MATHEMATICS OF OPERATIONS RESEARCH
Vol. 13, No. 4, November 1988
Printed in U.S.A.

THE LATTICE STRUCTURE OF THE SET OF STABLE MATCHINGS WITH MULTIPLE PARTNERS*

CHARLES BLAIR

University of Illinois at Urbana-Champaign

We continue recent work on the matching problem for firms and workers, and show that, for a suitable ordering, the set of stable matchings is a lattice.

1. Introduction. Gale and Shapley [2] considered the problem of matching finite sets of men and women to form couples. A matching was called stable if there did not exist a man in one couple and a woman in another who preferred each other to their current partners. [2] showed that for any preference orderings, there always exists at least one stable matching.

A striking feature of this model was pointed out in Knuth [4, pp. 92–93, attributed to J. H. Conway]. A natural partial ordering on the set of stable matchings has matching $f \geqslant$ matching g if every man is at least as happy with his partner in f as with his partner in g. [4] showed that this partial ordering gives a distributive lattice. In addition, (i) replacing "man" by "woman" in the definition has the effect of replacing " $\geqslant$ " by " $\leqslant$ ", (ii) the l.u.b. and g.l.b. operations in the lattice are simple: the l.u.b. of f, g is obtained by giving each man whichever of the partners he likes better, g.l.b. gives each man the partner he likes less.

In addition to its intrinsic interest, the lattice structure provides insight into the existence of stable outcomes that are simultaneously optimal either for the set of men or for the set of women. Such outcomes are surprising since the players of the same sex are competing against each other. This issue is discussed further in Roth [5].

Kelso and Crawford [3] considered a matching model in which the players on one side have multiple partners, e.g., a firm hires sets of workers, with each worker allowed to work for only one firm. Roth [5, 6] generalized this model to allow a worker to be employed by a set of firms, thus treating firms and workers symmetrically. In each of these models, existence of a stable multi-partner matching is established, provided the preferences of each player satisfy a "substitutability condition."

This paper considers the most general of these models, model III of [5], and investigates the extent to which the lattice structure present in the monogamous Gale-Shapley model is preserved. [5] showed that the l.u.b. and g.l.b. operations did not generalize. We show that the partial ordering in the monogamous case does generalize to give a lattice structure for the multi-partner case. However, the lattice is not necessarily distributive and the l.u.b. operation is nontrivial. [1] showed that the set of lattices which could occur in the monogamous case is precisely the set of finite distributive lattices. The problem of characterizing the lattices which can occur here is open.

*Received October 29, 1984; revised July 27, 1987.
AMS 1980 subject classification. Primary: 90A06.
IAOR 1973 subject classification. Main: Programming: Assignment.
OR/MS Index 1978 subject classification. Primary: 486 Networks/graphs/matching.
Key words. Matchings, lattice structure, preferences.

0364-765X/88/1304/0619$01.25

2. Description of the model and notation. We will have finite sets of firms F and workers W. These may be lumped together as the set of players $P = F \cup W$. A firm hires a set of workers, with a salary (also called a "job description" in [5]) for each. We will use S for the set of all possible salaries. Thus, a firm hires a subset of $W \times S$, i.e., a member of $2^{W \times S}$. A subset may be feasible for some firms and not for others. For each firm, there is a linear ordering over the feasible subsets, which indicates the firm's preferences. As in [5, note 7], we assume that a firm is never indifferent between two different feasible subsets. In the same way, each worker works for a subset of $F \times S$, and there are preference orderings associated with the feasible subsets for each worker.

Formally, a matching is a function f: $P \to 2^{P \times S}$ such that:

$$\text{If } (i, s) \in f(j), \text{ then } (j, s) \in f(i), \tag{2.1}$$

$$(i, s) \in f(j) \text{ and } (i, t) \in f(j) \text{ implies } s = t, \tag{2.2}$$

$$\text{If } i \in F,\ f(i) \text{ is a feasible subset of } W \times S, \tag{2.3}$$

$$\text{If } i \in W,\ f(i) \text{ is a feasible subset of } F \times S. \tag{2.4}$$

Intuitively, $(i, s) \in f(j)$ means (if $i \in F$) firm i hires worker j at salary s. Note that the values $f(i)$, $i \in F$, completely determine $f(i)$, $i \in W$, by (2.1).

