The Blocking Lemma for a Many-to-one Matching Model*

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December 2006

Abstract: The Blocking Lemma identifies a particular blocking pair for each non-stable and individually rational matching that is preferred by some agents of one side of the market to their optimal stable matching. Its interest lies in the fact that it has been an instrumental result to prove key results on matching. For instance, the fact that in the college admissions problem the workers-optimal stable mechanism is group strategy-proof for the workers and the strong stability theorem in the marriage model follow directly from the Blocking Lemma. However, it is known that the Blocking Lemma and its consequences do not hold in the general many-to-one matching model in which firms have substitutable preference relations. We show that the Blocking Lemma holds for the many-to-one matching model in which firms' preference relations are, in addition to substitutable, quota $q-$separable. We also show that the Blocking Lemma holds on a subset of substitutable preference profiles if and only if the workers-optimal stable mechanism is group strategy-proof for the workers on this subset of profiles.

*We are grateful to Flip Klijn, Howard Petith, William Thomson, an associate editor, and two referees for very helpful comments. The work of R. Martínez, A. Neme, and J. Oviedo is partially supported by Research Grant 319502 from the Universidad Nacional de San Luis (Argentina). The work of J. Massó is partially supported by the Spanish Ministry of Education and Science through grant EJ2005-01481/ECON and FEDER and through project CONSOLIDER-INGENIO 2010 (CSD2006-00016), and by the Generalitat de Catalunya through grant SGR2005-00454 and the Barcelona Economics Program (XREA). All authors acknowledge financial support from an AECI grant from the Spanish Ministry of Foreign Affairs.

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Journal of Economic Literature Classification Numbers: C78, D71, D78.
Keywords: Matching, Stability, Blocking Lemma.

1 Introduction

Two-sided, many-to-one matching models study assignment problems where a finite set of agents can be divided into two disjoint subsets: the set of institutions (called *firms*) and the set of individuals (called *workers*). Each firm has a preference relation on all subsets of workers and each worker has a preference relation on the set of firms plus the prospect of remaining unmatched. A preference profile is a list of preference relations, one for each agent. A matching assigns each firm with a subset of workers (possibly empty) in such a way that each worker can work for at most one firm. Given a preference profile a matching is called *stable* if all agents have acceptable partners (individual rationality) and there is no unmatched worker-firm pair who both would prefer to be matched to each other rather than staying with their current partners (pair-wise blocking).

The "college admissions model with substitutable preferences" is the name given by Roth and Sotomayor (1990) to the most general many-to-one model with ordinal preferences in which stable matchings exist. Each firm is restricted to have a substitutable preference relation on all subsets of workers; namely, each firm continues to want to employ a worker even if other workers become unavailable (Kelso and Crawford (1982) were the first to use this property in a more general model with money). For each substitutable preference profile the deferred-acceptance algorithms produce either the firms-optimal stable matching or the workers-optimal stable matching, depending on whether the firms or the workers make the offers. The firms (workers)-optimal stable matching is unanimously considered by all firms (respectively, workers) to be the best matching among all stable matchings.

A more specific many-to-one model, called the "college admissions problem" by Gale and Shapley (1962), supposes that firms have a maximum number of positions to be filled (their quota), and that each firm, given its ranking of individual workers, orders subsets of workers in a responsive manner; namely, for any two subsets that differ in only one worker a firm prefers the subset containing the most-preferred worker.[1] In this model the Blocking Lemma says the following. Fix a responsive preference profile. Suppose that the set of workers that strictly prefer an individually rational matching to the workers-

[1]Observe that the marriage model (*i.e.*, the one-to-one matching model) is a particular instance of the "college admissions problem" when all firms have quota one.

optimal stable matching is nonempty. Then, we can always find a firm and a worker with the following properties: (a) the firm and the worker block the individually rational matching, (b) the firm was hiring another worker who strictly prefers the individually rational matching to the workers-optimal stable matching, and (c) the worker (member of the blocking pair) considers the workers-optimal stable matching to be at least as good as the individually rational matching. The interest of the Blocking Lemma lies in the fact that it is an instrumental result to prove key results on matching. For instance, the fact that in the college admissions problem the workers-optimal stable mechanism is group strategy-proof for the workers (Dubins and Freedman, 1981)[2] and the strong stability theorem in the marriage model (Demange, Gale, and Sotomayor, 1987) follow directly from the Blocking Lemma. The first result says that if in centralized markets (like entry-level professional labor markets or the admission of students to colleges) a mechanism selects for each preference profile its corresponding workers-optimal stable matching then, no group of workers can never benefit by reporting untruthfully their preference relations. This is an important property and it becomes critical if the market has to be redesigned, in which case the declared preference profile conveys very valuable information. The second result says that every non-stable matching is either non-individually rational or we can identify a blocking pair (a firm and a worker) and another *stable* matching such that both members of the blocking pair weakly prefer to the original one.

It is known that the Blocking Lemma does not hold for the many-to-one matching model with substitutable preference profiles. The purpose of this paper is two-fold. First, we consider a weaker condition than responsiveness, called quota $q-$separability, that together with substitutability implies that the Blocking Lemma holds for all these preference profiles (Theorem 1).[3] A firm is said to have a separable preference relation over all subsets of

[2]To be precise, they show it for the marriage model, but their result can be extended to the college admissions problem. Some results concerning stability in the college admissions problem are immediate consequences of the fact that they hold for the marriage model. Each college is split into as many pieces as positions it has, so transforming the original many-to-one model into a one-to-one model. Responsiveness allows then the translation of stability from one model to another. See Roth and Sotomayor (1990) for a complete description of this procedure as well as for its applications. Observe that this reduction is possible only if preferences are responsive.

[3]We have already showed that if firms have substitutable and quota $q-$separable preference profiles then, (a) the set of unmatched agents is the same in all stable matchings (Martínez, Massó, Neme, and Oviedo, 2000), (b) the set of stable matchings has a lattice structure with two natural binary operations (Martínez, Massó, Neme, and Oviedo, 2001), (c) the workers-optimal stable matching is weakly Pareto optimal for the workers, relative to the set of individually rational matchings (Martínez, Massó, Neme,

workers if its partition between acceptable and unacceptable workers has the property that only adding acceptable workers makes any given subset of workers a better one. However, in many applications such as entry-level professional labor markets, separability alone does not seem very reasonable because firms usually have fewer openings (their quota) than the number of "good" workers looking for a job. In these cases it seems reasonable to restrict the preference relations of firms in such a way that the separability condition operates only up to their quota, considering unacceptable all subsets with higher cardinality. Moreover, while responsiveness seems the relevant property for extending an ordered list of individual students to a preference relation on all subsets of students, it is too restrictive, though, to capture some degree of complementarity among workers, which can be very natural in other settings. The quota $q-$separability condition permits greater flexibility in going from orders on individuals to orders on subsets. For instance, candidates for a job can be grouped together by areas of specialization. A firm with quota two may consider as the best subset of workers not the set consisting of the first two candidates on the individual ranking (which may have both the same specialization) but rather the subset composed of the first and fourth candidates in the individual ranking (*i.e.*, the first in each area of specialization).

Second, we show (in Theorem 2) that the Blocking Lemma holds on a subset of substitutable preference profiles (not necessarily quota $q-$separable) if and only if the workers-optimal stable mechanism is group strategy-proof for the workers on this subset of profiles. This means that, in contrast with what the literature has considered so far, the Blocking Lemma is more fundamental than just a key step to prove general results like group strategy-proofness of the workers-optimal stable mechanism for the workers. Observe that our former result (Martínez, Massó, Neme, and Oviedo, 2004) showing that the workers-optimal stable mechanism is group strategy-proof for the workers on the set of substitutable and quota $q-$separable preference profiles was proved assuming that the Blocking Lemma was true on the set of all these profiles. Hence, Theorem 2 and our former result does not imply that the Blocking Lemma holds on the set of all substitutable and quota $q-$separable preference profiles. Theorem 1 states that this is indeed the case.

The paper is organized as follows. In Section 2, which closely follows Martinez, Massó, Neme, and Oviedo (2004), we present the preliminary notation and definitions. In Sections

and Oviedo, 2004), and (d) the workers-optimal stable mechanism is group strategy-proof for the workers (Martínez, Massó, Neme, and Oviedo, 2004). This last result is proven assuming that the Blocking Lemma holds for all subsitutable and quota $q-$separable preference profiles; here, we are providing a proof that this is indeed the case.

