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NOTES ON A NEW COMPROMISE VALUE: THE χ-VALUE†

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We introduce and study a new compromise value for all weakly essential transferable utility games: the χ-value. It is closely related to the τ-value defined by Tijs in 1981. The main difference being the way we define the maximal aspiration a player may have in the game: instead of considering the marginal contribution to the grand coalition, we consider his maximal (among all coalitions) marginal contribution. We show that the class of games where the new value can be applied is larger.

KEY WORDS: Games, TU-games, compromise value, marginal contributions.

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1. INTRODUCTION

The purpose of this paper is to introduce a new compromise value for transferable utility games, that we call the χ-value. A compromise value chooses as the solution of the game the efficient vector lying in the segment between the vectors of maximal and minimal payoffs that each player may expect to obtain; that is, it is a compromise between their maximum and minimum aspirations. In bargaining problems, the Kalai-Smorodinsky solution (Kalai and Smorodinsky [9]) is already based on a compromise of this type. The prominent example of a compromise value for transferable utility games is the τ-value introduced by Tijs [12].

For every player, Tijs defines his maximum payoff as his marginal contribution to the grand coalition, and his minimum payoff as the maximum remainder he can obtain by going with a coalition of players and offering them their maximum payoff. The problem with the definition of the τ-value is that for some games (called by Tijs the class of non quasi-balanced games) it can not be applied.

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We propose, in Section 3, a new compromise value, the χ-value, by only modifying slightly the way the maximum aspiration is obtained in the definition of the τ-value. We propose the use of the maximal marginal contribution as the maximum aspiration for a player, while keeping the definition of the minimum one as the τ-value does, but using the new maximum payoff. In Proposition 1 we show that this minimum aspiration coincides with the vector of utilities associated to each player by the characteristic function. We find this property interesting, not only because it implies that the χ-value is defined in all (weakly essential) transferable utility games, but also because we obtain "endogenously" (using the new definition of maximal aspiration) that the minimum aspiration for each player is the value associated to each one of them by the characteristic function. It seems to us that this may be a good indication that the notion of maximum aspiration that we are proposing is sensible. At the end of the paper, in Section 5, we discuss an alternative (dual) way of defining the bounds to obtain an alternative compromise value.

Section 4 studies the relationships between the χ-value, the τ-value, the Shapley value and the Core of a game. An Appendix at the end of the paper contains a long proof (Proposition 2) omitted in the text. Section 2 below contains the notation and basic definitions.

2. PRELIMINARIES

A (cooperative) game with *transferable utility* is an ordered pair (N, v) where $N = \{1, \ldots, n\}$ is the set of *players* and $v{:}2^N \to \mathbb{R}$ is a function, the *characteristic function*, having the property that $v(\phi) = 0$. To each *coalition* $S \in 2^N$, v assigns a real number $v(S)$, the *worth* of coalition S. We will denote by $\#S$ the cardinality of coalition S. Given a *game* (N, v) we say that a vector $x \in \mathbb{R}^n$ is *efficient* if $\sum_{i \in N} x_i = v(N)$.

A game with transferable utility (N, v) is:

(a) *essential* if $\sum_{i \in N} v(\{i\}) < v(N)$ and *weakly essential* if $\sum_{i \in N} v(\{i\}) v \leqslant (N)$.
(b) *superadditive* if $v(S) + v(T) \leqslant v(S \cup T)$ for every $S, T \in 2^N$ such that $S \cap T = \phi$.
(c) *convex* if $v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$ for every $S, T \in 2^N$.

Let G be the class of weakly essential games, and let B be a subset of G. A *solution concept* on $B \subseteq G$ is a correspondence Φ assigning to every $(N, v) \in B$ a set of efficient vectors, that is, $\Phi(N, v) \subset \mathbb{R}^n$ and $x \in \Phi(N, v)$ implies $\sum_{i \in N} x_i = v(N)$. An example of a solution is the *Core*, defined by

$$C(N, v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geqslant v(S) \text{ for every } S \in 2^N \right\}.$$

A value on $B \subseteq G$ is a function ϕ that assigns to each $(N, v) \in B \subseteq G$ a unique efficient vector, that is, $\phi(N, v) \in \mathbb{R}^n$ and $\sum_{i \in N} \phi_i(N, v) = v(N)$. The Shapley value, $Sh(N, v)$ (Shapley [11]) is the most well-known example of a value.

