# On Strategy-proofness and the Salience of Single-peakedness*

Shurojit Chatterji[†] and Jordi Massó[‡]

July 10, 2015

Abstract: We consider strategy-proof social choice functions operating on a rich domain of preference profiles. We show that if the social choice function satisfies in addition tops-onlyness, anonymity and unanimity then the preferences in the domain have to satisfy a variant of single-peakedness (referred to as semilattice single-peakedness). We do so by deriving from the social choice function an endogenous partial order (a semilattice) from which the notion of a semilattice single-peaked preference can be defined. We also provide a converse of this main finding. Finally, we show how well-known restricted domains under which nontrivial strategy-proof social choice functions are admissible are semilattice single-peaked domains. Our characterization of a semi-lattice single-peaked domain may be viewed as a converse to the Gibbard-Satterthwaite theorem.

*We would like to thank Huaxia Zeng for helpful discussions and detailed comments on a preliminary version of the paper. Chatterji would like to acknowledge research support from SMU grant number C244/MSS13E001. Massó would like to acknowledge financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centers of Excellence in R&D (SEV-2011-0075) and grants ECO2008-0475-FEDER (Grupo Consolidado-C) and ECO2014-53051, and from the Generalitat de Catalunya, through the research grant SGR2014-515. We thank the participants of the IDGP 2015 Workshop *Institutions, Decisions and Governmental Practices: Theory, Simulations and Applications* held in UAB (Barcelona) and the Conference on *Economic Design* 2015 held in Istanbul Bilgi University, for helpful comments.

[†]Singapore Management University. School of Economics, 90 Stamford Road. Singapore 178903. E-mail: shurojitc@smu.edu.sg

[‡]Universitat Autònoma de Barcelona and Barcelona GSE. Departament d'Economia i d'Història Econòmica, Edifici B, UAB Campus. 08193, Cerdanyola del Vallès (Bellaterra). E-mail: jordi.masso@uab.es

*Keywords*: Strategy-proofness; Single-peakedness, Anonymity; Unanimity; Tops-onlyness; Semilattice.

*JEL Classification Numbers*: D71.

# 1 Introduction

Strategy-proofness plays a central role in mechanism design. A social choice function is strategy-proof if, for every preference profile, truthtelling is a dominant strategy in its induced game form. Hence, the potentially complex strategic decision problems of agents involved in a strategy-proof social choice function are extremely simple indeed. Whether or not an agent's strategy is dominant depends only on the preferences of the agent and not on the other agents' preferences. Under strategy-proofness the interlinked decisions become a collection of independent optimization problems. Thus, the use of a strategy-proof social choice function does not require (as any other solution concept related to Nash equilibrium would) any informational hypothesis about the beliefs that each agent has about the other agents' preferences, and the subsequent iteration of beliefs until the preference profile becomes common knowledge. However, the Gibbard-Satterthwaite theorem states that requiring truthful reporting of preferences in weakly dominant strategies implies dictatorship whenever preferences of agents are unrestricted. This fundamental result has directed subsequent research on social choice in the presence of private information towards suitably restricted domains of preferences which admit to the design of anonymous, and hence non-dictatorial strategy-proof social choice functions. Particularly prominent in this regard is the class of single-peaked preferences and its variants and the strategy-proof social choice functions characterized for such domains are extensions of the median voter scheme.[1] Single-peaked preferences are well known to have desirable properties in the context of aggregation theory. They also provide the underpinnings of many models in political and public economics.[2]

Single-peaked preferences have been specified by postulating an underlying structure on the set of alternatives that allows one to state for every triple $x, y$ and $z$ of alternatives, that $y$ is between $x$ and $z$, and so on, and the restriction imposed by single-peakedness is that if $x$ is top-ranked for a particular preference ordering, then $y$, by virtue of being in between $x$ and $z$, be ranked at least as high as $z$. This paper formulates a more general notion of single-peakedness in terms of a partial order on the set of alternatives with the property

[1]Single-peakedness was initially proposed by Black (1948) and Inada (1964). The surveys of Barberà (2001, 2010) and Sprumont (1995) contain several axiomatic characterizations of the median voter scheme and its extensions.

[2]See Austen-Smith and Banks (1999, 2005).

that every pair of alternatives possesses a supremum under the postulated partial order.[3] Our notion of single-peakedness requires that for any triple $x, y$ and $z$ of alternatives, a preference ordering that has $x$ as its top-ranked alternative should rank the supremum of the pair $(x, y)$ at least as high as the supremum of the pair $(z, y)$.[4]

Our main finding is that this notion of single-peakedness is *implied* by the existence of a strategy-proof and anonymous social choice function which is determined completely by the profile of the agents top-ranked alternatives (*i.e.*, it is tops-only), and satisfies additionally the innocuous requirement of unanimity, whenever such a social choice function can be defined for an even number of agents and the underlying domain satisfies a *richness* requirement.[5] Our approach reconstructs the partial order on alternatives in a natural way from the social choice function with the four stated properties. Our methodology applies to domains that allow the design of *well-behaved* social choice functions for some even number of voters. While this restriction to an even number of voters is somewhat awkward, we do not necessarily view it as a drawback of our approach, given that our intention is to reconstruct features of a domain of preferences that allows the design of well-behaved social choice functions for all societies; indeed while our methodology would not identify a domain that allows well-behaved social choice functions to be designed only for societies with an odd number of agents, one might argue that such a domain would not be attractive from a design perspective. The semilattice single peaked condition identified by our methodology suffices for the design of well-behaved strategy proof social choice functions for all, in particular odd, number of agents.

Fix a tops-only and unanimous social choice function. Assume the number of agents is two and let $x$ and $y$ be two alternatives. We say that $x \succeq y$ if and only if $x$ is chosen at any profile of preferences where one agent has $x$ as the top-ranked alternative and the other $y$. The assumed axioms of unanimity and anonymity imply that $\succeq$ is reflexive and antisymmetric respectively. Our requirement that the domain of preferences be rich ensures that $\succeq$ is transitive and that the social choice function must be of a particular form: at any profile of preferences, the social choice *is* the supremum of the pair of alternatives that are top-ranked by the two agents. Our definition of single-peakedness now obtains as a direct consequence of strategy-proofness. This methodology applies whenever the number of agents is even. A similar finding holds under an additional axiom of invariance when a social choice function with the aforementioned properties can be defined only for an odd number of agents. As a converse to our main finding, we show that any domain of preferences (there is no richness requirement) which is single-peaked with respect to a

[3]A partial order is a reflexive, antisymmetric and transitive binary relation.

[4]Later in the paper we explain this property and discuss why it may be seen as a weakening of single-peakedness.

[5]Most interesting rules identified in the restricted domain literature generate binary relations that allow interesting preference domains to satisfy our richness requirement.

partial order possessing the supremum property admits a strategy-proof, anonymous, and unanimous social choice function that is completely determined by the profile of the agents top-ranked alternatives, for any number of agents.

In the literature on social choice on restricted domains, there has been interest in formulating a sort of converse to the Gibbard-Satterthwaite theorem; a statement that would identify features of a domain that are implied by the design of a unanimous, strategy-proof social choice function that is "non-dictatorial". It has been conjectured that domain restrictions of the single-peaked variety and social choice functions of the median voter scheme form are salient in this regard.[6] We formalize a non-dictatorial social choice function using the axiom of anonymity and require additionally that the social choice function satisfy the tops-only property. For the complete domain, strategy-proofness and unanimity imply the tops-only property. Given that we work in a restricted domain setting with no structure on the set of alternatives, it does not appear feasible to derive the tops-only property as a consequence of strategy-proofness and unanimity; we accordingly impose it as an axiom. Our characterization of a semilattice single-peaked domain may be viewed as a converse to the Gibbard-Satterthwaite theorem since the semilattice structure on alternatives is implied by the axioms on the social choice function and a richness condition relative to the social choice function, and does not rely on any a priori structure on the set of alternatives or preferences (apart from the requirement that each preference has a unique top-ranked alternative).

## 1.1 Related Literature

An early formulation of a partially converse statement to the Gibbard-Satterthwaite theorem is Bogomolnaia (1998). In a model with finitely many alternatives and two agents, she identifies the features of any anonymous and tops-only social choice function under which the finite set of alternatives can be embedded into a finite dimensional euclidean space with a grid structure with the property that the social choice function takes the form of a (multi-dimensional) median voter scheme. This embedding depends crucially on the set of alternatives being finite. These features of tops-only and anonymous social choice functions are stated in terms of the same binary relation induced by a two agent tops-only and anonymous social choice function that we use in our paper, and are the following: (i) the binary relation is transitive and a semilattice and (ii) the social choice function is the supremum of the pair of alternatives that are the top-ranked alternatives of the two voters. These findings are extended to the three agent case under similar, but somewhat more demanding, hypothesis and she derives additionally that the domain of preferences

[6] Conjectures of this nature have been attributed by Barberà (2010) to Faruk Gul and referred to as Gul's conjecture.

must be multi-dimensional single-peaked on the set of alternatives. Our work extends this methodology in the following sense. We postulate a richness condition on the domain in terms of the binary relation on alternatives induced by a two agent social choice function satisfying our axioms and *derive* that the binary relation is transitive and that the social choice function has the supremum property. This is used to establish the salience of the supremum rule and a version of single-peaked preferences in a general setting with an arbitrary number of voters without requiring the set of alternatives to be finite. In particular, under our richness condition, the set of alternatives need not turn out to be embedded in a finite dimensional euclidean space with a grid structure as in Bogomolnaia (1998), but the characterization of the social choice function as a supremum rule on our version of a single-peaked domain remains valid.

More recently, work by Nehring and Puppe (2007a,b), and Chatterji, Sanver and Sen (2013), provide formulations of such a converse statement. Our paper complements these approaches and is closely related to the approach of these papers in that our axioms on the social choice function are similar. However, there are important differences in the scope of our model and our methodology. The richness condition in these papers is specified independently of the social choice function whose existence is postulated whereas in our paper the richness condition is specified in relation to the social choice function. But more importantly, the methodology in these papers relies also on the finiteness of the set of alternatives and on strict preferences. The approach of Nehring and Puppe (2007a,b) assumes a specific algebraic structures on the set of alternatives. The richness condition in Chatterji, Sanver and Sen (2013) is specified in terms of alternatives that appear as the first and second ranked alternatives in different preference orderings which makes it specific to a model with finitely many alternatives with strict preferences and also excludes the consideration of preferences commonly employed in the study of multidimensional models. Our formulation is more permissive in that we impose no finiteness requirement on the set of alternatives and, provided the top-ranked alternative is unique, we admit indifferences. As a consequence, our methodology is of necessity different and somewhat more direct than that of those papers. Our methods lead to a simple and a fairly general version of a statement to the effect that a particular form of single-peakedness is implied by strategy-proofness in conjunction with anonymity and other natural axioms and that this form of single-peakedness suffices for the design of social choice functions with these properties. Many prominent restricted domains of preferences studied in the literature appear as special cases of our formulation.