3 we present the Blocking Lemma and state, in Theorem 1, that it holds on the set of all substitutable and $q-$separable preference profiles. In Section 4 we state and prove the equivalence, on any subset of substitutable preference profiles (not necessarily quota $q-$separable), between the Blocking Lemma and group strategy-proofness of the workers-optimal stable mechanism for the workers. In Section 5 we conclude with an example of a substitutable and quota $q-$separable preference profile for which the symmetric Blocking Lemma for the firms does not hold. We collect all proofs in two Appendices at the end of the paper.

2 Preliminaries

There are two disjoint sets of *agents*, the set of n *firms* $F = \{f_1, ..., f_n\}$ and the set of m *workers* $W = \{w_1, ..., w_m\}$. Generic elements of both sets will be denoted, respectively, by f, $\overline{f}$, and $\widetilde{f}$, and by w, $\overline{w}$, and $\widetilde{w}$. Each worker $w \in W$ has a strict, transitive, and complete preference relation $P(w)$ over $F \cup \{\emptyset\}$, and each firm $f \in F$ has a strict, transitive, and complete preference relation $P(f)$ over 2^W. *Preference profiles* are $(n+m)$-tuples of preference relations and they are represented by $P = (P(f_1), ..., P(f_n); P(w_1), ..., P(w_m))$. Given a preference relation of a firm $P(f)$ the subsets of workers preferred to the empty set by f are called *acceptable*. Similarly, given a preference relation of a worker $P(w)$ the firms preferred by w to the empty set are called *acceptable*. Therefore, we are allowing for the possibility that firm f may prefer not to hire any worker rather than to hire unacceptable subsets of workers and that worker w may prefer to remain unemployed rather than to work for an unacceptable firm. To express preference relations in a concise manner, and since only acceptable partners will matter, we write acceptable partners in the order of decreasing preference. For instance,

$$\begin{aligned} P(f_i) &: w_1w_3, w_2, w_1, \emptyset \\ P(w_j) &: f_1, f_3, \emptyset \end{aligned}$$

mean that $\{w_1, w_3\} P(f_i) \{w_2\} P(f_i) \{w_1\} P(f_i) \emptyset$ and $f_1 P(w_j) f_3 P(w_j) \emptyset$.

A *market* is a triple (F, W, P), where F is a set of firms, W is a set of workers, and P is a preference profile. Given a market (F, W, P) the assignment problem consists of matching workers with firms, keeping the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

Definition 1 A *matching* μ is a mapping from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that for all $w \in W$ and $f \in F$:
(a) Either $|\mu(w)| = 1$ and $\mu(w) \subseteq F$ or else $\mu(w) = \emptyset$.
(b) $\mu(f) \in 2^W$.
(c) $\mu(w) = \{f\}$ if and only if $w \in \mu(f)$.

Condition (a) says that a worker can either be matched to at most one firm or remain unmatched. Condition (b) says that a firm can either hire a subset of workers or be unmatched. Finally, condition (c) states the bilateral nature of a matching in the sense that firm f hires worker w if and only if worker w works for firm f. We say that w and f are *unmatched* in a matching μ if $\mu(w) = \emptyset$ and $\mu(f) = \emptyset$. Otherwise, they are matched. A matching μ is said to be *one-to-one* if firms can hire at most one worker; namely, condition (b) in Definition 1 is replaced by: Either $|\mu(f)| = 1$ and $\mu(f) \subseteq W$ or else $\mu(f) = \emptyset$. The model in which all matchings are one-to-one is also known in the literature as the *marriage model*. The model in which all matchings are many-to-one (*i.e.*, they satisfy Definition 1) and firms have responsive preferences[4] is also known in the literature as the *college admissions problem* (Gale and Shapley, 1962). To represent matchings concisely we will follow the widespread notation where, for instance, given $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$,

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_3 w_4 & w_1 & \emptyset & w_2 \end{pmatrix}$$

represents the matching where firm f_1 is matched to workers w_3 and w_4, firm f_2 is matched to worker w_1, and firm f_3 and worker w_2 are unmatched. Given a matching μ and two subsets $F' \subseteq F$ and $W' \subseteq W$ we denote by $\mu(F')$ and $\mu(W')$ the sets $\{w \in W \mid \mu(w) \in F'\}$ and $\{f \in F \mid \exists w \in W' \text{ such that } w \in \mu(f)\}$, respectively; *i.e.*, $\mu(F') = \bigcup_{f \in F'} \mu(f)$ and $\mu(W') = \bigcup_{w \in W'} \mu(w)$.

Let P be a preference profile. Given a set of workers $S \subseteq W$, let $Ch(S, P(f))$ denote firm f's *most-preferred* subset of S according to its preference relation $P(f)$. Generically we will refer to this set as the *choice set*.

A matching μ is *blocked by worker* w if $\emptyset P(w) \mu(w)$. A matching μ is *blocked by firm* f if $\mu(f) \neq Ch(\mu(f), P(f))$. We say that a matching is *individually rational* if it is

[4] Roughly, for any two subsets of workers that differ in only one worker a firm prefers the subset containing the most-preferred worker. See Roth and Sotomayor (1990) for a precise and formal definition of responsive preferences as well as for a masterful and illuminating analysis of these models and an exhaustive bibliography.

not blocked by any individual agent. We will denote by $IR(P)$ the set of individually rational matchings. A matching μ is *blocked by a firm-worker pair* (f, w) if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup \{w\}, P(f))$, and $f P(w) \mu(w)$.

Definition 2 A matching μ is *stable* if it is not blocked by any individual agent nor any firm-worker pair.

Given a preference profile P, denote the set of stable matchings by $S(P)$. It is easy to construct examples of preference profiles with the property that the set of stable matchings is empty. These examples share the feature that at least one firm regards a subset of workers as being complements. This is the reason why the literature has focused on the restriction where workers are regarded as substitutes (Kelso and Crawford (1982) were the first to introduce the notion of substitutable preferences).

Definition 3 A firm f's preference relation $P(f)$ satisfies *substitutability* if for any set S containing workers w and w' $(w \neq w')$, if $w \in Ch(S, P(f))$ then $w \in Ch(S \setminus \{w'\}, P(f))$.

A preference profile P is *substitutable* if for each firm f, the preference relation $P(f)$ satisfies substitutability.

Blair (1988) shows that the choice set of substitutable preference relations have the following property.

Remark 1 *Let $P(f)$ be a substitutable preference relation and assume A and B are two subsets of workers. Then, $Ch(A \cup B, P(f)) = Ch(Ch(A, P(f)) \cup B, P(f))$.*

Kelso and Crawford (1982) shows that (in a more general model with money) if all firms have substitutable preference relations then: (1) the set of stable matchings is nonempty, and (2) firms unanimously agree that a stable matching μ_F is the best stable matching. Roth (1984) extends these results and shows that if all firms have substitutable preference relations then: (3) workers unanimously agree that a stable matching μ_W is the best stable matching,[5] and (4) the optimal stable matching for one side is the worst stable matching

[5]The matchings μ_F and μ_W are called, respectively, the *firms-optimal stable matching* and the *workers-optimal stable matching*. We are following the convention of extending preference relations from the original sets (2^W for the firms and $F \cup \{\emptyset\}$ for the workers) to the set of matchings. However, we now have to consider weak orderings since the matchings μ and μ' may associate to an agent the same partner. These orderings will be denoted by $R(f)$ and $R(w)$. For instance, to say that all firms prefer μ_F to any stable μ means that for every $f \in F$ we have that $\mu_F R(f) \mu$ for all stable μ (that is, either $\mu_F(f) = \mu(f)$ or else $\mu_F(f) P(f) \mu(f)$).

for the other side. That is, $S(P) \neq \emptyset$ and for all $\mu \in S(P)$ we have that $\mu_F R(f) \mu R(f) \mu_W$ for all $f \in F$ and $\mu_W R(w) \mu R(w) \mu_F$ for all $w \in W$.

The *deferred-acceptance algorithm*, originally defined by Gale and Shapley (1962) for the marriage model, produces either μ_F or μ_W depending on who makes the offers. At any step k of the algorithm in which firms make offers, a firm proposes itself to the choice set of the set of workers that have not already rejected it during the previous steps, while a worker accepts the offer of the best firm among the set of current offers plus the one made by the firm provisionally matched in the previous step (if any). The algorithm stops at step $\overline{K}$ at which all offers are accepted; the (provisional) matching then becomes definite and it is the firms-optimal stable matching μ_F. Symmetrically, at any step k of the algorithm in which workers make offers, a worker proposes himself to the best firm among the set of firms that have not already rejected him during the previous steps, while a firm accepts the choice set of the set of current offers plus that of the workers provisionally matched in the previous step (if any). The algorithm stops at step K at which all offers are accepted; the (provisional) matching then becomes definite and it is the workers-optimal stable matching μ_W.

