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A generalized assignment game

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Abstract

The game we propose in this paper is a natural extension of the "Assignment Game" of Shapley and Shubik [Shapley, L., Shubik, M., 1972. The assignment game I: the core. International Journal of Game Theory 1, 111–130] to the case where one seller owns a set of different objects instead of only one indivisible object. We prove that the core is nonempty and we study the structure of the set of core payoffs. We endow this set with a lattice structure under the partial ordering of the buyers. We show that, unlike other matching models, we cannot do the same for a dual partial ordering of the sellers.

Keywords: Matching; Assignment; Core; Lattice structure

JEL classification: C71; C78

1. Introduction

We study the interaction among a finite number of sellers and buyers in a market, which has the characteristic that each seller owns a set of possibly different objects, and each buyer wants to buy at most one object. The facts that agents in these markets belong to one of two disjoint sets (sellers and buyers), and there exists bilateral exchange allow us to study them as "two-sided matching markets". In particular, we study the simplest matching model of this type: there is only one seller in the market, since the extension to more than one seller does not add substantial information to the aim of this paper.

The worth of a potential transaction is given by a nonnegative real number associated to each possible pair of a buyer and an object. The seller in our model is allowed to own different objects.

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Hence, the gain that a buyer and the seller can share is not fixed in the sense that it depends on which object is sold. An outcome of this game specifies a matching between buyers and objects, and the price that each buyer pays to the seller. Therefore, we study a *many-to-one matching model with money*. This is a natural extension of the *one-to-one* “Assignment Game” (Shapley and Shubik, 1972), which is also studied in Roth and Sotomayor (1990). There are several papers such as Alcalde et al. (1998), and Sotomayor (1992, 1999a,b, in press) that extend the Assignment Game to many-to-one and many-to-many markets. All these works assume that all the objects a seller owns are homogeneous. Hence, the model we consider applies to markets that are not covered in the above mentioned papers, and the set of results we obtain constitutes an interesting contribution to the existing literature. The extension we propose also applies to labor markets with multidivisional firms: the seller in our model can be thought as a multiproduct firm or as a multidivisional firm with one vacancy per division, and the set of buyers as workers, where the salaries are determined explicitly within the model.

There are two natural questions one can address to: what partnerships can be expected to form in the market and how do the agents divide their gains? In this regard, one should use an appropriate solution concept for this class of games. In this paper we use *setwise stability* as our solution concept, which happens to coincide with the *core*.[1] We show that the core is nonempty by proving that we can associate to each competitive equilibrium a core outcome. We then study the structure of the set of core payoffs. Shapley and Shubik (1972) show that the core of every Assignment Game has a lattice structure, and they observe a polarization of interests between the two sides of the market within the core. For the many-to-one case in which all the objects owned by a seller are equal and preferences are additive, there exist core outcomes and an optimal core payoff for each side of the market (see Roth and Sotomayor, 1990). This lattice structure is also shared by the many-to-many model analyzed in Sotomayor (1999a). We prove that the set of core payoffs in our model has a similar structure to the one identified for the stable set in the many-to-one matching without money with substitutable preferences (see Martínez et al., 2001 and Blair, 1988 for a many-to-many model).[2] We show that the set of core payoffs does not have a dual lattice structure, and we endow this set with a lattice structure under the partial order of the buyers. We also show the existence of an optimal core payoff for each side of the market, which is the worst for the other side. We observe a polarization of interests only for the optimal core payoffs of each side of the market, but not for the whole set.

The paper is organized as follows. In Section 2, we present the model, and in the following section we show the existence of core payoffs and study the structure of the core.

2. The model

We consider a buyer–seller market consisting of m buyers and one seller. The seller owns a number of possibly different objects, and each buyer wants to buy at most one object. Formally, there is one finite set of buyers, P containing m agents, and there is one seller, s where $s \notin P$

[1] See Roth and Sotomayor (1990) for an excellent survey of the results in matching models. In the Assignment Game (Shapley and Shubik, 1972), and in the many-to-one models in Sotomayor (1992), there is also the coincidence of the core with the setwise stable set. Echenique and Oviedo (2006) and Sotomayor (1992, 1999a,b, in press) study extensions to many-to-many models where this coincidence is no longer valid.