The paper is organized as follows. Section 2 introduces basic definitions and notation while Section 3 contains the main results for the case of an even number of agents. In Section 4 we partially extend our results to the case of an odd number of agents. In Section 5 we relate our results to the large literature on domain restrictions for non-trivial strategy-proof social choice functions. Section 6 contains some final remarks and an appendix contains further analysis of the case of an odd number of agents and proofs of two results, omitted in the main text.

## 2 Basic definitions and notation

Let $N = \{1, ..., n\}$ be the finite set of *agents*, with $n \geq 2$, and $A$ be any set of *alternatives*. We do not assume any *a priori* structure on the set of alternatives. Each agent $i \in N$ has a preference (relation) $R_i \in \mathcal{D}$ over $A$, where $\mathcal{D}$ is a subset of complete, reflexive and transitive binary relations on $A$. The set $\mathcal{D}$ is referred to as the *domain* of preferences. For any $x, y \in A$, $xR_iy$ means that agent $i$ considers alternative $x$ to be at least as good as alternative $y$. Let $P_i$ and $I_i$ denote the strict and indifference relations induced by $R_i$ over $A$, respectively; namely, for any $x, y \in A$, $xP_iy$ if and only if $xR_iy$ and $\neg yR_ix$, and $xI_iy$ if and only if $xR_iy$ and $yR_ix$. We assume that for each $R_i \in \mathcal{D}$ there exists $t(R_i) \in A$, the *top* of $R_i$, such that $t(R_i)P_iy$ for all $y \in A\backslash\{t(R_i)\}$. For $x \in A$, let $R_i^x$ denote any preference in $\mathcal{D}$ with $t(R_i^x) = x$. Moreover, we assume that for each $x \in A$ the domain $\mathcal{D}$ contains at least one preference $R_i^x$. A *profile* $R = (R_1, ..., R_n) \in \mathcal{D}^n$ is an $n-$tuple of preferences, one for each agent. To emphasize the role of agent $i$ we will often write the profile $R$ as $(R_i, R_{-i})$.

A *social choice function* (SCF) is a mapping $f : \mathcal{D}^n \to A$ that assigns to every profile $R \in \mathcal{D}^n$ an alternative $f(R) \in A$.

A SCF $f : \mathcal{D}^n \to A$ is *tops-only* if for all $R, R' \in \mathcal{D}^n$ such that $t(R_i) = t(R_i')$ for all $i \in N$, $f(R) = f(R')$. Hence, a tops-only SCF $f : \mathcal{D}^n \to A$ can be written as $f : A^n \to A$. Accordingly, we will on occasion use the notation $f(t(R_1), ..., t(R_n))$ interchangeably with $f(R_1, ..., R_n)$.

A SCF $f : \mathcal{D}^n \to A$ is *unanimous* if for all $R \in \mathcal{D}^n$ and $x \in A$ such that $t(R_i) = x$ for all $i \in N$, $f(R) = x$.

To define an anonymous SCF on $\mathcal{D}^n$, for every profile $R \in \mathcal{D}^n$ and every one-to-one mapping $\sigma : N \to N$, define the profile $R^\sigma = (R_{\sigma(1)}, ..., R_{\sigma(n)})$ as the $\sigma-$permutation of $R$, where for all $i \in N$, $R_{\sigma(i)}$ is the preference that agent $\sigma(i)$ had in the profile $R$. A SCF $f : \mathcal{D}^n \to A$ is *anonymous* if for all one-to-one mappings $\sigma : N \to N$ and all $R \in \mathcal{D}^n$, $f(R^\sigma) = f(R)$.

A SCF $f : \mathcal{D}^n \to A$ is *strategy-proof* if for all $i \in N$, all $R \in \mathcal{D}^n$ and all $R_i' \in \mathcal{D}$,

$$f(R)R_if(R_i', R_{-i}).$$

A SCF $f$ is strategy-proof if for every agent at every preference profile $R$ truth-telling is a weakly dominant strategy in the direct revelation game induced by $f$ at $R$.

In this paper, in addition to strategy-proofness, we will require the SCF to satisfy anonymity. This is a key assumption in our analysis and is in some ways an opposite of dictatorship as the identity of no particular agent matters in determining the social outcome. The appeal of this axiom is obvious. In addition we will impose that the SCF also satisfy the tops-only requirement. This axiom simplifies considerably the specification of the SCF and is pervasive in the literature on the characterization of strategy-proof SCFs on restricted domains.[7] However, there are results on restricted domains establishing that tops-onlyness is a requirement of strategy-proofness together with an additional property like unanimity, efficiency or ontoness (see for instance Barberà, Sonnenschein and Zhou (1991) or Sprumont (1995)). But these approaches start from the very beginning with a given domain (often related to single-peakedness) whose structure is explicitly used in obtaining tops-onlyness as a requirement of strategy-proofness (and the additional property). Our difficulty to follow this approach is that we do not impose any structure on the domain of the SCF, except that it has to be rich. The axiom of unanimity is natural to impose and is mild as it follows as a consequence of strategy-proofness whenever the SCF is required to be onto the set of alternatives.

# 3 Results

## 3.1 Obtaining the induced binary relation

In this subsection we assume that $n = 2$ and indicate how to obtain a binary relation $\succeq$ from a tops-only SCF $f : \mathcal{D}^2 \to A$ and show that if the SCF satisfies in addition unanimity and anonymity, then $\succeq$ is reflexive and antisymmetric.[8] In doing so, we follow a procedure introduced by Bogomolnaia (1998).

Let $f : \mathcal{D}^2 \to A$ be a tops-only SCF. Define the binary relation $\succeq$ induced by $f$ over $A$ as follows: for all $x, y \in A$,

$$x \succeq y \text{ if and only if } f(x, y) = x. \tag{1}$$

A SCF aggregates individual preferences. A SCF can be seen as a systematic procedure specifying how a society resolves its members' disagreements. Hence, the binary relation $\succeq$ induced by a SCF $f$ over $A$ may be interpreted as the outcome of this procedure applied to the family of basic situations in which there are only two agents and two alternatives under consideration; the relation $x \succeq y$ reflects the fact that in this scenario the alternative

[7] Moulin (1984) and Berga (1998) indicate the difficulties of extending our results to a setting where agents' preferences may have several tops.

[8] A binary relation $\succeq$ over $A$ is *reflexive* if for all $x \in A$, $x \succeq x$, and it is *antisymmetric* if for all $x, y \in A$, $[x \succeq y \text{ and } y \succeq x] \Rightarrow [x = y]$.

$x$ prevails over $y$.[9] We will show later that if the SCF $f$ is strategy-proof, tops-only and anonymous, then its induced binary relation $\succeq$ is transitive, provided the domain of $f$ satisfies a richness condition. Here we note that the following result is immediate.

Remark 1 *Let $f : \mathcal{D}^2 \to A$ be a tops-only SCF and $\succeq$ be the binary relation induced by $f$ over $A$. If $f$ is unanimous, then $\succeq$ is reflexive. If $f$ is anonymous, then $\succeq$ is antisymmetric.*

## 3.2 Rich domain and semilattice single-peaked preferences

We now turn to a description of the domain of preferences that we characterize in this paper. First we present the notion of a rich domain on a set of alternatives endowed with a binary relation. Fix a binary relation $\succeq$ over $A$. Given two alternatives $x, y \in A$ with $y \succeq x$, define the set $[x, y]$ as

$$[x, y] = \{x, y\} \cup \{z \in A \mid y \succeq z \text{ and } z \succeq x\}.$$

If $x$ and $y$ are distinct alternatives and related by $\succeq$ as $y \succeq x$, then the set $[x, y]$ is obtained by adding to the set $\{x, y\}$ all alternatives in $A$ that "lie between" $x$ and $y$ according to $\succeq$ . For $y \not\succeq x$ define $[x, y] = \emptyset$.

Definition 1 *Fix a binary relation $\succeq$ over $A$. The domain $\mathcal{D}$ is* rich *on $(A, \succeq)$ if for all $x, y \in A$ with $[x, y] \neq \emptyset$ and $z \notin [x, y]$, there exist $R_i^x, R_i^y \in \mathcal{D}$ such that $yP_i^x z$ and $xP_i^y z$.*

Richness is a mild requirement. It says that for any pair of distinct alternatives $x$ and $y$ related by $\succeq$ and any alternative $z$ not lying between $x$ and $y$, a rich domain has to contain two preference relations with the properties that for one of the preferences $x$ is the top-ranked alternative and $y$ is strictly better than $z$, and for the other preference $y$ is the top-ranked alternative and $x$ is strictly better than $z$. Below we will illustrate the notion of rich domain by means of an example.

We now exhibit conditions under which $\succeq$ is transitive.[10]

Lemma 1 *Let $f : \mathcal{D}^2 \to A$ be a strategy-proof, tops-only and anonymous SCF. Let $\succeq$ be the binary relation induced by $f$ over $A$ and assume that $\mathcal{D}$ is rich on $(A, \succeq)$. Then, $\succeq$ is transitive.*

*Proof*: Assume the three distinct alternatives $x, y, z \in A$ are such that $x \succeq y$ and $y \succeq z$. We show that $x \succeq z$; namely, $f(x, z) = x$. First, suppose $f(x, z) = w \notin \{x, y\}$. By strategy-proofness, $f(x, w) = w$. Hence, $w \succeq x \succeq y$ and $w \notin [y, x] \neq \emptyset$. Since $\mathcal{D}$ is rich on $(A, \succeq)$,

[9] Since the binary relation is not required to be complete, it may be the case that neither alternative prevails over the other and $f(x, y)$ is a third alternative $z$.

[10] A binary relation $\succeq$ over $A$ is *transitive* if for all $x, y, z \in A$, $[x \succeq y \text{ and } y \succeq z] \Rightarrow [x \succeq z]$.

there exists $R_1^x \in \mathcal{D}$ such that $yP_1^x w$. But then,

$$f(y,z) = yP_1^x w = f(x,z),$$

a contradiction with strategy-proofness of $f$. Thus, $f(x,z) \in \{x,y\}$. Assume $f(x,z) = y$. But then, by strategy-proofness, $f(x,y) = y$, a contradiction with strategy-proofness of $f$. Hence $f(x,z) = x$ and $x \succeq z$. Thus, $\succeq$ is transitive. ■

A *partial order* $\succeq$ over $A$ is a reflexive, antisymmetric and transitive binary relation over $A$. A partial order $\succeq$ over $A$ is a *(join-)semilattice* if for all $(x,y) \in A \times A$, $\sup_{\succeq}(x,y)$ exists. We now turn to our notion of a single-peaked preference in this setting.

Definition 2 *Let $\succeq$ be a semilattice over $A$. The preference $R_i^x \in \mathcal{D}$ is* semilattice single-peaked on $(A,\succeq)$ *if for all $y,z \in A$, $\sup_{\succeq}(x,y) R_i^x \sup_{\succeq}(z,y)$.*

We say that a domain $\mathcal{D}$ is semilattice single-peaked on $(A,\succeq)$ if it is a subset of all semilattice single-peaked preferences on $(A,\succeq)$.