For $i \in P$, $A \subset P \times S$, $C_i(A)$ is the feasible subset of A that i likes best (the empty set is assumed to be feasible). Different preference orderings for i may yield the same C_i, but the C_i contain all the important information. Therefore we state the axioms in terms of C_i instead of in terms of preferences:

$$C_i(A) \subset A. \tag{2.5}$$

$$\text{If } C_i(A) \subset B \subset A, \text{ then } C_i(B) = C_i(A). \tag{2.6}$$

$$\text{If } (j, s) \in C_i(A), \text{ then } C_i(A) - (j, s) \subset C_i(A - (j, s)). \tag{2.7}$$

(2.5) and (2.6) are clearly essential properties for any function that chooses the most desirable subset of a set. (2.7) says that anything that is wanted in addition to (j, s) is still wanted if (j, s) is not available. In other words, members of $C_i(A)$ are wanted for their own sake, not because of potential benefits from interaction with other members.

We are using the minus sign for set-theoretic difference. A more traditional notation for $A - (j, s)$ might be $A \setminus \{(j, s)\}$. However, there is no ambiguity, since members of A cannot be added or subtracted.

This paragraph discusses the relation of our model to model III of [5]. It is not used in the rest of the paper. (2.7) is a special case of the substitutability condition [5, p. 383]. To see this, let $f_k = C_k(A)$, $g_k = A - (j, s)$ and let x_{ij} vary over all members of $C_k(A) - (j, s)$. Proposition 2.2 will show that (2.5)–(2.7) actually imply the substitutability condition (let $A = g_k \cup x_{ij}$, $B = f_k$). Our model is slightly more general than model III, since we do not assume the generalized salary condition [5, p. 382].

We shall make repeated use of the following consequences of (2.5)–(2.7).

PROPOSITION 2.1. $C_i(C_i(A)) = C_i(A)$.

PROOF. By (2.5) $C_i(A) \subset C_i(A) \subset A$. Then use (2.6). Q.E.D.

PROPOSITION 2.2. $C_i(A \cup B) \cap A \subset C_i(A)$.

PROOF. Induction on size of $(A \cup B) - A$. If size is zero, $C_i(A \cup B) = C_i(A)$. Otherwise, let $x \in (A \cup B) - A$. By induction hypothesis, $C_i(A \cup B - x) \cap A \subset$

$C_i(A)$. If $x \notin C_i(A \cup B)$, (2.6) implies $C_i(A \cup B) = C_i(A \cup B - x)$. If $x \in C_i(A \cup B)$, then (2.7) implies $C_i(A \cup B) \cap A \subset C_i(A \cup B - x) \cap A$. Q.E.D.

PROPOSITION 2.3. $C_i(A \cup B) = C_i(C_i(A) \cup B)$.

PROOF. From Proposition 2.2, $C_i(A \cup B) \cap A \subset C_i(A)$ and $C_i(A \cup B) \cap B \subset C_i(B) \subset B$. Thus $C_i(A \cup B) \subset C_i(A) \cup B \subset A \cup B$ and we may apply (2.6). Q.E.D.

If f is a matching, and $(j, s) \notin f(i)$, it may be the case that i would like to add (j, s) to his current involvements, possibly discarding some things that are presently part of $f(i)$. The formal relation corresponding to i wanting (j, s) is $(j, s) \in C_i(f(i) \cup (j, s))$.

We define f to be stable iff for all $i, j \in P$, $s \in S$:

$$C_i(f(i)) = f(i). \tag{2.8}$$

$$\text{If } (j, s) \in C_i(f(i) \cup (j, s)), \text{ then } C_j(f(j) \cup (i, s)) = f(j). \tag{2.9}$$

(2.8) says that no player wants to discard any of his partners in f. (2.9) follows immediately from (2.1) and (2.8) if $(j, s) \in f(i)$. In the interesting case, $(j, s) \notin f(i)$, the content of (2.9) is that if i wants (j, s), then j does not want (i, s). This is equivalent to the definition of stability in [5, p. 382].

If a matching is stable there is no way that a firm can make arrangements with a subset of the workers so that the firm and the subset are all happier than they were in the stable matching. We make this statement precise in Theorem 2.5.

LEMMA 2.4. *Let f and g be matchings with $C_i(f(i)) = f(i)$. If i strictly prefers $g(i)$ to $f(i)$ [i.e., $f(i) \neq g(i)$], then for some $(j, s) \in (g(i) \cup f(i)) - f(i)$, $(j, s) \in C_i(f(i) \cup (j, s))$.*

This says that if player i prefers his partners in matching g to his partners in matching f, there is a specific player-salary pair from g that he would like to add to those he gets under f.

PROOF. Since $g(i)$ is preferred to $f(i)$, and $f(i)$ is preferred to any of its proper subsets, $C_i(g(i) \cup f(i))$ is not a subset of $f(i)$. Let $(j, s) \in C_i(g(i) \cup f(i))$, $(j, s) \notin f(i)$. By Proposition 2.2, $(j, s) \in C_i(f(i) \cup (j, s))$. Q.E.D.