[2] In a many-to-many model where the gain of a given partnership is fixed (that is, it is independent of the “vacancy” chosen), Sotomayor (1992) proves that there might not exist the worst core payoff for a particular side of the market.

There is also a set Q of n objects. Let $P=\{p_1, p_2,..., p_m\}$ be the set of buyers. Generic buyers are denoted by p_i, p_k, etc. The payoff of buyer $p_i \in P$ will be denoted by u_i. Generic objects are denoted by q_j, q_h, etc. and the price of object $q_j \in Q$ is v_j. All the objects in Q are owned by seller s and we denote by $|Q|$ the number of objects.

Associated to each possible pair $(p_i, q_i) \in P \times Q$ there is a nonnegative real number, α_{ij}, which denotes the maximum price that buyer p_i is willing to pay for object q_j which is her reservation value. For simplicity, we assume, without loss of generality, that the reservation price of the seller for every object $q_j \in Q$ is zero. Therefore, α_{ij} denotes the potential gains from trade between buyer p_i and the seller if the object sold is q_j We denote by α the $m \times n$ matrix $(\alpha_{ij})_{i=1,..., m;\, j=1,..., n}$. We also assume that there are no monetary transfers among buyers. This is a natural assumption since we are studying a buyer–seller market, and hence the model allows only the conventional transfer of the purchase price from the successful buyers to the seller. Thus, if buyer p_i buys object q_j at a price v_j then the resulting payoffs are $u_i=\alpha_{ij}-v_j$ for the buyer, and v_j for the seller. The total payoff of the seller is the sum of all the prices of the objects he sells, which is denoted by w.[3] Agents' preferences are concerned only with their monetary payoffs. This implies that, for any pair of objects and a buyer, there is a pair of prices that makes the buyer indifferent between purchasing either of the objects. For technical convenience, we introduce one artificial null object, q_0 (also owned by seller s). Several buyers may buy this null object. This convention allows us to treat a buyer p_i that does not buy any object as if she has bought q_0. We assume that the value α_{i0} is zero to all buyers, and the price of the object q_0 is always zero. Hence, if buyer p_i buys q_0 she obtains a utility $u_i=\alpha_{i0}-v_0=0$.

Therefore, a market M is a quadruple (P, s, Q, α). We call this the "Generalized Assignment Game".

3. The core

The model we study is a many-to-one matching model with money where the gain of a given partnership of a buyer and the seller depends on the object bought. An outcome of this game specifies a matching between buyers and objects and the price that each buyer pays to the seller. Our main concern is to predict which outcomes are more likely to occur, that is, the core of the game. In what follows, we define a feasible matching and a feasible outcome of this model and some desired properties. And then we analyze the core.

Definition 1. A *feasible matching* μ for a market $M \equiv (P, s, Q, \alpha)$ is a function from the set $P \cup Q$ into the set $P \cup Q \cup \{q_0\}$ such that:

(i) For any $p_i \in P$, $\mu(p_i) \in Q \cup \{q_0\}$,
(ii) For any $q_j \in Q$, either $\mu(q_j) \in P$ or $\mu(q_j)=q_j$,
(iii) For any $(p_i, q_j) \in P \times Q$, $\mu(p_i)=q_j$ if and only if $\mu(q_j)=p_i$.

We say that buyer p_i is unmatched if $\mu(p_i)=q_0$. Similarly, we say that object q_j is unsold if $\mu(q_j)=q_i$.

[3]If we allowed to have reservation prices of the seller different from zero, say c_j for object $q_j \in Q$, then the potential gains from trade would be: $\max\{0;\, \alpha_{ij}-c_j\}$. Instead, we are normalizing the seller's utility of keeping one of the objects q_j to zero rather that c_j.