Single-peaked preferences embodies the idea that an alternative $y$ that is "closer" to the top $x$ of a preference ordering $R_i^x$ than is an alternative $z$, should be ranked at least as high as $z$. We now argue that semilattice single-peakedness embodies in some measure this idea in its treatment of those pairs of alternatives that arise as suprema under the semilattice $\succeq$ . Given a triple of alternatives $a,b,c$, we say that $b$ is "closer" to $a$ than is $c$ according to the semilattice $\succeq$, if $a \preceq b$ holds and $a \preceq c \preceq b$ (equivalently, $c \in [a,b]$) does not hold. Now consider any preference $R_i^x \in \mathcal{D}$ and consider any pair of alternatives $y,z$. Assume first that $\sup_{\succeq}(z,y) \succeq x$. Then we have $x \preceq \sup_{\succeq}(x,y) \preceq \sup_{\succeq}(z,y)$ holds, so that $\sup_{\succeq}(x,y)$ is closer to the top $x$ of $R_i^x$ than is $\sup_{\succeq}(z,y)$. Even when the condition $\sup_{\succeq}(z,y) \succeq x$ does not hold, we have at any rate that $\sup_{\succeq}(z,y) \notin [x, \sup_{\succeq}(x,y)]$ and here too $\sup_{\succeq}(x,y)$ is closer to the top $x$ of $R_i^x$ than is $\sup_{\succeq}(z,y)$. Indeed the condition of semilattice single-peakedness requires that in this situation, $\sup_{\succeq}(x,y)$ being closer to the top $x$, should be ranked by $R_i^x$ at least as high as $\sup_{\succeq}(z,y)$.

To better understand the notions of richness and semilattice single-peakedness on $(A,\succeq)$, it is convenient to look at the semilattice $(A,\succeq)$ as a partially directed graph. To make the argument more transparent assume $A$ is finite and that $\sup_{\succeq} A$ exists and is denoted by $\alpha$. Figure 1 below represents an example of such a semilattice $(A,\succeq)$ as a partially directed graph, where $A = \{x,y,z,\alpha,x_1,...,x_{13}\}$ and the direction of an arrow on the edge linking two alternatives indicates how they are related according to the partial order $\succeq$; for example, $x \longrightarrow y$ means that $y \succeq x$ (arrows that can be obtained from the transitivity of $\succeq$ are omitted).

First consider the pair of alternatives $\alpha, x$. Since $\alpha \succeq x$, the set $[x,\alpha]$ is non-empty and equals $\{x,y,z,x_2,x_3,x_4,\alpha\}$. The requirement of richness would for the set $[x,\alpha]$

then require there exist $R_i^x, R_i^\alpha \in \mathcal{D}$ such that $\alpha P_i^x v$ and $x P_i^\alpha v$ only for alternatives $v \in \{x_1, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$.

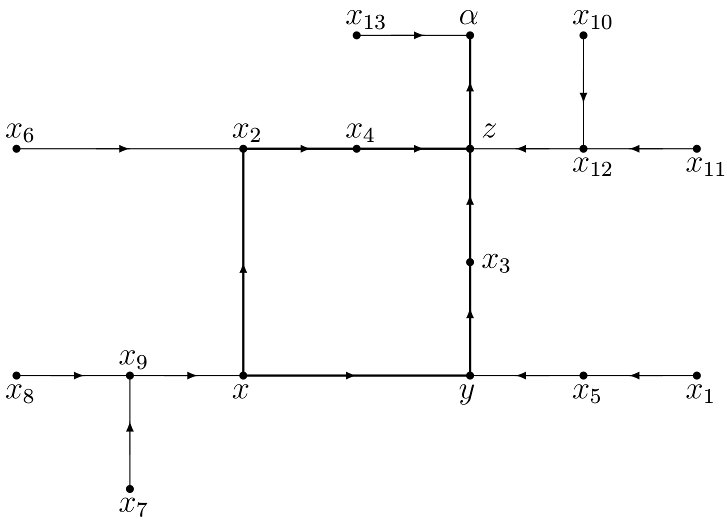


Figure 1

We next illustrate the restrictions implied by semilattice single-peakedness on a preference ordering where the alternative $x$ is top-ranked. The definition of semilattice single-peakedness imposes two sorts of restrictions on a preference relation $R_i^x$ (in addition to $xP_i^x y$ for all $y \neq x$). The first of these applies to alternatives that appear along any $\succeq -$path emanating from $x$. There are two such paths from $x$ to $\alpha$ (emphasized with bold type links); namely, $x \preceq y \preceq x_3 \preceq z \preceq \alpha$ and $x \preceq x_2 \preceq x_4 \preceq z \preceq \alpha$. Along such paths, we have classical single-peakedness. Thus, since the pair $y, z$ belong to the first path, we have $yR_i^x z$. Observe that $\sup_{\succeq}(x, y) = y$ and $\sup_{\succeq}(z, y) = z$. However, note that since the alternatives $x_3$ and $x_4$ belong to different paths, there is no restriction on the relative ranking of these two alternatives in $R_i^x$; indeed if one were to apply Definition 2 with $x_3, x_4$ playing the role of $y, z$ respectively, one only obtains $\sup_{\succeq}(x, x_3) = x_3 R_i^x z = \sup_{\succeq}(x_4, x_3)$.

The second restriction applies to alternatives that are not in a $\succeq -$path from $x$ to $\alpha$. Such alternatives are dispreferred to the "closest" alternative in the path; namely, if $w$ and $r$ are such that $x \preceq w \preceq \alpha$, $r \notin [x, \alpha]$, and $\sup_{\succeq}(x, r) = w$, then $wR_i^x r$ (observe that $\sup_{\succeq}(r, r) = r$). For instance in Figure 1, $yR_i^x x_5$ and $yR_i^x x_1$ but no condition is imposed on the preference between $x_5$ and $x_1$; moreover, take any $z', z'' \in \{x_{10}, x_{11}, x_{12}\}$ such that $z' \neq z''$ and observe that $\sup_{\succeq}(x, z') = z$, $\sup_{\succeq}(z'', z') = x_{12}$ and $\sup_{\succeq}(z', z') = z'$. Then, $zR_i^x x_{12}$ and $zR_i^x z'$.

Finally, we enumerate below the restrictions implied on a preference $R_i^x$ over $A$. By definition, we know that $xP_i^x y'$ for all $y' \notin A \backslash \{x\}$. Semilattice single-peakedness imposes the following relations among (few) pairs of alternatives (observe that in Figure 1, $z$ is the supremum of $A\backslash\{\alpha, x_{13}\}$):[11]

- $yR_i^x x_3 R_i^x z$ since $\sup_{\succeq}(x, y) = yR_i^x x_3 = \sup_{\succeq}(x_3, y)$ and $\sup_{\succeq}(x, x_3) = x_3 R_i^x z = \sup_{\succeq}(z, x_3)$.
- $x_2 R_i^x x_4 R_i^x z$ since $\sup_{\succeq}(x, x_2) = x_2 R_i^x x_4 = \sup_{\succeq}(x_4, x_2)$ and $\sup_{\succeq}(x, x_4) = x_4 R_i^x z = \sup_{\succeq}(z, x_4)$.
- $yR_i^x x_k$ for $k = 1, 5$ since $\sup_{\succeq}(x, x_k) = yR_i^x x_k = \sup_{\succeq}(x_k, x_k)$ (i.e., $x_k$ plays simultaneously the role of $y$ and $z$ in Definition 2).
- $x_2 R_i^x x_6$ since $\sup_{\succeq}(x, x_6) = x_2 R_i^x x_6 = \sup_{\succeq}(x_6, x_6)$ (i.e., $x_6$ plays simultaneously the role of $y$ and $z$ in Definition 2).
- $zR_i^x x_k$ for $k = 10, 11, 12$ since $\sup_{\succeq}(x, x_k) = zR_i^x x_k = \sup_{\succeq}(x_k, x_k)$.
- $\alpha R_i^x x_{13}$ since $\sup_{\succeq}(x, x_{13}) = \alpha R_i^x x_{13} = \sup_{\succeq}(x_{13}, x_{13})$ (i.e., $x_{13}$ plays simultaneously the role of $y$ and $z$ in Definition 2).

Observe that semilattice single-peakedness leaves freedom to $R_i^x$ on how it orders many pairs of alternatives. For instance, we have already noted that the relative ranking of the pair $x_3, x_4$ is not fixed. Consider next the path $x_7 \longrightarrow x_9 \longrightarrow x$. Here too, letting $x_7, x_9$ play the role of $y, z$ in Definition 2 does not lead to any restriction on the relative rankings of $x_7$ and $x_9$ in $R_i^x$ since $\sup_{\succeq}(x, x_9) = xR_i^x x_9 = \sup_{\succeq}(x_7, x_9)$.

### 3.3 Results for the case of $n$ even

We now proceed by first showing that any strategy-proof, tops-only and anonymous SCF $f : \mathcal{D}^2 \to A$ can be seen as the supremum of the binary relation $\succeq$ induced by $f$ over $A$, provided that the domain of $f$ is rich on $(A, \succeq)$.[12]

LEMMA **2** *Let $f : \mathcal{D}^2 \to A$ be a strategy-proof, tops-only and anonymous SCF. Let $\succeq$ be the binary relation induced by $f$ over $A$ and assume that $\mathcal{D}$ is rich on $(A, \succeq)$. Then, for all $x, y \in A$, $f(x, y) = \sup_{\succeq}(x, y)$.*

*Proof*: Let $x, y \in A$ and assume first that $x \neq y$. If $f(x, y) = x$, then $x \succeq y$ and $x = \sup_{\succeq}(x, y)$. Similarly if $f(x, y) = y$. Assume $f(x, y) = z \notin \{x, y\}$. By strategy-proofness,

[11] In addition to the relations derived from the transitivity of $R_i^x$, these are the *only* relations imposed on $R_i^x$ by semilattice single-peakedness.

[12] Subsection 6.2 contains an example of a set $A$ and a strategy-proof, tops-only, anonymous and unanimous SCF $f$ on a domain that is not rich on $(A, \succeq)$ with the property that $\succeq$ is not a semilattice and $f$ does not take the supremum form.

$f(z, y) = f(x, z) = z$. Hence, $z \succeq x$ and $z \succeq y$. Thus, $z$ is an upper bound of $(x, y)$. Assume $z \neq \sup_{\succeq}(x, y)$; namely, there exists $\bar{z} \in A$, $\bar{z} \neq z$, such that $\bar{z} \succeq x$ and $\bar{z} \succeq y$ and either $z \succeq \bar{z}$, or $z$ is not comparable to $\bar{z}$. In either case we have $\bar{z} \not\preceq z$ and hence, $z \notin [x, \bar{z}] \neq \emptyset$. Furthermore we have $f(\bar{z}, y) = \bar{z}$. Since $\mathcal{D}$ is rich on $(A, \succeq)$, there exists $R_1^x \in \mathcal{D}$ such that $\bar{z} P_1^x z$. But then,

$$f(\bar{z}, y) = \bar{z} P_1^x z = f(x, y),$$

a contradiction with strategy-proofness of $f$. Assume now that $x = y$ and $f(x, x) = z$. We want to show that $\sup_{\succeq}(x, x) = z$. Suppose not; *i.e.*, there exists $w \in A$ such that $w \succeq x$ and either $z \succeq w$ or $z$ is not comparable to $w$. In either case we have $w \not\preceq z$ and so $z \notin [x, w] \neq \emptyset$. Since $\mathcal{D}$ is rich on $(A, \succeq)$ there exists $R_1^x \in \mathcal{D}$ such that $w P_1^x z$. But then,

$$f(w, x) = w P_1^x z = f(x, x),$$

a contradiction with strategy-proofness of $f$. ■

We now extend our preliminary results to the case where $n$ is any positive even integer.