Tijs [12] introduced a value, called the τ-value, for a subset of essential games (the class of quasi-balanced games defined below). The idea behind it is to find a compromise between the maximum and the minimum amount that each player may expect to obtain in the game. Let $(N,v)\in G$ and $i\in N$ be given. Tijs defined $M_i^\tau(v)$, the *upper value of v for i*, as the marginal contribution of player i to the grand coalition, that is,

$$M_i^\tau(v)=v(N)-v(N\backslash\{i\}).$$

Using the vector of upper values $M^\tau(v)=(M_1^\tau(v),\ldots,M_n^\tau(v))$, he also defined $m_i^\tau(v)$, the *lower value of v for i*, as

$$m_i^\tau(v)=\max_{\substack{S\subseteq N\\ i\in S}}\left\{v(S)-\sum_{j\in S\backslash\{i\}}M_j^\tau(v)\right\}.$$

Player i may guarantee by himself this amount $m_i^\tau(v)$ by offering to the members of a coalition S (the one producing the maximum gain for him) $M_j^\tau(v)$, keeping for himself the remainder of $v(S)$.

Tijs defined the set of *Quasi-Balanced* games, $QB\subset G$, as the games $(N,v)\in G$ having the properties that

$$m^\tau(v)\leqslant M^\tau(v)\quad\text{and}\quad\sum_{i\in N}m_i^\tau(v)\leqslant v(N)\leqslant\sum_{i\in N}M_i^\tau(v).$$

For a game $(N,v)\in QB$, the *τ-value* of (N,v) is the unique efficient vector in the lineal segment having as extremes points $m^\tau(v)$ and $M^\tau(v)$. That is,

$$\tau(N,v)=m^\tau(v)+\alpha[M^\tau(v)-m^\tau(v)],$$

where α is such that $\sum_{i\in N}\tau_i(N,v)=v(N)$.

The τ-value has been extensively studied in the literature. There are several alternative characterizations of it (for example Tijs [13], Driessen [7], and Calvo, Tijs, Valenciano and Zarzuelo [6]). Bergantiños and Mendez-Naya [1] show how to implement it by subgame perfect equilibria in extensive form games. Moreover, the τ-value has generated a family of compromise values (see Tijs and Lipperts [14], van Heumen [8], Bondareva [2], van den Brink [4] and [5], Bondareva and Driessen [3]). Our work is clearly related to this literature which is surveyed by Tijs and Otten [15].

3. THE χ-VALUE

We think that one of the main objections to the τ-value is that the class of games where it is defined does not have a natural justification. It is selfjustified by imposing that the vector of maximum aspirations be larger than the vector of minimum ones and by requiring that the gain of cooperation, $v(N)$, be bounded by the sum of the maximum and minimum aspirations. That is, the value justifies the class. Example 1 below presents a non quasi-balanced superadditive game that does not have τ-value.

Example 1: Let (N,v) be a transferable utility game where $N=\{1,2,3\}, v(\{1\})=v(\{2\})=v(\{3\})=0, v(\{1,2\})=v(\{2,3\})=10, v(\{1,3\})=12$ and $v(\{1,2,3\})=15$. The maximal and minimal aspirations according to the τ-value are $M_1^\tau(v)=M_3^\tau(v)=$

$5, M_2^\tau(v)=3, m_1^\tau(v)=m_3^\tau(v)=7$ and $m_2^\tau(v)=5$. The game is not quasi-balanced since $m_i^\tau(v)>M_i^\tau(v)$ for every $i=1,2,3$ and also $\sum_{i=\{1,2,3\}} m_i^\tau(v)=19>15=v(\{1,2,3\})>\sum_{i=\{1,2,3\}} M_i^\tau(v)=13$.