Let P be a substitutable preference profile. In the proofs of Theorems 1 and 2 we will use properties of the deferred-acceptance algorithm in which workers make offers. For this reason we present the following notation. For all $f \in F$, the set

$$O(f) = \{w \in W \mid f R(w) \mu_W(w)\}$$

is the set of workers that make an offer to f along the deferred-acceptance algorithm in which workers make offers and whose outcome is matching μ_W. For any step $1 \leq k < K$, let $O^k(f)$ denote the set of workers that make an offer to f at k or at earlier steps. Obviously, for all $f \in F$,

$$O^1(f) \subseteq ... \subseteq O^k(f) \subseteq ... \subseteq O^K(f) = O(f).$$

Moreover, for all $f \in F$,

$$\mu_W(f) = Ch(O(f), P(f). \tag{1}$$

Example 1 below (taken from Martínez, Massó, Neme, and Oviedo (2004)) illustrates the deferred-acceptance algorithm in which workers make offers.

Example 1 Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the substitutable preference profile P, where

$$\begin{aligned} P(f_1) &: w_1w_2, w_2, w_1, w_4, \emptyset, \\ P(f_2) &: w_3, w_2w_4, w_1w_2, w_4, w_1, w_2, \emptyset, \\ P(f_3) &: w_4, w_1, w_3, \emptyset, \\ P(w_1) &: f_2, f_3, f_1, \emptyset, \\ P(w_2) &: f_2, f_1, \emptyset, \\ P(w_3) &: f_3, f_2, \emptyset, \\ P(w_4) &: f_2, f_1, f_3, \emptyset. \end{aligned}$$

The following table summarizes the 6 steps of the deferred-acceptance algorithm in which workers make offers with the corresponding offer sets, choice sets, and rejected workers for each of the three firms (we omit the brackets in the sets and write, for each f_i, O_i^k and Ch_i^k instead of $O^k(f_i)$ and $Ch(O^k(f_i), P(f_i))$, respectively).

	f_1			f_2			f_3		
k	O_1^k	Ch_1^k	rejects	O_2^k	Ch_2^k	rejects	O_3^k	Ch_3^k	rejects
1	$\emptyset$	$\emptyset$	–	w_1, w_2, w_4	w_2, w_4	w_1	w_3	w_3	–
2	$\emptyset$	$\emptyset$	–	w_1, w_2, w_4	w_2, w_4	–	w_1, w_3	w_1	w_3
3	$\emptyset$	$\emptyset$	–	w_1, w_2, w_3, w_4	w_3	w_2, w_4	w_1, w_3	w_1	–
4	w_2, w_4	w_2	w_4	w_1, w_2, w_3, w_4	w_3	–	w_1, w_3	w_1	–
5	w_2, w_4	w_2	–	w_1, w_2, w_3, w_4	w_3	–	w_1, w_3, w_4	w_4	w_1
6	w_1, w_2, w_4	w_1, w_2	–	w_1, w_2, w_3, w_4	w_3	–	w_1, w_3, w_4	w_4	–

Table 1

The algorithm terminates at step 6 (*i.e.*, $K = 6$), when no worker is rejected, and the provisional matching then becomes definite. For all $f \in F$, $O(f) = O^6(f)$ and $Ch(O(f), P(f)) = \mu_W(f)$; namely,

$$\mu_W = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2 & w_3 & w_4 \end{pmatrix}$$

is the workers-optimal stable matching. $\diamondsuit$

A firm f has a separable preference relation if the division between *good* workers ($\{w\}P(f)\,\emptyset$) and *bad* workers ($\emptyset P(f)\,\{w\}$) guides the ordering of subsets in the sense that adding a good worker leads to a *better* set, while adding a bad worker leads to a *worse* set.[6] Formally,

Definition 4 A firm f's preference relation $P(f)$ satisfies *separability* if for all $S \subseteq W$ and $w \notin S$ we have that $(S \cup \{w\}) P(f) S$ if and only if $\{w\} P(f) \emptyset$.

A preference profile P is *separable* if for each firm f, the preference relation $P(f)$ satisfies separability.

Remark 2 *All separable preference relations are substitutable. To see this, just note that if $P(f)$ is separable then, for every $S \subseteq W$, $Ch(S, P(f)) = \{w \in S \mid \{w\} P(f) \emptyset\}$. Moreover, the preference relation*

$$P(f) : w_1, w_1 w_2, w_2, \emptyset$$

shows that not all substitutable preference relations are separable.

Sönmez (1996) shows that if firms have separable preference relations then there exists a unique stable matching. A simple way to construct this unique stable matching μ is as follows: for each $w \in W$, let $\mu(w)$ be the maximal element, according to $P(w)$, on the set of firms for which w is an acceptable worker; *i.e.*, $\{f \in F \mid \{w\} P(f) \emptyset\}$. The stability of μ follows directly from separability of firms' preferences.

Here, we will assume that each firm f has, in addition to a substitutable and separable preference relation, a maximum number of positions to be filled: its quota q_f. This limitation may arise from, for example, technological, legal, or budgetary reasons. Since we are interested in stable matchings we introduce this restriction by incorporating it into the preference relation of the firm. The college admissions problem with responsive preference profiles (Gale and Shapley, 1962) incorporates the quota restriction of each firm by imposing a limit on the number of workers that a firm may admit. However, from the point of view of stability, this is equivalent to supposing that all sets of workers with cardinality larger than the quota are unacceptable for the firm. Therefore, even if the number of good workers for firm f is larger than its quota q_f, all sets of workers with cardinality strictly larger than q_f will be unacceptable. Formally,

Definition 5 A firm f's preference relation $P(f)$ over sets of workers is q_f-*separable* if:

[6] This condition has been extensively used in social choice; see, for instance, Barberà, Sonnenschein, and Zhou (1991). It has also been used in matching models; see, for instance, Alkan (2001), Dutta and Massó (1997), Ehlers and Klaus (2003), Martínez, Massó, Neme, and Oviedo (2000, 2001, and 2004), Papai (2000), and Sönmez (1996).

(a) For all $S \subsetneq W$ such that $|S| < q_f$ and $w \notin S$ we have that $(S \cup \{w\})P(f)S$ if and only if $\{w\}P(f)\emptyset$.
(b) $\emptyset P(f)\, S$ for all S such that $|S| > q_f$.[7]

We will denote by $q = (q_f)_{f \in F}$ the list of quotas and we will say that a preference profile P is quota $q-$separable if each $P(f)$ is quota q_f-separable. In principle we may have firms with different quotas. The case where all firms have quota 1$-$separable preference relations is equivalent, from the point of view of the set of stable matchings, to the marriage model. Hence, our set-up includes the marriage model as a particular case. In general, and given a list of quotas q, the sets of separable and quota $q-$separable preference relations are unrelated. Moreover, quota $q-$separability does not imply substitutability and the set of responsive preference relations is a strict subset of the set of quota q_f-separable and substitutable preference relations. There are quota $q-$separable preference profiles for which the set of stable matchings is empty (see Example 1 in Martínez, Massó, Neme, and Oviedo (2004)).

3 The Blocking Lemma

The Blocking Lemma is a statement relative to a substitutable preference profile. Given a substitutable preference profile P, the Blocking Lemma states that if the set of workers that strictly prefer an individually rational matching μ to μ_W is nonempty then, we can always find a blocking pair (f, w) of μ with the property that f was hiring at μ a worker strictly preferring μ to μ_W and w considers μ_W being at least as good as μ. Formally,

Definition 6 (The Blocking Lemma) Let P be a substitutable preference profile and let $\mu \in IR(P)$. Denote by $W' = \{w \in W \mid \mu(w)\, P(w)\, \mu_W(w)\}$ the set of workers who strictly prefer μ to μ_W. Assume W' is nonempty. We say that the *Blocking Lemma* holds at P if there exist $f \in \mu(W')$ and $w \in W \backslash W'$ such that the pair (f, w) blocks μ.

Let $\widetilde{\mathcal{P}}$ be a subset of substitutable preference profiles. We say that the Blocking Lemma holds on $\widetilde{\mathcal{P}}$ if it holds at all $P \in \widetilde{\mathcal{P}}$.

[7] For the purpose of studying the set of stable matchings, condition (b) in this definition could be replaced by the following condition: $|Ch(S, P(f))| \leq q_f$ for all S such that $|S| > q_f$. We choose condition (b) since it is simpler. Sönmez (1996) uses an alternative approach which consists of deleting condition (b) in the definition but then requiring in the definition of a matching that $|\mu(f)| \leq q_f$ for all $f \in F$. Notice that in his approach the set of separable preference relations of firm f is quota q_f-separable for all q_f.

