The division problem with maximal capacity constraints

Gustavo Bergantiños, Jordi Massó & Alejandro Neme

SERIEs
Journal of the Spanish Economic Association

ISSN 1869-4187
Volume 3
Combined 1-2

SERIEs (2012) 3:29-57
DOI 10.1007/s13209-011-0055-6


Volume 2 · Number 1–2 · March 2011

SpringerOpen

SERIEs

Journal of the Spanish Economic Association

Continued on the backside

Springer
SEPI FUNDACIÓN

Springer

SERIEs (2012) 3:29–57
DOI 10.1007/s13209-011-0055-6

SERIEs
Journal of the Spanish Economic Association

ORIGINAL ARTICLE

The division problem with maximal capacity constraints

Gustavo Bergantiños · Jordi Massó · Alejandro Neme

Received: 11 November 2010 / Accepted: 11 March 2011 / Published online: 25 March 2011

Abstract The division problem consists of allocating a given amount of an homogeneous and perfectly divisible good among a group of agents with single-peaked preferences on the set of their potential shares. A rule proposes a vector of shares for each division problem. Most of the literature has implicitly assumed that all divisions are feasible. In this paper we consider the division problem when each agent has a maximal capacity due to an objective and verifiable feasibility constraint

We thank an anonymous referee whose comments and suggestions helped us to write a better paper. The work of G. Bergantiños is partially supported by research grant ECO2008-03484-C02-01 from the Spanish Ministry of Science and Innovation and FEDER. Support for the research of J. Massó was received through the prize "ICREA Acadèmia" for excellence in research, funded by the Generalitat de Catalunya. He also acknowledges the support of MOVE (where he is an affiliated researcher), of the Barcelona Graduate School of Economics (where he is an affiliated professor), and of the Government of Catalonia, through grant SGR2009-419. His work is also supported by the Spanish Ministry of Science and Innovation through grants ECO2008-04756 (Grupo Consolidado-C) and CONSOLIDER-INGENIO 2010 (CDS2006-00016). The work of A. Neme is partially supported by the Universidad Nacional de San Luis through grant 319502 and by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) through grant PICT-02114.

G. Bergantiños
Research Group in Economic Analysis, Facultade de Económicas, Universidade de Vigo,
36310 Vigo (Pontevedra), Spain
e-mail: gbergant@uvigo.es

J. Massó (✉)
Departament d'Economia i d'Història Econòmica and CODE,
Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
e-mail: jordi.masso@uab.es

A. Neme
Instituto de Matemática Aplicada San Luis, Universidad Nacional de San Luis and CONICET,
Ejército de los Andes 950, 5700 San Luis, Argentina
e-mail: aneme@unsl.edu.ar

which imposes an upper bound on his share. Then each agent has a feasible interval of shares where his preferences are single-peaked. A rule has to propose to each agent a feasible share. We focus mainly on strategy-proof, efficient and consistent rules and provide alternative characterizations of the extension of the uniform rule that deals explicitly with agents' maximal capacity constraints.

Keywords Division problem · Single-peaked preferences · Uniform rule · Capacity constraints

JEL Classification D71

1 Introduction

For general social choice problems strategy-proofness is too demanding: only dictatorial rules are immune to strategic manipulation. This is a negative result because strategy-proof rules are appealing (they induce truth-telling independently of the information agents have about other agents' preferences) and because dictatorships are not interesting procedures to aggregate agents' preferences. However, in specific applications, the structure of the set of alternatives suggests that not all preferences are meaningful and admissible. But then, rules that were manipulable on the universal domain of preferences may become strategy-proof when applied only to those restricted domains. Sprumont (1991) is a prominent example of this approach.[1] Sprumont (1991) considers the division problem faced by a set of agents that have to share an amount of an homogeneous and perfectly divisible good. For instance, a group of agents participate in an activity that requires a fixed amount of labor (measured in units of time). Agents have a maximal number of units of time to contribute and consider working as being undesirable. Suppose that labor is homogeneous and the wage is fixed. Then, strictly monotonic and quasi-concave preferences on the set of bundles of money and leisure generate single-peaked preferences on the set of potential shares where the peak is the amount of working time associated to the optimal bundle and in both sides of the peak the preference is strictly monotonic, decreasing at its right and increasing at its left. Similarly, a group of agents join a partnership to invest in a project (an indivisible bond with a face value, for example) that requires a fixed amount of money (neither more nor less). Their risk attitudes and wealth induce single-peaked preferences on the amount to be invested. Finally, a group of firms with different sizes (i.e., maximal capacities) have to jointly undertake a project of a fixed size. Since they may be involved in other projects their preferences are single-peaked on their respective intervals of feasible shares of this project. In general, a (classical) *division problem* consists of a finite set of agents, a preference profile of declared list of single-peaked preferences on the interval $[0, +\infty)$, one for each agent, and the amount of the good to be allocated. Since single-peaked preferences reflect idiosyncratic characteristics of the agents, they have to be elicited by a rule that maps each division problem into a

[1] See also Barberà (1977), Moulin (1980), Barberà et al. (1991, 1998), Alcalde and Barberà (1994), Barberà and Jackson (1994) for different examples of this domain restriction approach.

vector of shares.[2] But in general, the sum of the peaks will be either larger or smaller than the total amount to be allocated. A positive or negative rationing problem emerges depending on whether the sum of the peaks exceeds or fails short the fixed amount. Rules differ from each other on how this rationing problem is resolved in terms of incentives, efficiency, fairness, consistency, monotonicity, etc.

However, almost all the literature on the division problem has implicitly neglected the fact that in many applications (like those described above) each agent i has a maximal capacity due to an objective and verifiable feasibility constraint which imposes an upper bound u_i on his share.[3] These upper bounds may come from capacity constraints, budget constraints or physical limits on the amount each agent can receive. We consider situations where these maximal capacity constraints are common knowledge.[4] Thus, we assume here that each agent i has a feasible interval of shares $[0, u_i]$ where his preferences are single-peaked. A rule has to propose to each agent a feasible share (i.e., a positive amount smaller or equal to u_i). We propose and axiomatically study a rule that solves the rationing problem when agents have maximal capacity constraints.

We are interested in rules that satisfy two classes of desirable properties. The first class is related to the behavior of the rule at a given division problem with maximal capacity constraints (i.e., the set of agents, their upper bounds and the amount to be shared are fixed). First, *strategy-proofness*. A rule is strategy-proof if no agent can profitably alter the rule's choice by misrepresenting his preferences. Namely, strategy-proofness guarantees that truth-telling is a weakly dominant strategy in the direct revelation game induced by the rule. Second, *efficiency*. A rule is efficient if it always selects Pareto optimal allocations. Efficiency guarantees that in solving the rationing problem (either positive or negative) no amount of the good is wasted. Third, *weak individual rationality from equal division*. A rule satisfies this property by choosing a Pareto improvement from the vector of equal shares, whenever this vector is feasible.[5] Weak individual rationality from equal division embeds to the rule a minimal egalitarian principal only broken if either the equal division is not feasible or to satisfy the bounds imposed by efficiency. Fourth, *equal treatment of equals*. A rule satisfies equal treatment of equals if it treats equally identical agents. Fifth, *weak envy-freeness*. A rule is weak envy-free if it does not generate feasible envies in the sense that there is no agent that by exchanging his shares with another agent, the new share is feasible and the agent is strictly better off.

[2] Since agents' preferences are single-peaked there are many non-dictatorial and strategy-proof rules. Barberà et al. (1997) indicates that the class of all strategy-proof rules for the classical division problem is extremely large.

[3] Kibris (2003), Bergantiños et al. (2011), Kim (2010) and Manjunath (2010) consider different types of restrictions on the set of feasible and/or acceptable shares. Moulin (1999) introduces maximal capacity constraints in the division problem to deal with the family of fixed path mechanisms. Ehlers (2002a,b) extends further Moulin (1999)'s results. At the end of this Introduction we will briefly describe their contributions and main differences with our approach.

[4] See Bergantiños et al. (2011) for an analysis where agents have intervals of acceptable shares that are private information.

[5] See Sönmez (1994) for an analysis of rules satisfying this property in the context of classical division problems without maximal capacity constraints.

The second class is related to the restrictions that the properties impose on a rule when comparing its proposal at different division problems with maximal capacity constraints (i.e., when either the set of agents, their upper bounds or the amount to be shared change). First, *consistency*. A rule is consistent if the proposed shares at a given problem coincide with the shares that the rule would propose at any smaller problem obtained after that a subset of agents, agreeing with the amounts the rule has assigned to them, leave the society taking with them their already assigned shares. Consistency guarantees that, in order to follow the rule's prescription at the reduced problem, the remaining agents do not have to reallocate their shares. Second, *upper bound monotonicity*. A rule is upper bound monotonic if roughly, it is monotonic with respect the vector of upper bounds. Third, *one-sided resource-monotonicity*. A rule is one-sided resource-monotonic if changes of the total amount to be allocated that do not change the sign of the rationing problem leave agents better off whenever the change of the amount makes the rationing problem less severe.[6]

Our results identify axiomatically a unique rule satisfying four different subsets of the set of properties that we have just presented. This rule is the *constrained uniform rule* which extends the uniform rule of the classical division problem to our setting. It tries to divide the good as equally as possible keeping the bounds imposed by efficiency and by the maximal capacity constraints as well. We first show in Proposition 1 that the constrained uniform rule satisfies all desirable properties we have described above and then, we present four characterizations of the rule. In all of them we extend characterizations of the uniform rule in the classical division problem by adding the property of upper bound monotonicity to the properties used in the previous results. In some of the characterizations we also need to adapt some classical properties to our setting. Theorem 1 states that the constrained uniform rule is the unique rule that satisfies strategy-proofness, efficiency, upper bound monotonicity and equal treatment of equals. Theorem 2 says that the constrained uniform rule is the unique rule that satisfies strategy-proofness, efficiency, upper bound monotonicity and weak envy-freeness. That is, Theorems 1 and 2 extend two existing characterizations of the uniform rule in the classical division problem: Ching (1994)'s characterization establishing that the uniform rule is the unique rule that satisfies strategy-proofness, efficiency and equal treatment of equals, and Sprumont (1991)'s characterization establishing that the uniform rule is the unique rule that satisfies strategy-proofness, efficiency and envy-freeness. Our two characterizations in Theorems 1 and 2 use upper bound monotonicity, which is a specific property of our setting with maximal capacity constraints, and the one in Theorem 2 adapts envy-freeness. Theorem 3 states that the constrained uniform rule is the unique rule that satisfies consistency, weak individual rationality from equal division, upper bound monotonicity and efficiency. Since by Ehlers (2002b) one-sided resource-monotonicity implies efficiency, we state as Corollary 1 that the constrained uniform rule is the unique rule that satisfies consistency, weak individual rationality from equal division, upper bound monotonicity and one-sided resource monotonicity. That is, Theorem 3 and Corollary 1 extend two existing characterizations of the uniform rule in the classical division problem: Dagan (1996)'s characterization estab-

[6] Ehlers (2002b) shows that one-sided resource-monotonicity implies efficiency.

lishing that the uniform rule is the unique rule that satisfies consistency, efficiency and individual rationality from equal division, and Sönmez (1994)'s characterization establishing that the uniform rule is the unique rule that satisfies consistency, individual rationality from equal division and one-sided resource-monotonicity. Our characterizations in Theorem 3 and Corollary 1 use the property of upper bound monotonicity and adapt individual rationality from equal division to our setting with maximal capacity constraints. We also show that in all characterizations the axioms are independent.