We propose a new compromise value, the χ-value, by using a different concept of a player maximum aspiration in a game. Instead of considering his marginal contribution to the grand coalition, as the τ-values does, it seems to us more appropriate to consider his maximal marginal contribution, which in general may be larger than the marginal contribution to the grand coalition. The vector of maximal marginal contributions was first proposed by Milnor [10], and used by van Heumen [8]. Besides, this modification has two important consequences. First, the corresponding minimum aspiration for player i, defined as the maximal remainder using the new definition, coincides with $v(\{i\})$ for the class of weakly essential games, and therefore the maximum aspirations are not smaller than the minimum ones for all weakly essential games. Secondly, and as a consequence of the first one, we can define a compromise value for all weakly essential games.

Formally, let (N,v) be a transferable utility game. For each $i \in N$, define *player i's maximum aspiration in the game* $(N,v), M_i^\chi(v)$, as his maximal marginal contribution; that is,

$$M_i^\chi(v)=\max_{\substack{S\subseteq N\\ i\in S}}\{v(S)-v(S\setminus\{i\})\}.$$

Notice that both definitions coincide in the class of convex games. Following Tijs idea of lower value, define *player i's minimum aspiration in the game* $(N,v), m_i^\chi(v)$, as the maximal remainder he can obtain after conceding to the other players their (new) maximum aspirations; that is,

$$m_i^\chi(v)=\max_{\substack{S\subseteq N\\ i\in S}}\left\{v(S)-\sum_{j\in S\setminus\{i\}} M_j^\chi(v)\right\}.$$

Player i can guarantee this amount for himself by offering to the members of a particular coalition S (the one generating their maximum remainder for him) their maximum aspirations, keeping for himself the remainder of $v(S)$. The difference between the definitions of $m_i^\chi(v)$ and $m_i^\tau(v)$ are the considered maximum aspirations.

First of all, it is easy to see that for every $S\subseteq N$, $v(S)\leqslant\sum_{j\in S} M_j^\chi(v)$. In particular, $v(N)\leqslant\sum_{i\in N} M_i^\chi(v)$. Therefore, for every $i\in N$,

$$m_i^\chi(v)=\max_{\substack{S\subseteq N\\ i\in S}}\left\{v(S)-\sum_{j\in S\setminus\{i\}} M_j^\chi(v)\right\}\leqslant\max_{\substack{S\subseteq N\\ i\in S}}\left\{v(S)-v(S\setminus\{i\})\right\}=M_i^\chi(v).$$

Proposition 1 below shows that using this new definition, the minimum aspiration for player i coincides, as it should, with the value associated to i by the characteristic function. van Heumen [8] proposed a compromise value (which coincides with the χ-value) by using the Milnor upper bound and imposing (exogenously) as lower bound the vector $(v(\{i\}))_{i\in N}$. It seems to us, that the endogenous derivation of $v(\{i\})$

as the minimum aspiration is an indication that our modified definition of maximum aspiration may be sensible.

Proposition 1: *Let (N, v) be a transferable utility game. Then, $m_i^\chi(v) = v(\{i\})$ for every $i \in N$.*

Proof: From the definition of $m_i^\chi(v)$ it follows that $m_i^\chi(v) \geqslant v(\{i\})$ just by taking $S = \{i\}$. To see that $m_i^\chi(v) \leqslant v(\{i\})$ it will be sufficient to show that $v(S) - \sum_{j \in S\setminus\{i\}} M_j^\chi(v) \leqslant v(\{i\})$ for every $S \subseteq N$ such that $i \in S$.

The proof is by induction on the number of players in the coalition S. If $S = \{i, j\}$, then

$$v(S) - \sum_{j \in S\setminus\{i\}} M_j^\chi(v) = v(\{i,j\}) - M_j^\chi(v) \leqslant v(\{i,j\}) - [v(\{i,j\}) - v(\{i\})] = v(\{i\}),$$

since $M_j^\chi(v) \geqslant v(\{i,j\}) - v(\{i\})$.

Assuming that the result is true if S contains $p \geqslant 2$ players (induction hypothesis), we will show that it is true for $p+1$ players.

