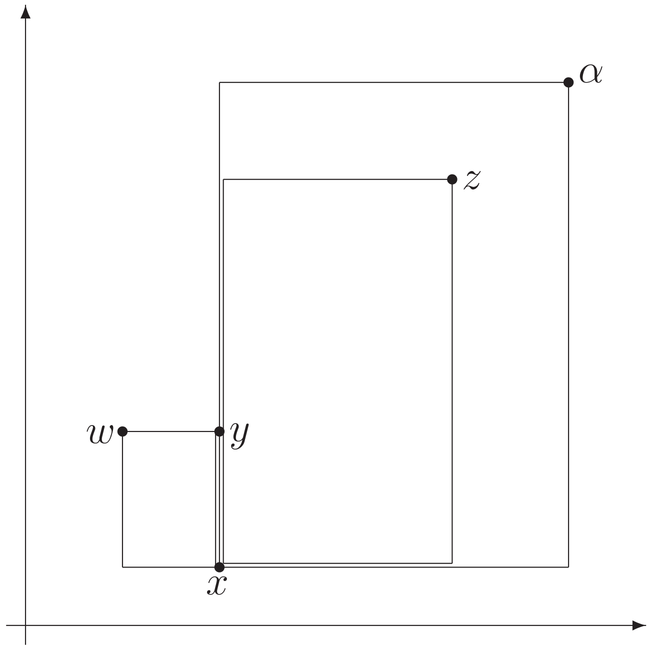

FIGURE 5

AN ILLUSTRATION OF THE PROPERTY OF MINIMAL BOXES CHARACTERIZING SEMILATTICE SINGLE-PEAKEDNESS

(2) The set of all multidimensional single-peaked preferences *MSP* is rich on $(A, \succeq)$, so that $(A, \succeq)$ is a semilattice and for all $x, y \in A, f(x, y) = \sup_{\succeq}(x, y)$.
(3) Let $x \in A$. The preference R_i^x is semilattice single-peaked on $(A, \succeq)$ if and only if for all $y, z \in A$ such that $y \in MB(x, \alpha) \cap MB(x, z), yR_i^x z$. We illustrate this fact in Figure 5 for the case $K = 2$ (see Section A.2 in the Appendix for a general proof of this statement). Observe that $y \in MB(x, \alpha), y \in MB(x, z), y \in MB(x, w)$, and $\sup_{\succeq}(x, y) = yR_i^x z = \sup_{\succeq}(z, y)$ and $\sup_{\succeq}(x, w) = yR_i^x w = \sup_{\succeq}(w, w)$.
(4) As in the unidimensional case, the set *MSP* is the intersection of all sets of semilattice single-peaked preferences, where each of these sets is associated with each of all possible values α in A.

4.3.2. *Voting by committees and separable preferences.* Barberà et al. (1991) show another example of a domain restriction, where, given a finite set of objects $\mathcal{K} = \{1, \ldots, K\}$, agents have to choose a subset (possibly empty) of $\mathcal{K}$. It can be described as follows: The set of alternatives is the family $2^{\mathcal{K}}$ of all subsets of $\mathcal{K}$ that can be identified with the K-dimensional hypercube $\{0, 1\}^K$. Namely, any set $X \in 2^{\mathcal{K}}$ can be described as the vector $x \in \{0, 1\}^K$ where, for each $k = 1, \ldots, K, x_k = 1$ if and only if $k \in X$. Barberà et al. (1991) identify the domain of separable preferences and characterize the class of strategy-proof SCFs with our properties as voting by quota. Separable preferences allow agents to evaluate objects as "good" or "bad." A voting by quota specifies an integer number for each object such that the object is included in the chosen subset if and only if the number of agents who declared it as being good is at least as large as this number. We proceed by specifying the details of this setup and briefly indicating the semilattice structure associated with it.

A (strict) preference R_i on A is said to be *separable* if adding an object to a given set makes the new set better if and only if the added object is good (as a singleton set, the object is preferred to the empty set). In the hypercube representation of $2^{\mathcal{K}}$, separability of R_i implies the following feature. Let x be the vector of 0s and 1s representing the best subset of objects according to R_i, and take any pair of vectors y and z of 0s and 1s (i.e., two subsets of objects Y and Z). From z, obtain x by iterating the following procedure: Take a coordinate of z that does not coincide with the corresponding coordinate of x and replace it by the coordinate of x, obtaining z'. Proceed similarly from z' until x is reached. Then, yR_iz if y is obtained in at least one of the steps of these procedures starting at z to obtain x. Let $\mathcal{S}$ be the set of all separable preferences on $\{0,1\}^K$.

For simplicity, we consider two agent SCFs. Following Barberà et al. (1991), an SCF $f : \mathcal{S}^2 \to \{0,1\}^K$ is *voting by quota* (not necessarily neutral) if there exists $q \in \{1,2\}^K$ such that for all $(R_1, R_2) \in \mathcal{S}^2$ and all $k = 1, \dots, K$,

$$f_k(R_1, R_2) = 1 \text{ if and only if } \#\{i \in N \mid t_k(R_i) = 1\} \geq q_k. \tag{6}$$

A characterization result in Barberà et al. (1991) implies that any strategy-proof, tops-only, anonymous, and unanimous SCF $f : \mathcal{S}^2 \to \{0,1\}^K$ is voting by quota. We indicate now how this setting relates to our result.

Let $f : \mathcal{S}^2 \to \{0,1\}^K$ be a voting by quota q. Let the binary relation $\succeq$ be induced from f using (1). The set of all separable preferences is rich on $(\{0,1\}^K, \succeq)$ and is a subset of semilattice single-peaked preferences on $(\{0,1\}^K, \succeq)$.

It turns out that the binary relation $\succeq$ has an interesting equivalent representation (which we formally verify in Section A.3 in the Appendix) that can be directly expressed using the quotas: For all $x, y \in \{0,1\}^K$,

$$x \succeq y \text{ if and only if } x \in MB(\alpha, y),$$

where $MB(\alpha, y)$ is as defined in Subsection 4.3.1 and the vector $\alpha \in \{0,1\}^K$ is as follows: For every $k = 1, \dots, K$, set

$$\alpha_k = \begin{cases} 1 & \text{if } q_k = 1 \\ 0 & \text{if } q_k = 2. \end{cases}$$

Moreover, it is easy to show that $\alpha = \sup_{\succeq}\{0,1\}^K$.