Several papers are closely related to the present one. First, Kibris (2003) studies the same division problem with maximal capacity constraints but instead he assumes free-disposability of the good. Then a rule assigns to each division problem with maximal capacity constraints a vector of shares satisfying the constraints and adding up *less* or equal than the total amount. Kibris (2003) characterizes an extension of the uniform rule to his setting with free-disposability. Second, Bergantiños et al. (2011) considers the division problem when agents' participation is voluntary. Each agent has an idiosyncratic interval of acceptable shares (which, in contrast with our setting here, is private information) where his preferences are single peaked. Then a rule proposes to each agent either to not participate at all or an acceptable share. Bergantiños et al. (2011) shows that strategy-proofness is too demanding in this setting. Then, they study a subclass of efficient and consistent rules and characterize extensions of the uniform rule that deal explicitly with agents' voluntary participation. Third, Kim (2010) characterizes, in the same setting than Bergantiños et al. (2011) with voluntary participation, a rule (called the generalized uniform rule) by the properties of efficiency, no-envy, separability and weak resource continuity. Fourth, Manjunath (2010) proposes a division problem where each agent's preferences are characterized by the top share and a *minimum* share in such a way that the agent is indifferent between any two quantities that are either below the minimum acceptable share or above the top share. Manjunath (2010) first shows that, under different fairness properties, strategy-proofness and efficiency are incompatible and second, he characterizes axiomatically different rules that solve the rationing problem in his setting. Finally, the division problem with maximal capacity constraints is also considered by Moulin (1999).[7] He characterizes the class of all fixed path mechanisms as the set of rules satisfying efficiency, strategy-proofness, consistency and resource monotonicity. The constrained uniform rule presented in this paper is the fixed path rationing method of Moulin (1999) using the main diagonal as path. Ehlers (2002a) presents a shorter proof of the main result in Moulin (1999) and Ehlers (2002b) extends it by showing that, for problems with strictly more than two agents, the class of all fixed path mechanisms coincides with the set of rules satisfying weak one-sided resource monotonicity, strategy-proofness and consistency.

The paper is organized as follows. In Sect. 2 we describe the model. In Sect. 3 we define several desirable properties that a rule may satisfy. In Sect. 4 we define the constrained uniform rule for the division problem with maximal capacity constraints and state several alternative axiomatic characterizations of this rule. Two Appendices at the end of the paper collect all omitted proofs.

7 In Moulin (1999) the maximal capacity constraints are justified on the basis of technical simplicity in order to define the priority rationing methods by an ordinary path and to define the duality operator that cuts the main proof in half.

2 The model

Let $t > 0$ be an amount of an homogeneous and perfectly divisible *good* that has to be allocated among a finite set N of *agents* according to their preferences. Since we will be considering situations where the amount of the good t and the finite set of agents may vary, let $\mathbb{N}$ be the set of positive integers and let $\mathcal{N}$ be the family of all non-empty and finite subsets of $\mathbb{N}$. The set of agents is then $N \in \mathcal{N}$ with cardinality n. In contrast with Sprumont (1991), we consider situations where each agent i has an objective and verifiable upper bound $u_i > 0$ on the amount he can receive. As we have already argued in the Introduction this bound may come from maximal capacity constraints, budget constraints or physical limits on the amount agent i can receive. These constraints may be different across agents. Thus, agent i's *preferences* $\succeq_i$ are a complete preorder (a complete, reflexive, and transitive binary relation) on the interval $[0, u_i]$ of i's feasible shares. Given a preference $\succeq_i$ let $\succ_i$ be the antisymmetric binary relation induced by $\succeq_i$ (i.e., for all $x_i, y_i \in [0, u_i]$, $x_i \succ_i y_i$ if and only if $y_i \succeq_i x_i$ does not hold) and let $\sim_i$ be the indifference relation induced by $\succeq_i$ (i.e., for all $x_i, y_i \in [0, u_i]$, $x_i \sim_i y_i$ if and only if $x_i \succeq_i y_i$ and $y_i \succeq_i x_i$). We assume that $\succeq_i$ is single-peaked on $[0, u_i]$ and we will denote by $p_i \in [0, u_i]$ agent i's *peak*. Formally, a preference $\succeq_i$ is *single-peaked* on $[0, u_i]$ if

(P.1) there exists $p_i \in [0, u_i]$ such that $p_i \succ_i x_i$ for all $x_i \in [0, u_i] \backslash \{p_i\}$;
(P.2) $x_i \succ_i y_i$ for any pair of shares $x_i, y_i \in [0, u_i]$ such that either $y_i < x_i \leq p_i$ or $p_i \leq x_i < y_i$.

A *profile* $\succeq_N = (\succeq_i)_{i \in N}$ is an $n-$tuple of preferences satisfying properties (P.1) and (P.2). Given a profile $\succeq_N$ and agent i's preferences $\succeq'_i$ we denote by $(\succeq'_i, \succeq_{N \backslash \{i\}})$ the profile where $\succeq_i$ has been replaced by $\succeq'_i$ and all other agents have the same preferences. Similarly, given a subset of agents $S \subseteq N$, we will write often $\succeq_N$ as $(\succeq_S, \succeq_{N \backslash S})$. When no confusion arises we denote the profile $\succeq_N$ by $\succeq$.

A division problem with maximal capacity constraints (a *problem* for short) is a 4-tuple $(N, u, \succeq, t)$ where $N \in \mathcal{N}$ is the set of agents, $u = (u_i)_{i \in N} \in \mathbb{R}^n_{++}$ is the vector of upper bounds, $\succeq = (\succeq_i)_{i \in N}$ is a profile and $t > 0$ is the amount of the good to be divided with the property that $t \leq \sum_{i \in N} u_i$; i.e., we assume that it is possible to divide t among all agents satisfying their maximal capacity constraints, otherwise the problem would not admit a solution. Let $\mathcal{P}$ be the set of all problems and consider the subclass of classical problems (as in Sprumont (1991)) $\mathcal{P}^{u=t} \equiv \{(N, u, \succeq, t) \in \mathcal{P} \mid u_i = t \text{ for all } i \in N\}$; namely, the upper bounds are irrelevant because agents can not receive more than t. Thus, we are extending the classical model. Whenever we refer to a problem $(N, u, \succeq, t)$ in $\mathcal{P}^{u=t}$ we will write it as $(N, \succeq, t)$.

The set of *feasible allocations* of problem $(N, u, \succeq, t) \in \mathcal{P}$ is

$$FA(N, u, \succeq, t) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i = t \text{ and for each } i \in N,\ x_i \in [0, u_i]\right\}.$$

Note that this set is never empty since we are assuming that $0 < t \leq \sum_{i \in N} u_i$.

3 Rules and their properties

A rule is a systematic way of solving the division problem by assigning to each problem one of its feasible allocations; that is, a *rule* f is a mapping on the set of problems with the property that for all $(N, u, \succeq, t) \in \mathcal{P}$, $f(N, u, \succeq, t) \in FA(N, u, \succeq, t)$.

Rules require each agent to report a preference. A rule is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully; namely, it induces truth-telling as a weakly dominant strategy in the direct revelation game generated by the rule.

(STRATEGY- PROOFNESS) A rule f is *strategy-proof* if for each problem $(N, u, \succeq_N, t) \in \mathcal{P}$, agent $i \in N$, and preference $\succeq_i'$ on $[0, u_i]$,

$$f_i(N, u, \succeq_N, t) \succeq_i f_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right).$$

Given a problem $(N, u, \succeq_N, t) \in \mathcal{P}$ we say that agent $i \in N$ *manipulates* f *at profile* $\succeq_N$ *via* $\succeq_i'$ on $[0, u_i]$ if $f_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t) \succ_i f_i(N, u, \succeq_N, t)$. Thus, a rule f is strategy-proof if no agent can manipulate it at any profile.

A rule is efficient if it always selects a Pareto optimal allocation.

(EFFICIENCY) A rule f is *efficient* if for each problem $(N, u, \succeq, t) \in \mathcal{P}$ there is no feasible allocation $(y_j)_{j\in N} \in FA(N, u, \succeq, t)$ with the property that $y_i \succeq_i f_i(N, u, \succeq, t)$ for all $i \in N$ and $y_j \succ_j f_j(N, u, \succeq, t)$ for some $j \in N$.

Is it immediate to see that the following remarks holds.

Remark 1 Let f be an efficient rule and let $(N, u, \succeq, t) \in \mathcal{P}$ be a problem.

(R.1) If $\sum_{i\in N} p_i \geq t$ then $f_i(N, u, \succeq, t) \leq p_i$ for all $i \in N$.
(R.2) If $\sum_{i\in N} p_i < t$ then $f_i(N, u, \succeq, t) \geq p_i$ for all $i \in N$.

A rule satisfies equal treatment of equals if two agents with identical preferences receive the same share. Observe that this property is weak because if two agents have different upper bounds they can not have the same preferences.

(EQUAL TREATMENT OF EQUALS) A rule f satisfies *equal treatment of equals* if for each problem $(N, u, \succeq, t) \in \mathcal{P}$, $\succeq_i = \succeq_j$ implies $f_i(N, u, \succeq, t) = f_j(N, u, \succeq, t)$.

A rule satisfies upper bound monotonicity if it is monotonic with respect to the vector of upper bounds in the following sense.

(UPPER BOUND MONOTONICITY) A rule f satisfies *upper bound monotonicity* if for any two problems $(N, u, \succeq, t), (N, u', \succeq', t) \in \mathcal{P}$ with the property that there is an agent $k \in N$ such that $u_k < u_k' \leq t$ and $\succeq_k$ coincides with $\succeq_k'$ on $[0, u_k]$, and for each $i \in N\setminus\{k\}$, $u_i = u_i'$ and $\succeq_i = \succeq_i'$ then,

$$f_i(N, u, \succeq, t) \geq \min\left\{f_i\left(N, u', \succeq', t\right), u_i\right\} \quad \text{for each } i \in N.$$

Observe that this condition above can be rewritten as

(i) $f_i(N, u, \succeq, t) \geq f_i\left(N, u', \succeq', t\right)$ for all $i \in N\setminus\{k\}$ and
(ii) $f_k(N, u, \succeq, t) \geq \min\left\{f_k\left(N, u', \succeq', t\right), u_k\right\}$.

Suppose that the upper bound of agent k decreases from u'_k to u_k whereas the other upper bounds remain the same. How should agents be affected? Condition (i) says that all agents but k should receive at least the same amount as before. Notice that (i) is related to a principle known in the literature as solidarity. Condition (ii) says that agent k should receive at least the same amount as before, when this amount is still feasible, or his new upper bound otherwise. From (i) and (ii) is easy to deduce that if the allocation proposed by the rule in the largest problem (u') belongs to the smallest problem (u) then, the allocation in the smallest problem must coincide with the one of the largest problem. Thus, upper bound monotonicity implies a principle known in the literature as independence of irrelevant alternatives.

A rule is consistent if the following requirement holds. Apply the rule to a given problem and assume that a subset of agents leave with their corresponding shares. Consider the new problem formed by the set of agents that remain with the same preferences (and upper bounds) that they had in the original problem and the total amount of the good minus the sum of the shares received by the subset of agents that already left. Then, the rule does not require to reallocate the shares of the remaining agents.