Gale and Sotomayor (1985) proved the Blocking Lemma for the marriage model (*i.e.*, the Blocking Lemma holds on the set of all profiles of preference relations in which each agent only orders the set of individual agents of the other side of the market plus the prospect of remaining unmatched). Using the decomposition described in Footnote 2, it is easy to see that the Blocking Lemma also holds for the college admission problem (*i.e.*, on the set of all responsive preference profiles). In Martínez, Massó, Neme, and Oviedo (2004) we exhibit an example (Example 1 above used to illustrate the deferred-acceptance algorithm in which workers make offers) of a substitutable preference profile P for which the Blocking Lemma does not hold at P. For completeness, we reproduce below why the Blocking Lemma does not hold at P.

Example 1 (Continued) Consider F, W, and the substitutable preference profile P of Example 1. As we have already found, the workers-optimal stable matching is

$$\mu_W = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2 & w_3 & w_4 \end{pmatrix}.$$

The individually rational matching

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_4 & w_1w_2 & w_3 \end{pmatrix}$$

has the property that for all $w \in W$, $\mu(w)P(w)\mu_W(w)$. Hence, $W \backslash W' = \emptyset$ since $W' = \{w \in W \mid \mu(w)P(w)\mu_W(w)\} = W$. Therefore, we can not find $w \in W \backslash W'$ and $f \in \mu(W')$ such that (f, w) blocks μ. Thus, the conclusion of the Blocking Lemma does not hold at P.

Theorem 1 *The Blocking Lemma holds on the set of all substitutable and quota $q-$separable preference profiles.*

The proof of Theorem 1 is in Appendix 1.

4 Blocking Lemma and Group Strategy-proofness

Many real matching markets are centralized. A centralized matching market consists of a clearinghouse that, after asking each agent to report a preference relation, proposes a matching. This defines a mechanism. A mechanism is stable if it proposes, for each reported preference profile, a stable matching.[8] Formally, let $\overline{\mathcal{P}}$ be a domain of preference profiles and let $\mathcal{M}$ be the set of all matchings. A *mechanism* $h : \overline{\mathcal{P}} \to \mathcal{M}$ maps each preference profile $P \in \overline{\mathcal{P}}$ to a matching $h(P) \in \mathcal{M}$. Therefore, $h(P)(f)$ is the set of workers assigned to f and $h(P)(w)$ is the firm assigned to w (if any) at preference profile $P \in \overline{\mathcal{P}}$ by mechanism h. A mechanism $h : \overline{\mathcal{P}} \to \mathcal{M}$ is *stable* if for all $P \in \overline{\mathcal{P}}$, $h(P) \in S(P)$.

Let $\mathcal{S}$ be the set of substitutable preference relations of any firm on 2^W and let $\mathcal{T}$ be the set of all preference relations of any worker on $F \cup \{\emptyset\}$. The set of all substitutable preference profiles can be written as the set $\mathcal{P} = \mathcal{S}^F \times \mathcal{T}^W$. Let $\widetilde{\mathcal{S}}$ be a subset of substitutable preference relations for any firm and let $\widetilde{\mathcal{P}} = \widetilde{\mathcal{S}}^F \times \mathcal{T}^W$.[9] To emphasize the role of a subset of workers $\widehat{W}$ we will write the preference profile $P \in \widetilde{\mathcal{P}}$ as $(P_{\widehat{W}}, P_{-\widehat{W}})$. Therefore, given $P \in \widetilde{\mathcal{P}}$, $\widehat{W} \subseteq W$, and $P'_{\widehat{W}} \in \mathcal{T}^{\widehat{W}}$, we write $(P'_{\widehat{W}}, P_{-\widehat{W}})$ to denote the preference profile P' obtained from P after replacing $P_{\widehat{W}} \in \mathcal{T}^{\widehat{W}}$ by $P'_{\widehat{W}} \in \mathcal{T}^{\widehat{W}}$. Mechanisms require each agent to report some preference relation. A mechanism is group strategy-proof for the workers if for all subsets of workers they can never obtain better partners by revealing their preference relations untruthfully. Formally,

Definition 7 A mechanism $h : \widetilde{\mathcal{P}} \to \mathcal{M}$ is *group strategy-proof for the workers* if for all preference profiles $P \in \widetilde{\mathcal{P}}$, all subsets of workers $\widehat{W} \subseteq W$, and all reports $P'_{\widehat{W}} \in \mathcal{T}^{\widehat{W}}$,

$$h(P)(w)\, R(w)\, h(P'_{\widehat{W}}, P_{-\widehat{W}})(w)$$

for all $w \in \widehat{W}$.

We say that $h_W : \widetilde{\mathcal{P}} \to \mathcal{M}$ is the *workers-optimal stable mechanism* if it always selects the workers-optimal stable matching; that is, for all $P \in \widetilde{\mathcal{P}}$, $h_W(P)$ is the workers-optimal stable matching relative to P. Theorem 2 below states that the Blocking Lemma and group strategy-proofness of the workers-optimal stable mechanism for the workers are equivalent on any subset of substitutable preference profiles $\widetilde{\mathcal{P}} = \widetilde{\mathcal{S}}^F \times \mathcal{T}^W$.

Theorem 2 *The Blocking Lemma holds on $\widetilde{\mathcal{P}}$ if and only if the workers-optimal stable mechanism $h_W : \widetilde{\mathcal{P}} \to \mathcal{M}$ is group strategy-proof for the workers.*

[8] The National Resident Matching Program is a very well-know example of a centralized entry-level professional labor market in the U.S.A. that uses a stable mechanism to match yearly around 20,000 medical students to hospital programs to undertake their medical internship (see Roth and Sotomayor (1990) for a description an analysis of this market). See Roth and Xing (1994) for a discussion of many centralized matching markets.

[9] Observe that the domain of preference relations of the workers is unrestricted while the domain of preference relations of the firms may be any subset of substitutable preference relations.

The proof of Theorem 2 is in Appendix 2 at the end of the paper.

5 Concluding Remark

One may ask whether or not the symmetric Blocking Lemma for the firms also holds on the same set of preference profiles for which the Blocking Lemma holds for the workers. The answer is negative because it does not even hold on the smaller subset of responsive preference profiles. Example 2 (taken from Roth (1984a)) contains a substitutable and quota $q-$separable preference profile P for which the Blocking Lemma for the firms does not hold at P.

Example 2 Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the $(2,1,1)-$separable and substitutable preference profile P,[10] where

$$\begin{aligned}
P(f_1) &: w_1w_2, w_1w_3, w_2w_3, w_1w_4, w_2w_4, w_3w_4, w_1, w_2, w_3, w_4, \emptyset, \\
P(f_2) &: w_1, w_2, w_3, w_4, \emptyset, \\
P(f_3) &: w_3, w_1, w_2, w_4, \emptyset, \\
P(w_1) &: f_3, f_1, f_2, \emptyset, \\
P(w_2) &: f_2, f_1, f_3, \emptyset, \\
P(w_3) &: f_1, f_3, f_2, \emptyset, \\
P(w_4) &: f_1, f_2, f_3, \emptyset.
\end{aligned}$$

The firms-optimal stable matching is

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_4 & w_2 & w_3 \end{pmatrix}.$$

Consider the individually rational matching

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_4 & w_2 & w_3 \end{pmatrix}.$$

The set of firms that prefer μ to μ_F is $F' \equiv \{f \in F \mid \mu(f)P(f)\mu_F(f)\} = \{f_1, f_3\}$. Thus, $F \backslash F' = \{f_2\}$. However, f_2 can only block μ together with w_1, but $\mu(w_1) = f_1 P(w_1) f_2$. Hence, the Blocking Lemma for the firms does not hold at P. Note that this profile is used in Roth (1984a) to show that, in the college admissions problem, the deferred-acceptance algorithm in which firms make offers is not strategy-proof for the firms. $\Diamond$

[10] Observe that P is indeed a responsive preference profile.

6 Appendix 1: Proof of Theorem 1

Through out all this appendix we will assume that P is a substitutable and quota $q-$separable preference profile and that $\mu \in IR(P)$. The set of workers who strictly prefer μ to μ_W will be denoted by $W' = \{w \in W \mid \mu(w) P(w) \mu_W(w)\}$ and we will assume that W' is nonempty.