Finally, it is easy to see that the set of separable preferences is the intersection of all semilattice single-peaked preferences as the quotas vary across all possible values.

5. FINAL REMARKS

We finish the article with some final remarks related to issues left aside during the presentation of the main results.

5.1. *Our Approach and Our Axioms.* An important feature of the Gibbard–Satterthwaite theorem is that it puts no a priori structure on the set of alternatives. The restricted domains literature typically proceeds by restricting the formulation to finitely many alternatives with strict preferences or by restricting attention to SCFs that satisfy continuity and are defined over continuous preferences. We do not assume any structure of the set of alternatives. Our approach relies on assuming a common domain of preferences for the agents. As shown by Le Breton and Weymark (1999, proposition 1), under the assumption of a common domain of preferences, strategy-proof SCFs are such that each preference has a nonempty set of maximal elements on the range of the SCF, and, furthermore, the SCFs satisfy an appropriate version of unanimity.[22] We specialize to the case where there is a unique maximal element for each preference. This assumption excludes models where the set of alternatives has private components

[22] These results do not assume any a priori structure on the set of alternatives. Their subsequent analysis assumes continuity of preferences.

since an agent would be indifferent among alternatives when the private components of others change.[23]

Our approach relies on the tops-only property that allows us to associate to each SCF the binary relation defined in (1). The anonymity of the SCF is another key axiom that guarantees the antisymmetry of the binary relation, whereas the richness assumption guarantees that it is a partial order. Specifically, our semilattice single-peakedness condition requires that the supremum of a pair not be ranked higher than the supremum of a particular pair and thus admits indifferences since it is the negation of strict preferences. This condition is predicated upon a partial order with the supremum property; the antisymmetry of this partial order is indispensable to this condition. These features of our approach allow us to sidestep the usual restrictions like finiteness of the set of alternatives and the strictness and/or continuity of preferences that were alluded to above.

We next discuss in greater detail the role of the tops-only axiom on the SCF. We begin with the tops-only property. The usual justification for this axiom is that it affords very significant descriptive and computational advantages. This property allows us to derive the binary relation on the set of alternatives from the SCF and thus is crucial to our analysis. What can be said if one drops this axiom? We present an example that suggests that it would be difficult to derive interesting domain restrictions without the tops-only axiom. The example specifies a preference domain that has no single-peaked-type structure but nonetheless admits a unanimous, anonymous, and strategy-proof SCF. This SCF is, however, not tops-only.

EXAMPLE 1. Let $A = \{x, y, z, w\}$ be the set of alternatives and $\mathcal{D}$ the domain of five strict preferences:

P^x	P^y	P^z	P'^z	P^w
x	y	z	z	w
y	x	y	w	z
w	w	w	x	x
z	z	x	y	y

Consider the strategy-proof, nontops-only, anonymous, and unanimous SCF $f : \mathcal{D}^2 \to A$ defined by the following table:

f	P^x	P^y	P^z	P'^z	P^w
P^x	x	y	y	w	w
P^y	y	y	y	w	w
P^z	y	y	z	z	w
P'^z	w	w	z	z	w
P^w	w	w	w	w	w

One may wonder whether the fact that no structure is implied by the existence of a unanimous, anonymous, strategy-proof SCF (as seen in Example 1) is driven by the feature that the domain contains too "few" preferences. It might be possible to formulate a concept of richness for nontops-only SCFs and make some progress on this issue, but the methodology would be very different and presumably much more complicated than the one considered in this article.

Another way to drop the tops-only axiom and still derive restrictions on domains would be to strengthen the richness requirement so that the tops-only property follows as a consequence of

[23] See, for instance, Sprumont (1991) and Barberà et al. (2016). Moulin (1984) and Berga (1998) indicate the difficulties of extending our results to a setting where agents' preferences may have several tops, even in models with pure public goods. Extending our analysis to these cases is interesting and left for future work.

strategy-proofness and unanimity. There indeed are results on restricted domains establishing that tops-onlyness is a requirement of strategy-proofness together with an additional property like unanimity, efficiency, or ontoness (see, e.g., Barberà et al., 1991; Sprumont, 1995; Weymark, 2008; Chatterji and Sen, 2011). But these approaches start from the very beginning with a given domain (often related to single-peakedness) and a set of alternatives whose structure is explicitly used in obtaining tops-onlyness as a requirement of strategy-proofness (and the additional property). We face two difficulties in following this approach. The first is that we do not impose any structure on the domain of the SCF except that it has to be rich and each preference has a unique top, and no structure on the set of alternatives. It would be worthwhile to identify conditions on domains at the level of generality of our model where unanimity and strategy-proofness would ensure the tops-only property. These conditions can along with our richness requirement be used to replicate our analysis without the tops-only axiom. This task is outside the scope of the present study. Second, this approach implies that *every* strategy-proof and unanimous SCF is tops-only. Our approach does not require the domain to be such that every strategy-proof and unanimous SCF is tops-only; we merely require that there is *one* with the tops-only property.

5.2. *Example of a Nonrich Domain.* Our methodology relies on establishing that a two-agent strategy-proof, tops-only, anonymous, and unanimous SCF $f : \mathcal{D}^2 \to A$ induces a semilattice over A and that f takes the supremum form; that is, for every $x, y \in A$, $\sup_{\succeq}(x, y)$ exists and $f(x, y) = \sup_{\succeq}(x, y)$. We show here that the rich domain condition is indispensable for this property. It is well known that for domains with few preferences, it is possible to define strategy-proof SCFs. The smaller the domain is the larger is the class of (often trivial) strategy-proof SCFs that might operate on it. The literature contains several alternative concepts of rich domains. However, all of them (as well as ours) are sufficient conditions that are far from being necessary. This applies also to our setting: It is possible to find a semilattice $(A, \succeq)$, a *nonrich* and semilattice single-peaked domain $\mathcal{D}$ on $(A, \succeq)$, and a strategy-proof, tops-only, anonymous, and unanimous SCF defined on it.