(Consistency) A rule f is *consistent* if for each problem $(N, u, \succeq_N, t) \in \mathcal{P}$, each subset of agents $S \subset N$ and each $i \in S$,

$$f_i(N, u, \succeq_N, t) = f_i\left(S, (u_j)_{j\in S}, \succeq_S, t - \sum_{j\in N\setminus S} f_j(N, u, \succeq_N, t)\right).$$

We next consider a weak version of envy-freeness. The basic principle under envy-freeness is that no agent can strictly prefer the share received by another agent; that is, in our setting, a rule would be envy-free if for each problem $(N, u, \succeq, t) \in \mathcal{P}$ and each pair of agents $i, j \in N$, $f_i(N, u, \succeq, t) \succeq_i f_j(N, u, \succeq, t)$ holds. However, the following example shows that when agents have maximal capacity constraints no rule is envy-free. Consider any problem $(N, u, \succeq, t) \in \mathcal{P}$ where $N = \{1, 2\}$, $u = (2, 10)$, $t = 10$, and $\succeq_2$ has the property that for each $x_1 \in [0, 2]$ and $y_2 \in [8, 10]$, $x_1 \succ_2 y_2$. Thus,

$$FA(N, u, \succeq, t) = \left\{(x_1, 10 - x_1) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 2\right\}.$$

Then, any rule f applied to any of these problems would generate envy (agent 2 would envy agent 1). Hence, no rule is envy-free. Thus, we consider a weaker version of envy-freeness which admits unfeasible envies and coincides with envy-freeness in the classical model.

(Weak Envy-freeness) A rule f is *weak envy-free* if for each problem $(N, u, \succeq, t) \in \mathcal{P}$ and each pair of agents $i, j \in N$ such that $f_j(N, u, \succeq, t) \succ_i f_i(N, u, \succeq, t)$, then the vector of shares $x = (x_k)_{k\in N}$, where $x_i = f_j(N, u, \succeq, t)$, $x_j = f_i(N, u, \succeq, t)$, and $x_k = f_k(N, u, \succeq, t)$ for all $k \in N\setminus\{i, j\}$ has the property that $x \notin FA(N, u, \succeq, t)$.

For the classical division problem Sönmez (1994) proposed the principle of individual rationality from equal division. A rule f is individually rational from equal division if all agents receive a share that is at least as good as the equal division share; namely, for each division problem $(N, u, \succeq, t) \in \mathcal{P}$, for all $i \in N$,

$$f_i(N, u, \succeq, t) \succeq_i \frac{t}{n}.$$

In a classical division problem equal division is always feasible but it may not be efficient. Precisely, this principle tries to make compatible equal division with efficiency by allowing for Pareto improvements from the equal division share. Observe that in our setting the allocation $(\frac{t}{n}, \ldots, \frac{t}{n})$ may not be feasible and/or there may not even exist a vector of feasible shares at which all agents are better off than at equal division. To see that, consider again the problem $(N, u, \succeq, t) \in \mathcal{P}$ where $N = \{1, 2\}, u = (2, 10)$, $p_2 = 5$, and $t = 10$. Thus,

$$FA(N, u, \succeq, t) = \{(x_1, 10 - x_1) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 2\}.$$

If f satisfies individual rationality from equal division then $f_2(N, u, \succeq, t) = 5$, which means that $f(N, u, \succeq, t) \notin FA(N, u, \succeq, t)$. Thus, when agents have maximal capacity constraints, this property is too strong (no rule satisfies it) and it can not be applied directly. However, and since we think that its content is appealing, we suggest to use the same principle whenever it is possible.

(Weak Individual Rationality from Equal Division) A rule f satisfies *weak individual rational from equal division* if for each problem $(N, u, \succeq, t) \in \mathcal{P}$ for which $\left(\frac{t}{n}, \ldots, \frac{t}{n}\right) \in FA(N, u, \succeq, t)$ then, for all $i \in N$,

$$f_i(N, u, \succeq, t) \succeq_i \frac{t}{n}.$$

The last property we want to consider is related to the behavior of the rule when the amount t to be shared changes. However, it only imposes conditions on the rule whenever the change of the amount to be shared does not change the sign of the rationing problem: if the good is scarce, an increase of the amount to be shared should make all agents better off and if the good is too abundant, a decrease of the amount to be shared should make all agents better off.

(One- sided Resource- monotonicity) A rule f satisfies *one-sided resource-monotonicity* if for all two problems $(N, u, \succeq, t), (N, u, \succeq, t') \in \mathcal{P}$ with the property that either $t \leq t' \leq \sum_{i \in N} p_i$ or $\sum_{i \in N} p_i \leq t' \leq t$ then $f_i(N, u, \succeq, t') \succeq_i f_i(N, u, \succeq, t)$ for all $i \in N$.

Ehlers (2002b) shows that one-sided resource monotonicity implies efficiency. We state this result as Remark 2 below.

Remark 2 Let f be an one-sided resource-monotonic rule. Then, f is efficient.

4 The constrained uniform rule and its characterizations

The uniform rule has played a central role in the classical division problem because it is the unique rule satisfying different sets of desirable properties. For instance, Sprumont

(1991) shows that the uniform rule is the unique rule satisfying (i) strategy-proofness, efficiency and anonymity, and (ii) strategy-proofness, efficiency and envy-free.[8]

The *uniform rule* U is defined as follows: for each division problem $(N, \succeq, t) \in \mathcal{P}^{u=t}$ and for each $i \in N$,

$$U_i(N, \succeq, t) = \begin{cases} \min\{\beta, p_i\} & \text{if } \sum_{j\in N} p_j \geq t \\ \max\{\beta, p_i\} & \text{if } \sum_{j\in N} p_j < t, \end{cases}$$

where β solves the equation $\sum_{j\in N}\min\{\beta, p_j\} = t$ if $\sum_{j\in N} p_j \geq t$ and solves the equation $\sum_{j\in N}\max\{\beta, p_j\} = t$ if $\sum_{j\in N} p_j < t$. Namely, U tries to allocate the good as equally as possible, keeping the efficient constraints binding (see Remark 1): if $\sum_{i\in N} p_i \geq t$ then $U_i(N, \succeq, t) \leq p_i$ for all $i \in N$, and if $\sum_{i\in N} p_i < t$ then $U_i(N, \succeq, t) \geq p_i$ for all $i \in N$.

Observe that when applied to division problems with maximal capacity constraints U is not a rule since at some problems it chooses non-feasible allocations. In the rest of this section we extend the uniform rule to our environment, state that it satisfies many desirable properties, and give four axiomatic characterizations. We define the *constrained uniform rule* F in the class of problems $\mathcal{P}$ as follows: for each problem $(N, u, \succeq, t) \in \mathcal{P}$ and each agent $i \in N$,

$$F_i(N, u, \succeq, t) = \begin{cases} \min\{\beta, p_i\} & \text{if } \sum_{j\in N} p_j \geq t \\ \min\{\max\{\beta, p_i\}, u_i\} & \text{if } \sum_{j\in N} p_j < t, \end{cases}$$

where β solves the equation $\sum_{j\in N}\min\{\beta, p_j\} = t$ if $\sum_{j\in N} p_j \geq t$ and solves the equation $\sum_{j\in N}\min\{\max\{\beta, p_j\}, u_j\} = t$ if $\sum_{j\in N} p_j < t$. Observe that for each problem $(N, u, \succeq, t) \in \mathcal{P}$ there exists such number β because the expression $\sum_{j\in N} F_j(N, u, \succeq, t)$ (which implicitly depends on β) is continuous and increasing on β and for $\beta = 0$ is smaller than t while for $\beta = \sum_{j\in N} u_j$ is larger or equal than t. Hence, by the Intermediate Value Theorem such β exists. The idea behind the rule is simple. When the amount of the good to be allocated is scarce, the peaks will already be the relevant upper limits on agents' shares, so the vector of upper bounds u is irrelevant, and the constrained uniform rule coincides with the uniform rule. However, when the amount of the good to be allocated is abundant, to satisfy feasibility the rule has to make sure that no agent receives more than his upper bound.

Proposition 1 below states that the constrained uniform rule satisfies all desirable properties presented in Sect. 3.

Proposition 1 *The constrained uniform rule satisfies strategy-proofness, efficiency, consistency, weak envy-freeness, equal treatment of equals, upper bound monotonicity, weak individual rationality from equal division and one-sided resource-monotonicity.*

Proof See Appendix 1. □

[8] See Ching (1992, 1994), Dagan (1996), Schummer and Thomson (1997), Sönmez (1994), and Thomson (1994a,b, 1995, 1997) for alternative characterizations of the uniform rule in the division problem. In the surveys on strategy-proofness of Barberà (1996, 2001, 2010) and Sprumont (1995) the division problem and the uniform rule plays a prominent role.

We are now ready to state the main results of the paper: four axiomatic characterizations of the constrained uniform rule with four different sets of independent properties. Theorems 1 and 2 use strategy-proofness while Theorem 3 and Corollary 1 use consistency.

Theorem 1 *The constrained uniform rule is the unique rule satisfying strategy-proofness, efficiency, upper bound monotonicity and equal treatment of equals. Moreover, the axioms are independent.*

Proof See Appendix 2. □

Theorem 2 *The constrained uniform rule is the unique rule satisfying strategy-proofness, efficiency, upper bound monotonicity and weak envy-freeness. Moreover, the axioms are independent.*

Proof See Appendix 2. □

Theorem 3 *The constrained uniform rule is the unique rule satisfying consistency, weak individual rational from equal division, upper bound monotonicity and efficiency. Moreover, the axioms are independent.*

Proof See Appendix 2. □

Theorem 3 and Remark 2 (i.e., Proposition 1 in Ehlers (2002b)) imply the following corollary.

Corollary 1 *The constrained uniform rule is the unique rule satisfying consistency, weak individual rational from equal division, upper bound monotonicity and one-sided resource-monotonicity. Moreover, the axioms are independent.*

Table 1 below illustrates the four characterizations of the constrained uniform rule in Theorems 1 (T.1), 2 (T.2) and 3 (T.3), and Corollary 1 (C.1) while Table 2 illustrates the four corresponding related characterizations of the uniform rule in classical division problems. Observe that Tables 1 and 2 have two differences. Table 1 has the property of upper bound monotonicity in the four columns (meaningless in classical division problems) while Table 2 has the original properties of individual rationality from equal division and envy-freeness in classical division problems.

Appendix 1. Proof of Proposition 1

(1) The constrained uniform rule F satisfies efficiency.[9] Fix a problem $(N, u, \succeq, t) \in \mathcal{P}$. We consider two cases.

1. $\sum_{j \in N} p_j < t$. Assume that there exists a vector of feasible shares $x = (x_i)_{i \in N} \in FA(N, u, \succeq, t)$ with the property that $x_i \succeq_i F_i(N, u, \succeq, t)$ for all $i \in N$. We prove that $x = F(N, u, \succeq, t)$. Let $i \in N$ be arbitrary. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\max\{\beta, p_i\}, u_i\}$. We consider three cases.

[9] To prove that F is efficient we do not use Proposition 1 in Ehlers (2002b) since our proof that F satisfies one-sided resource-monotonicity will rely on the fact that F is efficient.

Table 1 With maximal capacity constraints

	(T.1)	(T.2)	(T.3)	(C.1)
Sp	X	X		
E	X	X	X	
C			X	X
Wirfed			X	X
Ete	X			
Wef		X		
OsRm				X
Ubm	X	X	X	X

Table 2 Classical division problems

	Ching	Sprumont	Dagan	Sönmez
Sp	X	X		
E	X	X	X	
C			X	X
Irfed			X	X
Ete	X			
Ef		X		
OsRm				X
Ubm				

(a) $F_i(N, u, \succeq, t) = p_i$. Since $x_i \succeq_i F_i(N, u, \succeq, t)$, it follows that $x_i = p_i$.