Given a strategy-proof, tops-only and anonymous SCF $g : \mathcal{D}^n \to A$ where $n$ is a positive even integer, let $N_1 = \{1, ..., \frac{n}{2}\}$ and let $N_2 = \{\frac{n}{2} + 1, ..., n\}$. Define an SCF $f : \mathcal{D}^2 \to A$ by setting, for all $(R_1, R_2) \in \mathcal{D}^2$, $f(R_1, R_2) = g(\bar{R})$ where $\bar{R} \in \mathcal{D}^n$ is such that $\bar{R}_j = R_1$ for all $j \in N_1$ and $\bar{R}_j = R_2$ for all $j \in N_2$. We note the following fact which appears as Proposition 2 in Chatterji, Sanver and Sen (2013).

FACT **1** *Let $\mathcal{D}$ be an arbitrary domain and let $n$ be a positive even integer. Suppose there exists a strategy-proof, tops-only and anonymous SCF $g : \mathcal{D}^n \to A$. Then SCF $f : \mathcal{D}^2 \to A$, defined by setting, for all $(R_1, R_2) \in \mathcal{D}^2$, $f(R_1, R_2) = g(\bar{R})$ where $\bar{R} \in \mathcal{D}^n$ is such that $\bar{R}_j = R_1$ for all $j \in N_1$ and $\bar{R}_j = R_2$ for all $j \in N_2$, is strategy-proof, tops-only and unanimous. Moreover, if $g$ is unanimous, then so is $f$.*

Lemmata 1 and 2 do not require that the SCF should be unanimous. If the SCF $f$ in Lemmata 1 and 2 is unanimous then the binary relation $\succeq$ induced by $f$ over $A$ is reflexive. Moreover, if the SCF $g$ in Fact 1 is unanimous, then so is $f$. From now on we will be interested only in unanimous SCFs.

In view of Fact 1, we say that a strategy-proof, tops-only, anonymous and unanimous SCF $g : \mathcal{D}^n \to A$, where $n$ is a positive even integer, *induces a binary relation $\succeq$ over $A$*, where it is understood that $\succeq$ is the binary relation induced by $f$ over $A$ where $f$ is *induced* from $g$ by "cloning" the first $\frac{n}{2}$ agents as agent 1 and the remaining as agent 2. We now state our principal finding.

PROPOSITION **1** *Let $g : \mathcal{D}^n \to A$ be a strategy-proof, tops-only, anonymous and unanimous SCF where $n$ is a positive even integer. Let $\succeq$ be the binary relation induced by $g$ over $A$ and* *assume that $\mathcal{D}$ is rich on $(A, \succeq)$. Then, (i) $\succeq$ is a semilattice over $A$, (ii) for all $x, y \in A$, $f(x, y) = \sup_{\succeq}(x, y)$, where $f$ is induced from $g$, and (iii) $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq)$.*

*Proof*: The proofs of (i) and (ii) follow from Lemmata 1 and 2 respectively. To show that the condition specified in Definition 2 holds, observe that by Lemma 2 and strategy-proofness, $f(x, y) = \sup_{\succeq}(x, y) R_1^x \sup_{\succeq}(z, y) = f(z, y)$. ■

We next show that a semilattice single-peaked domain admits a strategy-proof, tops-only, anonymous and unanimous SCF for an arbitrary number of agents.

PROPOSITION **2** *Let $\mathcal{D}$ be a semilattice single-peaked domain on the semilattice $(A, \succeq)$. Then, there exists a strategy-proof, tops-only, anonymous and unanimous SCF $f : \mathcal{D}^n \to A$ for all $n$, which when $n$ is even is such that $\succeq$ is induced by $f$ over $A$.*

*Proof*: We first establish the following induction step: Suppose for $k \geq 2$, $\sup_{\succeq}(x_1, ..., x_k)$ exists for every set $\{x_1, ..., x_k\}$ of $k$ distinct alternatives. Then for any alternative $x_{k+1} \notin \{x_1, ..., x_k\}$, $\sup_{\succeq}(x_1, ..., x_{k+1})$ exists and is given by $\sup_{\succeq}\big(\sup_{\succeq}(x_1, ..., x_k), x_{k+1}\big)$.

To verify this step, let $y = \sup_{\succeq}(x_1, ..., x_k)$. By the induction hypothesis, $\sup_{\succeq}(y, x_{k+1})$ exists and is denoted $w$. Since $\succeq$ is transitive, $w$ is an upper bound for $(x_1, ..., x_{k+1})$. Suppose there exists $v \in A \backslash \{w\}$ such that $v$ is an upper bound for $(x_1, ..., x_{k+1})$. Then it must be that $v \succeq y$ since $y = \sup_{\succeq}(x_1, ..., x_k)$. We also have $v \succeq x_{k+1}$. These imply that $v$ is an upper bound for $(y, x_{k+1})$. But since $\sup_{\succeq}(y, x_{k+1})$ exists and is $w$, we must have $v \succeq w$ and so $w = \sup_{\succeq}(x_1, ..., x_{k+1})$.

Given a preference profile $R \in \mathcal{D}^n$, let $G(R) = \{x_1, ..., x_k\}, k \leq n$, be the set of distinct alternatives such that for each $t = 1, ..., k, x_t = t(R_i)$ for some $i \in N$.

For every $R \in \mathcal{D}^n$, define

$$f(R) = \sup_{\succeq} G(R). \tag{2}$$

Since $\succeq$ is a semilattice, the induction step verified earlier implies that $f$ is well-defined. By construction, $f$ is tops-only, anonymous and unanimous. We next show that $f$ is strategy-proof. Given $R \in \mathcal{D}^n$ and $i \in N$, let $G(R_{-i}) = G(R) \backslash \{t(R_i)\}$ and observe that $f(R_i, R_{-i}) = \sup_{\succeq}(t(R_i), \sup_{\succeq} G(R_{-i}))$. To show that $f$ is strategy-proof, we wish to show for arbitrary $R_i^x \in \mathcal{D}$ and $R_i^z \in \mathcal{D}$, $z \in A \backslash \{x\}$,

$$f(R_i^x, R_{-i}) = \sup_{\succeq}(x, \sup_{\succeq} G(R_{-i})) R_i^x \sup_{\succeq}(z, \sup_{\succeq} G(R_{-i})) = f(R_i^z, R_{-i}). \tag{3}$$

By the definition of $f$ in (2) and the definition of semilattice single-peakedness, (3) holds.

It is straightforward to verify that when $n$ is a even positive integer, $\succeq$ is induced by $f$ as defined in (2). ■

# 4 A partial extension to the case of $n$ odd

We consider in this section an extension of our results to the case where the domain $\mathcal{D}$ is assumed to admit a SCF $g : \mathcal{D}^n \to A$ which is strategy-proof, tops-only, anonymous and unanimous where $n \geq 3$ is a positive odd integer. We will do so by introducing an additional axiom.[13] This axiom requires that the SCF satisfy an *invariance* requirement across two profiles of preferences where agents tops are either of two alternatives $x$ or $y$, when the number of agents with top $x$ and top $y$ differ by exactly one across the two profiles.

DEFINITION **3** *The SCF $g : \mathcal{D}^n \to A$, where $n \geq 3$ is a positive odd integer, satisfies* invariance *if for every $x, y \in A$, for every $i \in N$ and every pair of preference profiles of the form $(R_i, R_{-i}), (R_i', R_{-i})$ where $t(R_i) = x$ and $t(R_i') = y$, and $R_{-i}$ is any subprofile where $\frac{n-1}{2}$ agents have $x$ as their top and $\frac{n-1}{2}$ have $y$ as their top, it is the case that $g(R_i, R_{-i}) = g(R_i', R_{-i})$.*

Let $g : \mathcal{D}^n \to A$ be a tops-only SCF, where $n \geq 3$ is a positive odd integer. Define the binary relation $\succ_o$ induced by $g$ over $A$ as follows: for all $(x, y) \in A \times A$, let $(\underbrace{x, ..., x}_{\frac{n+1}{2}}, \underbrace{y, ..., y}_{\frac{n-1}{2}})$ denote a profile of top-ranked alternatives where the first $\frac{n+1}{2}$ agents have $x$ as the top and the remaining $\frac{n-1}{2}$ agents have $y$ as the top and define

$$x \succeq_o y \text{ if and only if } g(\underbrace{x, ..., x}_{\frac{n+1}{2}}, \underbrace{y, ..., y}_{\frac{n-1}{2}}) = x. \tag{4}$$

REMARK **2** *Let $g : D^n \to A$, where $n \geq 3$ is positive odd integer, be a tops-only SCF and $\succeq_o$ be the binary relation induced by $g$ over $A$. If $g$ is anonymous, and satisfies invariance, then $\succeq_o$ is antisymmetric. If $g$ is unanimous, then $\succeq_o$ is reflexive.*

REMARK **3** *The analogues of Lemmata 1 and 2 can be proved analogously for $\succeq_o$ as well by standard arguments. We omit the details.*

Finally we obtain as Proposition 3 the extension of Proposition 1 to the case where $n \geq 3$ is a positive odd integer and the SCF satisfies in addition invariance.

PROPOSITION **3** *Let $g : \mathcal{D}^n \to A$ be a strategy-proof, tops-only, anonymous and unanimous SCF that satisfies invariance where $n \geq 3$ is a positive odd integer. Let $\succeq_o$ be the binary relation induced by $g$ over $A$ and assume that $\mathcal{D}$ is rich on $(A, \succeq_o)$. Then, (i) $\succeq_o$ is a semilattice over $A$, (ii) for all $x, y \in A$, $g(\underbrace{x, ..., x}_{\frac{n+1}{2}}, \underbrace{y, ..., y}_{\frac{n-1}{2}}) = \sup_{\succeq_o}(x, y)$, and (iii) $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq_o)$.*

[13] We consider in Appendix 7.1 a version of our analysis without this axiom.