Let $S = \{i_1, \ldots, i_p, i\}$ be any set with cardinality $p+1$. Then,

$$\begin{aligned} v(S) - \sum_{j \in S\setminus\{i\}} M_j^\chi(v) &= v(S) - \sum_{j=\{1,\ldots,p\}} M_{i_j}^\chi(v) \\ &\leqslant v(S) - [v(S) - v(S\setminus\{i_p\})] - \textstyle\sum_{j=\{1,\ldots,p-1\}} M_{i_j}^\chi(v) \\ &= v(S\setminus\{i_p\}) - \sum_{j=\{1,\ldots,p-1\}} M_{i_j}^\chi(v) \\ &= v(S\setminus\{i_p\}) - \sum_{j \in (S\setminus\{i_p\})\setminus\{i\}} M_j^\chi(v). \end{aligned}$$

Since $S\setminus\{i_p\}$ has p players, we can conclude that the above expression is smaller or equal than $v(\{i\})$, and therefore,

$$v(S) - \sum_{j \in S\setminus\{i\}} M_j^\chi(v) \leqslant v(\{i\}) \text{ for every } S \subseteq N \text{ such that } i \in S.$$

Hence, $m_i^\chi(v) = v(\{i\})$ for every $i \in N$. ∎

An important consequence of Proposition 1 is that, using the new vectors of aspirations, we can define a compromise value for all weakly essential games. That is,

Corollary: *If (N, v) is a weakly essential game, then $\sum_{i \in N} m_i^\chi(v) \leqslant v(N)$.*

We can now define the χ-value for every weakly essential game. Consider $(N, v) \in G$; define the *χ-value* of (N, v), denoted by $\chi(N, v)$, as the unique efficient vector in the lineal segment having as extreme points $m^{\chi}(v)$ and $M^{\chi}(v)$. That is,

$$\chi(N,v) = m^{\chi}(v) + \alpha[M^{\chi}(v) - m^{\chi}(v)],$$

where α is such that $\sum_{i\in N}\chi_i(N,v) = v(N)$.

To illustrate the definition, consider again Example 1.

Example 1 (continued): The maximal and minimal aspirations according to the χ-value are $M_1^{\chi}(v) = M_3^{\chi}(v) = 12, M_2^{\chi}(v) = 10$ and $m_i^{\chi}(v) = v(\{i\}) = 0$ for every $i = 1,2,3$. After the corresponding caluculations we find that $\chi(N,v) = (90/17, 75/17, 90/17)$ and $Sh(N,v) = (16/3, 13/3, 16/3)$. None of this two imputations belong to the Core of (N,v) since its core is empty.

4. THE χ-VALUE, THE τ-VALUE, THE SHAPLEY VALUE AND THE CORE

In this section, we study the relationship of the χ-value with the τ-value, the Shapley value and the Core.

In the class of convex games the τ-value and the χ-value coincide since the grand coalition is the coalition where every player has the bigger marginal contribution. Therefore, the vectors of upper values and maximum aspirations coincide. It is well known that, in this class of games, the Shapley value belongs to the Core but the τ-value may not.

Example 2 below illustrates that the χ-value may belong to the Core in games where the τ-value does not.

Example 2: Let (N,v) be a transferable utility game with $N = \{1,2,3,4,5\}$, $v(\{i\}) = 0$ for every $i \in N$, $v(S) = 40$ if $\#S = 2, v(S) = 100$ if $\#S = 3$ and $S \neq \{1,2,3\}$, $v(\{1,2,3\}) = 190, v(S) = 200$ if $\#S = 4$ and $v(N) = 300$. The vectors of upper and lower values are $M^{\tau}(v) = (100,100,100,100,100)$ and $m^{\tau}(v) = (0,0,0,0,0)$. Therefore, $\tau(N,v) = (60,60,60,60,60)$ which does not belong to $C(N,v)$ since $\tau_1(N,v), + \tau_2(N,v) + \tau_3(N,v) = 180 < 190 = v(\{1,2,3\})$. However, $M^{\chi}(v) = (150,150,150,100,100)$ and $m^{\chi}(v) = (0,0,0,0,0)$, implying that $\chi(N,v) = (69.23, 69.23, 69.23, 46.15, 46.15)$ which does belong to $C(N,v)$.

Example 3 shows a game with the property that the χ-value belongs to the Core but the Shapley value does not.