(b) $F_i(N, u, \succeq, t) = u_i \neq p_i$. Suppose that $x_i < u_i$. As $\sum_{j \in N} x_j = \sum_{j \in N} F_j(N, u, \succeq, t) = t$, there exists $k \in N$ such that $x_k > F_k(N, u, \succeq, t)$. Again, by the definition F, we have three different possibilities for $F_k(N, u, \succeq, t)$. We obtain a contradiction in each of them.

i. $F_k(N, u, \succeq, t) = u_k$. Then, $x_k > u_k$, which contradicts that $x \in FA(N, u, \succeq, t)$.

ii. $F_k(N, u, \succeq, t) = p_k$. Then, $x_k > p_k$, which contradicts that $x_k \succeq_k F_k(N, u, \succeq, t)$.

iii. $F_k(N, u, \succeq, t) = \beta$ and $p_k < \beta < u_k$. Then, $\beta < x_k$, which contradicts, by single-peakedness, that $x_k \succeq_k F_k(N, u, \succeq, t)$.

Thus, $x_i = u_i$.

(c) $F_i(N, u, \succeq, t) = \beta > p_i$. Since $x_i \succeq_i F_i(N, u, \succeq, t)$, it follows, by single-peakedness, that $x_i \leq \beta$. Suppose that $x_i < \beta$. As $\sum_{j \in N} x_j = \sum_{j \in N} F_j(N, u, \succeq, t) = t$, there exists $k \in N$ such that $x_k > F_k(N, u, \succeq, t)$. Using arguments similar to those used in case (b) we obtain a contradiction.

Thus, $x_i = \beta$.

2. $\sum_{j \in N} p_j \geqslant t$. Assume that there exists a vector of feasible shares $x = (x_i)_{i \in N} \in FA(N, u, \succeq, t)$ with the property that $x_i \succeq_i F_i(N, u, \succeq, t)$ for all $i \in N$. We prove

that $x = F(N, u, \succeq, t)$. Let $i \in N$ be arbitrary. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\beta, p_i\}$. We consider two cases.

(a) $F_i(N, u, \succeq, t) = p_i$. Since $x_i \succeq_i F_i(N, u, \succeq, t)$, it follows that $x_i = p_i$.

(b) $F_i(N, u, \succeq, t) = \beta < p_i$. Since $x_i \succeq_i F_i(N, u, \succeq, t)$, it follows, by single-peakedness, that $x_i \geq \beta$. Suppose that $x_i > \beta$. As $\sum_{j \in N} x_j = \sum_{j \in N} F_j(N, u, \succeq, t) = t$, there exists $k \in N$ such that $x_k < F_k(N, u, \succeq, t)$. We have two possibilities for $F_k(N, u, \succeq, t)$. We obtain a contradiction in each of them.

i. $F_k(N, u, \succeq, t) = p_k$. Then, $x_k < p_k$, which contradicts that $x_k \succeq_k F_k(N, u, \succeq, t)$.

ii. $F_k(N, u, \succeq, t) = \beta$ and $\beta < p_k$. Then, $x_k < \beta$, which contradicts, by single-peakedness, that $x_k \succeq_k F_k(N, u, \succeq, t)$.

Thus, $x_i = \beta$.

(2) The constrained uniform rule F satisfies strategy-proofness. Fix a problem $(N, u, \succeq, t) \in \mathcal{P}$. We consider two cases.

1. $\sum_{j \in N} p_j < t$. Let $i \in N$ and the preferences $\succeq_i'$ on $[0, u_i]$ be arbitrary. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\max\{\beta, p_i\}, u_i\}$. We prove that $F_i(N, u, \succeq, t) \succeq_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$. We consider three cases.

(a) $F_i(N, u, \succeq, t) = u_i$. If $\beta < u_i$ then $F_i(N, u, \succeq, t) = u_i$ and $p_i \leq u_i$ imply that $p_i = u_i$. Hence, $F_i(N, u, \succeq, t) = p_i \succeq_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$. Assume $\beta \geq u_i$. Observe that $u_i + \sum_{j \in N\setminus\{i\}} F_j(N, u, \succeq, t) = t$. Since F is efficient, by (R.2), $p_j \leq F_j(N, u, \succeq, t)$ for all $j \neq i$. Hence, $u_i + \sum_{j \in N\setminus\{i\}} p_j \leq t$. Thus, since $p_i' \leq u_i$, $p_i' + \sum_{j \in N\setminus\{i\}} p_j \leq t$. Therefore, $p_i' \leq u_i \leq \beta = \beta'$, where β' is such that

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) = \min\left\{\max\left\{\beta', p_i'\right\}, u_i\right\}.$$

Thus,

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) = u_i = F_i\left(N, u, \succeq, t\right).$$

(b) $F_i(N, u, \succeq, t) = p_i < u_i$. Then,

$$F_i\left(N, u, \succeq, t\right) = p_i \succeq_i F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right).$$

(c) $F_i(N, u, \succeq, t) = \beta$ and $p_i < \beta < u_i$. We consider five cases.

i. $p_i' \leq p_i$. Then, $\sum_{j \in N\setminus\{i\}} p_j + p_i' < t$ and $\beta' = \beta$. Thus,

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) = F_i\left(N, u, \succeq, t\right).$$

ii. $p_i < p_i' \leq \beta$. Then, $\beta' = \beta$ holds again and

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) = F_i\left(N, u, \succeq, t\right).$$

iii. $p_i < \beta < p_i'$ and $\sum_{j \in N\setminus\{i\}} p_j + p_i' < t$. Then, $\beta' \leq \beta$ and

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) = p_i' > \beta = F_i\left(N, u, \succeq, t\right) > p_i.$$

Hence, by single-peakedness, $F_i(N, u, \succeq, t) \succ_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$.

iv. $p_i < \beta < p_i'$ and $p_i' + \sum_{j \in N\setminus\{i\}} p_j = t$. Thus, by efficiency of F,

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) = p_i' > \beta = F_i\left(N, u, \succeq, t\right) > p_i.$$

Hence, by single-peakedness, $F_i(N, u, \succeq, t) \succ_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$.

v. $p_i < \beta < p_i'$ and $p_i' + \sum_{j \in N\setminus\{i\}} p_j > t$. Since

$$p_i + \sum_{j \in N\setminus\{i\}} p_j < \beta + \sum_{j \in N\setminus\{i\}} F_j\left(N, u, \succeq, t\right) = t$$

and F is efficient, by (R.2), $t - \sum_{j \in N\setminus\{i\}} p_j \geq \beta$. In the other hand,

$$p_i' + \sum_{j \in N\setminus\{i\}} p_j > t = \sum_{j \in N} F_j\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right).$$

Hence, $F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t) = t - \sum_{j \in N\setminus\{i\}} F_j(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$. Since F is efficient, by (R.1), $F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t) \geq t - \sum_{j \in N\setminus\{i\}} p_j$. Thus,

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) \geq t - \sum_{j \in N\setminus\{i\}} p_j \geq \beta = F_i\left(N, u, \succeq, t\right) > p_i.$$

Hence, by single-peakedness, $F_i(N, u, \succeq, t) \succeq_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$.

2. $\sum_{j \in N} p_j \geq t$. Let $i \in N$ and $\succeq_i'$ on $[0, u_i]$ be arbitrary. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\beta, p_i\}$. We prove that $F_i(N, u, \succeq, t) \succeq_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$. We consider four cases.
   (a) $F_i(N, u, \succeq, t) = p_i$. Then, $F_i(N, u, \succeq, t) = p_i \succeq_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$.
   (b) $F_i(N, u, \succeq, t) = \beta < p_i$ and $p_i' \geq p_i$. Then, $\sum_{j \in N\setminus\{i\}} p_j + p_i' \geq t$, and $\beta = \beta'$, where now β' is such that $F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t) = \min\{\beta', p_i'\}$. Thus,

$$F_i\left(N, u, \succeq, t\right) = F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right).$$

   (c) $F_i(N, u, \succeq, t) = \beta < p_i$, $p_i' < p_i$, and $\sum_{j \in N\setminus\{i\}} p_j + p_i' \geq t$. Then,

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) \leq \beta = F_i\left(N, u, \succeq, t\right) < p_i.$$

   Hence, by single-peakedness, $F_i(N, u, \succeq, t) \succeq_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$.

(d) $F_i(N, u, \succeq, t) = \beta < p_i, p_i' < p_i$, and $\sum_{j \in N\setminus\{i\}} p_j + p_i' < t$. Since $\sum_{j \in N\setminus\{i\}} p_j + p_i > \sum_{j \in N\setminus\{i\}} F_j(N, u, \succeq, t) + \beta = t$ and F is efficient, by (R.1), $t - \sum_{j \in N\setminus\{i\}} p_j \leq \beta$. In the other hand, since $\sum_{j \in N\setminus\{i\}} p_j + p_i' < \sum_{j \in N} F_j(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t) = t$ and F is efficient, by (R.2), $t - \sum_{j \in N\setminus\{i\}} p_j \geq F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$. Thus,

$$F_i\left(N, u, \left(\succeq_i', \succeq_{N\setminus\{i\}}\right), t\right) \leq t - \sum_{j \in N\setminus\{i\}} p_j \leq \beta = F_i\left(N, u, \succeq, t\right) < p_i.$$

Hence, by single-peakedness, $F_i(N, u, \succeq, t) \succeq_i F_i(N, u, (\succeq_i', \succeq_{N\setminus\{i\}}), t)$.

(3) The fact that the constrained uniform rule F satisfies equal treatment of equals follows immediately from its definition.
(4) The constrained uniform rule F satisfies upper bound monotonicity. Let $(N, u, \succeq, t)$ and $(N, u', \succeq', t)$ be two problems in $\mathcal{P}$ with the property that there is an agent $k \in N$ such that $u_k < u_k' \leq t$ and $\succeq_k$ coincides with $\succeq_k'$ on $[0, u_k]$, and for each $i \in N\setminus\{k\}$, $u_i = u_i'$ and $\succeq_i = \succeq_i'$. We consider two cases.

1. $\sum_{j \in N} p_j' < t$. By the definition of F, $F_i(N, u', \succeq', t) = \min\{\max\{\beta', p_i'\}, u_i'\}$ for all $i \in N$. Since $p_i = p_i'$ for all $i \in N\setminus\{k\}$ and $p_k \leq p_k'$, $\sum_{j \in N} p_j < t$. Then, $F_i(N, u, \succeq, t) = \min\{\max\{\beta, p_i\}, u_i\}$ for all $i \in N$ where $\beta \geq \beta'$. Hence, for all $i \in N\setminus\{k\}$, $\max\{\beta', p_i\} \leq \max\{\beta, p_i\}$. Thus, since $F_i(N, u', \succeq', t) \leq u_i' = u_i$ for all $i \in N\setminus\{k\}$,

$$F_i\left(N, u, \succeq, t\right) \geq \min\left\{F_i\left(N, u', \succeq', t\right), u_i\right\} \quad \text{for all } i \in N\setminus\{k\}.$$

We show that $F_k(N, u, \succeq, t) \geq \min\{F_k(N, u', \succeq', t), u_k\}$ by considering four cases.

(a) $F_k(N, u', \succeq', t) \leq u_k$. By the definition of F it is easy to see that $F(N, u, \succeq, t) = F(N, u', \succeq', t)$.