The example below exhibits a domain $\mathcal{D}$, a strategy-proof, tops-only, anonymous, and unanimous SCF $f : \mathcal{D}^2 \to A$ whose induced partial order $\succeq$ over A is not a semilattice because for some $x, y \in A$, $\sup_{\succeq}(x, y)$ does not exist (and hence $f(x, y) \neq \sup_{\succeq}(x, y)$), and where $\mathcal{D}$ is not a rich domain on $(A, \succeq)$.

EXAMPLE 2. Let $A = \{x, y, z, \bar{z}, w\}$ be the set of alternatives and $\mathcal{D}$ the domain of five strict preferences:

P^x	P^y	P^z	$P^{\bar{z}}$	P^w
x	y	z	$\bar{z}$	w
z	z	w	w	x
$\bar{z}$	$\bar{z}$	$\bar{z}$	z	y
w	w	x	y	z
y	x	y	x	$\bar{z}$

Consider the strategy-proof, tops-only, anonymous, and unanimous SCF $f : \mathcal{D}^2 \to A$ defined by the following table:

f	x	y	z	$\bar{z}$	w
x	x	z	z	$\bar{z}$	w
y	z	y	z	$\bar{z}$	w
z	z	z	z	w	w
$\bar{z}$	$\bar{z}$	$\bar{z}$	w	$\bar{z}$	w
w	w	w	w	w	w

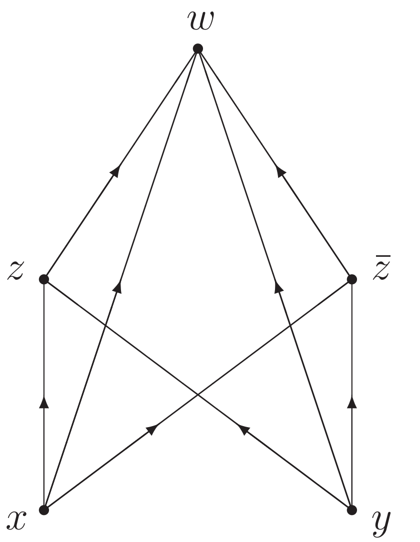

FIGURE 6

A PARTIAL ORDER THAT IS NOT A SEMILATTICE

The partial order $\succeq$ induced by f can be represented by Figure 6.

This partial order $\succeq$ is not a semilattice since $\sup_{\succeq}(x, y)$ does not exist. But the domain $\mathcal{D}$ is not rich on $(A, \succeq)$ since $z \notin [x, \bar{z}] \neq \emptyset$ and there does not exist any $\hat{P}^x \in \mathcal{D}$ such that $x\hat{P}^x\bar{z}\hat{P}^x z$; observe that there are other missing preferences, for instance, any $\hat{P}^{\bar{z}}$ such that $\bar{z}\hat{P}^{\bar{z}}x\hat{P}^{\bar{z}}z$.

5.3. *Relation to Other Concepts of Single-Peakedness.* Nehring and Puppe (2007a, 2007b, 2010) start with an abstract algebraic structure of a property space on a finite set of alternatives and a concept of "betweenness" and use it to define the concept of generalized single-peakedness. The necessity part of their characterization is similar to our analysis in spirit and shows that if there exists an onto, strategy-proof, anonymous, and neutral SCF on a rich domain of generalized single-peaked preferences induced by a property space, then this property space is a median space.[24] The concepts of generalized single-peakedness and semilattice single-peakedness are related but independent of each other. For instance, the complete domain, which never appears in our analysis, is a generalized single-peaked domain.

The domain of preferences we characterize is closer in spirit to semi-single-peaked domains. Semilattice single-peakedness extends the concept of semi-single-peakedness in at least three directions. The key differences are that the set of alternatives may be infinite and preferences admit indifferences. The concept of semi-single-peakedness is built upon an undirected graph that is necessarily a tree. The concept of semilattice single-peakedness can be illustrated via a directed graph (which need not be a tree when viewed as an undirected graph by ignoring the direction). Finally, the threshold (as described in Subsection 4.2) does not have to be an alternative; for instance, when $A = (0, 1) \subset \mathbb{R}$ and the partial order $\succeq$ is the natural order $>$ on real numbers (a semilattice on $(0, 1)$), then $1 \notin A$ would play the role of the threshold in Chatterji et al.'s (2013) construction. We show below that the analysis of Chatterji et al. (2013) is not implied by our analysis restricted to finitely many alternatives.

[24] See Bogomolnaia (1998) for representations of median voter schemes using medians on median graphs.

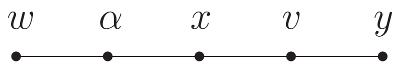

FIGURE 7

A STRONGLY PATH-CONNECTED GRAPH

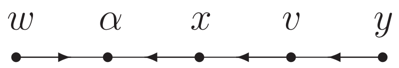

FIGURE 8

THE SEMILATTICE OF THE MEDIAN VOTER RULE F

EXAMPLE 3. Let $A = \{w, \alpha, x, v, y\}$ be the set of alternatives. We consider the following domain $\mathcal{D}$ of exactly eight strict preferences given below:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
w	α	α	x	x	v	v	y
α	w	x	α	v	x	y	v
x	x	w	w	α	α	x	x
v	v	v	v	w	w	α	α
y	y	y	y	y	y	w	w

This domain is strongly path connected in the terminology of Chatterji et al. (2013) and consequently satisfies their richness condition. The concept of a strongly path connected domain can be seen as follows: The alternatives w and α are said to be strongly connected since there exist two preference orderings P_1 and P_2 that rank the alternatives x, v, and y identically, whereas the positions of w and α, the top two ranked alternatives, are reversed across the two orderings. Likewise α and x, x and v, and finally v and y are strongly connected. One now associates to this domain a graph whose vertices are the five alternatives and where two vertices are an edge if and only if they are strongly connected. A domain is said to be strongly path connected if this graph is a connected graph. The domain $\mathcal{D}$ specified above is indeed a strongly path-connected domain. Figure 7 depicts this strongly path-connected graph.