THEOREM 2.5. *Suppose f is stable and $i \in F$. There does not exist a nonempty $T \subset W$ and a matching g such that:* (i) *$g(k) \subset f(k)$ for $k \in F - i$.* (ii) *i strictly prefers $g(i)$ to $f(i)$.* (iii) *If $j \in W - T$ and $(i, s) \in g(j)$, then $(i, s) \in f(j)$.* (iv) *For all $j \in T$, j strictly prefers $g(j)$ to $f(j)$.*

PROOF. By (ii) and Lemma 2.4 we would have $(j, s) \in C_i(f(i) \cup (j, s))$, $(j, s) \in g(i)$, $(j, s) \notin f(i)$. By (2.1) $(i, s) \in g(j)$ and $(i, s) \notin f(j)$. By (iii) $j \in T$, so $g(j)$ is strictly preferred to $f(j)$ and $C_j(f(j) \cup g(j))$ is not a subset of $f(j)$, by (2.8). By (i), $g(j) \subset f(j) \cup (i, s)$, so $(i, s) \in C_j(f(j) \cup g(j)) = C_j(f(j) \cup (i, s))$. Since $(j, s) \in C_i(f(i) \cup (j, s))$, (2.9) would require $C_j(f(j) \cup (i, s)) = f(j)$. But this is contradicted by $(i, s) \notin f(j)$. Q.E.D.

Thus firm i and workers T cannot all become happier by changing the matching among themselves, even if we allow members of T to resign from firms, and allow i to lay off nonmembers of T. The proof of Theorem 2.5 can be repeated with F and W exchanged to show no worker can form a profitable coalition with a subset of the firms.

However, the following example shows that we may have a unique stable matching, but the possibility that there are coalitions involving sets of firms and workers to produce a (nonstable) matching in which all members of the coalition are happier. Indeed, in our example, the coalition consists of all firms and workers.

EXAMPLE 2.6. In this example and all subsequent ones, numbers will be firms, small letters will be workers, and S will consist of one member. Here we have three firms and three workers with the following preferences:

1	2	3	a	b	c
a, b	b, c	a, c	1,2	2,3	1,3
b, c	a, c	a, b	2,3	1,3	1,2
a	b	c	1	2	3
b	a	a	2	1	1
c	c	b	3	3	2

Thus the first choice for firm 1 would be to hire a and b. Its worst choice (other than hiring nobody) would be to hire c.

It is easy to verify that the C_i given by these preferences satisfy (2.5)–(2.7) and that the matching in which 1 hires a, 2 hires b and 3 hires c is stable. It can also be shown that this is the only stable matching. (b will never work for 1 in a stable matching, since he would always prefer to work for 2 instead, etc.)

However, the unstable matching in which 1 hires b and c, 2 hires a and c, and 3 hires a and b makes everybody happier. In the monogamous case, the stable matches correspond to the core of the game—those outcomes such that no coalition could make a binding contract among themselves to improve things for all members. This example shows that, in a multipartner setting, the core may be empty. Further assumptions about the preferences are needed if we want the core to be the set of stable outcomes here.

3. Existence of a stable matching.

LEMMA 3.1. *Suppose the matching given by* $f(i) = C_i(A_i)$, $i \in F$, $A_i \subset W \times S$ *satisfies the condition*:

$$\text{(3.1)} \qquad \textit{If } (j, s) \in (W \times S) - A_i, \textit{ then } C_j(f(j) \cup (i, s)) \subset f(j)$$

[(3.1) *says that if* $(j, s) \in A_i$, *then* j *does not want* (i, s)].

Then, for any $L \in W$, *the new matching* $f'(i) = C_i(A_i')$ *where*

$$A_i' = A_i - (L, s) \quad \textit{if } (i, s) \in f(L) - C_L(f(L)),$$

$$= A_i \qquad \textit{otherwise},$$

satisfies (3.1) *with* f *replaced by* f', A_i *by* A_i'.

PROOF. If $j \neq L$, $(j, s) \notin A_i'$ implies $(j, s) \notin A_i$. Since $(j, s) \notin A_i$, $(j, s) \notin f(i)$, hence [(2.1)] $(i, s) \notin f(j)$. By (3.1), $(i, s) \notin C_j(f(j) \cup (i, s))$. By (2.7), $(j, s) \in C_k(A_k)$ implies $(j, s) \in C_k(A_k')$ for all $k \in F$. Hence $(j, s) \in f(k)$ implies $(j, s) \in f'(k)$, so [(2.1)] $f(j) \subset f'(j)$. We apply Proposition 2.2 with $A = f(j) \cup (i, s)$, $B = f'(j) - f(j)$ to conclude that $(i, s) \notin C_j(f(j) \cup (i, s))$ implies $(i, s) \notin C_j(f'(j) \cup (i, s))$, hence $C_j(f'(j) \cup (i, s)) \subset f'(j)$.