In the proof of the Blocking Lemma for the marriage model (*i.e.*, $q_f = 1$ for all $f \in F$), the firm f that, together with a worker w, blocks μ is matched at μ because $f \in \mu(W')$; that is, f fills its quota at μ (*i.e.*, $|\mu(f)| = q_f = 1$). In the proof of the Blocking Lemma for the many-to-one model, the firm f that, together with a worker w, blocks μ is also matched at μ because $f \in \mu(W')$, but now it will be necessary to consider separately the case in which f does not fill its quota at μ (*i.e.*, $|\mu(f)| < q_f$) from the case in which f fills its quota at μ (*i.e.*, $|\mu(f)| = q_f$). Proposition 1 considers the case where $|\mu(f)| < q_f$. For the case where $|\mu(f)| = q_f$ the proof of the Blocking Lemma will also be decomposed, as in the marriage model, into two propositions depending on whether or not $\mu(W') = \mu_W(W')$ holds (Proposition 2 for the simple case where they are different and Proposition 3 for the more involved case where they are equal). However, before proving these three propositions, we prove a series of three lemmata that will be used in the proof of all three propositions since they hold regardless of whether or not $\mu(W') = \mu_W(W')$ and whether or not all firms in $\mu(W')$ fill their quota at μ.

Lemma 1 *For each $f \in \mu(W')$, $|\mu_W(f)| = q_f$.*

Proof. Assume otherwise and let $f' \in \mu(W')$ be such that $|\mu_W(f')| < q_{f'}$. Since W' is nonempty and $f' \in \mu(W')$ there exists $w' \in W'$ such that $f' = \mu(w') P(w') \mu_W(w')$, which implies that $w' \notin \mu_W(f')$. Moreover, $\mu \in IR(P)$, quota $q_{f'}-$separability of $P(f')$, and $|\mu_W(f')| < q_{f'}$ imply $w' \in Ch(\mu_W(f') \cup \{w'\}, P(f'))$. Thus, (f', w') blocks μ_W, which is a contradiction. ■

Lemma 2 *Assume there exist $f \in \mu(W')$ and $w \in Ch(\mu(f) \cup \mu_W(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}$. Then $w \in W \backslash W'$, and (f, w) blocks μ.*

Proof. Since $w \in Ch(\mu(f) \cup \mu_W(f), P(f))$ and $w \notin [\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]$, we have that either $w \in W'$ and $w \in \mu(f) \backslash \mu_W(f)$ or $w \in W \backslash W'$ and $w \in \mu_W(f) \backslash \mu(f)$. Assume $w \in W'$ and $w \in \mu(f) \backslash \mu_W(f)$. Then,

$$f = \mu(w) P(w) \mu_W(f). \tag{2}$$

Moreover, $w \in Ch(\mu(f) \cup \mu_W(f), P(f))$ implies, by substitutability of $P(f)$, that $w \in Ch(\mu_W(f) \cup \{w\}, P(f))$, which together with (2) imply that (f, w) blocks μ_W. Therefore, we can assume that $w \in W \backslash W'$ and $w \in \mu_W(f) \backslash \mu(f)$. Then,

$$f = \mu_W(w) P(w) \mu(w). \tag{3}$$

Moreover, $w \in Ch(\mu(f) \cup \mu_W(f), P(f))$ implies, by substitutability of $P(f)$, that $w \in Ch(\mu(f) \cup \{w\}, P(f))$, which together with (3) imply that (f, w) blocks μ. ■

By Remark 1 and condition (1), Lemma 2 can also be stated as follows.

Remark 3 *Assume there exist* $f \in \mu(W')$ *and* $w \in Ch(\mu(f) \cup O(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}$. *Then* $w \in W \backslash W'$, *and* (f, w) *blocks* μ.

Lemma 3 *Assume there exists* $f \in \mu(W')$ *such that* $|\mu(f) \cap W'| \neq |\mu_W(f) \cap W'|$. *Then, there exist* $\widetilde{f} \in \mu(W')$ *and* $w \in W \backslash W'$ *such that* $(\widetilde{f}, w)$ *blocks* μ.

Proof. It follows from Claims 1 and 2 below. ■

Claim 1 Assume there exists $f \in \mu(W')$ such that $|\mu(f) \cap W'| > |\mu_W(f) \cap W'|$. Then, there exists $w \in W \backslash W'$ such that (f, w) blocks μ.

Proof of Claim 1. Assume $f \in \mu(W')$. We will first show that $|\mu(f) \cap W'| > |\mu_W(f) \cap W'|$ implies that there exists $w \in Ch(\mu(f) \cup \mu_W(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}$. To see this, first observe that, by Lemma 1, $|\mu_W(f)| = q_f$. Moreover,

$$|\mu_W(f)| = |\mu_W(f) \cap W'| + |\mu_W(f) \cap (W \backslash W')| = q_f$$

and

$$|\mu(f)| = |\mu(f) \cap W'| + |\mu(f) \cap (W \backslash W')| \leq q_f. \tag{4}$$

By hypothesis and (4), $|\mu_W(f) \cap W'| + |\mu(f) \cap (W \backslash W')| < q_f$. By $|\mu_W(f)| = q_f$, $\mu_W(f) \subseteq \mu(f) \cup \mu_W(f)$, and q_f-separability of $P(f)$, $|Ch(\mu(f) \cup \mu_W(f), P(f))| = q_f$. Hence, there exists

$$w \in Ch(\mu(f) \cup \mu_W(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}.$$

By Lemma 2, (f, w) blocks μ. ■

Claim 2 Assume there exists $f \in \mu(W')$ such that $|\mu(f) \cap W'| < |\mu_W(f) \cap W'|$. Then, there exist $\widetilde{f} \in \mu(W')$ and $w \in W \backslash W'$ such that $(\widetilde{f}, w)$ blocks μ.

PROOF OF CLAIM 2. By definition, $W' = \cup_{f \in \mu(W')}(\mu(f) \cap W')$. Thus, $\cup_{f \in \mu(W')}(\mu(f) \cap W') \subseteq W'$. Therefore,

$$|W'| = \sum_{\overline{f} \in \mu(W')} \left|\mu(\overline{f}) \cap W'\right| \geq \sum_{\overline{f} \in \mu(W')} \left|\mu_W(\overline{f}) \cap W'\right|.$$

Hence, by the hypothesis that there exists $f \in \mu(W')$ such that $|\mu(f) \cap W'| < |\mu_W(f) \cap W'|$, there exists $\widetilde{f} \in \mu(W')$ with the property that $\left|\mu(\widetilde{f}) \cap W'\right| > \left|\mu_W(\widetilde{f}) \cap W'\right|$. This $\widetilde{f}$ satisfies the hypothesis of Claim 1, and hence, there exists $w \in W \backslash W'$ such that $(\widetilde{f}, w)$ blocks μ. ■

Proposition 1 *Assume $f \in \mu(W')$ is such that $|\mu(f)| < q_f$.*[11] *Then, there exist $\widetilde{f} \in \mu(W')$ and $w \in W \backslash W'$ such that $(\widetilde{f}, w)$ blocks μ.*

Proof. Let $f \in \mu(W')$ be such that $|\mu(f)| < q_f$. By Lemma 1, $|\mu_W(f)| = q_f$. We consider two cases:

CASE 1. $|\mu(f) \cap W'| \neq |\mu_W(f) \cap W'|$. By Lemma 3, there exist $\widetilde{f} \in \mu(W')$ and $w \in W \backslash W'$ such that $(\widetilde{f}, w)$ blocks μ.

CASE 2. $|\mu(f) \cap W'| = |\mu_W(f) \cap W'|$. Then, $|\mu(f) \cap (W \backslash W')| < |\mu_W(f) \cap (W \backslash W')|$ since $|\mu(f)| < q_f = |\mu_W(f)|$. Hence, there exists $w \in (\mu_W(f) \cap (W \backslash W')) \setminus (\mu(f) \cap (W \backslash W'))$. In particular, $w \in (\mu_W(f) \cap (W \backslash W')) \backslash \mu(f)$. Therefore, since $w \notin W'$, $w \notin \mu(f)$, and $w = \mu_W(f)$ we have that

$$fP(w)\mu(w). \tag{5}$$

Moreover, by quota q_f-separability of $P(f)$ and the individual rationality of μ_W, w is a good worker for f. Hence, $|\mu(f)| < q_f$ implies

$$w \in Ch(\mu(f) \cup \{w\}, P(f)). \tag{6}$$

Conditions (5) and (6) say that $f \in \mu(W')$ and $w \in W \backslash W'$ are such that (f, w) blocks μ. ■

Proposition 2 *Assume $\mu(W') \neq \mu_W(W')$ and, for all $f \in \mu(W')$, $|\mu(f)| = q_f$. Then, there exist $f \in \mu(W')$ and $w \in W \backslash W'$ such that (f, w) blocks μ.*

Proof. Consider the following two cases:

CASE 1. There exists $f \in \mu(W') \backslash \mu_W(W')$. Thus, there is $w' \in W'$ such that $w' \in \mu(f)$, implying $|\mu(f) \cap W'| \geq 1$. Moreover, $f \notin \mu_W(W')$ implies that $|\mu_W(f) \cap W'| = 0$.