Now consider a median voter rule $f : \mathcal{D}^2 \to A$, where the fixed ballot is located at α and consider the partial order $\succeq$ associated with this SCF f as defined by (1). Namely, $\alpha \succeq w$ and $\alpha \succeq x \succeq v \succeq y$. Figure 8 depicts this semilattice.

Observe that $[y, x]$ is nonempty and $w \notin [y, x]$. This article's concept of richness requires that there exist a preference ordering where x is the top-ranked alternative and where y is ranked above w. This condition is violated by P_4 and P_5 above. Thus, the richness condition of Chatterji et al. (2013) does not imply that our richness condition will necessarily be satisfied. The converse is also true since our concept of richness can be applied to multidimensional models with multidimensional single-peaked (or separable) preferences that are excluded by strongly path connected domains.[25] Thus, the two concepts of richness and consequently the results of the two papers are independent.

Savaglio and Vannucci (2014) consider a social choice setting where the set of alternatives is a distributive lattice $(A, \leq)$ from which a latticial ternary betweenness relation is defined: z lies between x and y if and only if $x \wedge y \leq z \leq x \vee y$, where the binary operations $\wedge$ and $\vee$ are

[25] Chatterji et al. (2016) characterize single-peaked preferences on a tree (as defined by Demange, 1983) on strongly path-connected domains using random SCFs.

the infimum and the supremum taken according to $\leq$, respectively. Agents' preferences satisfy some unimodality conditions that are consistent with this latticial ternary betweenness relation. They study and characterize strategy-proof SCFs on such domains. Note that our setting admits semilattices that are not necessarily lattices (the infimum of pairs of alternatives may not exist) and, more importantly, we do not start by assuming a specific structure on the set of alternatives but rather we obtain it as the consequence of the existence of a strategy-proof, tops-only, anonymous, and unanimous SCF on a rich domain of preferences.

Reffgen (2015) considers the case where agents' preferences are single-peaked with respect to different linear orders and characterizes strategy-proof SCFs. These typically violate either anonymity or unanimity or both, and, hence, we cannot obtain his analysis as a special case of our approach as we did in Section 4 for other formulations of single-peakedness.

5.4. *Characterization of All Strategy-Proof SCFs and Group Strategy-Proofness.* Our results indicate that the supremum rule is prominent in the class of strategy-proof, tops-only, anonymous, and unanimous SCFs. On an arbitrary (rich or otherwise) domain of semilattice single-peaked preferences, the supremum rule is shown in Proposition 2 to possess the aforementioned properties. On the other hand, any SCF with these properties induces, under the hypothesis of richness, a two-agent SCF that coincides with the supremum rule. A complete characterization of all SCFs that are strategy-proof, tops-only, anonymous, and unanimous on an arbitrary domain of semilattice single-peaked preferences is outside the scope of the present study.

There are semilattice single-peaked domains for which strategy-proof SCFs need not be efficient (see, e.g., voting by quota in Subsection 4.3.2), and hence these SCFs are not group strategy-proof. In particular, a profile of semilattice single-peaked preferences need not satisfy the indirect sequential inclusion condition of Barberà et al. (2010), which guarantees that strategy-proof SCFs are group strategy-proof. One can expect therefore that the domain implications of replacing strategy-proofness by group strategy-proofness will be considerably more stringent. We leave the identification of such domains for future work.

APPENDIX

In this appendix, we first consider the case where the domain $\mathcal{D}$ is assumed to admit a strategy-proof, tops-only, anonymous, and unanimous SCF $g : \mathcal{D}^n \to A$, where n is a positive odd integer. We subsequently include proofs of assertions omitted in the text and end with a remark on the case where the set of alternatives is finite.

A.1. *The Case of an Odd Number of Agents.* We illustrate, using an example, the sort of restrictions on the domain of preferences that are implied by the existence of a strategy-proof, tops-only, anonymous, and unanimous SCF with an odd number of agents.

EXAMPLE A.1. Assume the set of alternatives is the unit interval in the real line ordered with the linear order $>$, *that is,* $A = [0, 1]$ and the SCF $g : \mathcal{D}^3 \to A$ is a three agent SCF that takes the form

$$g(P_1, P_2, P_3) = med\{t(P_1), t(P_2), t(P_3), 0, 1\}, \text{ for all } (P_1, P_2, P_3) \in \mathcal{D}^3,$$

that is, a median voter rule with two fixed ballots located at the extremes. Trivially, g is tops-only, anonymous, and unanimous. Moreover, using Moulin (1980) and Berga and Serizawa (2000), it can be easily verified that in this case, the strategy-proofness of g requires that $\mathcal{D}$ be single-peaked on $[0, 1]$.[26] In particular, any weaker restriction of semilattice single-peakedness (as in Figure 2) will never suffice for the strategy-proofness of g.

[26] See Subsection 3.2 for the formal definition of single-peakedness and the median.

The example above assumed a special structure on the set of alternatives and a very special form of the SCF. The domain implications for the general case of an arbitrary $g : \mathcal{D}^n \to A$ where n is a positive odd integer is not attempted here. The purpose of the example is to indicate that the domain implication arising from the hypothesis that there exists an SCF with the required properties for an odd number of voters is likely to be considerably more restrictive than semilattice single-peakedness. Proposition 2 illustrates that semilattice single-peakedness suffices for the design of a well-behaved SCF for all numbers of voters; the additional restrictions (e.g., of single-peakedness as in the example above) implied by the hypothesis of the existence of a well-behaved SCF for an odd number of voters are in a sense spurious. We devote the remainder of this section to identifying a class of SCFs that are defined on an odd number of voters and where the restriction on the domain implied by the axioms is exactly semilattice single-peakedness.

We will do so by introducing an additional axiom.[27] This axiom requires that the SCFs satisfy an *invariance* requirement across two profiles of preferences where agents tops are either of two alternatives x or y, when the number of agents with top x and top y differ by exactly 1 across the two profiles.