Definition 2. A *feasible outcome*, denoted by (u, v, μ), is a vector of utilities (or payoffs) for buyers, $u \in \Re^m_+$, a price vector, $v \in \Re^n_+$, and a feasible matching, μ such that:[4]

(i) $u_i = \alpha_{i\mu(p_i)} - v_{\mu(p_i)}$, for every $p_i \in P$, and
(ii) $v \in \Re^n_+$ is such that every unsold object has zero price.[5]

We say that the vector (u, v) is compatible with μ. If (u, v, μ) is a feasible outcome, then the total payoff of the seller is $w = \sum_{q_j \in Q} v_j$ and we call (u, w) a payoff vector.[6]

Definition 3. A feasible matching μ is *optimal* for a market M if it maximizes the gain of the whole set of agents. That is, if for all feasible matchings μ' we have:

$$\sum_{\substack{p_i \in P \\ q_j = \mu(p_i)}} \alpha_{ij} \geq \sum_{\substack{p_i \in P \\ q_j = \mu'(p_i)}} \alpha_{ij}.$$

The concept of cooperative equilibrium for a matching model is setwise stability. This concept of stability requires that, given a feasible outcome, no group of agents induces an instability. We say that a group of agents (a coalition) induces an instability if, by making new trades only among themselves, possibly dissolving some transactions and possibly keeping some, can all obtain a strictly higher payoff. Note that we do not allow for side payments among agents within the same side of the market, and hence any coalition that induces an instability is formed by the seller and a group of buyers. Thus in this model, the setwise stability is equivalent to the strong core, and given that the seller only cares about his total payoff, and payoffs and prices can be adjusted continuously, the strong core and the core coincide.

Definition 4. Denote by T the coalition formed by seller s and a group of buyers T_p where $|T_p| \leq |Q|$. A feasible outcome (u, v, μ) is a *core outcome* if there do not exist any coalition T and any feasible matching $\hat{\mu}$ such that

$$\sum_{\substack{p_i \in T_p \\ \hat{\mu}(p_i) = q_j}} \alpha_{ij} > \sum_{q_j \in Q} v_j + \sum_{p_i \in T_p} u_i.$$

If such a coalition T and a matching $\hat{\mu}$ exist, we say that they block the feasible outcome (u, v, μ) and we call T a blocking coalition.

Definition 5. A payoff vector $(u, w) \in \Re^m_+ \times \Re_+$ is a *core payoff* for a market M if there exist a vector of prices $v \in \Re^n_+$ and a feasible matching μ such that (u, v, μ) is a core outcome and $w = \sum_{q_j \in Q} v_j$. We define the *core* as the set of all core outcomes.

[4]We sometimes abuse notation by writing $\alpha_{i\mu(p_i)}$ instead of α_{ij}, where $q_j = \mu(p_i)$. Same occurs for $\alpha_{\mu(q_j)j}$.
[5]This assumption simplifies notation and some of the definitions and results. Moreover, since in the Assignment Game each seller is identified with his object, there is no sense in assuming that the seller may have an unsold object with a positive price.
[6]By definition, at a feasible outcome, $u_i \geq 0$ for every buyer and $v_j \geq 0$ for every object. This implies that no buyer can obtain a higher utility by becoming unmatched, and the seller cannot obtain a higher payoff by leaving some of his objects unsold. This implies that a feasible outcome is individually rational.

An important remark is that a matching at any core outcome is optimal: if the matching was not optimal, then the coalition formed by all agents would block the given outcome.

In the following proposition we prove that the core is the Cartesian Product of the set of core payoffs and the set of optimal matchings, for a compatible price vector.[7] This result will be useful in the last subsection of the paper:

Proposition 1. *Take any core payoff (u, w) and any optimal matching μ'. Then there exists a vector of prices v' such that (u, v', μ') is a core outcome and $w = \sum_{q_j \in Q} v_j'$.*

Proof. Since (u, w) is a core payoff, there exist a vector of prices and an optimal matching μ such that (u, v, μ) is a core outcome and $w = \sum_{q_j \in Q} v_j$. Since both μ and μ' are optimal,

$$\sum_{\substack{p_i \in P \\ q_j = \mu'(p_i)}} \alpha_{ij} = \sum_{\substack{p_i \in P \\ q_j = \mu(p_i)}} \alpha_{ij} = \sum_{p_i \in P} u_i + \sum_{q_j \in Q} v_j. \tag{1}$$