[11] Note that $f \in \mu(W')$ and $|\mu(f)| < 1$ are incompatible in the marriage model.

Therefore, $|\mu(f) \cap W'| \neq |\mu_W(f) \cap W'|$. Hence, by Lemma 3, the conclusion of Proposition 2 holds.

CASE 2. There exists $f \in \mu_W(W') \backslash \mu(W')$. Thus, there is $w' \in W'$ such that $w' \in \mu_W(f)$, implying $|\mu_W(f) \cap W'| \geq 1$. Moreover, $f \notin \mu(W')$ implies $|\mu(f) \cap W'| = 0$. Therefore, $|\mu(f) \cap W'| \neq |\mu_W(f) \cap W'|$. Hence, by Lemma 3, the conclusion of Proposition 2 holds. ■

Finally, Proposition 3 below states that the conclusion of the Blocking Lemma holds for all remaining cases where $\mu(W') = \mu_W(W')$.

Proposition 3 *Assume $\mu(W') = \mu_W(W')$ and, for all $f \in \mu(W')$, the following three properties hold:*
(a) $|\mu(f)| = q_f$.
(b) $|\mu(f) \cap W'| = |\mu_W(f) \cap W'|$.
(c) $Ch(\mu(f) \cup O(f), P(f)) \backslash \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\} = \emptyset$.
Then, there exist $\overline{f} \in \mu(W')$ and $\overline{w} \in W \backslash W'$ such that $(\overline{f}, \overline{w})$ blocks μ.

Proof. By (c), for all $f \in \mu(W')$, $Ch(\mu(f) \cup O(f), P(f)) \subseteq \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}$. We want to show that this inclusion holds with equality. We will do it by showing that the two sets have the same cardinality.

By (a), quota $q-$separability of $P(f)$, and $\mu(f) \subseteq \mu(f) \cup O(f)$, $|Ch(\mu(f) \cup O(f), P(f))| = q_f$. Thus,

$$\begin{aligned} |(\mu_W(f) \cap W') \cup (\mu(f) \cap (W \backslash W'))| &\leq |\mu_W(f) \cap W'| + |\mu(f) \cap (W \backslash W')| \\ &= |\mu(f) \cap W'| + |\mu(f) \cap (W \backslash W')| \qquad \text{by (b)} \\ &= |\mu(f)| \\ &= q_f \qquad \text{by (a).} \end{aligned}$$

Hence,

$$Ch(\mu(f) \cup O(f), P(f)) = \{[\mu_W(f) \cap W'] \cup [\mu(f) \cap (W \backslash W')]\}. \tag{7}$$

CLAIM 3 For all $w \in W'$, $\mu_W(w) \neq \emptyset$.

PROOF OF CLAIM 3 By definition, $W' = \cup_{f \in \mu(W')} \mu(f)$. Hence, $W' = (\cup_{f \in \mu(W')} \mu(f)) \cap W'$. Because all workers can work at most for one firm and since (b) holds,

$$|W'| = \sum_{f \in \mu(W')} |\mu(f) \cap W'| = \sum_{f \in \mu(W')} |\mu_W(f) \cap W'|. \tag{8}$$

If there would exist $w \in W'$ such that $\mu_W(w) = \emptyset$ then, $|W'| > \sum_{f \in \mu(W')} |\mu_W(f) \cap W'|$, which would contradict (8). ■

For each step $1 \leq k \leq K$ of the deferred-acceptance algorithm in which workers make offers, define the set T_k as the set of pairs of firms and workers in $\mu(W')$ and W', respectively, with the property that w is acceptable for f, w makes an offer to f at k or at earlier steps, and f rejects w at step k. Namely,

$$T_k = \left\{(f, w) \in \mu(W') \times W' \mid w \in Ch(\{w\}, P(f)) \cap O^k(f) \text{ and } w \notin Ch\left(O^k(f), P(f)\right)\right\}. \tag{9}$$

Set $T_0 = \emptyset$. Since for all $1 < k \leq K$ and all $f \in F$, $O^{k-1}(f) \subseteq O^k(f)$ holds, substitutability of $P(f)$ implies that, for all $1 \leq k \leq K$, $T_{k-1} \subseteq T_k$. Furthermore, there exists $1 \leq \bar{k} \leq K$ such that $T_{\bar{k}} \neq \emptyset$. To see that, note that: (i) each $w \in W'$ makes an offer to $f = \mu(w)$ at (some) step $\underline{k}$ of the algorithm ($w \in O^{\underline{k}}(f)$), (ii) w is acceptable for f ($w \in Ch(\{w\}, P(f))$, and (iii) since $w \notin \mu_W(f)$, f rejects w at (some) step $\bar{k} \geq \underline{k}$ ($w \notin Ch(O^{\underline{k}}(f), P(f))$. Therefore, step k_M where

$$k_M = \max\{1 \leq k \leq K \mid T_k \backslash T_{k-1} \neq \emptyset\}$$

is well-defined. By Claim 3, $T_K \backslash T_{K-1} = \emptyset$. Hence, and since $T_0 = \emptyset$, $1 \leq k_M < K$.

Claim 4 For all $f \in \mu(W')$ and all $k \geq k_M$, $\left|O^k(f)\right| > q_f$.

Proof of Claim 4 By definition of T_{k_M}, for all $f \in \mu(W')$ and all $w \in \mu(f) \cap W'$, $(f, w) \in T_{k_M}$. Since $\mu \in IR(P)$, w is acceptable for f. Hence, by quota q_f-separability of $P(f)$, $w \notin Ch\left(O^{k_M}(f), P(f)\right)$ implies $\left|O^k(f)\right| > q_f$ for all $k \geq k_M$ because $O^k(f) \supseteq O^{k_M}(f)$ for all $k \geq k_M$. ■

By definition of k_M and (1), for all $f \in \mu(W')$ and all $k \geq k_M$,

$$Ch\left(O^k(f), P(f)\right) \cap W' \subseteq \mu_W(f) \cap W' = Ch(O(f), P(f)) \cap W'. \tag{10}$$

Let $(f_1, w_1) \in \mu(W') \times W'$ be a pair such that $(f_1, w_1) \in T_{k_M} \backslash T_{k_M-1}$. By Claim 3, $\mu_W(w_1) \neq \emptyset$. Hence, there exists $f_2 \in \mu(W')$ such that

$$w_1 \in \mu_W(f_2). \tag{11}$$

Claim 5 There exists $k^* > k_M$ such that $w_1 \in O^{k^*}(f_2) \backslash O^{k^*-1}(f_2)$ and $w_1 \in Ch\left(O^{k^*}(f_2), P(f_2)\right)$.

Proof of Claim 5. By (11) and the fact that $w_1 \in W'$, $w_1 \in \mu_W(f_2) \cap W'$. Hence, there exists k^* such that $w_1 \in Ch\left(O^{k^*}(f_2), P(f_2)\right)$. Note that $k^* > k_M$ because at step k_M, w_1 was rejected by firm f_1. ■

Claim 6 $Ch\left(O^{k_M}(f_2) \cup \mu(f_2), P(f_2)\right) \setminus \left\{\left[\mu_W(f_2) \cap W' \cap O^{k_M}(f_2)\right] \cup \left[\mu(f_2) \cap W \backslash W'\right]\right\} \neq \emptyset$.