Remark 4 *Part (iii) of the Proposition establishes that $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq_o)$. Consequently, by an application of Proposition 2 there exists a strategy-proof, tops-only, anonymous and unanimous SCF $f : \mathcal{D}^n \to A$ for all $n$. Furthermore, we note that the SCF constructed in the proof of Proposition 2 also satisfies invariance.*

# 5 Related literature

In this section we relate our results to the large literature on restricted domains. The starting point of this approach is to assume that the set of alternatives $A$ has a particular structure (for instance, $A$ is a linearly ordered set). Using this structure one can define a meaningful domain restriction on preferences over $A$ (for instance, single-peakedness) under which non-trivial strategy-proof SCFs can be defined (for instance, the median voter scheme). Our Proposition 2 (and its proof) follows partially this approach. We start by hypothesizing that the set $A$, together with the binary relation $\succeq$, is a semilattice from which we define the domain $\mathcal{D}$ of semilattice single-peaked preferences on $(A, \succeq)$. We then show that there exists a strategy-proof, tops-only, anonymous and unanimous SCF $f$ on the domain $\mathcal{D}$ which, when $n$ is a positive even integer, is such that $\succeq$ is induced by $f$ over $A$. We want to emphasize however that our main contribution is Proposition 1, which follows a very different approach. Without assuming any structure on the set of alternatives $A$, we suppose that there is a strategy-proof, tops-only, anonymous and unanimous SCF $g$ on a given domain $\mathcal{D}$ of preferences over $A$. Following Bogomolnaia (1998) we show how to identify using condition (1) a binary relation $\succeq$ over $A$. Then, provided that the domain $\mathcal{D}$ is rich on $(A, \succeq)$, we prove that $(A, \succeq)$ is a semilattice, the domain $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq)$ and $g$ can be obtained as the supremum rule of a two-agents SCF $f$ induced from $g$. Hence, the semilattice structure on $A$ follows from the existence of a SCF satisfying the desirable properties without imposing any condition on $A$ whatsoever. We now relate with more detail our results to some representative results of the restricted domains literature.

## 5.1 Single-peaked preferences on a line

The most significant domain restriction is single-peakedness, originally proposed by Black (1948) and studied by Moulin (1980). Following the latter, assume that the set of alternatives is the unit interval in the real line endowed with the linear order $>$; *i.e.*, $A = [0, 1]$. A preference $R_i$ is *single-peaked* on $A$ if there exists a unique alternative $t(R_i) \in A$ such that, for all $x \in A \backslash \{t(R_i)\}$, $t(R_i) P_i x$ and for all $x, y \in A$, $x R_i y$ whenever either $t(R_i) \geq x > y$ or $y > x \geq t(R_i)$. Let $\mathcal{SP}$ be the set of all single-peaked preferences on $A$. Following Moulin (1980), a SCF $f : \mathcal{SP}^2 \to A$ is a *median voter scheme* if there exists a fixed ballot $\alpha \in A$

such that for all $(R_1, R_2) \in \mathcal{SP}^2$,[14]

$$f(R_1, R_2) = med_{>}(t(R_1), t(R_2), \alpha).$$

A characterization result in Moulin (1980) implies that any strategy-proof, tops-only, anonymous and unanimous SCF $f : \mathcal{SP}^2 \to A$ is a median voter scheme. We relate this setting with our result.

Assume the SCF $f : \mathcal{SP}^2 \to A$ is strategy-proof, tops-only, anonymous and unanimous. Let $\alpha$ be its associated fixed ballot and $\succeq$ be the semilattice obtained from $f$ using (1). Then, the following facts can be verified.

(a) The binary relation $\succeq$ induced from $f$ using (1) is as follows: if either $y < x \leq \alpha$ or $\alpha \leq x < y$ then $x \succeq y$ and if $x > \alpha > y$ then $x \not\succeq y$ and $y \not\succeq x$. Figure 2 below gives a geometric representation of this semilattice.


Figure 2

(b) For all $x, y \in A$, $f(x, y) = \sup_{\succeq}(x, y) = med_{>}\{x, y, \alpha\}$.

(c) The domain $\mathcal{SP}$ is rich on $(A, \succeq)$ and it is a strict subset of the set of all semilattice single-peaked preferences on $(A, \succeq)$.[15]

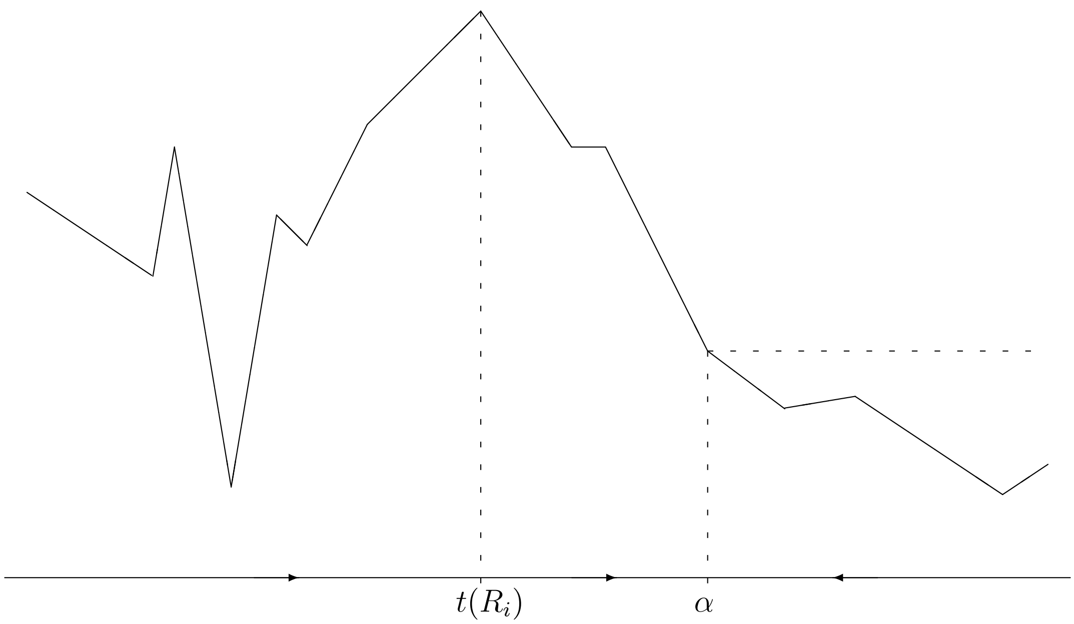


Figure 3

[14]Given a list of $K$ real numbers $(x_1, ..., x_K)$, where $K$ is a positive odd integer, define $med_{>}(x_1, ..., x_K) = y$, where $y \in \mathbb{R}$ is such that $\#\{t \in \{1, ..., K\} \mid x_t \geq y\} = \#\{t \in \{1, ..., K\} \mid x_t \leq y\} = \frac{K+1}{2}$.

[15]In fact, the set $\mathcal{SP}$ is the intersection of all sets of semilattice single-peaked preferences, where each of these sets is associated to each of all possible values $\alpha$ in $A$.

Figure 3 illustrates a semilattice single-peaked preference $R_i$ on $(A, \succeq)$ when $\sup_{\succeq} A = \alpha$. Observe four features of $R_i$. First, $R_i$ is far from being single-peaked on $A$. Second, $R_i$ is monotonically (not necessarily strictly) decreasing on the segment $[t(R_i), \alpha]$, and hence single-peaked on it. Third, no condition is imposed between pairs on $[0, t(R_i))$. Fourth, $\alpha R_i x$ for each alternative $x \in (\alpha, 1]$ and no condition is imposed between pairs of alternatives on this segment.

## 5.2 Semi single-peaked preferences

The notion of single-peakedness on a tree was introduced by Demange (1983) and studied further by Danilov (1994).[16] A weaker notion called *semi single-peakedness* was introduced in Chatterji, Sanver and Sen (2013). It can be described as follows. Assume that the set of alternatives $A$ is a finite tree; *i.e.*, for every pair of alternatives (nodes) $x, y \in A$, there is a *unique* path $p$ linking them, denoted $\langle x, y \rangle$. Two alternatives $x, y$ are directly linked if $\langle x, y \rangle = \{x, y\}$.[17] Given alternatives $x, y, z \in A$, let $\pi(z, \langle x, y \rangle)$ denote the projection of $z$ on the path $\langle x, y \rangle$ which is defined as the *unique* alternative $w \in A$ such that $\langle x, z \rangle \cap \langle y, z \rangle = \langle w, z \rangle$. A path $p$ is "maximal" if it cannot be extended by adding more edges at either one of the two ends. Fix a particular alternative $\alpha$ on the tree $A$ (call it *the threshold*), and use it to specify a threshold on *every* maximal path $p$, denoted $\lambda(p)$, as $\lambda(p) = \pi(\alpha, p)$. Thus, for every maximal path $p$, if it contains the alternative $\alpha$, set $\lambda(p) = \alpha$; otherwise, the threshold $\lambda(p)$ is the *unique* alternative that lies on every path from an alternative on $p$ to $\alpha$.

Given $A$ with a tree structure and the threshold $\alpha \in A$ define the binary relation $\succeq$ as follows: for all $x, y \in A$,

$$x \succeq y \text{ if and only if } x = \pi(\alpha, \langle x, y \rangle),$$

which is equivalent, using the direct graphic expression to, $x \succeq y$ if and only if $x \in \langle \alpha, y \rangle$.

Then, the following facts can be verified.

(a) $(A, \succeq)$ is a semilattice.

(b) The set of strict semilattice single-peaked preferences on $(A, \succeq)$ coincides with the set of *semi single-peaked preferences on the tree $A$ with respect to the threshold $\alpha$* where, according to Chatterji, Sanver and Sen (2013), a preference $R_i$ belongs to the latter set if for all $x, y \in A$:

[16] Savaglio and Vannucci (2014) extends the analysis to graphs that are not necessarily trees. We further comment on this paper in Subsection 6.3. Schummer and Vohra (2002) also study a model where the set of alternatives is possibly infinite and arranged as a graph. They consider separately the case where the graph is a tree and the case where the graph has cycles. They characterize strategy-proof and onto SCFs assuming preferences are Euclidean, which satisfy our richness condition.

[17] See Example 2 in Subsection 6.3 for a description of how to, given a domain of preferences, directly link two alternatives.

(i) $\left[x, y \in p \text{ such that } x, y \in \langle t(R_i), \lambda(p)\rangle \text{ and } x \in \langle t(R_i), y\rangle\right] \Rightarrow [xR_iy]$, and
(ii) $\left[x \in p \text{ such that } \lambda(p) \in \langle t(R_i), x\rangle\right] \Rightarrow [\lambda(p)R_ix]$.

(c) The two person SCF $f$, where for all $x, y \in A$,

$$f(x, y) = \pi(\alpha, \langle x, y\rangle) \tag{5}$$

is strategy-proof, tops-only, anonymous and unanimous on the domain of semi single-peaked preferences on the tree $A$ with respect to the threshold $\alpha$. Moreover, $f(x, y) = \sup_{\succeq}(x, y)$. Figure 4 below illustrates this construction.

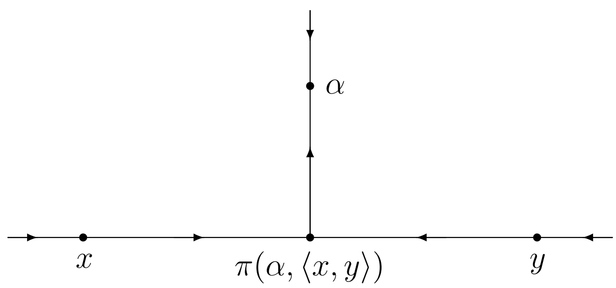


Figure 4

(d) The domain of all semi single-peaked preferences on the tree $A$ with respect to the threshold $\alpha$ is rich on $(A, \succeq)$.

## 5.3 Multidimensional models

In many social choice problems alternatives are multidimensional. To describe an alternative one has to specify the level reached in each of its attributes. Our setting includes also these cases. Border and Jordan (1983) and Barberà, Gul, and Stacchetti (1993) are prototypical examples of this approach, and they can be seen as extensions of Moulin (1980). We first relate our results with the main ones contained in these two papers and second, with the results in Barberà, Sonnenschein and Zhou (1991) on voting by quota, the case when each attribute can take only two possible values.