Define v' as follows:

$$v_j' = \begin{cases} \alpha_{ij} - u_i & \text{if } \mu'(q_j) = p_i \\ 0 & \text{if } \mu'(q_i) = q_j. \end{cases}$$

Hence, (u, v', μ') is a feasible outcome. By definition of v', condition (1), and the fact that $\sum_{p_i \in P} \mu_i + \sum_{q_j \in Q} v_j' = \sum_{p_i \in P q_j = \mu'(p_i)} \alpha_{ij}$, $\sum_{q_j \in Q} v_j' = \sum_{q_j \in Q} v_j = w$ holds. Now, we prove by contradiction that there is no blocking coalition for (u, v', μ'). Suppose that there exist a coalition $T = \{s\} \cup T_p$ and a feasible matching $\hat{\mu}$ such that

$$\sum_{\substack{p_i \in T_p \\ q_j = \hat{\mu}(p_i)}} \alpha_{ij} > w + \sum_{p_i \in T_p} u_i.$$

But this implies that (u, v, μ) is also blocked by coalition T, which is a contradiction. Also note that prices are positive by the definition of a feasible outcome. □

3.1. Nonemptiness

In this subsection we prove the nonemptiness of the core. We define a competitive equilibrium for a given market and we show that we can associate to each competitive equilibrium a core outcome. A similar argument is used in Sotomayor (1992), but in a very different scenario. Since the set of competitive equilibria is an instrument to prove the nonemptiness of the core, it is interesting to study the relationship between these two sets for a given market.

In our setting, a competitive equilibrium consists of a vector of prices and a feasible matching. The competitive equilibrium property implies that, given the prices of the objects, each buyer maximizes utility and trade takes place under a feasible matching. Following the definition used in the Assignment Game, we define a competitive equilibrium as follows.

Given a vector of prices $v \in \Re^n_+$ let $D_i(v)$ denote the *demand set of buyer p_i*, which is defined as the nonempty set of all objects that maximize p_i's utility given v, i.e., $D_i(v)=$

[7]In the Assignment Game (Shapley and Shubik, 1972) it is shown that the set of stable outcomes is the Cartesian product of the set of stable payoffs and the set of optimal matchings.

$\{q_j \in Q;\ \alpha_{ij} - v_j \geq \alpha_{ih} - v_h$, for all $q_h \in Q\}$. The price vector $v \in \Re^n_+$ is *competitive* if each buyer can be matched with an object in her demand set, that is, if there exists a feasible matching μ such that $\mu(p_i) \in D_i(v)$ for all p_i in P. Such a matching μ is said to be *competitive* for the prices v. The pair (v, μ) is a *competitive equilibrium* if v is competitive, μ is competitive for v, and $v_j = 0$ for any unsold object q_j.

The following proposition states the relationship between the core of a given market M and the set of competitive equilibria.

Proposition 2. *Let a pair (v, μ) be a competitive equilibrium for a market M. Then, the feasible outcome (u, v, μ), where $\mu_i = \alpha_{i\mu(p_i)} - v_{\mu(p_i)}$, for all $p_i \in P$, is a core outcome.*

Proof. Suppose that, given a competitive equilibrium (v, μ) in M, the feasible outcome (u, v, μ) is not a core outcome in M. We prove that the pair (v, μ) is not a competitive equilibrium.

Since (u, v, μ) is not a core outcome, there exists a coalition T, formed by the seller s and a subset of buyers T_p, and a feasible matching μ' such that $\sum_{q_j \in Q \mu'(q_j) \in P} v'_j > \sum_{q_j \in Q \mu(q_j) \in P} v_j$ and $\alpha_{i\mu'(p_i)} - v'_{\mu'(p_i)} > \alpha_{\mu(p_i)}$, for every $p_i \in T_p$, where v' is the new vector of prices.