Proof of Claim 6. By (7) and $f_2 \in \mu(W')$,

$$Ch(\mu(f_2) \cup O(f_2), P(f_2)) = \{[\mu_W(f_2) \cap W'] \cup [\mu(f_2) \cap (W \backslash W')]\}. \tag{12}$$

Let $w \in \mu(f_2) \cap (W \backslash W')$. By (12), $w \in Ch(\mu(f_2) \cup O(f_2), P(f_2))$. By substitutability of $P(f_2)$, $O^{k_M}(f_2) \subseteq O(f_2)$, and the fact that $\mu(f_2) \cap (W \backslash W') \subseteq \mu(f_2)$, $w \in Ch(\mu(f_2) \cup O^{k_M}(f_2), P(f_2))$. Hence,

$$[\mu(f_2) \cap (W \backslash W')] \subseteq Ch\left(O^{k_M}(f_2) \cup \mu(f_2), P(f_2)\right).$$

Moreover, $[\mu_W(f_2) \cap W' \cap O^{k_M}(f_2)] \subseteq \mu_W(f_2) \cap W'$. Let $w \in [\mu_W(f_2) \cap W' \cap O^{k_M}(f_2)]$. Then, $w \in \mu_W(f_2) \cap W'$. By (12), $w \in Ch(\mu(f_2) \cup O(f_2), P(f_2))$. By substitutability of $P(f_2)$, $w \in Ch\left(O^{k_M}(f_2) \cup \mu(f_2), P(f_2)\right)$. Hence,

$$[\mu_W(f_2) \cap W' \cap O^{k_M}(f_2)] \subseteq Ch\left(O^{k_M}(f_2) \cup \mu(f_2), P(f_2)\right).$$

By (a), $|\mu(f_2)| = q_{f_2}$. Moreover, $\mu(f_2) \subseteq O^{k_M}(f_2) \cup \mu(f_2)$ holds trivially. By individual rationality of μ and quota $q_{f_2}-$separability of $P(f_2)$,

$$\left|Ch\left(O^{k_M}(f_2) \cup \mu(f_2), P(f_2)\right)\right| = q_{f_2}. \tag{13}$$

By Claim 5, $w_1 \notin O^{k_M}(f_2)$. Hence, $w_1 \in \mu_W(f_2) \cap W'$ implies

$$|[\mu(f_2) \cap (W \backslash W')]| + \left|[\mu_W(f_2) \cap W' \cap O^{k_M}(f_2)]\right| < q_{f_2}. \tag{14}$$

To see that (14) holds, observe that, by (a), $|[\mu(f_2) \cap (W \backslash W')]| + |[\mu(f_2) \cap W']| = |\mu(f_2)| = q_{f_2}$. By (b), $|\mu(f_2) \cap (W \backslash W')| + |\mu_W(f_2) \cap W'| = q_{f_2}$. Since $w_1 \in \mu_W(f_2) \cap W'$ and $w_1 \notin O^{k_M}(f_2)$ we have that $\left|\mu_W(f_2) \cap W' \cap O^{k_M}(f_2)\right| < |\mu_W(f_2) \cap W'|$. Hence, (14) holds. Thus, by (13), the statement of Claim 6 holds. ■

Let

$$\widetilde{w} \in Ch\left(O^{k_M}(f_2) \cup \mu(f_2), P(f_2)\right) \setminus \left\{\left[\mu_W(f_2) \cap W' \cap O^{k_M}(f_2)\right] \cup [(\mu(f_2) \cap W \backslash W']\right\} \neq \emptyset. \tag{15}$$

By Claim 6, such $\widetilde{w}$ exists. We will show that $\widetilde{w} \notin W'$. Assume otherwise; *i.e.*, $\widetilde{w} \in W'$. If $\widetilde{w} \in \mu(f_2)$ then $\widetilde{w} \in O^{k_M}(f_2)$. Hence, $\widetilde{w} \in \mu_W(f_2)$. Thus, $\mu(\widetilde{w}) = f_2 = \mu_W(\widetilde{w})$, which contradicts that $\widetilde{w} \in W'$. If $\widetilde{w} \in O^{k_M}(f_2)$ then $\widetilde{w} \in \mu_W(f_2)$. Thus, $\widetilde{w} \in \mu_W(f_2) \cap W' \cap O^{k_M}(f_2)$. But (15) says that $\widetilde{w} \notin \mu(f_2) \cap W' \cap O^{k_M}(f_2)$, a contradiction. Hence, $\widetilde{w} \in W'$.

By (15), $\widetilde{w} \notin \mu(f_2) \cap (W \backslash W')$. Since $\widetilde{w} \notin W'$,

$$\widetilde{w} \notin \mu(f_2). \tag{16}$$

By (15) again, $\widetilde{w} \in Ch(O^{k_M}(f_2) \cup \mu(f_2), P(f_2))$. Hence, by substitutability of $P(f_2)$,

$$\widetilde{w} \in Ch(\mu(f_2) \cup \{\widetilde{w}\}, P(f_2)). \tag{17}$$

By $\widetilde{w} \in Ch(O^{k_M}(f_2) \cup \mu(f_2), P(f_2))$ and (16), $\widetilde{w} \in O^{k_M}(f_2)$. Hence,

$$f_2 R(\widetilde{w}) \mu_W(\widetilde{w}). \tag{18}$$

Moreover, by $\widetilde{w} \notin W'$,

$$\mu_W(\widetilde{w}) R(\widetilde{w}) \mu(\widetilde{w}), \tag{19}$$

and by (16), $\mu(\widetilde{w}) \neq f_2$. Hence, by (18) and (19),

$$f_2 P(\widetilde{w}) \mu(\widetilde{w}). \tag{20}$$

Thus, since $f_2 \in \mu(W')$ and $\widetilde{w} \notin W'$, (16), (17) and (20) say that the conclusion of the Blocking Lemma holds. ■

7 Appendix 2: Proof of Theorem 2

$\Longrightarrow$) Assume that the Blocking Lemma holds on $\widetilde{\mathcal{P}} = \widetilde{\mathcal{S}}^F \times \mathcal{T}^W$ and suppose, to get a contradiction, that $h_W : \widetilde{\mathcal{P}} \to \mathcal{M}$ is not strategy-proof for the workers; namely, there exist $P \in \widetilde{\mathcal{P}}$, a nonempty subset of workers $\widehat{W}$, and $P'_{\widehat{W}} \in \mathcal{T}^{\widehat{W}}$ such that for all $w \in \widehat{W}$,

$$\mu'_W(w) = h_W(P'_{\widehat{W}}, P_{-\widehat{W}})(w)\, P(w)\, h_W(P)(w) = \mu_W(w).$$

We first show that $\mu'_W \in IR(P)$. Since $\mu'_W \in S(P'_{\widehat{W}}, P_{-\widehat{W}})$ and $P'(i) = P(i)$ for all $i \in (F \cup W) \backslash \widehat{W}$, $\mu'_W(i) R'(i) \emptyset$ if $i \in W \backslash W'$ and $\mu'_W(i) = Ch(\mu'_W(i), P'(i))$ if $i \in F$. Moreover, for all $w \in \widehat{W}$, $\mu'_W(w) P(w) \mu_W(w) R(w) \emptyset$. Hence, $\mu'_W \in IR(P)$. Since all matchings $\mu \in S(P)$ have the property that $\mu_W R(w) \mu$ for all $w \in W$ and there exists at least one $w \in \widehat{W}$ with $\mu'_W(w) P(w) \mu_W(w)$, we conclude that $\mu'_W \notin S(P)$. Since $\emptyset \neq \widehat{W} \subseteq W' = \{w \in W \mid \mu'_W(w) P(w) \mu_W(w)\}$ and the Blocking Lemma holds at $P \in \widetilde{\mathcal{P}}$, there exists a pair $(\overline{f}, \overline{w})$, where $\overline{f} \in \mu(W')$ and $\overline{w} \in W \backslash W'$ such that $(\overline{f}, \overline{w})$ blocks μ'_W at P. But $\overline{w} \in W \backslash W'$ implies $P'(\overline{w}) = P(\overline{w})$. Thus, $(\overline{f}, \overline{w})$ blocks μ at $(P'_{\widehat{W}}, P_{-\widehat{W}})$, contradicting that $\mu'_W \in S(P'_{\widehat{W}}, P_{-\widehat{W}})$.

$\Longleftarrow$) Assume that $h_W : \widetilde{\mathcal{P}} \to \mathcal{M}$ is group strategy-proof for the workers and suppose, to get a contradiction, that the Blocking Lemma does not hold on $\widetilde{\mathcal{P}}$; namely, there exist $P \in \widetilde{\mathcal{P}}$ and an individually rational matching μ such that

$$W' = \{w \in W \mid \mu(w)\, P(w)\, \mu_W(w) = h_W(P)\} \neq \emptyset.$$

For all $w' \in W'$, consider the preference relation $P'(w) \in \mathcal{T}$, where

$$P'(w) : \mu(w), \emptyset.$$

Since h_W is group strategy-proof for the workers, there exists $w' \in W'$ such that

$$h_W(P'_{W'}, P_{-W'})(w') = \emptyset;$$

otherwise, if for all $w \in W'$, $h_W(P'_{W'}, P_{-W'})(w) = \mu(w)$, W' would manipulate h_W at P. Let $f' = \mu(w')$. Then, $f' \in \mu(W')$.