### 5.3.1 Multidimensional single-peaked preferences

Assume the set of alternatives $A$ is a Cartesian product of subsets of real numbers; *i.e.*,

$$A = \prod_{k=1}^{K} A_k,$$

where, for each $k = 1, ..., K$, $A_k \subseteq \mathbb{R}$ can be finite or infinite.[18] Define the $L_1$−norm in $A$ as follows: for every $x \in A$,

$$\|x\| = \sum_{k=1}^{K} |x_k| .$$

Given $x, y \in A$, let

$$MB(x, y) = \{z \in A \mid \|x - y\| = \|x - z\| + \|z - y\|\}$$

be the minimal box containing $x$ and $y$.

A preference $R_i \in \mathcal{D}$ is *multidimensional single-peaked on* $A$ if, for all $y \in MB(x, t(R_i))$, $yR_ix$ holds (namely, alternatives that lie on a $L_1$−path going from $x$ to $t(R_i)$ should be ranked at least as high as $x$).[19] Let $\mathcal{MSP}$ be the set of all such preferences on $A$.

General results in Border and Jordan (1983) and Barberà, Gul, and Stacchetti (1993) imply the following characterization. Let $f : \mathcal{MSP}^2 \to A$ be an anonymous and unanimous SCF. Then, $f$ is strategy-proof if and only if there exists a vector of fixed ballots $\alpha = (\alpha_1, ..., \alpha_K) \in A$ such that for all $(R_1, R_2) \in \mathcal{MSP}^2$ and $k = 1, ..., K$,

$$f_k(R_1, R_2) = med_{>}(t_k(R_1), t_k(R_2), \alpha_k).$$

Consider a SCF $f : \mathcal{MSP}^2 \to A$ and let $\succeq$ be the semilattice obtained from $f$ using (1). Furthermore, assume $f$ is strategy-proof, anonymous and unanimous, and let $\alpha \in A$ be its associated vector of fixed ballots. The following facts hold.

(a) For all $x, y \in A$,

$$x \succeq y \text{ if and only if } x \in MB(y, \alpha).$$

Hence, for each $k \in \{1, ..., K\}$ the pair $(A_k, \succeq_k)$ is a semilattice where $\succeq_k$ is defined as follows: for $x_k, y_k \in A_k$,

$$x_k \succeq_k y_k \text{ if and only if either } y_k \leq x_k \leq \alpha_k \text{ or } \alpha_k \leq x_k \leq y_k.$$

This means that the semilattice $(A, \succeq)$ can be equivalently described by the family of semilattices $\{(A_k, \succeq_k)\}_{k=1}^{K}$.

(b) Let $x \in A$. The preference $R_i^x$ is semilattice single-peaked on $(A, \succeq)$ if and only if for all $y, z \in A$ such that $y \in MB(x, \alpha) \cap MB(x, z)$, $yR_i^x z$. We illustrate it in Figure 5 below for the case $K = 2$ (see Appendix 7.2 for a general proof of this statement). Observe that $y \in MB(x, \alpha)$, $y \in MB(x, z)$, $y \in MB(x, w)$, and $\sup_{\succeq}(x, y) = yR_i^x z = \sup_{\succeq}(z, y)$ and $\sup_{\succeq}(x, w) = yR_i^x w = \sup_{\succeq}(w, w)$.

[18] Border and Jordan (1983) study the infinite case while Barberà, Gul and Stacchetti (1993) study the finite case.

[19] It is possible to show that the set of star-shaped and separable preferences on $A$ (defined in Border and Jordan (1983)) coincides with the set of multidimensional single-peaked preferences on $A$.

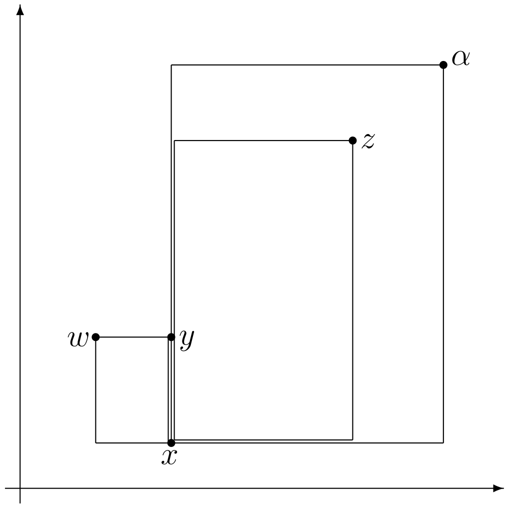


Figure 5

(c) Let $f : \mathcal{MSP}^2 \to A$ be a strategy-proof, tops-only, anonymous and unanimous SCF and let $\alpha = (\alpha_1, ..., \alpha_K) \in A$ be its associated vector of fixed ballots. Then, for all $x, y \in A, f(x, y) = (\sup_{\succeq_1}(x_1, y_1), ..., \sup_{\succeq_K}(x_K, y_K))$.

(d) The set of all multidimensional single-peaked preferences $\mathcal{MSP}$ is rich on $(A, \succeq)$.[20]

### 5.3.2 Voting by committees and separable preferences

Barberà, Sonnenschein and Zhou (1991) contains another example of a domain restriction that can be described in a multidimensional setting. It is as follows. Let $\mathcal{K} = \{1, ..., K\}$ be a finite set of objects. Agents have to choose a subset of $\mathcal{K}$ (possibly empty). Hence, the set of alternatives is the family $2^{\mathcal{K}}$ of all subsets of $\mathcal{K}$ which can be identified with the $K-$dimensional hypercube $\{0, 1\}^K$. Namely, any set $X \in 2^{\mathcal{K}}$ can be described as the vector $x \in \{0, 1\}^K$ where, for each $k = 1, ..., K$, $x_k = 1$ if and only if $k \in X$.

A (strict) preference $R_i$ on $A$ is said to be *separable* if adding an object to a given set makes the new set better if and only if the added object is good (as a singleton set, the object is preferred to the empty set). In the hypercube representation of $2^{\mathcal{K}}$, separability of $R_i$ means the following. Let $x$ be the vector of zeros and ones representing the best subset of objects according to $R_i$, and take any pair of vectors $y$ and $z$ of zeros and ones (*i.e.*, two subsets of objects $Y$ and $Z$). From $z$ obtain $x$ by iterating the following procedure. Take a coordinate of $z$ that does not coincide with the corresponding coordinate of $x$, and replace it by the coordinate of $x$, obtaining $z'$. Proceed similarly from $z'$, until $x$ is reached. Then, $yR_iz$ if and only if $y$ is obtained in one of the steps for some of these procedures starting at $z$ to obtain $x$. Let $\mathcal{S}$ be the set of all separable preferences on $\{0, 1\}^K$.

For simplicity we consider two agent SCFs. Following Barberà, Sonnenschein and Zhou (1991) a SCF $f : \mathcal{S}^2 \to \{0, 1\}^K$ is *voting by quota* (not necessarily neutral) if there exists

[20] As in the unidimensional case, the set $\mathcal{MSP}$ is the intersection of all sets of semilattice single-peaked preferences,where each of these sets is associated to each of all possible values $\alpha$ in $A$.

$q \in \{1,2\}^K$ such that for all $(R_1, R_2) \in \mathcal{S}^2$ and all $k = 1, ..., K$,

$$f_k(R_1, R_2) = 1 \text{ if and only if } \#\{i \in N \mid t_k(R_i) = 1\} \geq q_k. \tag{6}$$

A characterization result in Barberà, Sonnenschein and Zhou (1991) implies that any strategy-proof, tops-only, anonymous and unanimous SCF $f : \mathcal{S}^2 \to \{0,1\}^K$ is voting by quota. We indicate now how to relate this setting with our result.

Let $f : \mathcal{S}^2 \to \{0,1\}^K$ be a voting by quota $q$ and from it, define the vector $\alpha \in \{0,1\}^K$ as follows: for every $k = 1, ..., K$, set

$$\alpha_k = \begin{cases} 1 & \text{if } q_k = 1 \\ 0 & \text{if } q_k = 2. \end{cases}$$

Next, define the binary relation $\succeq$ over $\{0,1\}^K$ as follows: for all $x, y \in \{0,1\}^K$,

$$x \succeq y \text{ if and only if } x \in MB(\alpha, y).$$

We show here that $\alpha = \sup_{\succeq}\{0,1\}^K$. To see that, fix $k \in \{1, ..., K\}$ and consider any $x$ and $y$ such that $x_k = 1$ and $y_k = 0$. Assume $q_k = 1$. Then, $f_k(x, y) = 1$ and $1 \succeq_k 0$. Hence, $\sup_{\succeq_k}\{0,1\} = 1 = \alpha_k$. Assume now that $q_k = 2$. Then, $f_k(x, y) = 0$ and $0 \succeq_k 1$. Hence, $\sup_{\succeq_k}\{0,1\} = 0 = \alpha_k$. In Appendix 7.3 we show that $\succeq$ is induced by $f$ over $A$ by condition (1).

Finally, it is easy to see that the set of all separable preferences is rich on $(\{0,1\}^K, \succeq)$.

# 6 Final remarks

We finish the paper with some final remarks related to issues left aside during the presentation of the main results.

## 6.1 Example of a non rich domain

Our methodology relies on establishing that a two agent strategy-proof, tops-only, anonymous and unanimous SCF $f : \mathcal{D}^2 \to A$ induces a semilattice over $A$ and that $f$ takes the supremum form, *i.e.*, for every $x, y \in A$, $\sup_{\succeq}(x, y)$ exists and $f(x, y) = \sup_{\succeq}(x, y)$. We show here that the rich domain condition is indispensable for this property. The example below exhibits a domain $\mathcal{D}$, a strategy-proof, tops-only, anonymous and unanimous SCF $f : \mathcal{D}^2 \to A$ whose induced partial order $\succeq$ over $A$ is not a semilattice because for some $x, y \in A$, $\sup_{\succeq}(x, y)$ does not exist (and hence $f(x, y) \neq \sup_{\succeq}(x, y)$), and where $\mathcal{D}$ is not a rich domain on $(A, \succeq)$.

**Example 1:** Let $A = \{x, y, z, \bar{z}, w\}$ be the set of alternatives and $\mathcal{D}$ the domain of five strict preferences:

| $P^x$ | $P^y$ | $P^z$ | $P^{\bar{z}}$ | $P^w$ |
|---|---|---|---|---|
| $x$ | $y$ | $z$ | $\bar{z}$ | $w$ |
| $z$ | $z$ | $w$ | $w$ | $x$ |
| $\bar{z}$ | $\bar{z}$ | $\bar{z}$ | $z$ | $y$ |
| $w$ | $w$ | $x$ | $y$ | $z$ |
| $y$ | $x$ | $y$ | $x$ | $\bar{z}$ |

.