(b) $F_k(N, u', \succeq', t) = p_k' > u_k$. Then, since $\succeq_k$ is single-peaked on $[0, u_k]$ and coincides with $\succeq_k'$ on $[0, u_k]$, $p_k = u_k \leq p_k'$. Hence,

$$F_k\left(N, u, \succeq, t\right) = \min\left\{\max\left\{\beta, u_k\right\}, u_k\right\} = u_k.$$

Thus, $F_k(N, u, \succeq, t) = \min\{F_k(N, u', \succeq', t), u_k\}$.

(c) $F_k(N, u', \succeq', t) = \beta' > u_k$. Then, $\max\{\beta', p_k'\} = \beta' \leq u_k'$. Since $\beta \geq \beta'$ and $p_k' \geq p_k$, $\max\{\beta, p_k\} = \beta$. Hence,

$$F_k\left(N, u, \succeq, t\right) = \min\left\{\beta, u_k\right\} = u_k \geq \min\left\{F_k\left(N, u', \succeq', t\right), u_k\right\}.$$

(d) $F_k(N, u', \succeq', t) = u_k' > u_k$. Then, $u_k' \leq \beta' = \max\{\beta', p_k'\}$. Since $\beta \geq \beta'$, $u_k' > u_k$, and $p_k' \geq p_k$,

$$F_k\left(N, u, \succeq, t\right) = \min\left\{\beta, u_k\right\} = u_k \geq \min\left\{F_k\left(N, u', \succeq', t\right), u_k\right\}.$$

2. $\sum_{i\in N} p'_i \geq t$. By the definition of F, $F_i(N, u', \succeq', t) = \min\{\beta', p'_i\}$ for all $i \in N$. Assume first that $\sum_{i\in N} p_i < t$. By efficiency of F, (R.2) implies that $p_i \leq F_i(N, u, \succeq, t)$ for all $i \in N$. Consider $i \in N\backslash\{k\}$. Then, by (R.1) and $p_i = p'_i$, $F_i(N, u', \succeq', t) \leq p'_i = p_i \leq F_i(N, u, \succeq, t) \leq u_i$. Thus, $F_i(N, u, \succeq, t) \geq \min\{F_i(N, u', \succeq', t), u_i\}$ for all $i \in N\backslash\{k\}$. Assume now that $\sum_{i\in N} p_i \geq t$. Then, since $p_k \leq p'_k$, $\beta \geq \beta'$. Therefore, $u_i \geq F_i(N, u, \succeq, t) = \min\{\beta, p_i\} \geq \min\{\beta', p'_i\} = F_i(N, u', \succeq', t)$ for all $i \in N\backslash\{k\}$. Thus,

$$F_i(N, u, \succeq, t) \geq \min\left\{F_i\left(N, u', \succeq', t\right), u_i\right\} \quad \text{for each } i \in N\backslash\{k\}.$$

To show that $F_k(N, u, \succeq, t) \geq \min\{F_k(N, u', \succeq', t), u_k\}$ holds we consider three cases.

(a) $F_k(N, u', \succeq', t) \leq u_k$. By the definition of F it is easy to see that $F(N, u, \succeq, t) = F(N, u', \succeq', t)$.

(b) $F_k(N, u', \succeq', t) > u_k$ and $\sum_{i\in N} p_i < t$. Since F is efficient, by (R.1), $F_k(N, u', \succeq', t) \leq p'_k$. Then, $p'_k > u_k$ and hence, since $\succeq'_k$ coincides with $\succeq_k$ on $[0, u_k]$, $p_k = u_k$. Again, since F is efficient, by (R.2), $F_k(N, u, \succeq, t) \geq p_k$. Thus, $F_k(N, u, \succeq, t) = u_k$ and $F_k(N, u, \succeq, t) = \min\{F_k(N, u', \succeq', t), u_k\}$.

(c) $F_k(N, u', \succeq', t) > u_k$ and $\sum_{i\in N} p_i \geq t$. Since $p_k < p'_k$, $\beta \geq \beta'$. Thus, $F_k(N, u', \succeq', t) \leq \beta' \leq \beta$. Hence,

$$F_k(N, u, \succeq, t) \min\{\beta, u_k\} = u_k = \min\{F_k\left(N, u', \succeq', t\right), u_k\}.$$

(5) The fact that the constrained uniform rule F satisfies consistency follows from the proof of Lemma 4 in Bergantiños et al. (2011).

(6) The constrained uniform rule F satisfies weak envy-freeness. Let $(N, u, \succeq, t) \in \mathcal{P}$, $j \in N$, and $k \in N\backslash\{j\}$ be such that $F_k(N, u, \succeq, t) \succ_j F_j(N, u, \succeq, t)$. We consider two cases.

1. $\sum_{i\in N} p_i < t$. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\max\{\beta, p_i\}, u_i\}$ for all $i \in N$. Since $F_k(N, u, \succeq, t) \succ_j F_j(N, u, \succeq, t)$, $F_j(N, u, \succeq, t) \neq p_j$. Thus, $\max\{\beta, p_j\} = \beta > p_j$. Hence, $F_j(N, u, \succeq, t) = \min\{\beta, u_j\} \leq \beta$. By single-peakedness, $F_k(N, u, \succeq, t) < F_j(N, u, \succeq, t)$. By (R.2), $p_k \leq F_k(N, u, \succeq, t)$. Hence, $\max\{\beta, p_k\} = \beta$ and $F_k(N, u, \succeq, t) = \min\{\beta, u_k\} = u_k < F_j(N, u, \succeq, t) \leq u_j$. Therefore, $F_j(N, u, \succeq, t) \notin [0, u_k]$. Thus, $(x_i)_{i\in N} \notin FA(N, u, \succeq, t)$ where $x_j = F_k(N, u, \succeq, t)$, $x_k = F_j(N, u, \succeq, t)$, and $x_i = F_i(N, u, \succeq, t)$ for $i \in N\backslash\{k, j\}$.
2. $\sum_{i\in N} p_i \geq t$. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\beta, p_i\}$ for all $i \in N$. Since $F_k(N, u, \succeq, t) \succ_j F_j(N, u, \succeq, t)$, $F_j(N, u, \succeq, t) \neq p_j$. Thus, $F_j(N, u, \succeq, t) = \beta < p_j$. By single-peakedness, $F_k(N, u, \succeq, t) > \beta$, a contradiction with $F_k(N, u, \succeq, t) = \min\{\beta, p_k\}$.

(7) The constrained uniform rule F satisfies weak individual rationality from equal division. Let $(N, u, \succeq, t) \in \mathcal{P}$ be such that $(\frac{t}{n}, \ldots, \frac{t}{n}) \in FA(N, u, \succeq, t)$. Thus, $\frac{t}{n} \leq u_i$ for all $i \in N$. We consider two cases.

1. $\sum_{i\in N} p_i < t$. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\max\{\beta, p_i\}, u_i\}$ for all $i \in N$. Thus, $\beta \leq \frac{t}{n}$. Let $i \in N$. We consider two cases.
   (a) $p_i \geq \beta$. Then, $F_i(N, u, \succeq, t) = p_i \succeq_i \frac{t}{n}$.
   (b) $p_i < \beta$. Then, $F_i(N, u, \succeq, t) = \min\{\beta, u_i\} \leq \beta \leq \frac{t}{n}$. By (R.2), $p_i \leq F_i(N, u, \succeq, t)$. By single-peakedness, $F_i(N, u, \succeq, t) \succeq_i \frac{t}{n}$.
2. $\sum_{i\in N} p_i \geq t$. By the definition of F, $F_i(N, u, \succeq, t) = \min\{\beta, p_i\}$ for all $i \in N$. Thus, $\beta \geq \frac{t}{n}$. We consider two cases.
   (a) $p_i \geq \beta$. Then, by single-peakedness, $F_i(N, u, \succeq, t) = \beta \succeq_i \frac{t}{n}$.
   (b) $p_i < \beta$. Then, $F_i(N, u, \succeq, t) = p_i \succeq_i \frac{t}{n}$.

(8) The constrained uniform rule F satisfies one-sided resource-monotonicity. Let $(N, u, \succeq, t)$ and $(N, u, \succeq, t')$ be two problems in $\mathcal{P}$. We consider two cases.

1. Assume that $t \geq t' \geq \sum_{i\in N} p_i$. Then, by the definition of F, for all $i \in N$,

$$F_i(N, u, \succeq, t) = \min\left\{\max\left\{\beta, p_i\right\}, u_i\right\} \quad \text{and}$$
$$F_i\left(N, u, \succeq, t'\right) = \min\left\{\max\left\{\beta', p_i\right\}, u_i\right\}.$$

   Hence, $\beta' \leq \beta$ and since F is efficient, by (R.2), for all $i \in N$,

$$p_i \leq F_i\left(N, u, \succeq, t'\right) \leq F_i(N, u, \succeq, t).$$

   By single-peakedness, $F_i(N, u, \succeq, t') \succeq_i F_i(N, u, \succeq, t)$.
2. It is similar to the previous case an we omit it.

This ends the proof of Proposition 1.

Appendix 2. Proofs of Theorems 1, 2, and 3

A2.1. Proof of the characterization in Theorem 1

By Proposition 1, the constrained uniform rule F satisfies strategy-proofness, efficiency, upper bound monotonicity and equal treatment of equals.

Let f be a rule satisfying strategy-proofness, efficiency, upper bound monotonicity and equal treatment of equals. Thus, f also satisfies strategy-proofness, efficiency, and equal treatment of equals in $\mathcal{P}^{u=t}$. Ching (1994) shows that the uniform rule U is the unique rule satisfying strategy-proofness, efficiency, and equal treatment of equals in $\mathcal{P}^{u=t}$. Thus, for all $(N, u, \succeq, t) \in \mathcal{P}^{u=t}$, $f(N, u, \succeq, t) = U(N, u, \succeq, t)$ and by definition of F, $F(N, u, \succeq, t) = U(N, u, \succeq, t)$. Hence, $f = F$ in $\mathcal{P}^{u=t}$.

Let $(N, u, \succeq, t) \in \mathcal{P}$ be an arbitrary problem. We want to show that $f(N, u, \succeq, t) = F(N, u, \succeq, t)$. Consider any problem $(N, u^t, \succeq^t, t) \in \mathcal{P}^{u=t}$ where for each $i \in N$, $u_i^t = t$, $\succeq_i^t$ coincides with $\succeq_i$ on $[0, u_i]$ and $p_i^t = p_i$. Since $f = F$ in $\mathcal{P}^{u=t}$,

$$f\left(N, u^t, \succeq^t, t\right) = F\left(N, u^t, \succeq^t, t\right). \tag{1}$$

We consider two cases.