Example 3: Let (N,v) be a transferable utility game with $N = \{1,2,3,4\}$, $v(\{i\}) = 0$ for every $i \in N$, $v(\{1,4\}) = v(\{2,4\}) = v(\{3,4\}) = 2$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 10$, $v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 12$, $v(\{1,2,3\}) = 110$ and $v(N) = 112$. The vectors of maximum and minimum aspirations are $M^{\chi}(v) = (100,100,100,2)$ and $m^{\chi}(v) = (0,0,0,0)$. Therefore, $\chi(N,v) = (36.77, 36.77, 36.77, 0.66)$ which belongs to $C(N,v)$. However, $Sh(N,v) = \{36.58, 36.58, 36.58, 1.25)$ does not belong to $C(N,v)$ since $Sh_1(N,v) + Sh_2(N,v) + Sh_3(N,v) = 109.75 < 110 = v(\{1,2,3\})$.

It is known that the τ-value belongs to the Core for every balanced game with three players. However, for superadditive games with three players we have the following general result.

Proposition 2: *Let* (N, v) *be a superadditive game with* $\#N = 3$. *Then* $[\chi(N, v) \in C(N, v)] \Rightarrow [\mathrm{Sh}(N, v) \in C(N, v)]$.

Proof: See the Appendix.

It is also easy to find examples showing that for non-superadditive games, even with three players, anything is possible. Also notice that a consequence of Proposition 2 is that there are games where the τ-value does belong to the Core while the χ-value does not, since for three-person games the τ-value always belongs to the Core but the Shapley value does not.

5. FINAL REMARKS

Before finishing the paper we would like to make three remarks. First, about well-known properties that the χ-value satisfies. It is easy to show that it satisfies Individual Rationality, Symmetry, Dummy, Strategic Equivalence, Standard for 2 and $M^{\chi}(v)$-Proportionality. Moreover, it is possible to uniquely characterize the χ-value, in the same line that Tijs [13] did for the τ-value, by the properties of Strategic Equivalence and $M^{\chi}(v)$-Proportionality. However, we do not see it as a very illuminating characterization since the proportionality axiom is already incorporated in the definition of the solution.

Second, about the possibility of using alternative bounds in the definition of a compromise value. Consider its dual approach; that is, obtain the χ^*-value as the compromise between the lower bound defined by $m_i^*(v) = v(\{i\})$ for every $i \in N$, and the upper bound defined by

$$M_i^*(v) = \max_{\substack{S \subseteq N \\ i \in S}} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} m_j^*(v) \right\}.$$

Then, for zero-normalized ($v(\{i\}) = 0$ for every $i \in N$) games,

$$\chi_i^*(N, v) = \frac{v(N)}{\#N} \quad \text{for all } i \in N.$$

Third, about the possibility of defining the compromise value on the dual game. Consider the dual game (N, v^*) defined as follows:

$$v^*(S) = v(N) - v(N \setminus S) \quad \text{for all } S \subseteq N.$$

It is easy to show that the upper bounds used in the definition of the χ-value coincide in the two games; that is, $M^{\chi}(v^*) = M^{\chi}(v)$. Morever, since $v^*(N) = v(N)$ and $v^*(\{i\}) = v(N) - v(N \setminus \{i\}) = M_i^{\tau}(v)$ for all $i \in N$, we have that if $\sum_{i \in N} M_i^{\tau}(v) \leqslant v(N)$ then (N, v^*) is weakly essential and by Proposition 1, $m^{\chi}(v^*) = M^{\tau}(v)$; hence the χ-value on (N, v^*) is well defined in this case.

APPENDIX

Proposition 2: *Let (N,v) be a superadditive game with $\#N=3$. Then* $[\chi(N,v)\in C(N,v)]\Rightarrow[\mathrm{Sh}(N,v)\in C(N,v)]$.