DEFINITION A.1. The SCF $g : \mathcal{D}^n \to A$, where $n \geq 3$ is a positive odd integer, satisfies *invariance* if for every $x, y \in A$, for every $i \in N$, and for every pair of preference profiles of the form $(R_i, R_{-i}), (R_i', R_{-i}) \in \mathcal{D}^n$ where $t(R_i) = x$ and $t(R_i') = y$ and R_{-i} is any subprofile where $\frac{n-1}{2}$ agents have x as their top and $\frac{n-1}{2}$ have y as their top, it is the case that $g(R_i, R_{-i}) = g(R_i', R_{-i})$.

Let $g : \mathcal{D}^n \to A$ be a tops-only SCF, where $n \geq 3$ is a positive odd integer. Define the binary relation $\succ_o$ induced by g over A as follows: For all $(x, y) \in A \times A$, let $(\underbrace{x, \dots, x}_{\frac{n+1}{2}}, \underbrace{y, \dots, y}_{\frac{n-1}{2}})$ denote a profile of top-ranked alternatives where the first $\frac{n+1}{2}$ agents have x as the top and the remaining $\frac{n-1}{2}$ agents have y as the top and define

$$x \succeq_o y \text{ if and only if } g(\underbrace{x, \dots, x}_{\frac{n+1}{2}}, \underbrace{y, \dots, y}_{\frac{n-1}{2}}) = x. \tag{A.1}$$

REMARK A.1. Let $g : D^n \to A$, where $n \geq 3$ is positive odd integer, be a tops-only SCF, and $\succeq_o$ be the binary relation induced by g over A. If g is anonymous and satisfies invariance, then $\succeq_o$ is antisymmetric. If g is unanimous, then $\succeq_o$ is reflexive.

REMARK A.2. The analogs of Lemmas 1 and 2 can be proved analogously for $\succeq_o$ as well by standard arguments. We omit the details.

Finally, we obtain as Proposition 4 the extension of Proposition 3 to the case where $n \geq 3$ is a positive odd integer and the SCF satisfies in addition invariance.

PROPOSITION A.1. *Let $g : \mathcal{D}^n \to A$ be a strategy-proof, tops-only, anonymous, and unanimous SCF that satisfies invariance where $n \geq 3$ is a positive odd integer. Let $\succeq_o$ be the binary relation induced by g over A and assume that $\mathcal{D}$ is rich on $(A, \succeq_o)$. Then, (i) $\succeq_o$ is a semilattice over A, (ii) for all $x, y \in A$, $g(\underbrace{x, \dots, x}_{\frac{n+1}{2}}, \underbrace{y, \dots, y}_{\frac{n-1}{2}}) = \sup_{\succeq_o}(x, y)$, and (iii) $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq_o)$.*

[27] We consider in Subsection A.1.1 a version of our analysis without this axiom.

REMARK A.3. Part (iii) of the proposition establishes that $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq_o)$. Consequently, by an application of Proposition 2, there exists a strategy-proof, tops-only, anonymous, and unanimous SCF $f : \mathcal{D}^n \to A$ for all n. Furthermore, we note that the SCF constructed in the proof of Proposition 2 also satisfies invariance.

EXAMPLE A.1 (CONTINUED). The median voter rule with two fixed ballots located at the extremes considered in Example 4 yields the most restrictions on preferences in the class of all median voter rules with three agents for the case where A is the unit interval ordered with $>$. Indeed, any median voter rule defined by positioning the two fixed ballots at distinct alternatives other than 0 and 1 would impose less restriction on preferences and would admit domains of preferences that would only strictly contain the set of single-peaked preferences on $[0, 1]$. The least restrictive case would arise when the two fixed ballots are located on the same alternative; the induced SCF would then satisfy invariance, and the domain restriction implied would be exactly semilattice single-peakedness.

We first illustrate the content of the invariance axiom by exhibiting for well-known settings SCFs that satisfy it and SCFs that do not.

Consider in the Moulin (1980) setting the SCF $f : SP^3 \to [0, 1]$ that for all $(x, y, z) \in [0, 1]^3$, $f(x, y, z) = med_>\{x, y, z, \alpha_1, \alpha_2\}$, where $\alpha_1, \alpha_2 \in [0, 1]$. Then, f satisfies invariance if and only if $\alpha_1 = \alpha_2$.

Consider the Barberà et al. (1991) setting with $n = 3$ and $K = 2$. The SCF $f : \mathcal{S}^3 \to \{0, 1\}^2$ defined by quota $q = (q_1, q_2)$ satisfies invariance if and only if $q_1 \neq 2$ and $q_2 \neq 2$.

Clearly, in the Moulin (1980) and in the Barberà et al. (1991) settings, there are many instances of well-studied SCFs that satisfy all our requirements but violate invariance. But in both cases, there indeed exist some SCFs that satisfy invariance in addition to the properties we have imposed in this article.[28] In what follows, we provide a brief account of the picture without assuming invariance.

A.1.1 *Odd number of agents without invariance.* We continue the analysis of an odd number of agents by not imposing invariance of the SCF. At the end of this subsection, we introduce invariance in order to understand its role in Proposition 4.

We restrict attention to the case $n = 3$. By the cloning method employed in Fact 1, we can induce such an SCF whenever one exists for an odd number of agents that is divisible by 3. Let $f : \mathcal{D}^3 \to A$ be a strategy-proof, tops-only, anonymous, and unanimous SCF. Fix $x \in A$ and define, following a procedure also introduced by Bogomolnaia (1998), $g_x : \mathcal{D}^2 \to A$ by setting, for each pair $y, z \in A$, $g_x(y, z) = f(x, y, z)$. Then, g_x is a strategy-proof, tops-only, and anonymous SCF. Note that we cannot deduce that g_x is unanimous since $g_x(y, y) = y$ does not follow from the assumed unanimity of f. Let $\succeq_x$ be the binary relation induced by g_x over A using (1). Remark 1 applies, and the binary relation $\succeq_x$ is antisymmetric but cannot be assumed reflexive since $g_x(y, y) = y$ is not guaranteed.

We will therefore consider binary relations that are antisymmetric and transitive (which will follow from the richness axiom we introduce below) and refer to them as orders. The following definitions generalize our concepts of richness and semilattice single-peakedness to the case at hand.