1. $\sum_{i\in N} p_i < t$. Assume, without loss of generality, that $u_1 \leq u_2 \leq \cdots \leq u_n$. Consider any problem $(N, u^{t,1}, \succeq^{t,1}, t) \in \mathcal{P}$ where $u_1^{t,1} = u_1$ and $\succeq_1^{t,1} = \succeq_1$ and for each $i \in N\backslash\{1\}$, $u_i^{t,1} = u_i^t = t$ and $\succeq_i^{t,1} = \succeq_i^t$.
We first prove that $f(N, u^{t,1}, \succeq^{t,1}, t) = F(N, u^{t,1}, \succeq^{t,1}, t)$. We consider two cases.
   (a) $F_1(N, u^t, \succeq^t, t) \leq u_1$. By (1), $f_1(N, u^t, \succeq^t, t) = F_1(N, u^t, \succeq^t, t) \leq u_1 = u_1^{t,1}$. Since f and F satisfy upper bound monotonicity, for all $i \in N$,

$$f_i\left(N, u^{t,1}, \succeq^{t,1}, t\right) \geq \min\left\{f_i\left(N, u^t, \succeq^t, t\right), u_i^{t,1}\right\} = f_i\left(N, u^t, \succeq^t, t\right) \quad \text{and}$$
$$F_i\left(N, u^{t,1}, \succeq^{t,1}, t\right) \geq \min\left\{F_i\left(N, u^t, \succeq^t, t\right), u_i^{t,1}\right\} = F_i\left(N, u^t, \succeq^t, t\right).$$

   Then,

$$f\left(N, u^{t,1}, \succeq^{t,1}, t\right) = f\left(N, u^t, \succeq^t, t\right) = F\left(N, u^t, \succeq^t, t\right) = F\left(N, u^{t,1}, \succeq^{t,1}, t\right).$$

   (b) $F_1(N, u^t, \succeq^t, t) > u_1$. By (1), $f_1(N, u^t, \succeq^t, t) > u_1$. Since f and F satisfy upper bound monotonicity,

$$f_1\left(N, u^{t,1}, \succeq^{t,1}, t\right) = F_1\left(N, u^{t,1}, \succeq^{t,1}, t\right) = u_1. \tag{2}$$

   Notice that (2) holds independently of the preferences $(\succeq_i')_{i\in N\backslash\{1\}}$ of the other agents, as long as $\sum_{i\in N\backslash\{1\}} p_i' + p_1 < t$. This statement will be used implicitly in the rest of the proof. Moreover, by the definition of F, for each $i \in N\backslash\{1\}$,

$$F_i\left(N, u^{t,1}, \succeq^{t,1}, t\right) = \max\{\beta, p_i\}. \tag{3}$$

   Suppose that $f(N, u^{t,1}, \succeq^{t,1}, t) \neq F(N, u^{t,1}, \succeq^{t,1}, t)$. Then, there exists $i_1 \in N\backslash\{1\}$ such that

$$f_{i_1}\left(N, u^{t,1}, \succeq^{t,1}, t\right) > \max\{\beta, p_{i_1}\}. \tag{4}$$

   Let $k \in N\backslash\{1\}$ be such that $p_k \leq p_i$ for all $i \in N\backslash\{1\}$. We consider three cases and find a contradiction in each of them.
      i. For all $i \in N\backslash\{1\}$, $\max\{\beta, p_i\} = p_i > \beta$. By (2), (3), and feasibility of F, $\sum_{i\in N} F_i(N, u^{t,1}, \succeq^{t,1}, t) = u_1 + \sum_{i\in N\backslash\{1\}} p_i = t$. Hence, by (2) and efficiency of f, for all $i \in N\backslash\{1\}$,

$$f_i\left(N, u^{t,1}, \succeq^{t,1}, t\right) = p_i.$$

      Thus, by hypothesis, $f_{i_1}(N, u^{t,1}, \succeq^{t,1}, t) = \max\{\beta, p_{i_1}\}$, a contradiction with (4).

ii. There exists $j' \in N\backslash\{1\}$ such that $\max\{\beta, p_{j'}\} = \beta$ and $k \neq i_1$ (we will consider the remaining case where $k = i_1$ later in case iii). Hence, $\max\{\beta, p_k\} = \beta$. Note that $k \neq 1, k \neq i_1 \neq 1$, and $\succeq_k^{t,1}$ and $\succeq_{i_1}^{t,1}$ are preferences on $[0, t]$. Let $\succeq'_{i_1} = \succeq_k^{t,1}$. In order to simplify the notation we omit $N, u^{t,1}$, and t in the definition of a problem. Since $p_k \leq p_{i_1}$ and f is efficient, by (R.2), $p_k \leq p_{i_1} \leq f_{i_1}(\succeq^{t,1})$. If $p_{i_1} \leq f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) < f_{i_1}(\succeq^{t,1})$ then, by single-peakedness, i_1 manipulates f at $\succeq^{t,1}$ via $\succeq'_{i_1}$. Thus,

$$f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) \geq f_{i_1}\left(\succeq^{t,1}\right) > \max\left\{\beta, p_{i_1}\right\} \tag{5}$$

or

$$p_k = p'_{i_1} \leq f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) < p_{i_1}.$$

Since f satisfies equal treatment of equals,

$$f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) = f_k(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}). \tag{6}$$

By (2), and since $\sum_{i\in N\backslash\{1,i_1\}} p_i + p_1 + p_{i_1} < t$,

$$f_1(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) = F_1(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) = u_1 \tag{7}$$

Assume that $N = \{1, k, i_1\}$. If $f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) > \max\left\{\beta, p_{i_1}\right\}$ then,

$$\begin{aligned} t &= \textstyle\sum_{i\in N} f_i(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) \\ &> u_1 + \max\left\{\beta, p_{i_1}\right\} + \max\left\{\beta, p_{i_1}\right\} && \text{by (5) and (6)} \\ &\geq u_1 + \max\left\{\beta, p_k\right\} + \max\left\{\beta, p_{i_1}\right\} && \text{by definition of } k,\ p_k \leq p_{i_1} \\ &= \textstyle\sum_{i\in N} F_i\left(\succeq^{t,1}\right), && \text{by (2) and (3)} \end{aligned}$$

which contradicts that F is a rule. Assume now that $p'_{i_1} \leq f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) < p_{i_1}$. If $f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) \succ_{i_1}^{t,1} f_{i_1}(\succeq^{t,1})$ then i_1 manipulates f at $\succeq^{t,1}$ via $\succeq'_{i_1}$. Hence, $f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) \preceq_{i_1}^{t,1} f_{i_1}(\succeq^{t,1})$. Consider now any preference $\widehat{\succeq}_{i_1}$ on $[0, t]$ such that $\widehat{p}_{i_1} = p_{i_1}$ and $f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) \widehat{\succ}_{i_1}^{t,1} f_{i_1}(\succeq^{t,1})$. Since f is efficient, (R.2) implies that $p_{i_1} = \widehat{p}_{i_1} \leq f_{i_1}(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}})$. Assume $f_{i_1}(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}}) < f_{i_1}(\succeq^{t,1})$, then, by single-peakedness, i_1 manipulates f at $\succeq^{t,1}$ via $\widehat{\succeq}_{i_1}$. If $f_{i_1}(\succeq^{t,1}) < f_{i_1}(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}})$, then i_1 manipulates f at $(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j\in N\backslash\{i_1\}})$ via $\succeq_{i_1}^{t,1}$. Hence,

$f_{i_1}(\succeq^{t,1}) = f_{i_1}(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}})$. Moreover, by (2), $f_1(\succeq^{t,1}) = f_1(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) = u_1$. Thus, $f(\succeq^{t,1}) = f(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}})$. Hence, i_1 manipulates f at $(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}})$ via $\succeq'_{i_1}$ because $f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) \succ_{i_1}^{t,1} f_{i_1}(\widehat{\succeq}_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) = f_{i_1}(\succeq^{t,1})$.
Thus, $N \neq \{1, k, i_1\}$ and $f_{i_1}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) = f_{i_1}(\succeq^{t,1})$. By (3), (4), (7), and since

$$f(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) \in FA(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}),$$

there exists $i_2 \in N\backslash\{1, k, i_1\}$ such that

$$f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) < \max\left\{\beta, p_{i_2}\right\}.$$

Since f satisfies efficiency, (R.2) implies that

$$p_{i_2} \leq f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) < \beta. \tag{8}$$

Let $\succeq'_{i_2} = \succeq_k^{t,1}$. Again, by efficiency of f and the definition of k,

$$f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) \geq p_{i_2} \geq p_k \quad \text{and}$$
$$f_{i_2}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}) \geq p_k.$$

Assume that $N = \{1, k, i_1, i_2\}$. We want to show that $f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) = f_{i_2}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}})$. Assume otherwise. If $f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}}) < f_{i_2}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}})$, i_2 manipulates f at profile $((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}})$ via $\succeq_{i_2}^{t,1}$, a contradiction with strategy-proofness of f. Assume now that $f_{i_2}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}) < f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}})$. First, if $p_{i_2} \leq f_{i_2}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}})$ then i_2 manipulates f at profile $(\succeq'_{i_1}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1\}})$ via $\succeq'_{i_2}$, a contradiction with strategy-proofness of f. Assume now that $p_k = p'_{i_2} \leq f_{i_2}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}) < p_{i_2}$. Since f satisfies equal treatment of equals,

$$\begin{aligned} f_{i_2}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}) &= f_{i_1}((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}) \\ &= f_k((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}). \end{aligned}$$

By (2) (and the comment after (2)),

$$\begin{aligned} &f_1((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}) \\ &\quad = F_1((\succeq'_j)_{j \in \{i_1, i_2\}}, (\succeq_j^{t,1})_{j \in N\backslash\{i_1, i_2\}}) = u_1. \end{aligned}$$

Then,

$$\begin{aligned} t &= \sum_{i\in N} f_i((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) \\ &= u_1 + 3f_{i_2}((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) \\ &< u_1 + 3\beta \\ &\le u_1 + \max\{\beta, p_k\} + \max\left\{\beta, p_{i_1}\right\} + \max\left\{\beta, p_{i_2}\right\} \\ &= \sum_{i\in N} F_i\left(\succeq^{t,1}\right), \end{aligned}$$

which contradicts that F is a rule, where the strict inequality above follows from the hypothesis that $f_{i_2}((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) < f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1\}})$ and (8), and the last equality follows from (2) and (3). Thus,

$$f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1\}}) = f_{i_2}((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}). \quad (9)$$

Then, by equal treatment of equals and efficiency of f,

$$\begin{aligned} f_{i_1}((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) &= f_{i_2}((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) \\ &= f_k((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}). \end{aligned}$$

Since $N = \{1, k, i_1, i_2\}$,

$$\begin{aligned} t &= \textstyle\sum_{i\in N} f_i((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) \\ &= u_1 + 3f_{i_2}((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) && \text{by (2) and equal treatment} \\ &= u_1 + 3f_{i_2}(\succeq'_{i_1}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1\}}) && \text{by (9)} \\ &< u_1 + 3\beta && \text{by (8)} \\ &\le u_1 + \max\{\beta, p_k\} + \max\left\{\beta, p_{i_1}\right\} + \max\left\{\beta, p_{i_2}\right\} \\ &= \textstyle\sum_{i\in N} F_i\left(\succeq^{t,1}\right), && \text{by (2) and (3)} \end{aligned}$$

which contradicts that F is a rule. Thus, $N \neq \{1, k, i_1, i_2\}$. Now, there exists $i_3 \in N\setminus\{1, k, i_1, i_2\}$ such that

$$f_{i_3}((\succeq'_j)_{j\in\{i_1,i_2\}}, (\succeq_j^{t,1})_{j\in N\setminus\{i_1,i_2\}}) > \max\{\beta, p_{i_3}\}.$$

We can now proceed as in the case of i_1 (see condition (4)), and since N is finite we obtain a contradiction in a finite number of steps.

iii. There exists $j' \in N\setminus\{1\}$ such that $\max\{\beta, p_{j'}\} = \beta$ and $k = i_1$. Thus, by (4), there exists $i_2 \in N\setminus\{1, i_1\}$ such that $f_{i_2}(\succeq^{t,1}) < \max\{\beta, p_{i_2}\}$. Let

$\succeq'_{i_2} = \succeq^{t,1}_{i_1}$. Using arguments similar to those used in case ii above with i_2 we can prove that

$$\begin{aligned} f_{i_1}(\succeq'_{i_2}, (\succeq^{t,1}_j)_{j\in N\setminus\{i_2\}}) &= f_{i_2}(\succeq'_{i_2}, (\succeq^{t,1}_j)_{j\in N\setminus\{i_2\}}) \\ &= f_{i_2}\left(\succeq^{t,1}\right) \\ &< \max\left\{\beta, p_{i_2}\right\} \end{aligned}$$

and $N \neq \{1, i_1, i_2\}$. Thus, there exists $i_3 \in N\setminus\{1, i_1, i_2\}$ such that

$$f_{i_3}(\succeq'_{i_2}, (\succeq^{t,1}_j)_{j\in N\setminus\{i_2\}}) > \max\left\{\beta, p_{i_3}\right\}.$$

Since N is finite, using arguments similar to those used in case ii, we obtain a contradiction.