Consider the strategy-proof, tops-only, anonymous and unanimous SCF $f : \mathcal{D}^2 \to A$ defined by the following table:

| $f$ | $x$ | $y$ | $z$ | $\bar{z}$ | $w$ |
|---|---|---|---|---|---|
| $x$ | $x$ | $z$ | $z$ | $\bar{z}$ | $w$ |
| $y$ | $z$ | $y$ | $z$ | $\bar{z}$ | $w$ |
| $z$ | $z$ | $z$ | $z$ | $w$ | $w$ |
| $\bar{z}$ | $\bar{z}$ | $\bar{z}$ | $w$ | $\bar{z}$ | $w$ |
| $w$ | $w$ | $w$ | $w$ | $w$ | $w$ |

.

The partial order $\succeq$ induced by $f$ can be represented by the following figure:

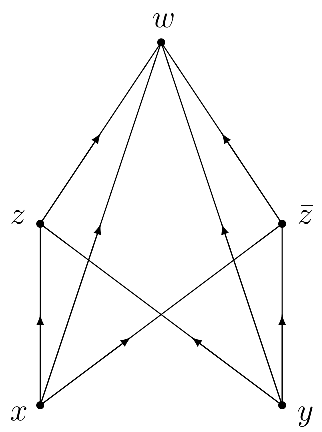


Figure 6

This partial order $\succeq$ is not a semilattice since $\sup_{\succeq}(x, y)$ does not exist. But the domain $\mathcal{D}$ is not rich on $(A, \succeq)$ since $z \notin [x, \bar{z}] \neq \emptyset$ and there does not exist any $\hat{P}^x \in \mathcal{D}$ such that $x\hat{P}^x\bar{z}\hat{P}^x z$; observe that there are other missing preferences, for instance any $\hat{P}^{\bar{z}}$ such that $\bar{z}\hat{P}^{\bar{z}}x\hat{P}^{\bar{z}}z$. $\square$

## 6.2 Finite set of alternatives

We identify in the Corollary below a set of necessary conditions on any strategy-proof, tops-only, anonymous and unanimous SCF which applies to the case where the set of alternatives is finite and $n$ is an even positive integer. This is obtained by application of Proposition 1 with a result on two agent SCFs from Bogomolnaia (1998).

COROLLARY *Let $g : \mathcal{D}^n \to A$ be a strategy-proof, tops-only, anonymous and unanimous SCF where $n \geq 2$ is an even positive integer and $A$ is finite. Let $\succeq$ be the binary relation induced by $g$ over $A$ and assume that $\mathcal{D}$ is rich on $(A, \succeq)$. Then, (i) $A \subseteq \{0,1\}^K$ for some positive integer $K$, (ii) there exists $\alpha \in A$ such that $f_k(x_k, y_k) = med_>(x_k, y_k, \alpha_k)$ for $k \in \{1, ..., K\}$ and $x, y \in A$, where the SCF $f : \mathcal{D}^2 \to A$ is induced by $g$, (iii) $\mathcal{D}$ is semilattice single-peaked on $(A, \succeq)$, or equivalently,* for all $x, y, z \in A$, $[y \in MB(x, \alpha) \cap MB(x, z)] \Rightarrow [yR_i^x z]$.

## 6.3 Relation to other notions of single-peakedness

Nehring and Puppe (2007a,b) start with an abstract algebraic structure of a property space on a finite set of alternatives and a notion of "betweenness", and use it to define the notion of generalized single-peakedness. The necessity part of their characterization is similar to our analysis in spirit and shows that if there exists an onto, strategy-proof, anonymous and neutral social choice function on a rich domain of generalized single-peaked preferences induced by a property space, then this property space is a median space.[21] The notions of generalized single-peakedness and semilattice single-peakedness are related but independent of each other. For instance, the complete domain, which never appears in our analysis is a generalized single-peaked domain.

The domain of preferences we characterize is closer in spirit to semi single-peaked domains. Semilattice single-peakedness extends the notion of semi single-peakedness in at least three directions. The key differences are that the set of alternatives may be infinite and preferences admit indifferences. The notion of semi single-peakedness is built upon an undirected graph which is necessarily a tree. The notion of semilattice single-peakedness can be illustrated via a directed graph (which need not be a tree when viewed as an undirected graph by ignoring the direction). Finally, the threshold (as described in Subsection 5.2) does not have to be an alternative; for instance, when $A = (0,1) \subset \mathbb{R}$ and the partial order $\succeq$ is the natural order $>$ on real numbers (a semilattice on $(0,1)$) then, $1 \notin A$ would play the role of the threshold in Chatterji, Sanver and Sen (2013)'s construction. We show below that the analysis of Chatterji, Sanver and Sen (2013) is not implied by our analysis restricted to finitely many alternatives.

[21] See Bogomolnaia (1998) for characterizations of median voter schemes using medians on median graphs.

**Example 2:** Let $A = \{w, \alpha, x, v, y\}$ be the set of alternatives. We consider the following domain $\mathcal{D}$ of exactly eight strict preferences given below:

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ | $P_8$ |
|---|---|---|---|---|---|---|---|
| $w$ | $\alpha$ | $\alpha$ | $x$ | $x$ | $v$ | $v$ | $y$ |
| $\alpha$ | $w$ | $x$ | $\alpha$ | $v$ | $x$ | $y$ | $v$ |
| $x$ | $x$ | $w$ | $w$ | $\alpha$ | $\alpha$ | $x$ | $x$ |
| $v$ | $v$ | $v$ | $v$ | $w$ | $w$ | $\alpha$ | $\alpha$ |
| $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $w$ | $w$ |

.

This domain is strongly path connected in the terminology of Chatterji, Sanver and Sen (2013) and consequently satisfies their richness condition. The notion of a strongly path connected domain can be seen as follows. The alternatives $w$ and $\alpha$ are said to be strongly connected since there exist two preference orderings $P_1$ and $P_2$ which rank the alternatives $x, v$ and $y$ identically while the positions of $w$ and $\alpha$, the top two ranked alternatives are reversed across the two orderings. Likewise $\alpha$ and $x$, $x$ and $v$, and finally $v$ and $y$ are strongly connected. One now associates to this domain a graph whose vertices are the five alternatives and where two vertices are an edge if and only if they are strongly connected. A domain is said to be strongly path connected if this graph is a connected graph. The domain $\mathcal{D}$ specified above is indeed a strongly path connected domain. Figure 7 below depicts this strongly path connected graph.

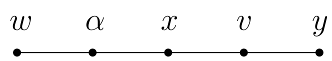


Figure 7

Now consider a median voter scheme $f : \mathcal{D}^2 \to A$, where the fixed ballot is located on $\alpha$ and consider the partial order $\succeq$ associated with this SCF $f$ as defined by (1). Namely, $\alpha \succeq w$ and $\alpha \succeq x \succeq v \succeq y$. Figure 8 below depicts this semilattice.

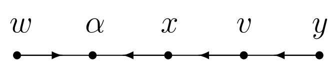


Figure 8

Observe that $[y, x]$ is non-empty and $w \notin [y, x]$. This paper's notion of richness requires that there exist a preference ordering where $x$ is the top ranked alternative and where $y$ is ranked above $w$. This condition is violated by $P_4$ and $P_5$ above. Thus the richness condition of Chatterji, Sanver and Sen (2013) does not imply that our richness condition will necessarily be satisfied. The converse is also true since our notion of richness can be applied to multidimensional models with separable preferences which are excluded by strongly path connected domains.[22] Thus the two notions of richness and consequently the results of the two papers are independent. □

Savaglio and Vannucci (2014) consider a social choice setting where the set of alternatives is a distributive lattice $(A, \leq)$ from which a latticial ternary betweenness relation is defined: $z$ lies between $x$ and $y$ if and only if $x \wedge y \leq z \leq x \vee y$, where the binary operations $\wedge$ and $\vee$ are the infimum and the supremum taken according to $\leq$, respectively. Agents' preferences satisfy some unimodality conditions, that are consistent with this latticial ternary betweenness relation. They study and characterize strategy-proof SCFs on such domains. Note that our setting admits semilattices that are not necessarily lattices (the infimum of pairs of alternatives may not exist) and more importantly, we do not start by assuming an specific structure on the set of alternatives but rather we obtain it as the consequence of the existence of a strategy-proof, tops-only, anonymous and unanimous SCF on a rich domain of preferences.

## 6.4 Invariance

We first illustrate the content of the invariance axiom by exhibiting for well-known settings, SCFs that satisfy it and SCFs that do not.

Consider in the Moulin (1980) setting the SCF $f : \mathcal{SP}^3 \to [0,1]$ that for all $(x, y, z) \in [0,1]^3$, $f(x, y, z) = med_{>}\{x, y, z, \alpha_1, \alpha_2\}$, where $\alpha_1, \alpha_2 \in [0,1]$. Then, $f$ satisfies invariance if and only if $\alpha_1 = \alpha_2$.

Consider the Barberà, Sonnenschein and Zhou (1991) setting with $n = 3$ and $K = 2$. The SCF $f : \mathcal{S}^3 \to \{0,1\}^2$ defined by quota $q = (q_1, q_2)$ satisfies invariance if and only if $q_1 \neq 2$ and $q_2 \neq 2$.

Clearly, in the Moulin (1980) and in the Barberà, Sonnenschein and Zhou (1991) settings there are many instances of well studied SCFs that satisfy all our requirements but violate invariance. But in both cases, there indeed exists some SCFs which satisfy invariance in addition to the properties we have imposed in this paper. This leads to the following question. Suppose a domain admits a strategy-proof, tops-only, anonymous and unanimous SCF for an odd number of agents. Does it imply that there exists a SCF which in addition to satisfying all the foregoing properties also satisfies invariance, or does one need additional conditions on the domain in order to ensure invariance? The single-peaked domain and the multi-dimensional version of it alluded to above are domains where this property is gotten without any additional conditions. A formal statement that resolves this question is of obvious interest to us, but we are unable to establish any version of it here. In the

[22] Chatterji, Sen and Zeng (2014) characterize single-peaked preferences on a tree (as defined by Demange (1983)) on strongly path connected domains using random social choice functions.

appendix, we provide a brief account of the picture without assuming invariance.

### 6.5 Characterization of all strategy-proof SCFs

Our results indicate that the supremum rule is prominent in the class of strategy-proof, tops-only, anonymous and unanimous SCFs. On an arbitrary (rich or otherwise) domain of semilattice single-peaked preferences, the supremum rule shown in Proposition 2 to possess the aforementioned properties. On the other hand, any SCF with these properties, induces, under the hypothesis of richness, a two agent SCF that coincides with the supremum rule. A complete characterization of all SCFs that are strategy-proof, tops-only, anonymous and unanimous on an arbitrary domain of semilattice single-peaked preferences is outside the scope of the present study.

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# 7 Appendix

## 7.1 Odd number of agents without invariance

We consider here the case where the domain under consideration is known to admit a strategy-proof, tops-only, anonymous and unanimous SCF for an odd number of agents. We do so without imposing invariance on the SCF. At the end of this subsection we introduce invariance in order to understand its role in Proposition 3 in the main text.