DEFINITION A.2. Let A be an arbitrary set. A family of orders $\{\succeq_r\}_{r\in A}$ over A is given. The domain $\mathcal{D}$ is *rich* on $(A, \{\succeq_r\}_{r\in A})$ if for any $y, z, w \in A$, if $[y, z]_{\succeq_x}$ is nonempty for some $x \in A$ and $w \notin [y, z]_{\succeq_x}$, then there exist $R_i^y, R_i^z \in \mathcal{D}$ such that $zP_i^y w$ and $yP_i^z w$.

DEFINITION A.3. Let $\{\succeq_r\}_{r\in A}$ be a family of orders over A. The domain $\mathcal{D}$ is *order-family single-peaked* on $(A, \{\succeq_r\}_{r\in A})$ if for all $x, y, z, w \in A$ and all $R_i^x, R_i^z \in \mathcal{D}$,

[28] However, the example in appendix A of Nehring and Puppe (2010) exhibits a domain that admits a unique strategy-proof, tops-only, anonymous, and unanimous SCF for an odd number of agents that does not satisfy invariance.

(i) $\sup_{\succeq_w}(x, y) R_i^x \sup_{\succeq_w}(z, y)$ and
(ii) $\sup_{\succeq_z}(x, y) R_i^z \sup_{\succeq_w}(x, y)$.

To see why conditions (i) and (ii) in Definition A.3 follow from strategy-proofness, consider $x, y, z, w \in A$ and any $R_2^x, R_1^z \in \mathcal{D}$. Then (i) follows from $\sup_{\succeq_w}(x, y) = f(w, x, y) R_2^x f(w, z, y) = \sup_{\succeq_w}(z, y)$ and (ii) follows from $\sup_{\succeq_z}(x, y) = f(z, x, y) R_1^z \sup_{\succeq_w}(x, y) = f(w, x, y)$.

The proofs of Lemmas 1 and 2 do not require that the two-agent SCF under consideration satisfy unanimity. These lemmas apply here if the domain $\mathcal{D}$ is rich on $(A, \{\succeq_r\}_{r\in A})$ (in the sense of Definition A.2). We omit the details. Consequently, analogously to Proposition 3, we obtain here for all $x, y, z \in A$, $g_x(y, z) = \sup_{\succeq_x}(y, z)$ and $\mathcal{D}$ is order-family single-peaked on $(A, \{\succeq_r\}_{r\in A})$.

To summarize, if a domain $\mathcal{D}$ admits a three-agent SCF satisfying strategy-proofness, tops-onlyness, anonymity, and unanimity and the richness condition is satisfied, then $\mathcal{D}$ is order-family single-peaked on $(A, \{\succeq_r\}_{r\in A})$. However, we have been unable to evaluate whether this concept of single-peakedness would suffice for the design of a strategy-proof SCF satisfying tops-onlyness, anonymity, and unanimity. This is the principal difficulty in extending our analysis for an even number of agents in Section 2 to the case of an odd number of agents.

We are, however, able to design an SCF with the required four properties if we introduce additionally a concept of invariance of the family of orders. We express invariance in terms of the family of orders as follows: We say that the family of orders $\{\succeq_r\}_{r\in A}$ satisfies *order-invariance* if $\sup_{\succeq_x}(x, y) = \sup_{\succeq_y}(x, y)$ for all pairs (x, y). This condition would be implied by the existence of an SCF defined for an odd number of agents that satisfies strategy-proofness, tops-onlyness, anonymity, unanimity, and invariance in the sense of Definition A.1.

We may now define a two-agent SCF in the following manner; for any pair (x, x) of alternatives, define $f(x, x) = x$, whereas for any pair (x, y), $x \neq y$, of alternatives, define $f(x, y) = \sup_{\succeq_x}(x, y) = \sup_{\succeq_y}(x, y)$. It is evident that this SCF satisfies anonymity, and unanimity and is tops-only. This SCF will also satisfy strategy-proofness whenever $\mathcal{D}$ is order-family single-peaked on $(A, \{\succeq_r\}_{r\in A})$. Indeed, we have $f(x, y) = \sup_{\succeq_y}(x, y) R_i^x \sup_{\succeq_y}(z, y) = f(z, y)$ by (i) of Definition A.3. This verification of strategy-proofness uses the invariance of the family of orders in a central way and breaks down without it.

A.2. *Multidimensional Semilattice Single-Peakedness.* We now prove that in the multidimensional model, the following characterization of semilattice single-peaked preferences holds.

The preference R_i^x is semilattice single-peaked on $(A, \succeq)$ if and only if for all $y, z \in A$ such that $y \in MB(x, \alpha) \cap MB(x, z)$, $y R_i^x z$.

First, we show that if R_i^x is semilattice single-peaked on $(A, \succeq)$, then for all $y, z \in A$ such that $y \in MB(x, \alpha) \cap MB(x, z)$, $y R_i^x z$. Since $y \in MB(x, \alpha)$, it is true that $y \succeq x$, and hence, $\sup_{\succeq}(x, y) = y$. Moreover, $y \in MB(x, z)$, implies that, for each $k = 1, \ldots, K$, either $x_k \leq y_k \leq z_k$ or $z_k \leq y_k \leq x_k$. Before proceeding, we identify from $(A, \succeq)$ a family of semilattices $(A_k, \succeq_k)$, one for each component k, which correspond to the semilattice of the one-dimensional case. For each $k \in \{1, \ldots, K\}$, the pair $(A_k, \succeq_k)$ is a semilattice where $\succeq_k$ is defined as follows: For $x_k, y_k \in A_k$,

$$x_k \succeq_k y_k \text{ if and only if either } y_k \leq x_k \leq \alpha_k \text{ or } \alpha_k \leq x_k \leq y_k.$$

Assume without loss of generality that $x_k \leq y_k \leq z_k$. Since $x_k \leq y_k \leq \alpha_k$, $\sup_{\succeq k}(x_k, z_k) = \sup_{\succeq_k}(y_k, z_k) = \alpha_k$ if $\alpha_k \leq z_k$ and $\sup_{\succeq k}(x_k, z_k) = \sup_{\succeq_k}(y_k, z_k) = z_k$ otherwise. Hence, $\sup_{\succeq}(x, z) = \sup_{\succeq}(y, z)$. By semilattice single-peakedness, we know that $\sup_{\succeq}(x, z) R_i^x \sup_{\succeq}(z, z)$. Thus, $\sup_{\succeq}(y, z) R_i^x z$. Since $\sup_{\succeq}(x, y) R_i^x \sup_{\succeq}(z, y)$ by semilattice single-peakedness, we have $y R_i^x z$ as required.