Proof: Without loss of generality we will assume that (N,v) is a zero-normalized game. Suppose that $C(N,v)\neq 0$; therefore, we have that

$$2v(\{1,2,3\})\geqslant v(\{1,2\})+v(\{1,3\})+v(\{2,3\}). \tag{1}$$

By definition, the Shapley value can be written as:

$$\mathrm{Sh}_1(N,v)=1/3(v(\{1,2,3\})-v(2,3\}))+(1/6)v(\{1,2\})+(1/6)\,v(\{1,3\}) \tag{2}$$

$$\mathrm{Sh}_2(N,v)=1/3(v(\{1,2,3\})-v(1,3\}))+(1/6)v(\{1,2\})+(1/6)\,v(\{2,3\}) \tag{3}$$

$$\mathrm{Sh}_3(N,v)=1/3(v(\{1,2,3\})-v(1,2\}))+(1/6)v(\{1,3\})+(1/6)\,v(\{2,3\}). \tag{4}$$

Since (N,v) is superadditive we have that $\mathrm{Sh}(N,v)\in C(N,v)$ if and only if $\mathrm{Sh}_1(N,v)+\mathrm{Sh}_2(N,v)\geqslant v(\{1,2\})$, $\mathrm{Sh}_1(N,v)+\mathrm{Sh}_3(N,v)\geqslant v(\{1,3\})$, and $\mathrm{Sh}_2(N,v)+\mathrm{Sh}_3(N,v)\geqslant v(\{2,3\})$. Therefore, using (2) and (3) in the expression $\mathrm{Sh}_1(N,\ v)+Sh_2(N,v)\geqslant v(\{1,2\})$ we obtain that

$$v(\{1,2,3\})\geqslant v(\{1,2\})+(1/4)v(\{1,3\})+(1/4)v(\{2,3\}). \tag{5}$$

By a similar argument, it is easy to see that

$$v(\{1,2,3\})\geqslant v(\{1,3\})+(1/4)v(\{1,2\})+(1/4)v(\{2,3\}) \tag{6}$$

$$v(\{1,2,3\})\geqslant v(\{2,3\})+(1/4)v(\{1,2\})+(1/4)v(\{1,3\}). \tag{7}$$

Assume that $\mathrm{Sh}(N,v)\notin C(N,v)$, we will show that $\chi(N,v)\notin C(N,v)$. If $\mathrm{Sh}(N,v)\notin C(N,v)$ it means that at least one of the inequalities in (5), (6) or (7) is not verified. Assume it is (5) (the other cases are done similarly), that is, $v(\{1,2,3\})<v(\{1,2\})+(1/4)v(\{1,3\})+(1/4)v(\{2,3\})$. Without loss of generality suppose $v(\{1,3\})\geqslant v(\{2,3\})$ and $v(\{1,3\})>0$ (if $v(\{1,3\})=0$ then the Core is empty because $v(\{1,2,3\})<v(\{1,2\})$). Then, by definition, $M_3^\chi(v)=v(\{1,3\})$ since $v(\{1,2,3\})-v(\{1,2\})<(1/2)v(\{1,3\})$. We will distinguish, and successively eliminate, four possible cases:

Case 1: $M_2^\chi(v)\neq v(\{1,2\})$. Suppose that $M_2^\chi(v)=v(\{2,3\})>v(\{1,2\})$. By (1) and $\mathrm{Sh}_1(N,v)+\mathrm{Sh}_2(N,v)<v(\{1,2\})$, we have that

$$4(v(\{1,2,3\})-v(\{1,2\}))\leqslant 2v(\{1,2,3\})-v(\{1,2\}),$$

and therefore,

$$v(\{1,2\})\geqslant(2/3)v(\{1,2,3\}). \tag{8}$$

But this is a contradiction with $C(N,v)\neq\phi$, since

$$v(\{1,3\})\geqslant v(\{2,3\})>v(\{1,2\})\geqslant(2/3)v(\{1,2,3\})$$

implies that (1) is violated.

Suppose that $M_2^\chi(v) = v(\{1,2,3\}) - v(\{1,3\}) > v(\{1,2\})$. Since $v(\{2,3\}) \leqslant v(\{1,3\})$ we have that

$$v(\{1,2,3\}) - v(\{2,3\}) \geqslant v(\{1,2,3\}) - v(\{1,3\}) > v(\{1,2\}),$$

but then, $2v(\{1,2,3\}) - v(\{1,3\}) - v(\{2,3\}) > 2v(\{1,2\})$, which implies that $v(\{1,2,3\}) - v(\{1,2\}) > (1/2)(v(\{1,3\}) + v(\{2,3\}))$ contradicting $\mathrm{Sh}_1(N,v) + \mathrm{Sh}_2(N,v) < v(\{1,2\})$.