Thus, we have proved that $f(N, u^{t,1}, \succeq^{t,1}, t) = F(N, u^{t,1}, \succeq^{t,1}, t)$ holds. Consider the problem $(N, u^{t,2}, \succeq^{t,2}, t) \in \mathcal{P}$ where $u^{t,2}_2 = u_2$ and $\succeq^{t,2}_2 = \succeq_2$, and $u^{t,2}_i = u^{t,1}_i$ and $\succeq^{t,2}_i = \succeq^{t,1}_i$ for each $i \in N\setminus\{2\}$.

We now prove that $f(N, u^{t,2}, \succeq^{t,2}, t) = F(N, u^{t,2}, \succeq^{t,2}, t)$ by considering two cases.

(a) $F_2(N, u^{t,1}, \succeq^{t,1}, t) \leq u_2$. Since f and F satisfy upper bound monotonicity, using arguments similar to those used in case (a) above, we can deduce that $f(N, u^{t,2}, \succeq^{t,2}, t) = F(N, u^{t,2}, \succeq^{t,2}, t)$.

(b) $F_2(N, u^{t,1}, \succeq^{t,1}, t) > u_2$. Using arguments similar to those used in case (b) above we obtain that $f(N, u^{t,2}, \succeq^{t,2}, t) = F(N, u^{t,2}, \succeq^{t,2}, t)$.

In general, for each $j = 3, \ldots, n$ we consider the problem $(N, u^{t,j}, \succeq^{t,j}, t) \in \mathcal{P}$ where $u^{t,j}_j = u_j$ and $\succeq^{t,j}_j = \succeq_j$, and $u^{t,j}_i = u^{t,j-1}_i$ and $\succeq^{t,j}_i = \succeq^{t,j-1}_i$ for each $i \in N\setminus\{j\}$. Using arguments similar to those used previously we can prove that $f(N, u^{t,j}, \succeq^{t,j}, t) = F(N, u^{t,j}, \succeq^{t,j}, t)$.

Since $(N, u^{t,n}, \succeq^{t,n}, t) = (N, u, \succeq, t)$, we have that

$$f(N, u, \succeq, t) = F(N, u, \succeq, t).$$

2. $\sum_{i\in N} p_i \geq t$. Let $(N, u^{t,1}, \succeq^{t,1}, t) \in \mathcal{P}$ be defined as above. Using arguments similar to those used in 1.a we can prove that $f(N, u^{t,1}, \succeq^{t,1}, t) = F(N, u^{t,1}, \succeq^{t,1}, t)$. Let $j \in \{2, \ldots, n\}$. Applying similar arguments to the problems $(N, u^{t,j-1}, \succeq^{t,j-1}, t)$ and $(N, u^{t,j}, \succeq^{t,j}, t)$ we can prove that $f(N, u^{t,j}, \succeq^{t,j}, t) = F(N, u^{t,j}, \succeq^{t,j}, t)$. Since $(N, u^{t,n}, \succeq^{t,n}, t) = (N, u, \succeq, t)$, we have that $f(N, u, \succeq, t) = F(N, u, \succeq, t)$.

This ends the proof of Theorem 1.

A2.2. Proof of the characterization in Theorem 2

By Proposition 1 we know that the constrained uniform rule F satisfies strategy-proofness, efficiency, upper bound monotonicity and weak envy-freeness.

By Theorem 1 it is sufficient to prove that if f satisfies efficiency and weak envy-freeness then f satisfies equal treatment of equals. Suppose that f is efficient and it does not satisfy equal treatment of equals. There exists $(N, u, \succeq, t) \in \mathcal{P}$ and $i, j \in N$ such that $u_i = u_j, \succeq_i = \succeq_j$, and $f_i(N, u, \succeq, t) \neq f_j(N, u, \succeq, t)$. Since f satisfies efficiency, by (R.1) or (R.2), $f_i(N, u, \succeq, t)$ and $f_j(N, u, \succeq, t)$ are at the same side of the common peak $p_i = p_j$. Hence, by single-peakedness, either $f_i(N, u, \succeq, t) \succ_j f_j(N, u, \succeq, t)$ or $f_j(N, u, \succeq, t) \succ_i f_i(N, u, \succeq, t)$. Suppose that $f_i(N, u, \succeq, t) \succ_j f_j(N, u, \succeq, t)$ (the other case is similar and we omit it). Consider the allocation $(x_k)_{k \in N}$ where $x_i = f_j(N, u, \succeq, t), x_j = f_i(N, u, \succeq, t)$, and $x_k = f_k(N, u, \succeq, t)$ for all $k \in N \backslash \{i, j\}$. Since $u_i = u_j$ and $f(N, u, \succeq, t) \in FA(N, u, \succeq, t)$ we deduce that $x \in FA(N, u, \succeq, t)$, which means that f does not satisfy weak envy-freeness.

□

A2.3. Proof of the characterization in Theorem 3

By Proposition 1, the constrained uniform rule F satisfies consistency, weak individual rationality from equal division, upper bound monotonicity and efficiency.

Let f be a rule satisfying consistency, weak individual rationality from equal division, upper bound monotonicity and efficiency. Thus, f also satisfies consistency, weak individual rationality from equal division and efficiency in $\mathcal{P}^{u=t}$. Moreover, weak individual rationality from equal division and individual rationality from equal division coincide in $\mathcal{P}^{u=t}$. Thus, f satisfies consistency, individual rationality from equal division and efficiency in $\mathcal{P}^{u=t}$. Dagan (1996) shows that the uniform rule U is the unique rule satisfying consistency, individual rationality from equal division and efficiency in $\mathcal{P}^{u=t}$. Thus, for all $(N, u, \succeq, t) \in \mathcal{P}^{u=t}$, $f(N, u, \succeq, t) = U(N, u, \succeq, t)$ and by definition of F, $F(N, u, \succeq, t) = U(N, u, \succeq, t)$. Hence, $f = F$ in $\mathcal{P}^{u=t}$.

Let $(N, u, \succeq, t) \in \mathcal{P}$ be an arbitrary problem. We want to show that $f(N, u, \succeq, t) = F(N, u, \succeq, t)$. Consider any problem $\left(N, u^t, \succeq^t, t\right) \in \mathcal{P}^{u=t}$ where for each $i \in N$, $u_i^t = t$, $\succeq_i^t$ coincides with $\succeq_i$ on $[0, u_i]$, and $p_i^t = p_i$. Since $f = F$ in $\mathcal{P}^{u=t}$,

$$f\left(N, u^t, \succeq^t, t\right) = F\left(N, u^t, \succeq^t, t\right). \tag{10}$$

We consider two cases.

1. $\sum_{i \in N} p_i < t$. Assume, without loss of generality, that $u_1 \leq u_2 \leq \cdots \leq u_n$. We proceed by induction on the number of agents n. If $n = 1$, then $f_1(N, u, \succeq, t) = F_1(N, u, \succeq, t) = t$. Assume that $f(N, u, \succeq, t) = F(N, u, \succeq, t)$ holds when $n \leq m$. We prove that it also holds when $n = m + 1$.
   We first prove that $f_1(N, u, \succeq, t) = F_1(N, u, \succeq, t)$. Consider the problems $\left(N, u^{t,j}, \succeq^{t,j}, t\right) \in \mathcal{P}$, $j = 1, \ldots, n$ defined as in the proof of Theorem 1. We consider two cases.
   (a) $F_1\left(N, u^t, \succeq^t, t\right) \leq u_1$. Then, because of the definition of F, it is easy to deduce that $F_i\left(N, u^t, \succeq^t, t\right) \leq u_i$ for all $i \in N$. Since $F_1\left(N, u^t, \succeq^t, t\right) \leq u_1 = u_1^{t,1}$, by upper bound monotonicity we deduce $f\left(N, u^{t,1}, \succeq^{t,1}, t\right) = F\left(N, u^{t,1}, \succeq^{t,1}, t\right)$.

Since $F_i\left(N,u^t,\succeq^t,t\right) \leq u_i$ for all $i \in N$ and $f\left(N,u^{t,1},\succeq^{t,1},t\right) = F\left(N,u^{t,1},\succeq^{t,1},t\right)$, $F_2\left(N,u^{t,1},\succeq^{t,1},t\right) \leq u_2 = u_2^{t,2}$. By upper bound monotonicity we deduce $f\left(N,u^{t,2},\succeq^{t,2},t\right) = F\left(N,u^{t,2},\succeq^{t,2},t\right)$.

Using arguments similar to those used previously we can prove that for $j = 3,\ldots,n$,

$$f\left(N,u^{t,j},\succeq^{t,j},t\right) = F\left(N,u^{t,j},\succeq^{t,j},t\right).$$

Since $\left(N,u^{t,n},\succeq^{t,n},t\right) = (N,u,\succeq,t)$, we have that $f(N,u,\succeq,t) = F(N,u,\succeq,t)$.

(b) Assume now that $F_1\left(N,u^t,\succeq^t,t\right) > u_1$. By upper bound monotonicity

$$f_1\left(N,u^{t,1},\succeq^{t,1},t\right) = F_1\left(N,u^{t,1},\succeq^{t,1},t\right) = u_1. \tag{11}$$

We only prove that $F_1\left(N,u^{t,2},\succeq^{t,2},t\right) = u_1$ since to prove that $f_1\left(N,u^{t,2},\succeq^{t,2},t\right) = u_1$ is similar and we omit it. We consider two cases.

i. $F_2\left(N,u^{t,1},\succeq^{t,1},t\right) \leq u_2$. By upper bound monotonicity, $F\left(N,u^{t,2},\succeq^{t,2},t\right) = F\left(N,u^{t,1},\succeq^{t,1},t\right)$. Hence, by (11), $F_1\left(N,u^{t,2},\succeq^{t,2},t\right) = u_1$.

ii. $F_2\left(N,u^{t,1},\succeq^{t,1},t\right) > u_2$. By upper bound monotonicity and (11),

$$F_1\left(N,u^{t,2},\succeq^{t,2},t\right) \geq F_1\left(N,u^{t,1},\succeq^{t,1},t\right) = u_1.$$

Since $F\left(N,u^{t,2},\succeq^{t,2},t\right) \in FA\left(N,u^{t,2},\succeq^{t,2},t\right)$ and $u_1^{t,2} = u_1$, $F_1\left(N,u^{t,2},\succeq^{t,2},t\right) = u_1$ holds.

Using arguments similar to those used previously we can prove that for $j = 3,\ldots,n$,

$$f_1\left(N,u^{t,j},\succeq^{t,j},t\right) = F_1\left(N,u^{t,j},\succeq^{t,j},t\right) = u_1.$$

Since $\left(N,u^{t,n},\succeq^{t,n},t\right) = (N,u,\succeq,t)$, we have that $f_1(N,u,\succeq,t) = F_1(N,u,\succeq,t)$.