Let $\mu'_W = h_W(P'_{W'}, P_{-W'})$ and $\mu_W = h_W(P)$ be the corresponding workers-optimal stable matchings in $S(P'_{W'}, P_{-W'})$ and $S(P)$, respectively. Let $P' = (P'_{W'}, P_{-W'})$. For all $f \in F$, we denote by

$$O_P(f) = \{w \in W \mid f R(w)\, \mu_W(w)\}$$

and

$$O_{P'}(f) = \{w \in W \mid f R'(w)\, \mu'_W(w)\}$$

the set of worker that make an offer to f along the deferred-acceptance algorithm in which workers make offers applied to P and P', respectively. Let $O^k_P(f)$ denote the set of all workers that make an offer to f along the deferred acceptance algorithm applied to P at k or at earlier steps. Then,

$$O^1_P(f) \subseteq ... \subseteq O^k_P(f) \subseteq ... \subseteq O^K_P(f) = O_P(f).$$

Similarly,

$$O^1_{P'}(f) \subseteq ... \subseteq O^k_{P'}(f) \subseteq ... \subseteq O^{K'}_{P'}(f) = O_{P'}(f).$$

Then, for all $f \in F$, $\mu_W(f) = Ch(O_P(f), P(f))$ and $\mu'_W(f) = Ch(O_{P'}(f), P(f))$.

Claim 7 For all $w \notin W'$, $\mu'_W(w)\, R(w)\, \mu_W(w)$.

Proof of Claim 7. Assume otherwise; *i.e.,* there exists $\overline{w} \notin W'$ such that $\mu_W(\overline{w})\, P(\overline{w})\, \mu'_W(\overline{w})$. Let $k_m \leq K'$ be the first step in the deferred-acceptance algorithm in which workers make offers applied to P' in which there exist a firm and a worker $w \in \mu_W(f)$ such that w is rejected by f. Namely,

$$k_m = \min\left\{k \leq K' \mid \exists(f,w) \text{ s. t. } w \in O_{P'}^{k}(f),\ w \in \mu_W(f),\ \text{and } w \notin Ch\left(O_{P'}^{k}(f), P(f)\right)\right\}. \tag{21}$$

Note that $P(\overline{w}) = P'(\overline{w})$. The assumption that $\mu_W(\overline{w})\, P(\overline{w})\, \mu'_W(\overline{w})$ implies that there exits $1 \leq k < K'$ such that $\overline{w}$ is rejected by $\overline{f} = \mu_W(\overline{w})$ along the deferred-acceptance algorithm in which workers make offers applied to P'; *i.e.*, $\overline{f} = \mu_W(\overline{w})$, $\overline{w} \in O_{P'}^{k}\left(\overline{f}\right)$ and $\overline{w} \notin Ch\left(O_{P'}^{k}\left(\overline{f}\right), P\left(\overline{f}\right)\right)$. Let $\overline{f}_1$ and $\overline{w}$ be the pair satisfying (21) at step k_m. Assume $O_{P'}^{k_m}(\overline{f}_1) \subseteq O_P(\overline{f}_1)$. Then,

$$\overline{w} \in \mu_W(\overline{f}_1) = Ch(O_P(\overline{f}_1), P(\overline{f}_1))$$

and

$$w \notin Ch(O_{P'}^{k_m}(\overline{f}_1), P'(\overline{f}_1)),$$

which contradicts substitutability of $P(\overline{f}_1)$ since $P(\overline{f}_1) = P'(\overline{f}_1)$. Hence, $O_{P'}^{k_m}(\overline{f}_1) \nsubseteq O_P(\overline{f}_1)$. Let $\overline{w}_1$ be such that

$$\overline{w}_1 \in O_{P'}^{k_m}(\overline{f}_1) \tag{22}$$

and

$$\overline{w}_1 \notin O_P(\overline{f}_1). \tag{23}$$

By the definition of $O_P(\overline{f}_1)$, (23) implies that

$$\mu_W(\overline{w}_1)P(\overline{w}_1)\overline{f}_1. \tag{24}$$

By (22),

$$\overline{f}_1 R'(\overline{w}_1)\mu'_W(\overline{w}_1). \tag{25}$$

If $\overline{w}_1 \in W'$ then, by definition of $P'(\overline{w}_1)$, $\overline{f}_1 = \mu(\overline{w}_1)$. Hence, $\overline{f}_1 = \mu(\overline{w}_1)P(\overline{w}_1)\mu_W(\overline{w}_1)$, contradicting (24). Then, $\overline{w}_1 \notin W'$. Hence, $P'(\overline{w}_1) = P(\overline{w}_1)$. Thus, (25) can be written as

$$\overline{f}_1 R(\overline{w}_1)\mu'_W(\overline{w}_1). \tag{26}$$

Thus, (24) and (26) imply

$$\mu_W(\overline{w}_1)P(\overline{w}_1)\overline{f}_1 R(\overline{w}_1)\mu'_W(\overline{w}_1)R(\overline{w}_1)\emptyset.$$

Hence, there exists f_2 such that $f_2 = \mu_W(\overline{w}_1)$. Therefore, there exists $\widehat{k} \geq k_m$ such that $\overline{w}_1 \in O_{P'}^{\widehat{k}}(f_2)$, $\overline{w}_1 \in \mu_W(f_2)$ and $\overline{w}_1 \notin Ch(O_{P'}^{\widehat{k}}(f_2), P(f_2))$. Since $f_2 P(\overline{w}_1)\overline{f}_1$ and $\overline{w}_1 \in$

$O^{\widehat{k}}_{P'}(f_2)$, $\overline{w}_1$ makes an offer to f_2 along the deferred-acceptance algorithm in which workers make offers applied to P', and $\overline{w}_1$ is rejected. But later, $\overline{w}_1$ makes an offer to $\overline{f}_1$. Hence, $\widehat{k} < k_m$. A contradiction. Thus, for all $w \notin W'$, $\mu'_W(w)R(w)\mu_W(w)$. ■

Claim 8 Let $w' \in W'$ be such that $h_W(P'_{W'}, P_{-W'})(w') = \emptyset$ and $\mu(w') = f'$. Then, $Ch(\mu(f') \cup O_{P'}(f'), P(f')) \nsubseteq \mu(f')$.

Proof of Claim 8. Assume $Ch(\mu(f') \cup O_{P'}(f'), P(f')) \subseteq \mu(f')$. Since $\mu \in IR(P)$, $Ch(\mu(f') \cup O_{P'}(f'), P(f')) = \mu(f')$. Hence, by assumption,

$$w' \in \mu(f') = Ch(\mu(f') \cup O_{P'}(f'), P(f')),$$

and, by substitutability of $P(f')$,

$$w' \in Ch(O_{P'}(f') \cup \{w'\}, P(f')).$$

By definition of $P'(w')$, $w' \in O_{P'}(f')$. Hence, $w' \in \mu'_W(f')$, which contradicts that $\mu'_W(w') = \emptyset$. Thus, $Ch(\mu(f') \cup O_{P'}(f'), P(f')) \nsubseteq \mu(f')$. ■

Let $w_1 \in Ch(\mu(f') \cup O_{P'}(f'), P(f')) \backslash \mu(f')$. Assume $w_1 \in W'$. Since $w_1 \notin \mu(f')$, $w_1 \in O_{P'}(f')$. Therefore, $w_1 \in O_{P'}(f') \cap W'$. But $w_1 \in W'$ means that $P'(w_1) : \mu(w_1), \emptyset$. Hence, w_1 makes an offer to f' along the deferred-acceptance algorithm in which workers make offers applied to P' (*i.e.*, $w_1 \in O_{P'}(f')$). Hence, by definition of $P'(w_1)$, $w_1 \in \mu(f')$, a contradiction. Thus, $w_1 \notin W'$. By substitutability of $P(f')$,

$$w_1 \in Ch(\mu(f') \cup \{w_1\}, P(f')) \tag{27}$$

and

$$w_1 \in \mu'_W(f') = Ch(O_{P'}(f'), P(f')). \tag{28}$$

Since $w_1 \notin W'$, $\mu_W(w_1)R(w_1)\mu(w_1)$. By (28),

$$f' = \mu'_W(w_1) R(w_1) \mu_W(w_1) R(w_1) \mu(w_1). \tag{29}$$

By assumption, $\mu(w_1) \neq f'$. Thus, (29) implies

$$f' P(w_1)\mu(w_1). \tag{30}$$

Conditions (27) and (30) imply that (f', w_1) blocks μ, and $f' = \mu(W')$ and $w_1 \notin W'$. Hence, the Blocking Lemma holds at P. ■

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