Case 2: $M_1^\chi(v) = M_2^\chi(v) = v(\{1,2\})$. Then, by definition of the χ-value,

$$\chi(N,v) = \frac{v(\{1,2,3\})}{v(\{1,2\}) + v(\{1,2\}) + v(\{1,3\})} \cdot (v(\{1,2\}), v(\{1,2\}), v(\{1,3\})).$$

Therefore,

$$\chi_3(N,v) = \frac{v(\{1,2,3\})}{2v(\{1,2\}) + v(\{1,3\})} \cdot v(\{1,3\}).$$

It is a straightforward exercise to show that $\chi_3(N,v) > v(N) - v(\{1,2\})$. Therefore, by efficiency $\chi_1(N,v) + \chi_2(N,v) < v(\{1,2\})$, implying that $\chi(N,v) \notin C(N,v)$.

Case 3: $M_1^\chi(v) = v(\{1,2,3\}) - v(\{2,3\})$ and $M_2^\chi(v) = v(\{1,2\})$. Then, by the definition of the χ-value,

$$\chi(N,v) = \frac{v(\{1,2,3\}) \cdot (v(\{1,2,3\}) - v(\{2,3\}), v(\{1,2\}), v(\{1,3\}))}{v(\{1,2\}) + v(\{1,2,3\}) + v(\{1,3\}) - v(\{2,3\})}.$$

Therefore,

$$\begin{aligned}\chi_3(N,v) &= \frac{v(\{1,2,3\}) \cdot v(\{1,3\})}{v(\{1,2\}) + v(\{1,2,3\}) + v(\{1,3\}) - v(\{2,3\})} \\ &\geqslant \frac{4v(\{1,2,3\})^2 - 4v(\{1,2,3\})\, v(\{1,2\}) - v(\{1,2,3\})\, v(\{2,3\})}{3v(\{1,2,3\}) - 2v(\{2,3\})},\end{aligned}$$

since, by $\mathrm{Sh}_1(N,v) + \mathrm{Sh}_2(N,v) < v(\{1,2\})$, we have that $v(\{1,3\}) > 4v(\{1,2,3\}) - 4v(\{1,2\}) - v(\{2,3\})$, and by (1), we have that $v(\{1,2\}) + v(\{1,2,3\}) + v(\{1,3\}) - v(\{2,3\})$ is smaller or equal to $3v(\{1,2,3\}) - 2v(\{2,3\})$. To see that $\chi(N,v) \notin C(N,v)$ it is sufficient to see that $\chi_3(N,v) > v(\{1,2,3\}) - v(\{1,2\})$. After some algebra it is easy to see that this condition is equivalent to

$$v(\{1,2,3\})\ (v(\{1,2,3\}) - v(\{1,2\})) \geqslant v(\{2,3\})\ (2v(\{1,2\}) - v(\{1,2,3\})),$$

which is true, since $v(\{1,2,3\}) \geqslant 2v(\{1,2\}) - v(\{1,2,3\})$ and the fact that $M_1^\chi(v) = v(\{1,2,3\}) - v(\{2,3\})$ implies that $v(\{1,2,3\}) - v(\{2,3\}) \geqslant v(\{1,2\})$.

Case 4: $M_2^\chi(v) = v(\{1,2\})$ and $M_1^\chi(v) = v(\{1,3\})$. By hypothesis, we have that

$$\chi(N,v) = \frac{v(\{1,2,3\})}{2v(\{1,3\}) + v(\{1,2\})} \cdot (v(\{1,3\}), v(\{1,2\}), v(\{1,3\})).$$

If $\chi(N,v) \in C(N,v)$ then $\chi_1(N,v) + \chi_3(N,v) \geqslant v(\{1,3\})$, which implies that

$$\frac{v(\{1,2,3\})2v(\{1,3\})}{2v(\{1,3\}) + v(\{1,2\})} \geqslant v(\{1,3\}).$$

Therefore, since $v(\{1,3\}) > v(\{1,2\})$ and (8), we have that

$$2v(\{1,2,3\}) \geqslant 2v(\{1,3\}) + v(\{1,2\}) > 2v(\{1,2,3\}),$$

which is a contradiction. ■

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