This implies that there exists $q_j \in Q$ with $\mu'(q_j) \in T_p$ (denote $\mu'(q_j)$ by p_i), such that, either $v'_j > 0$ and $\mu(q_i) = q_j$, or $v'_j > v_j$, $\mu(q_j) \in P$, and $\mu'(q_j) \neq \mu(q_j)$. In both cases, we must have $u'_i > u_i$. Therefore, for the pair (p_i, q_j) we have that $u_i + v_j < \alpha_{ij}$. But this implies that $\mu(p_i) \notin D_i(v)$, because there exists $q_j \in Q$ such that $u_i = \alpha_{i\mu(p_i)} - v_{\mu(p_i)} < \alpha_{ij} - v_j$. And this is a contradiction to the fact that the pair (v, μ) is a competitive equilibrium. □

We provide the following example to see that we may have a core outcome, (u, v, μ), where the pair (v, μ) is not a competitive equilibrium[8].

Example 1. Consider the market $M = (P, s, Q, \alpha)$ with $Q = \{q_1, q_2\}$, $P = \{p_1\}$, and $\alpha_{11} = 5$, $\alpha_{12} = 4$. The feasible outcome (u, v, μ), where $u = 1$, $v = (4, 0)$ and $\mu(p_1) = q_1$, is a core outcome since $\alpha_{12} = 4 < 5 = \alpha_{11}$. But the pair (v, μ) is not a competitive equilibrium since v is not competitive: $\alpha_{11} - v_1 = 1 < 4 = \alpha_{12} - v_2$, which implies that $q_1 \notin D_1(v) = q_2$.

As an immediate consequence of Proposition 2, and using the existence result of Shapley and Shubik for the set of competitive equilibria in the Assignment Game, the following theorem holds.

Theorem 1. *A core outcome exists for every given market $M \equiv (P, s, Q, \alpha)$.*

Proof. We can obtain a one-to-one market (Assignment Game) from a given many-to-one market M (Generalized Assignment Game) by considering the same sets of objects and buyers, the same α, and assuming that each object is owned by a different seller. Note that the set of competitive equilibria coincides for both markets. Shapley and Shubik (1972) prove the existence of the set of competitive equilibria for the Assignment Game. This result, together with the result in Proposition 2, show the existence of a core outcome for every given market M. □

3.2. Structure

In this subsection we analyze the structure of the set of core payoffs. In particular, we propose two binary operations to endow this set with a lattice structure under the partial order of the buyers, and we prove that it is not possible to endow the core with a dual lattice structure. This

[8] Sotomayor (in press) studies a many-to-many model where the set of competitive equilibria is smaller than the stable set.

special lattice structure indicates that agents on the same side have same interests over the agents on the other side, although each side competes for the same agents, and there is a polarization of interests between the two sides. Also, many algorithms used to obtain stable matchings are based on this structure. For the Generalized Assignment Game, we also observe this conflict of interests between the two sides of the market, but only for specific outcomes. This duality is not observed with respect to the whole set of core outcomes.

Following Simmons (1963), a lattice is a partially ordered set any two of whose elements have a least upper bound and a greatest lower bound in the set. When each of the possible subsets of the set has a least upper bound and a greatest lower bound in the set, we say that the lattice is complete. This is equivalent to the set being convex and compact (see Sotomayor, 1999a).

Let us define the partial orders $\geq_P$ and $\geq_s$.[9] For any two core payoffs (u, w) and (u', w'), $(u, w) \geq_P (u', w')$ if $u_i \geq u'_i$ for all p_i in P, and $(u, w) \geq_s (u', w')$ if $w \geq w'$. Note that these two partial orders are not dual in our model.