The remaining case is $j = L$. If $(i, s) \notin f(L)$, then $A_i' = A_i$, so $f(i) = f'(i)$ and $(i, s) \notin f'(L)$. If $(i, s) \in C_L(f(L))$, we again have $A_i' = A_i$, so $(i, s) \in f'(L)$. If $(i, s) \in f(L) - C_L(f(L))$, $(L, s) \notin A_i'$ so $(i, s) \notin f'(L)$. Thus we have established $f'(L) = C_L(f(L))$.

If $(L, s) \notin A_i'$, then either $(L, s) \notin A_i$ or $(i, s) \in f(L) - C_L(f(L))$. In the first case, (3.1) gives $C_L(f(L) \cup (i, s)) =$ [Proposition 2.3], $C_L(f'(L) \cup (i, s)) \subset f(L)$. Since $(L, s) \notin A_i$ implies $(L, s) \notin f(i)$ and [(2.1)] $(i, s) \notin f(L)$, we have $(i, s) \notin$

$C_L(f'(L) \cup (i, s))$, hence $C_L(f'(L) \cup (i, s)) \subset f'(L)$. In the second case $f'(L) = C_L(f(L)) \subset f'(L) \cup (i, s) \subset f(L)$ implies [(2.6)] $C_L(f'(L) \cup (i, s)) = f'(L)$. Q.E.D.

THEOREM 3.2. *There exists a stable matching.*

PROOF. We begin by showing that if $f(i) = C_i(A_i)$ for all $i \in F$ and (3.1) is satisfied, then for all $i, j \in P$:

(3.2) If $(j, s) \in C_i(f(i) \cup (j, s))$ then $C_j(f(j) \cup (i, s)) \subset f(j)$

[(3.2) is essentially the half of the stability condition (2.9) with i a firm, j a worker]. (3.2) is immediate if $(j, s) \in f(i)$, since (2.1) implies $(i, s) \in f(j)$. For $i \in F$, if $(j, s) \in C_i(f(i) \cup (j, s)) =$ [Proposition 2.3] $C_i(A_i \cup (j, s))$ and $(j, s) \notin f(i)$, then $(j, s) \notin A_i$. By (3.1), $C_j(f(j) \cup (i, s)) \subset f(j)$.

If $i \in W$, $(j, s) \notin f(i)$, $(j, s) \in C_i(f(i) \cup (j, s))$, then (3.1) with i and j exchanged implies $(i, s) \in A_j$, so $f(j) = C_j(A_j) \subset f(j) \cup (i, s) \subset A_j$. By (2.6) $C_j(f(j) \cup (i, s)) = f(j)$.

If f satisfies (3.2) and $C_j(f(j)) = f(j)$ then the conclusion of (3.2) can be strengthened to $C_j(f(j) \cup (i, s)) = f(j)$. Thus if $f(i) = C_i(A_i)$ for all $i \in F$ and f satisfies (3.1), the only way that f could fail to be stable is if $C_L(f(L)) \neq f(L)$ for some $L \in W$.

To construct the stable matching, we begin by taking $A_i = W \times S$ for all $i \in F$, and $f(i) = C_i(A_i)$. (3.1) is satisfied vacuously. If f is not stable, there is an $L \in W$ with $f(L) \neq C_L(f(L))$. We apply Lemma 3.1 to obtain A'_i, f'. Since f' satisfies (3.1), either f' is stable or there is an L' with $f'(L') \neq C_{L'}(f(L'))$. Lemma 3.1 can then be applied to obtain f''. This process can be repeated each time there is a nonstable matching. Since the sum of the sizes of the A_i decreases each time, we must eventually terminate with a stable matching. Q.E.D.

The idea behind this proof is the same as that in the original construction [2] of a stable matching in the monogamous case. To begin, all the players on one side are given their first choice. Then the preferences of the other players are consulted. It is established that certain players cannot be matched with certain others. Then one attempts to match the players on one side with their first choices among the remaining possibilities, and so forth. It is also the same as the construction in [6, pp. 53–54], except that there the A'_i delete several members from A_i simultaneously. We have given the detailed proof here partly to stress the importance of the C_i (as opposed to the preferences), partly because we wish to show later that the matching constructed here is the best stable matching for the firms.

4. The lattice structure of the set of stable matchings. In the monogamous case, the lattice is obtained by defining one stable matching as $\geqslant$ another if every man is at least as happy in the first as in the second. A partial ordering on multi-partner matchings could be obtained by replacing "man" by "firm." In Example 5.1, we show that this ordering is not a lattice.