We restrict attention to the case $n = 3$. By the cloning method employed in Fact 1, we can induce such an SCF whenever one exists for an odd number of agents that is divisible by 3. Let $f : \mathcal{D}^3 \to A$ be strategy-proof, tops-only, anonymous and unanimous SCF. Fix $x \in A$ and define, following a procedure also introduced by Bogomolnaia (1998), $g_x : \mathcal{D}^2 \to A$ by setting, for each pair $y, z \in A$, $g_x(y, z) = f(x, y, z)$. Then $g_x$ is a strategy-proof, tops-only and anonymous SCF. Note that we cannot deduce that $g_x$ is unanimous since $g_x(y, y) = y$ does not follow from the assumed unanimity of $f$. Let $\succeq_x$ be the binary relation induced by $g_x$ over $A$ using (1). Remark 1 applies and the binary relation $\succeq_x$ is antisymmetric but cannot be assumed reflexive since $g_x(y, y) = y$ is not guaranteed.

We will therefore consider binary relations that are antisymmetric and transitive (which will follow from the richness axiom we introduce below) and refer to them as orders. The following definitions generalize our notions of richness and semilattice single-peakedness to the case at hand.

Definition 4 *Let $A$ be an arbitrary set. A family of orders $\{\succeq_r\}_{r\in A}$ over $A$ is given. The domain $\mathcal{D}$ is* rich *on $(A, \{\succeq_r\}_{r\in A})$ if for any $y, z, w \in A$, if $[y, z]_{\succeq_x}$ is non-empty for some $x \in A$ and $w \notin [y, z]_{\succeq_x}$, then there exist $R_i^y, R_i^z \in \mathcal{D}$ such that $zP_i^y w$ and $yP_i^z w$.*

Definition 5 *Let $\{\succeq_r\}_{r\in A}$ be a family of orders over $A$. The domain $\mathcal{D}$ is order-family single-peaked on $(A, \{\succeq_r\}_{r\in A})$ if for all $x, y, z, w \in A$ and all $R_i^x, R_i^z \in \mathcal{D}$,*
*(i)* $\sup_{\succeq_w}(x, y) R_i^x \sup_{\succeq_w}(z, y)$ *and*
*(ii)* $\sup_{\succeq_z}(x, y) R_i^z \sup_{\succeq_w}(x, y)$.

The proofs of Lemmata 1 and 2 do not require that the two-agent SCF under consideration satisfy unanimity. These Lemmata apply here if the domain $\mathcal{D}$ is rich on $(A, \{\succeq_r\}_{r\in A})$ (in the sense of Definition 4). We omit the details. Consequently, analogously to Proposition 1, we obtain here for all $x, y, z \in A$, $g_x(y, z) = \sup_{\succeq_x}(y, z)$ and $\mathcal{D}$ is order-family single-peaked on $(A, \{\succeq_r\}_{r\in A})$.

To summarize, if a domain $\mathcal{D}$ admits a three agent SCF satisfying strategy-proofness, tops-onlyness, anonymity and unanimity and the richness condition is satisfied, then $\mathcal{D}$ is order-family single-peaked on $(A, \{\succeq_r\}_{r\in A})$. However, this notion of single-peakedness does not suffice for the design of a strategy-proof SCF satisfying tops-onlyness, anonymity and unanimity. This is the principal difficulty in extending our analysis for an even number of agents in Section 2 to the case of an odd number of agents.

We are however able to design an SCF with the required four properties if we introduce additionally a notion of invariance of the family of orders. We express invariance in terms of the family of orders as follows. We say that the family of orders $\{\succeq_r\}_{r\in A}$ satisfies *order-invariance* if $\sup_{\succeq_x}(x,y) = \sup_{\succeq_y}(x,y)$ for all pairs $(x,y)$. This condition would be implied by the existence of a SCF defined for an odd number of agents that satisfies strategy-proofness, tops-onlyness, anonymity, unanimity and invariance in the sense of Definition 3.

We may now define a two agent SCF in the following manner; for any pair $(x,x)$ of alternatives, define $f(x,x) = x$, while for any pair $(x,y)$, $x \neq y$, of alternatives, define $f(x,y) = \sup_{\succeq_x}(x,y) = \sup_{\succeq_y}(x,y)$. It is evident that this SCF satisfies anonymity, unanimity and is tops-only. This SCF will also satisfy strategy-proofness whenever $\mathcal{D}$ is order-family single-peaked on $(A,\{\succeq_r\}_{r\in A})$. Indeed we have $f(x,y) = \sup_{\succeq_y}(x,y)R_i^x \sup_{\succeq_y}(z,y) = f(z,y)$ by (i) of Definition 5. This verification of strategy-proofness uses the invariance of the family of orders in a central way and breaks down without it.

## 7.2 Multidimensional semilattice single-peakedness

We now prove that in the multidimensional model the following characterization of semilattice single-peaked preferences holds:

The preference $R_i^x$ is semilattice single-peaked on $(A,\succeq)$ if and only if for all $y,z\in A$ such that $y\in MB(x,\alpha)\cap MB(x,z)$, $yR_i^xz$.

First, we show that if $R_i^x$ is semilattice single-peaked on $(A,\succeq)$, then for all $y,z\in A$ such that $y\in MB(x,\alpha)\cap MB(x,z), yR_i^xz$. Since $y\in MB(x,\alpha)$, it is true that $y\succeq x$ and hence, $\sup_{\succeq}(x,y) = y$. Moreover, $y\in MB(x,z)$, implies that, for each $k=1,...,K$, either $x_k\le y_k\le z_k$ or $z_k\le y_k\le x_k$. Assume without loss of generality that $x_k\le y_k\le z_k$. Since $x_k\le y_k\le \alpha_k$, $\sup_{\succeq k}(x_k,z_k) = \sup_{\succeq_k}(y_k,z_k) = \alpha_k$ if $\alpha_k\le z_k$ and $\sup_{\succeq k}(x_k,z_k) = \sup_{\succeq_k}(y_k,z_k) = z_k$ otherwise. Hence, $\sup_{\succeq}(x,z) = \sup_{\succeq}(y,z)$. By semilattice single-peakedness, we know that $\sup_{\succeq}(x,z)R_i^x\sup_{\succeq}(z,z)$. Thus, $\sup_{\succeq}(y,z)R_i^xz$. Since $\sup_{\succeq}(x,y)R_i^x\sup_{\succeq}(z,y)$ by semilattice single-peakedness, we have $yR_i^xz$ as required.

Conversely, we show that if for all $y,z\in A$ such that $y\in MB(x,a)\cap MB(x,z)$, $yR_i^xz$, then $R_i^x$ is semilattice single-peaked on $(A,\succeq)$. Given $y,z\in A$, to show that $\sup_{\succeq}(x,y)R_i^x\sup_{\succeq}(z,y)$, it suffices to show that $\sup_{\succeq}(x,y)\in MB(x,\alpha)\cap MB(x,\sup_{\succeq}(z,y))$. Since $\sup_{\succeq}(x,y)\succeq x$, it is evident that $\sup_{\succeq}(x,y)\in MB(x,\alpha)$. Next, to show that $\sup_{\succeq}(x,y)\in MB(x,\sup_{\succeq}(z,y))$, we simplify the notation and let $\sup_{\succeq}(x,y) = w$ and $\sup_{\succeq}(z,y) = w'$. We know that for each $k\in\{1,...,K\}$, $w_k = med_>(x_k,y_k,\alpha_k)$ and $w'_k = med_>(z_k,y_k,\alpha_k)$. Assume without loss of generality that $x_k\le y_k$. Consider three situations: (i) $w_k = x_k$, (ii) $w_k = \alpha_k$ and (iii) $w_k = y_k$. In situation (i), it is evident that either $x_k \leq w_k \leq w'_k$ or $w'_k \leq w_k \leq x_k$. In situation (ii), we know that $x_k \leq \alpha_k \leq y_k$. Consequently, $w'_k = med_>(x_k, y_k, \alpha_k) \geq \alpha_k = w_k$. Hence, $x_k \leq w_k \leq w'_k$. In situation (iii), we know that $\alpha_k \geq y_k$. Consequently, $w'_k = med_>(x_k, y_k, \alpha_k) \geq y_k = w_k$. Hence, $x_k \leq w_k \leq w'_k$. In conclusion, $w_k$ is always in the middle of $x_k$ and $w'_k$ for all $k \in \{1, ..., K\}$. Therefore, $\sup_{\succeq}(x, y) \in MB(x, \sup_{\succeq}(z, y))$ as required.

## 7.3 The binary relation $\succeq$ in voting by committees

We show that in the voting by committees model, the binary relation $\succeq$, obtained by setting for all $x, y \in \{0, 1\}^K$,

$$x \succeq y \text{ if and only if } x \in MB(\alpha, y),$$

is induced by $f$ over $A$ by condition (1).

Assume $x \succeq y$. We want to show that $f(x, y) = x$; *i.e.*, $f_k(x, y) = x_k$ for all $k = 1, ..., K$. Take an arbitrary $k \in \{1, ..., K\}$ and assume first that $f_k(x, y) = 1$. Since $f$ is voting by quota, $x_k + y_k \neq 0$. If $x_k + y_k = 2$ then, $x_k = 1$ and $f_k(x, y) = x_k$. Assume now that $x_k + y_k = 1$. Since $f$ is voting by quota, $q_k = 1$, and by the definition of $\alpha$, $\alpha_k = 1$. To obtain a contradiction, suppose $x_k = 0$. Since, by the definition of $\succeq$, $x \in MB(\alpha, y)$ holds, we have that $y_k = 0$, a contradiction with the assumption that $x_k + y_k = 1$. Assume now that $f_k(x, y) = 0$. Then, $x_k + y_k < q_k$. If $x_k = 0$ then $f_k(x, y) = x_k$, which is what we wanted to prove. If $x_k = 1$ then $q_k = 2$, $\alpha_k = 0$ and $y_k = 0$. Hence, $x \notin MB(\alpha, y)$. Thus, $x \not\succeq y$, a contradiction. Since $k$ was arbitrary, $f(x, y) = x$.

To prove the other implication in the definition of $x \succeq y$ by (1) assume $f(x, y) = x$. We want to show that $x \succeq y$. Take an arbitrary $k \in \{1, ..., K\}$. Suppose first that $x_k = 0$. If $y_k = 1$ then $q_k = 2$ and $\alpha_k = 0$. Namely, (i) $x_k = \alpha_k = 0$ and $y_k = 1$. If $y_k = 0$ then either $q_k = 1$, in which case $\alpha_k = 1$, or $q_k = 2$, in which case $\alpha_k = 0$. Namely, either (ii) $x_k = y_k = 0$ and $\alpha_k = 1$ or (iii) $x_k = y_k = \alpha_k = 0$. Suppose now that $x_k = 1$. If $y_k = 1$ then either $q_k = 1$, in which case $\alpha_k = 1$, or $q_k = 2$, in which case $\alpha_k = 0$. Namely, either (iv) $x_k = y_k = \alpha_k = 1$ or (v) $x_k = y_k = 1$ and $\alpha_k = 0$. If $y_k = 0$ then $q_k = 1$, in which case $\alpha_k = 1$. Namely, (vi) $x_k = \alpha_k = 1$ and $y_k = 0$. Hence, (i) to (vi) hold for an arbitrary $k \in \{1, ..., K\}$. Thus, $x \in MB(\alpha, y)$, and by definition of $\succeq$, $x \succeq y$ holds.