Lemma 1. *Let (u, w) and (u', w') be two core payoffs. If $(u, w) \geq_P (u', w')$ then $(u', w') \geq_s (u, w)$.*

Proof. Note that, since (u, w) and (u', w') are core payoffs, for any optimal matching μ, there exist v and v' such that (u, v, μ) and (u', v', μ) are core outcomes, $w = \sum_{q_j \in Q} v_j$, and $w' = \sum_{q_j \in Q} v_j{}'$. Assume $u_i \geq u'_i$ for all p_i in P, which implies $v_{\mu(p_i)} \leq v'_{\mu(p_i)}$. Therefore, $w \leq w'$. □

Remark 1. Let (u, w) and (u', w') be two core payoffs. If $(u, w) \geq_s (u', w')$, we do not necessarily have $(u, w) \geq_P (u', w')$.

Proof. Let $M=(P, s, Q, \alpha)$ be such that $Q=\{q_1, q_2\}$, $P=\{p_1, p_2\}$, and $\alpha_{11}=5$, $\alpha_{12}=4$, $\alpha_{21}=2$, $\alpha_{22}=3$. The payoff vectors $(u, w)=((2, 0), 6)$ and $(u', w')=((1, 2), 5)$ are core payoffs. On the one hand, $w > w'$ so $(u, w) \geq_s (u', w')$. On the other hand, $u'_1 < u_1$ but $u'_2 < u_2$, so we cannot say that $(u, w) \geq_P (u', w')$. □

As a direct consequence of the above lemma, the set of core payoffs is not a dual lattice in the usual sense.[10] Given two core payoffs, if we let buyers choose their best (worst) and we let the seller choose his worst (best) payoff, the resulting payoff may not even be feasible.

In what follows, we endow the set of core payoffs with a lattice structure under the partial order of the buyers. We define the least upper bound (greatest lower bound) of two given core payoffs as the one where *only* the buyers choose their best (worst) among the two.[11]

Definition 6. Let (u, w) and (u', w') be two core payoffs and let μ be an optimal matching. We define $\bar{\mu}$ and $\underline{w}(\mu)$ as follows:

(i) For every $p_i \in P$, $\bar{u}_i = \max\{u_i, u'_i\}$.
(ii) For s, $\underline{w}(\mu) = \sum_{q_j \in Q}(\min\{v_j, v_j{}'\})$, where v and v' are such that (u, v) and (u', v') are compatible with μ, $w = \sum_{q_j \in Q} v_j$, and $w' = \sum_{q_j \in Q} v'_j$.[12]

[9] These two binary relations are also preorders since they are reflexive. Moreover, $\geq_s$ is a complete preorder.

[10] A similar result can be found in Martínez et al. (2001) for matching games without money where agents have substitutable preferences.

[11] If we let *only* the seller choose between two core payoffs, we obtain the same result. But this is no longer true if we had more than one seller, where we are not able to endow the set of core payoffs with a lattice structure under the partial order of the sellers.

[12] We will prove later (Lemma 2) that this sum of prices coincides for any optimal matching.

Similarly, replacing max by min and min by max, we define $\underline{u}$ and $\overline{w}(\mu)$.

Proposition 3. *Let (u, w) and (u′, w′) be two core payoffs and let μ be an optimal matching. Then, the vectors of payoffs ($\overline{u}$, $\underline{w}(\mu)$) and ($\underline{u}$, $\overline{w}(\mu)$) are core payoffs.*

Proof. Take the optimal matching μ and the vector of prices $\underline{v}$ such that $\underline{v}_j=\min\{v_j, v_j'\}$, where v and v' are such that (u, v) and (u', v') are compatible with μ, $w=\sum_{q_j\in Q} v_j$, and $w'=\sum_{q_j\in Q} v_j'$. We prove that $(\overline{u}, \underline{v}, \mu)$ is a core outcome. First, in (a), we prove that it is a feasible outcome using the fact that (u, v, μ) and (u', v', μ) are feasible since they are core outcomes by Proposition 1. In (b), we prove that the outcome cannot be blocked by any coalition.