Instead, we will define $f \geqslant g$ only if each firm wishes to keep its partners in f, even if all the partners in g were also made available, and would not wish to add any new partners. We will show that this more restrictive partial ordering is a lattice. Since it clearly specializes to the standard definition in the monogamous case, it seems to be the appropriate generalization. Formally, we define:

DEFINITION 4.1. Let f, g be matchings which satisfy $C_i(f(i)) = f(i)$ and $C_i(g(i)) = g(i)$ for all $i \in F$. Define $f \geqslant g$ iff $C_i(f(i) \cup g(i)) = f(i)$ for all $i \in F$. $f \geqslant f$ follows from the restriction on the domain of definition (the need for this restriction was pointed out by a referee).

PROPOSITION 4.2. *If $f \geqslant g$ and $g \geqslant h$, then $f \geqslant h$.*

PROOF. For $I \in F$, $C_i(f(i) \cup h(i)) = C_i((C_i(f(i) \cup g(i))) \cup h(i)) =$ [Proposition 2.3] $C_i(f(i) \cup g(i) \cup h(i)) = C_i(f(i) \cup C_i(g(i) \cup h(i))) = C_i(f(i) \cup g(i)) = f(i)$. Q.E.D.

It is immediate that $f \geqslant g$ and $g \geqslant f$ implies $f = g$, so we have a partial ordering on a set of matchings which [by (2.8)] includes the stable ones. In Theorem 4.5 we show that replacing F by W in Definition 4.1 replaces $\geqslant$ by $\leqslant$ (compare with results about polarization of interest in [5, Theorems 4, 5, and 9]).

LEMMA 4.3. *Suppose f is stable and, for all $i \in W$, $C_i(f(i) \cup g(i)) = g(i)$. Then, for all $j \in F$, $C_j(f(j) \cup g(j)) = f(j)$.*

PROOF. Let $j \in F$, $(i, s) \in g(j)$. By (2.1) $(j, s) \in g(i)$. $C_i(f(i) \cup g(i)) = g(i)$ and Proposition 2.2. imply $(j, s) \in C_i(f(i) \cup (j, s))$. By (2.9), $C_j(f(j) \cup (i, s)) = f(j)$. By Proposition 2.2, $C_j(f(j) \cup g(j)) \cap (f(j) \cup (i, s)) \subset f(j)$. Since this is true for all $(i, s) \in g(j)$, $C_j(f(j) \cup g(j)) \subset f(j) \subset f(j) \cup g(j)$. By (2.6), $C_j(f(j) \cup g(j)) = C_j(f(j)) =$ [(2.8)] $f(j)$. Q.E.D.

If we exchange F with W and f with g in the previous argument, we obtain:

LEMMA 4.4. *Suppose g is stable and, for all $i \in F$, $C_i(f(i) \cup g(i)) = f(i)$. Then, for all $j \in W$, $C_j(f(j) \cup g(j)) = g(j)$.*

Lemmas 4.3 and 4.4 combined give:

THEOREM 4.5. *If f and g are stable, then $f \geqslant g$ iff, for all $i \in W$, $C_i(f(i) \cup g(i)) = g(i)$.*

The Gale-Shapley procedure [2] constructs a stable matching that is maximal in its lattice order. Since the construction in Theorem 3.2 is the multi-partner generalization, it should construct the maximal member in this ordering.

THEOREM 4.6. *Let g be the stable matching constructed in Theorem 3.2. If h is stable, $g \geqslant h$.*

PROOF. Recall that a sequence of matchings $f(i) = C_i(A_i)$, $f'(i) = C_i(A_i')$, etc. is constructed. We first show by induction that, if h is stable $h(i) \subset A_i$, $h(i) \subset A_i'$, etc. for all $i \in F$ and all $A_i, A_i', A_i'', \ldots$ constructed.

At the beginning, $A_i = W \times S$, so $h(i) \subset A_i$ is immediate. If we assume $h(i) \subset A_i$ and let A_i' be obtained as in Lemma 3.1, then $h(i) \subset A_i'$ is immediate unless $(i, s) \in f(L) - C_L(f(L))$.

If $f(k) = C_k(A_k)$ and $h(k) \subset A_k$, then $C_k(f(k) \cup h(k)) =$ [Proposition 2.3] $C_k(A_k \cup h(k)) = C_k(A_k) = f(k)$ for all $k \in F$. By Lemma 4.4, $C_L(f(L) \cup h(L)) = h(L)$. If we apply Proposition 2.2 with $A = f(L)$, $B = h(L)$ we obtain $f(L) \cap h(L) \subset C_L(f(L))$.