Conversely, we show that if for all $y, z \in A$ such that $y \in MB(x, a) \cap MB(x, z)$, $y R_i^x z$, then R_i^x is semilattice single-peaked on $(A, \succeq)$. Given $y, z \in A$, to show that $\sup_{\succeq}(x, y) R_i^x \sup_{\succeq}(z, y)$, it suffices to show that $\sup_{\succeq}(x, y) \in MB(x, \alpha) \cap MB(x, \sup_{\succeq}(z, y))$. Since $\sup_{\succeq}(x, y) \succeq x$, it is evident that $\sup_{\succeq}(x, y) \in MB(x, \alpha)$. Next, to show that $\sup_{\succeq}(x, y) \in MB(x, \sup_{\succeq}(z, y))$, we simplify the notation and let $\sup_{\succeq}(x, y) = w$ and $\sup_{\succeq}(z, y) = w'$. We know that for each $k \in \{1, \ldots, K\}$, $w_k = med_{>}(x_k, y_k, \alpha_k)$ and $w'_k = med_{>}(z_k, y_k, \alpha_k)$. Assume without loss of generality that $x_k \leq y_k$. Consider three situations: (i) $w_k = x_k$, (ii) $w_k = \alpha_k$, and (iii) $w_k = y_k$. In situation (i), it is evident that either $x_k \leq w_k \leq w'_k$ or $w'_k \leq w_k \leq x_k$. In situation (ii), we know that $x_k \leq \alpha_k \leq y_k$. Consequently, $w'_k = med_{>}(x_k, y_k, \alpha_k) \geq \alpha_k = w_k$. Hence, $x_k \leq w_k \leq w'_k$. In situation (iii), we know that $\alpha_k \geq y_k$. Consequently, $w'_k = med_{>}(x_k, y_k, \alpha_k) \geq y_k = w_k$. Hence, $x_k \leq w_k \leq w'_k$. In conclusion, w_k is always in the middle of x_k and w'_k for all $k \in \{1, \ldots, K\}$. Therefore, $\sup_{\succeq}(x, y) \in MB(x, \sup_{\succeq}(z, y))$ as required.

A.3. *The Binary Relation $\succeq$ in Voting by Quota.* We show that in the voting by quota model, the binary relation $\succeq$, obtained by setting for all $x, y \in \{0, 1\}^K$,

$$x \succeq y \text{ if and only if } x \in MB(\alpha, y),$$

is induced by f over A by condition (1).

Assume $x \succeq y$. We want to show that $f(x, y) = x$; that is, $f_k(x, y) = x_k$ for all $k = 1, \ldots, K$. Take an arbitrary $k \in \{1, \ldots, K\}$ and assume first that $f_k(x, y) = 1$. Since f is voting by quota, $x_k + y_k \neq 0$. If $x_k + y_k = 2$, then $x_k = 1$ and $f_k(x, y) = x_k$. Assume now that $x_k + y_k = 1$. Since f is voting by quota, $q_k = 1$, and by the definition of α, $\alpha_k = 1$. To obtain a contradiction, suppose $x_k = 0$. Since, by the definition of $\succeq$, $x \in MB(\alpha, y)$ holds, we have that $y_k = 0$, a contradiction with the assumption that $x_k + y_k = 1$. Assume now that $f_k(x, y) = 0$. Then, $x_k + y_k < q_k$. If $x_k = 0$, then $f_k(x, y) = x_k$, which is what we wanted to prove. If $x_k = 1$, then $q_k = 2, \alpha_k = 0$, and $y_k = 0$. Hence, $x \notin MB(\alpha, y)$. Thus, $x \not\succeq y$, a contradiction. Since k was arbitrary, $f(x, y) = x$.

To prove the other implication in the definition of $x \succeq y$ by (1), assume $f(x, y) = x$. We want to show that $x \succeq y$. Take an arbitrary $k \in \{1, \ldots, K\}$. Suppose first that $x_k = 0$. If $y_k = 1$, then $q_k = 2$ and $\alpha_k = 0$. Namely, (i) $x_k = \alpha_k = 0$, and $y_k = 1$. If $y_k = 0$, then either $q_k = 1$, in which case $\alpha_k = 1$, or $q_k = 2$, in which case $\alpha_k = 0$. Namely, either (ii) $x_k = y_k = 0$ and $\alpha_k = 1$ or (iii) $x_k = y_k = \alpha_k = 0$. Suppose now that $x_k = 1$. If $y_k = 1$, then either $q_k = 1$, in which case $\alpha_k = 1$, or $q_k = 2$, in which case $\alpha_k = 0$. Namely, either (iv) $x_k = y_k = \alpha_k = 1$ or (v) $x_k = y_k = 1$ and $\alpha_k = 0$. If $y_k = 0$, then $q_k = 1$, in which case $\alpha_k = 1$. Namely, (vi) $x_k = \alpha_k = 1$ and $y_k = 0$. Hence, (i) to (vi) hold for an arbitrary $k \in \{1, \ldots, K\}$. Thus, $x \in MB(\alpha, y)$, and by definition of $\succeq$, $x \succeq y$ holds.

A.4. *Finite Set of Alternatives.* We identify in the corollary below a set of necessary conditions on any strategy-proof, tops-only, anonymous, and unanimous SCF that applies to the case where the set of alternatives is finite and n is an even positive integer. This is obtained by combining Proposition 3 with a result on two-agent SCFs from Bogomolnaia (1998).