(a) For every (p_i, q_i) such that $\mu(p_i)=q_j$, either $\overline{u}_i=u_i$ or $\overline{u}_i=u_i'$. In the first case, by feasibility of (u, v, μ) and (u', v', μ), $u_i+v_j=u_i'+v_j'=\alpha_{ij}$. Therefore, $u_i>u_i'$ implies that $v_j\leq v_j'$, and we must have $\underline{v}_j=v_j$. Hence, $\overline{u}_i+\underline{v}_j=u_i+v_j=\alpha_{ij}$. The proof for the second case is similar. For s, $\underline{w}(\mu)=\sum_{q_j\in Q}(\min\{v_j, v_j'\})=\sum_{q_j\in Q}\underline{v}_j$, by definition.

(b) We check that there does not exist any coalition formed by the seller and a group of buyers that blocks the feasible outcome $(\overline{u}, \underline{v}, \mu)$. Note that the buyers are optimally allocated, and cannot reorganize in a profitable way under μ. Otherwise, (u, v, μ) and (u', v', μ) would also be blocked by this coalition, and, hence, they would not be core outcomes. Consider a coalition T, formed by the seller s, and a group of buyers, T_p, where, at least one buyer, $p_i\in T_p$, was unmatched under μ, and there exists a buyer p_k, that was buying q_j under μ and, in the blocking coalition, he is assigned to object q_h, with $\mu(q_h)\in P$ with price v_h.[13] Noting this, if $(\overline{u}, \underline{v}, \mu)$ was not a core outcome, it must be the case that $\alpha_{ij}+\alpha_{kh}>\overline{u}_i+\underline{v}_h>\alpha_{kj}$. Either $\alpha_{ij}+\alpha_{kh}>\overline{u}_i+\underline{v}_h+\alpha_{kj}=\overline{u}_i+v_h+\alpha_{kj}\geq u_i+v_h+\alpha_{kj}$, which is a contradiction with (u, v, μ) being a core outcome or $\alpha_{ij}+\alpha_{kh}>\overline{u}_i+v^{-}{}_h+\alpha_{kj}=\overline{u}_i+v_h'+\alpha_{kj}\geq u_i'+\alpha_{kj}$, which is a contradiction with (u', v', μ) being a core outcome.

Similar reasoning follows for $(\underline{u}, \overline{v}, \mu)$. □

The following lemma shows that the definition of $\overline{w}(\mu)$ for any two core payoffs does not depend on the matching μ.

Lemma 2. *Let (u, w) and (u′, w′) be two core payoffs and let μ and μ̂ be two optimal matchings. Assume that v, v′, v̂ and v̂′ are such that (u, v) and (u′, v′) are compatible with μ, and (u, v̂) and (u′, v̂′) are compatible with μ̂. Then,*

$$\sum_{q_j\in Q}(\min\{v_j, v_j'\})=\sum_{q_j\in Q}(\min\{\hat{v}_j, \hat{v}_j'\})$$

and

$$\sum_{q_j\in Q}(\max\{v_j, v_j'\})=\sum_{q_j\in Q}(\max\{\hat{v}_j, \hat{v}_j'\}).$$

Proof. Denote $\underline{w}'=\sum_{q_j\in Q}(\min\{v_j, v_j'\})$ and $\underline{w}''=\sum_{q_j\in Q}(\min\{\hat{v}_j, \hat{v}_j'\})$. Following the proof of Proposition 3, $(\overline{u}, v^m, \mu)$ and $(\overline{u}, \hat{v}^m, \hat{\mu})$ are core outcomes, with $v_j^m=\min\{v_j, v_j'\}$ and $\hat{v}_j^m=\min$

[13] Note that the rest of possible cases where the blocking coalition is formed by the seller and one buyer can be analyzed using a similar argument.

$\{\hat{v}_j, \hat{v}'_j\}$ for every $q_j \in Q$. Also by Proposition 1, there exists a vector of prices v^* such that $(\overline{u}, v^*, \hat{\mu})$ is a core outcome. But by feasibility of $\hat{\mu}$ it must be the case that $\overline{u}_i = \alpha_{i\hat{\mu}(p_i)} - v^*_{\hat{\mu}(p_i)}$ for every $p_i \in P$. This implies that $v^*_j = \hat{v}^m_j$ for every $q_i \in Q$, and therefore, $\underline{w}' = \underline{w}''$

Similarly, we can prove this property for the maximum prices. □

Now, for any two core payoffs (u, w) and (u', w') we can properly denote by $\underline{w}$ the minimum gain that seller s can get at any optimal matching, and by $\overline{u}_i$ the maximum payoff for buyer p_i. Moreover, we can state the main result of the paper:

Theorem 2. *The set of core payoffs forms a complete lattice under the partial order* $\geq_P$.