Thus if $(i, s) \in f(L) - C_L(f(L))$, then $(i, s) \notin h(L)$, hence $(L, s) \notin h(i)$ and $h(i) \subset A_i$ implies $h(i) \subset A_i - (L, s) = A_i'$. The stable matching g from Theorem 3.2 is thus constructed so that $g(i) = C_i(A_i)$ for A_i such that $h(i) \subset A_i$. As in the preceding paragraph, this implies $C_i(g(i) \cup h(i)) = C_i(A_i \cup h(i)) = C_i(A_i) = g(i)$, so $g \geqslant h$. Q.E.D.

Next we show that the partial ordering is a lattice. We wish to construct the least upper bound on stable matchings f and g. The matching which gives every firm i its most preferred subset of $f(i) \cup g(i)$ has most of the desired properties but is not necessarily stable. However, a sequence of modifications can be performed which terminate in a stable least upper bound.

We can obtain the greatest lower bound on f and g by exchanging F and W throughout this construction, which (Theorem 4.5) replaces $\geqslant$ by $\leqslant$.

For technical reasons, we will be concerned with the set of firms which, given a matching, are interested in a specified worker. For f a matching, $j \in W$ define

$$I(j, f) = \{(i, s) \mid (j, s) \in C_i(f(i) \cup (j, s))\} \tag{4.1}$$

PROPOSITION 4.7. *If $C_i(f(i)) = f(i)$ for all $i \in F$, then $f(k) \subset I(k, f)$ for all $k \in W$.*

PROOF. If $(i, s) \in f(k)$, then [(2.1)] $(k, s) \in f(i) = f(i) \cup (k, s) = C_i(f(i) \cup (k, s))$, so $(i, s) \in I(k, f)$. Q.E.D.

PROPOSITION 4.8. *If $f(i) = C_i(A_i)$ and $g(i) = C_i(A'_i)$, where $A_i \subset A'_i$ for all $i \in F$, then $I(k, g) \subset I(k, f)$ for all $k \in W$.*

PROOF. If $(i, s) \in I(k, g)$, $(k, s) \in C_i(g(i) \cup (k, s)) =$ [(Proposition 2.3)] $C_i(A'_i \cup (k, s))$. By Proposition 2.2, $(k, s) \in C_i(A_i \cup (k, s)) = C_i(f(i) \cup (k, s))$, hence $(i, s) \in I(k, f)$. Q.E.D.

Next we show that our first matching has several desirable properties.

LEMMA 4.9. *Let $h(i) = C_i(f(i) \cup g(i))$, $i \in F$*

(1) If e is stable, $e \geqslant f$, and $e \geqslant g$, then $e \geqslant h$.

(2) $h(k) \subset C_k(I(k, h))$ for all $k \in W$.

(3) $C_k(h(k)) = h(k)$ for all $k \in W$.

[*(2) says that no worker will ever want to give up his current partners, even if he starts working for new firms.*]

PROOF OF (1). $C_i(e(i) \cup h(i)) =$ [Proposition 2.3] $C_i(e(i) \cup f(i) \cup g(i)) =$ [$e \geqslant f$ and Proposition 2.3] $C_i(e(i) \cup g(i)) = e(i)$.

PROOF OF (2). Since f is stable, $(i, s) \in I(k, f)$ implies [(2.9)] $C_k(f(k) \cup (i, s)) = f(k)$. By Proposition 4.7, $f(k) \subset I(k, f)$.

From Proposition 2.2, $C_k(I(k, f)) \cap (f(k) \cup (i, s)) \subset C_k(f(k) \cup (i, s)) = f(k)$. Since this holds for all $(i, s) \in I(k, f)$, $C_k(I(k, f)) \subset f(k) \subset I(k, f)$, so (2.6) gives $C_k(I(k, f)) = C_k(f(k)) = f(k)$. By Proposition 4.8, $I(k, h) \subset I(k, f)$, so Proposition 2.2 gives $f(k) \cap I(k, h) \subset C_k(I(k, h))$. By Proposition 4.7, $h(k) \subset I(k, h)$, so $f(k) \cap h(k) \subset C_k(I(k, h))$. The same argument with g instead of f establishes $g(k) \cap h(k) \subset C_k(I(k, h))$. Since $h(k) \subset f(k) \cup g(k)$, (2) is established.

PROOF OF (3). By Proposition 2.2 with $A = h(k)$, $A \cup B = I(k, h)$, (2) implies $h(k) \subset C_k(h(k))$, hence (2.5) $h(k) = C_k(h(k))$. Q.E.D.

Our next lemma describes the modifications used and shows that the desired properties are preserved. Intuitively, we look at all the firms who are interested in a specific worker and have that worker join the new firms he find most attractive.

LEMMA 4.10. *Let $h(i) = C_i(A_i)$ be any matching which satisfies properties* (1)–(3) *of Lemma* 4.9. *Fix $j \in W$. Let*

$A'_i = A_i \cup (j, s)$ if $(i, s) \in C_j(I(j, h))$,

$A'_i = A_i$ otherwise.