COROLLARY A.1. *Let $(\mathcal{D}, A, \succeq)$ be such that $\mathcal{D}$ is rich on $(A, \succeq)$ and A is finite. If there exists a strategy-proof, tops-only, anonymous, and unanimous SCF $g : \mathcal{D}^n \to A$, where $n \geq 2$ is an even positive integer that induces $\succeq$ over A, then (i) $A \subseteq \{0, 1\}^K$ for some positive integer K, (ii) there exists $\alpha \in A$ such that $f_k(x_k, y_k) = med_{>}(x_k, y_k, \alpha_k)$ for $k \in \{1, \ldots, K\}$ and $x, y \in A$, where the SCF $f : \mathcal{D}^2 \to A$ is induced by g, and (iii) $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq)$ or, equivalently, for all $x, y, z \in A$, $[y \in MB(x, \alpha) \cap MB(x, z)] \Rightarrow [yR_i^x z]$.*

REFERENCES

AUSTEN-SMITH, D., AND J. BANKS, *Positive Political Theory I: Collective Preference* (Ann Arbor: University of Michigan Press, 1999).

———, AND ———, *Positive Political Theory II: Strategy and Structure* (Ann Arbor: University of Michigan Press, 2005).

BARBERÀ, S., "An Introduction to Strategy-Proof Social Choice Functions," *Social Choice and Welfare* 18 (2001), 619–53.

———, "Strategy-Proof Social Choice," in K. J. Arrow, A. K. Sen, and K. Suzumura, eds., *Handbook of Social Choice and Welfare* (Amsterdam: North-Holland, 2010), Volume 2, Chapter 25, 731–831.
———, D. Berga, and B. Moreno, "Individual versus Group Strategy-Proofness: When Do They Coincide?" *Journal of Economic Theory* 145 (2010), 1648–74.
———, ———, and ———, "Group Strategy-Proofness in Private Good Economies," *American Economic Review* 106 (2016), 1073–99.
———, F. Gul, and E. Stacchetti, "Generalized Median Voter Schemes and Committees," *Journal of Economic Theory* 61 (1993), 262–89.
———, J. Massó, and A. Neme, "Maximal Domains of Preferences Preserving Strategy-Proofness for Generalized Median Voter Schemes," *Social Choice and Welfare* 16 (1999), 321–36.
———, H. F. Sonnenschein, and L. Zhou, "Voting by Committees," *Econometrica* 59 (1991), 595–609.
Berga, D., "Strategy-Proofness and Single-Plateaued Preferences," *Mathematical Social Sciences* 35 (1998), 105–20.
———, and S. Serizawa, "Maximal Domain for Strategy-Proof Rules with One Public Good," *Journal of Economic Theory* 90 (2000), 39–61.
Black, D., "On the Rationale of Group Decision Making," *Journal of Political Economy* 56 (1948), 23–34.
Bogomolnaia, A., "A Characterization of Median Voter Schemes," in Ph.D. Dissertation, Universitat Autònoma de Barcelona, 1998, *Medians and Lotteries: Strategy-Proof Social Choice Rules for Restricted Domains*, Chapter 4, 45–73.
Border, K., and J. S. Jordan, "Straightforward Elections, Unanimity and Phantom Voters," *Review of Economic Studies* 50 (1983), 153–70.
Chatterji, S., and A. Sen, "Tops-Only Domains," *Economic Theory* 46 (2011), 255–82.
———, R. Sanver, and A. Sen, "On Domains That Admit Well-Behaved Strategy-Proof Social Choice Functions," *Journal of Economic Theory* 148 (2013), 1050–73.
Ching, S., and S. Serizawa, "A Maximal Domain for the Existence of Strategy-Proof Rules," *Journal of Economic Theory* 78 (1998), 157–66.
Danilov, V. I., "The Structure of Non-manipulable Social Choice Rules on a Tree," *Mathematical Social Sciences* 27 (1994), 123–31.
Demange, G., "Single Peaked Orders on a Tree," *Mathematical Social Sciences* 3 (1983), 389–96.
Hatsumi, K., D. Berga, and S. Serizawa, "A Maximal Domain for Strategy-Proof and No-Vetoer Rules in the Multi-object Choice Model," *International Journal of Game Theory* 43 (2014), 153–68.
Inada, K., "A Note on the Simple Majority Decision Rule," *Econometrica* 32 (1964), 525–31.
Le Breton, M., and J. Weymark, "Strategy-Proof Social Choice with Continuous Separable Preferences," *Journal of Mathematical Economics* 32 (1999), 47–85.
Moulin, H., "On Strategyproofness and Single-Peakedness," *Public Choice* 35 (1980), 437–55.
———, "Generalized Condorcet Winners for Single Peaked and Single Plateau Preferences," *Social Choice and Welfare* 1 (1984), 127–47.
Nehring, K., and C. Puppe, "The Structure of Strategy-Proof Social Choice–Part I: General Characterization and Possibility Results on Median Spaces," *Journal of Economic Theory* 135 (2007a), 269–305.
———, and ———, "Efficient and Strategy-Proof Voting Rules: A Characterization," *Games and Economic Behavior* 59 (2007b), 132–53.
———, and ———, "Abstract Arrowian Aggregation," *Journal of Economic Theory* 145 (2010), 467–94.
Reffgen, A., "Strategy-Proof Social Choice on Multiple and Multi-dimensional Single-Peaked Domains," *Journal of Economic Theory* 157 (2015), 349–83.
Savaglio, E., and S. Vannucci, "Strategy-Proofness and Single Peakedness in Bounded Distributive Lattices," mimeo, Cornell University Library, 2014.
Schummer, J. P., and R. Vohra, "Strategy-Proof Location on a Network," *Journal of Economic Theory* 104 (2002), 405–28.
Serizawa, S., "Power of Voters and Domain of Preferences Where Voting by Committees Is Strategy-Proof," *Journal of Economic Theory* 67 (1995), 599–608.
Sprumont, Y., "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule," *Econometrica* 59 (1991), 509–19.
———, "Strategy-Proof Collective Choice in Economic and Political Environments," *Canadian Journal of Economics* 28 (1995), 68–107.
Weymark, J., "Strategy-Proofness and the Tops-Only Property," *Journal of Public Economic Theory* 10 (2008), 7–26.