Proof. By Proposition 3, every two core payoffs (u, w) and (u', w') have a supremum, denoted by $(\overline{u}, \underline{w})$, and an infimum, denoted by $(\underline{u}, \overline{w})$, under the partial order $\geq_P$, and $(\underline{u}, \overline{w})$ and $(\overline{u}, \underline{w})$ are an upper-bound and a lower bound under $\geq_s$. This directly shows that the set of core payoffs is a lattice under the partial order $\geq_P$. For this to be a complete lattice, we show that this set is convex and compact. By Proposition 1, the set of core payoffs is the same for any optimal matching. Let μ be an optimal matching. The set of core payoffs is the solution of a system of linear non-strict inequalities associated with μ, and hence it is closed and convex. The boundedness follows from the fact that for all matched pairs (p_i, q_j) under μ, $0 \leq u_i \leq \alpha_{ij}$ and $0 \leq v_j \leq \alpha_{ij}$. Hence, the set of core payoffs is convex and compact, and therefore it forms a complete lattice under the partial order $\geq_P$. □

We can go further and analyze a common result in the matching literature: the existence of a polarization of interests between the two sides of the market within the set of core payoffs. In our case, we only have this polarization between the optimal core payoffs for each side, given that the two partial orders are not dual in this model.

Definition 7. A core payoff (u^*, w_*) is called *P-optimal* if for any core payoff (u, w), $(u^*, w_*) \geq_P (u, w)$. A core payoff (u^*, w_*) is called *s-optimal* if for any core payoff (u, w), $(u^*, w_*) \geq_s (u, w)$.

Proposition 4. *There exist a unique P-optimal core payoff* (u^*, w_*) *and a unique s-optimal core payoff* (u_*, w^*). *Moreover, for any core payoff* (u, w), $(u^*, w_*) \geq_P (u, w) \geq_P (u_*, w^*)$ *and* $(u_*, w^*) \geq_s (u, w) \geq_s (u^*, w_*)$. *There exists a unique s-optimal core payoff with symmetrical properties.*

Proof. The existence of a unique P-optimal core payoff (u^*, w_*) is guaranteed by Theorem 2, since every complete lattice has a unique maximal element. This, together with Lemma 1, guarantees that (u^*, w_*) is such that $(u, w) \geq_s (u^*, w_*)$ for any core payoff (u, w). Also, there exists a unique minimal element (u_*, w^*) that is the least preferred for every buyer given the partial order $\geq_P$. This implies that (u_*, w^*) is such that $(u, w) \geq_P (u_*, w^*)$ for any core payoff (u, w). But by Lemma 1, $(u_*, w^*) \geq_s (u, w)$, which verifies the existence of a unique s-optimal core payoff that is the least preferred by the buyers. □

Finally, we comment on the relationship between the optimal core payoffs and the competitive equilibrium payoffs by means of the following example:

Example 2. Let $M \equiv (P, s, Q, \alpha)$ with $Q = \{q_1, q_2\}$, $P = \{p_1\}$, and $\alpha_{11} = 5$, $\alpha_{12} = 4$. The set of core payoffs for this market is $G = \{(u_1, w) \in \Re^2_+ | u_1 + w = 5\}$ and the set of competitive equilibria is $C = \{(v, u) | v_1 \in [0, 1] \text{ and } \mu(p_1) = q_1\}$. We observe that the seller can gain more in the core than in the set of competitive equilibria, while the buyer's optimal core payoff coincides with the buyer's optimal payoff within the set of competitive equilibria.[14]

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[14] This coincidence depends on the fact that, in this model, the buyers can buy at most one object.