[*$C_j(I(j, h))$ has at most one (i, s) for each i.*]

Then $h'(i) = C_i(A'_i)$ also satisfies properties (1)–(3), *with h replaced by h'.*

PROOF OF (1). We will show that for any stable e, $e \geqslant h$ implies $e \geqslant h'$.

Since $e \geqslant h$, $C_i(e(i) \cup h(i)) = e(i)$. If $(j, s) \in e(i)$, Proposition 2.2 implies $(j, s) \in C_i(h(i) \cup (j, s))$, hence $(i, s) \in I(j, h)$. Thus $e(j) \subset I(j, h)$.

If $A'_i = A_i$, $C_i(e(i) \cup h'(i)) = e(i)$ is easy. If $A_i \neq A'_i = A_i \cup (j, s)$, $(i, s) \in C_j(I(j, h))$ implies [Proposition 2.2] $(i, s) \in C_j(e(j) \cup (i, s))$. Since e is stable, (2.9) implies $C_i(e(i) \cup (j, s)) = e(i)$. Thus $C_i(e(i) \cup h'(i)) =$ [Proposition 2.3] $C_i(e(i) \cup A_i \cup (j, s)) = C_i(e(i) \cup A_i) = C_i(e(i) \cup h(i)) = e(i)$.

PROOF OF (2). If $k \in W - j$, $h'(k) \subset h(k) \subset C_k(I(k, h))$. By Propositions 4.7 and 4.8, $h'(k) \subset I(k, h') \subset I(k, h)$. So $h'(k) \subset C_k(I(k, h)) \cap I(k, h') \subset$ [Proposition 2.2] $C_k(I(k, h'))$.

If $(i, s) \in h'(j) - h(j)$ then $(i, s) \in C_j(I(j, h))$, by definition of A_i'. If $(i, s) \in h(j)$, then $(i, s) \in C_j(I(j, h))$. Thus $h'(j) \subset C_j(I(j, h))$. As above, $h'(j) \subset I(j, h') \subset I(j, h)$ so $h'(j) \subset C_j(I(j, h'))$ by Proposition 2.2.

PROOF OF (3). The proof that (2) implies (3) in Lemma 4.9 is valid for any h.

THEOREM 4.11. *There is a stable least upper bound on f and g.*

PROOF. We construct a sequence of matchings $h, h', h'', \ldots$ beginning with $h(i) = C_i(f(i) \cup g(i))$ and using Lemma 4.10. Each of the matchings satisfies properties (1)–(3). Property (1) ensures that the matchings are $\leqslant$ any stable upper bound on f and g. Since $A_i \subset A_i'$ in Lemma 4.10, it is easy to see that $f, g \leqslant h \leqslant h' \leqslant h'' \leqslant \ldots$. Property (3) implies that the matchings all satisfy (2.8). Thus, if the current matching h is not stable, there are i, j, s with $(i, s) \in I(j, h)$ and $C_j(h(j) \cup (i, s)) = h(j) \cup (i, s) \neq h(j)$. We apply Lemma 4.10. A_i' has the new member (j, s) so the sum of the sizes of A_m' for all $m \in F$ has increased. The finiteness of P and S implies this process eventually terminates in a stable matching. (Another way to see that the process halts is to note that every firm is at least as happy in h' as in h, and at least one firm is strictly happier.) Q.E.D.

Thus we have established that the stable matchings are a lattice. It is puzzling that, although the existence proof Theorem 3.2 is essentially the same as in the monogamous case, the lattice operations are substantially more complicated. As previously mentioned, the multipartner lattice is not necessarily distributive (example 5.2), although the monogamous ones are. This suggests the two situations are fundamentally different, but some further insight would be helpful.

5. Two examples.

Recall our conventions used in Example 2.6.

As mentioned at the beginning of §4, a partial ordering in which $f \geqslant g$ iff every firm prefers its set of partners under f to its partners under g is not a lattice.

EXAMPLE 5.1. We consider 13 firms and 12 workers with the following preferences:

1	2	3	4	5	6	7
a, b	e	d	h	j	h	k
a, d	d	e	a	h	j	a
b, d						
b	8	9	10	11	12	13
$\cdots$	L	k	m, p	q	p	n, q
	k	L	b, n, p	p	q	m, q
			m			n
			$\cdots$			$\cdots$

a	b	d	e	h	j	k	L
4	10	2	3	5	6	8	9
7	1	1,3	2	4,6	5	7,9	8
1		1		$\cdots$		$\cdots$	
		3					
		m	n	p	q		
		13	10	11	12		
		10	13	10,12	11,13		
				$\cdots$	$\cdots$		

The ... indicates that there are other subsets on the preference lists for these players which are worse than those shown. The exact choice of these is not important for this example. However, some care is necessary so that (2.7) is satisfied. For example the list for 10 could be completed by the six subsets b, n; n, p; b, p; b; n; p.

It can be verified that the following two matchings are stable:

S1: 1 hires b, $2d$, $3e$, $4a$, $5h$, $6j$, $7k$, $8L$, $9k$, $10m$, $11p$, $12q$, $13n$.

S2: 1 hires b and d, $2e$, $3d$, $4h$, $5j$, $6h$, $7a$, $8k$, $9L$, $10m$, $11p$, $12q$, $13n$.

We wish to look at stable matchings in which each firm is at least as happy as it is with either S1 or S2. It is clear that in such a matching firms 2 through 9 each must get their first choice. Firm 1 must get b and d, a and d, or a and b. It can be verified that a stable matching is given by:

UB1: $1ab$, $2e$, $3d$, $4h$, $5j$, $6h$, $7k$, $8L$, $9k$, $10m$, $11p$, $12q$, $13n$.

Except for firm 1, UB1 gives each firm the better of the partners in S1 and S2. If we change UB1 by having firm 1 hire ad, it will not be stable because 1 will want b and b will want 1. Similarly, if 1 hires bd, 1 and a want each other.

Thus UB1 is a minimal stable upper bound in the partial ordering described in the first paragraph of §4. However another stable upper bound in this ordering is:

UB2: $1ad$, $2e$, $3d$, $4h$, $5j$, $6h$, $7k$, $8L$, $9k$, $10bnp$, $11q$, $12p$, $13mq$.

Since 1 is happier in UB1 than in UB2, UB1 is not less than UB2 in the proposed partial ordering. Thus UB1 is a minimal upper bound on S1, S2 but not a least upper bound.

Thus the proposed partial ordering does not give a lattice. [With the partial ordering given by Definition 4.1, UB2 is not comparable to S2 because $C_1(a, b, d) = a, b$.]

EXAMPLE 5.2. The lattice given by definition 4.1 may not be distributive. Suppose we have 7 firms and 10 workers with preferences:

1	2	3	4	5	6	7
a	b	c	d	e	f	g
bcd	h	i	j	h	i	j
...	ae	af	ag			
	...	...	...			

a	b	c	d	e	f	g	h	i	j
2,3,4	1	1	1	2	3	4	5	6	7
2,3	2	3	4	5	6	7	2	3	4
3,4									
2,4									
1,2									
1,3									
1,4									
1									
...									

Three stable matchings are:

M1: $1bcd$, $2ae$, $3af$, $4j$, $5h$, $6i$, $7g$,

M2: $1bcd$, $2ae$, $3i$, $4ag$, $5h$, $6f$, $7j$,

M3: $1bcd$, $2h$, $3af$, $4ag$, $5e$, $6i$, $7j$.

The least upper bound on M1 and M2 cannot have worker a hired by firms 3 or 4. Thus, worker a will want to work for firm 1. This means 1 will not want b, c, d, who will want to work for 2, 3, 4. Since e, f, g have no alternative employment, they will work for 5, 6, 7. Thus the least upper bound has every firm get its first choice. Since this matching is the maximal element, the greatest lower bound of this matching and M3 is M3.

However, Theorem 4.5 and reasoning similar to the previous paragraph shows that the greatest lower bound of M1 and M3 is the matching in which each worker gets his first choice [a works for 2, 3, 4; etc.]. This matching is the minimal element, and it is also the greatest lower bound on M2 and M3.

Thus computing upper bound followed by lower bound gives M3, while lower followed by upper gives the minimal element, so distributivity fails.

Acknowledgements. I wish to thank Alvin Roth for many helpful discussions. The referee pointed out numerous mistakes and obscurities in a previous draft.

References

[1] Blair, C. (1984). Every Finite Distributive Lattice is a Set of Stable Matchings. *J. Combin. Theory* Ser. A **37** 353–356.

[2] Gale, D. and Shapley, L. (1962). College Admissions and the Stability of Marriage. *Amer. Math. Monthly* **69** 9–15.

[3] Kelso, A. and Crawford, V. (1982). Job Matching, Coalition Formation, and Gross Substitutes. *Econometrica* **50** 1483–1504.

[4] Knuth, D. (1976). *Marriage Stables.* Montreal University Press.

[5] Roth, A. (1985). Conflict and Coincidence of Interest in Job Matching. *Math. Oper. Res.* **10** 379–389.

[6] ______. (1984) Stability and Polarization of Interests in Job Matching. *Econometrica* **52** 47–57.

DEPARTMENT OF BUSINESS ADMINISTRATION, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, CHAMPAIGN, ILLINOIS 61820