Since f and F satisfy consistency, for each $i \in N\backslash\{1\}$,

$$f_i(N,u,\succeq,t) = f_i(N\backslash\{1\},\left(u_j\right)_{j\in N\backslash\{1\}},\left(\succeq_j\right)_{j\in N\backslash\{1\}},t - f_1(N,u,\succeq,t)) \text{ and}$$
$$F_i(N,u,\succeq,t) = F_i(N\backslash\{1\},\left(u_j\right)_{j\in N\backslash\{1\}},\left(\succeq_j\right)_{j\in N\backslash\{1\}},t - F_1(N,u,\succeq,t)).$$

Since $f_1(N,u,\succeq,t) = F_1(N,u,\succeq,t)$ and the induction hypothesis,

$$\begin{aligned} &f_i(N\backslash\{1\},\left(u_j\right)_{j\in N\backslash\{1\}},\left(\succeq_j\right)_{j\in N\backslash\{1\}},t - f_1(N,u,\succeq,t)) \\ &\quad = F_i(N\backslash\{1\},\left(u_j\right)_{j\in N\backslash\{1\}},\left(\succeq_j\right)_{j\in N\backslash\{1\}},t - F_1(N,u,\succeq,t)). \end{aligned}$$

Thus, for all $i \in N \backslash \{1\}$,

$$f_i\left(N, u, \succeq, t\right) = F_i\left(N, u, \succeq, t\right).$$

2. $\sum_{i \in N} p_i \geq t$. Since f coincides with F in $\mathcal{P}^{u=t}$ and F is efficient, by (R.1), we have that for all $i \in N$

$$f_i\left(N, u^t, \succeq^t, t\right) = F_i\left(N, u^t, \succeq^t, t\right) \leq p_i^t = p_i \leq u_i.$$

Since f and F satisfy upper bound monotonicity, for all $i \in N$,

$$f_i\left(N, u, \succeq, t\right) \geq \min\left\{f_i\left(N, u^t, \succeq^t, t\right), u_i\right\} = f_i\left(N, u^t, \succeq^t, t\right) \quad \text{and}$$
$$F_i\left(N, u, \succeq, t\right) \geq \min\left\{F_i\left(N, u^t, \succeq^t, t\right), u_i\right\} = F_i\left(N, u^t, \succeq^t, t\right).$$

Then,

$$f\left(N, u, \succeq, t\right) = f\left(N, u^t, \succeq^t, t\right) = F\left(N, u^t, \succeq^t, t\right) = F\left(N, u, \succeq, t\right).$$

This ends the proof of Theorem 3.

A2.4. The independence of the axioms

Theorem 1 • *Efficiency is independent of the other properties.*
Define the rule f^1 as follows. Let $(N, u, \succeq, t) \in \mathcal{P}$. For each $i \in N$,

$$f_i^1\left(N, u, \succeq, t\right) = \min\left\{\alpha, u_i\right\},$$

where α is such that $\sum_{i \in N} f_i\left(N, u, \succeq, t\right) = t$.
The rule f^1 satisfies strategy-proofness, equal treatment of equals and upper bound monotonicity but fails efficiency.

• *Strategy-proofness is independent of the other properties.*
Define the rule f^2 as follows. Let $(N, u, \succeq, t) \in \mathcal{P}$. For each $i \in N$,

$$f_i^2\left(N, u, \succeq, t\right) = \begin{cases} p_i + \min\left\{\alpha, u_i - p_i\right\} & \text{if } \sum_{i \in N} p_i < t \\ \min\left\{\alpha, p_i\right\} & \text{if } \sum_{i \in N} p_i \geq t, \end{cases}$$

where α is such that $\sum_{i \in N} f_i\left(N, u, \succeq, t\right) = t$.
The rule f^2 satisfies efficiency, equal treatment of equals and upper bound monotonicity but fails strategy-proofness.

• *Equal treatment of equals is independent of the other properties.*
Define f^3 as the priority rule given by the order $(1, 2, \ldots, n)$. Let $(N, u, \succeq, t) \in \mathcal{P}$. In order to define f^3 formally, we consider two cases.

1. $\sum_{i \in N} p_i \geq t$. *Take* k *as the unique number satisfying that* $\sum_{i=1}^{k} p_i \leq t < \sum_{i=1}^{k+1} p_i$. *For each* $i \in N$,

$$f_i^3(N, u, \succeq, t) = \begin{cases} p_i & \text{if } i \leq k \\ t - \sum_{i=1}^{k} p_i & \text{if } i = k+1 \\ 0 & \text{if } i > k+1. \end{cases}$$

2. $\sum_{i \in N} p_i < t$. *Take* k *as the unique number satisfying that* $\sum_{i=1}^{k+1} p_i + \sum_{i=k+2}^{n} u_i \leq t < \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} u_i$. *For each* $i \in N$,

$$f_i^3(N, u, \succeq, t) = \begin{cases} p_i & \text{if } i \leq k \\ t - \sum_{i=1}^{k} p_i - \sum_{i=k+2}^{n} u_i & \text{if } i = k+1 \\ u_i & \text{if } i > k+1. \end{cases}$$

The rule f^3 *satisfies efficiency, strategy-proofness and upper bound monotonicity but fails equal treatment of equals.*

- *Upper bound monotonicity is independent of the other properties.*
Define the rule f^4 *inspired by the Constrained Equal Losses rule in bankruptcy (see Thomson 2003). Let* $(N, u, \succeq, t) \in \mathcal{P}$. *For each* $i \in N$,

$$f_i^4(N, u, \succeq, t) = \begin{cases} \max\{u_i - \alpha, p_i\} & \text{if } \sum_{i \in N} p_i < t \\ \min\{u_i - \alpha, p_i\} & \text{if } \sum_{i \in N} p_i \geq t, \end{cases}$$

where α *is such that* $\sum_{i \in N} f_i(N, u, \succeq, t) = t$.
The rule f^4 *satisfies efficiency, strategy-proofness and equal treatment of equals but fails upper bound monotonicity.*

Theorem 2 • *Efficiency is independent of the other properties.*
The rule f^1 *just defined satisfies strategy-proofness, weak envy-freeness and upper bound monotonicity but fails efficiency.*

- *Strategy-proofness is independent of the other properties.*
Given a problem $(N, u, \succeq, t) \in \mathcal{P}$ *let* $EWE(N, u, \succeq, t)$ *be the set of all feasible allocations satisfying efficiency and weak envy-freeness at* $(N, u, \succeq, t)$.
Consider first the case with two agents. Let R *be the rule in which agent 1 chooses the allocation he prefers most in* $EWE(N, u, \succeq, t)$.
Consider now the general case. Let $S(N, u, \succeq, t) = \{i \in N \mid u_i > 0\}$ *and define for each* $i \in N$,

$$f_i^5(N, u, \succeq, t) = \begin{cases} 0 & \text{if } |S| = 2 \text{ and } i \notin S \\ R_i(S, u_S, \succeq_S, t) & \text{if } |S| = 2 \text{ and } \{1, i\} \subset S \\ F_i(N, u, \succeq, t) & \text{otherwise}. \end{cases}$$

The rule f^5 *satisfies efficiency, weak envy-freeness and upper bound monotonicity but fails strategy-proofness.*

- *Weak envy-freeness is independent of the other properties.*
  The rule f^3 just defined satisfies efficiency, strategy-proofness and upper bound monotonicity but fails weak envy-freeness.
- *Upper bound monotonicity is independent of the other properties.*
  Define the rule f^6 as follows. Let $(N, u, \succeq, t) \in \mathcal{P}$. Define $S = \left\{i \in N \mid F_i\left(N, u, \succeq, t\right) \neq u_i\right\}$ and take any $\varepsilon > 0$ satisfying $\sum_{i \in N} p_i < t < \sum_{i \in N} (u_i - \varepsilon)$ whenever $\sum_{i \in N} p_i < t < \sum_{i \in N} u_i$. For each $i \in N$,

$$f_i^6 (N, u, \succeq, t) = \begin{cases} \max\{u_i - \varepsilon, p_i\} & \text{if } \sum_{i \in N} p_i < t < \sum_{i \in N} u_i \text{ and } F_i(N, u, \succeq, t) = u_i \\ F_i\left(S, u_S, \succeq_S, t - \sum_{N \setminus S} \max\{u_i - \varepsilon, p_i\}\right) & \text{if } \sum_{i \in N} p_i < t < \sum_{i \in N} u_i \text{ and } F_i(N, u, \succeq, t) \neq u_i \\ F_i(N, u, \succeq, t) & \text{otherwise.} \end{cases}$$

  The rule f^6 satisfies efficiency, strategy-proofness and weak envy-freeness but fails upper bound monotonicity.

Theorem 3
- *Efficiency is independent of the other properties.*
  The rule f^1 defined above satisfies consistency, weak individual rationality from equal division and upper bound monotonicity but fails efficiency.
- *Consistency is independent of the other properties.*
  Given a problem $(N, u, \succeq, t) \in \mathcal{P}$ let $EI(N, u, \succeq, t)$ be the set of feasible allocations satisfying efficiency and weak individual rationality from equal division at $(N, u, \succeq, t)$.
  Consider first the case with two agents. Let $\widehat{R}$ be the rule in which agent 1 chooses the allocation he prefers most in $EI(N, u, \succeq, t)$.
  Consider now the general case. For each $i \in N$,

$$f_i^7 (N, u, \succeq, t) = \begin{cases} \widehat{R}_i (N, u, \succeq, t) & \text{if } |N| = 2 \\ F_i (N, u, \succeq, t) & \text{otherwise.} \end{cases}$$

  The rule f^7 satisfies weak individual rationality from equal division, upper bound monotonicity and efficiency but fails consistency.
- *Weak individual rationality from equal division is independent of the other properties.*
  The rule f^3 defined above satisfies consistency, upper bound monotonicity and efficiency but fails weak individual rationality from equal division.
- *Upper bound monotonicity is independent of the other properties.*
  Given a problem $(N, u, \succeq, t) \in \mathcal{P}$, let π^u be an order of the set of agents satisfying two properties. First, if $u_i < u_j$, then $\pi_i^u < \pi_j^u$. Second, if $u_i = u_j$, then $\pi_i^u < \pi_j^u$ if and only if $i < j$. Let f^8 be the sequential dictatorial rule given by the order π^u where agents must choose an allocation in $EI(N, u, \succeq, t)$.
  The rule f^8 satisfies consistency, weak individual rationality from equal division and efficiency but fails upper bound monotonicity.

Corollary 1
- *One-sided resource-monotonicity is independent of the other properties.*

The rule f^1 satisfies consistency, weak individual rationality from equal division and upper bound monotonicity but fails one-sided resource-monotonicity.

- *Consistency is independent of the other properties.*
  The rule f^7 just defined satisfies weak individual rationality from equal division, upper bound monotonicity and one-sided resource-monotonicity but fails consistency.
- *Weak individual rationality from equal division is independent of the other properties.*
  The rule f^3 defined in Appendix A1.2 satisfies consistency, upper bound monotonicity and one-sided resource-monotonicity but fails weak individual rationality from equal division.
- *Upper bound monotonicity is independent of the other properties.*
  The rule f^8 just defined satisfies consistency, weak individual rationality from equal division and one-sided resource-monotonicity but fails upper bound monotonicity.

References

Alcalde J, Barberà S (1994) Top dominance and the possibility of strategy-proof solutions to matching markets. Econ Theory 4:417–435

Barberà S (1977) The manipulation of social choice mechanisms that do not leave 'too much' to chance. Econometrica 45:1573–1588

Barberà S (1996) Notes on strategy-proof social choice functions. In: Arrow K, Sen A, Suzumura K (eds) Social choice re-examinated. MacMillan, London. French version: Sur les fonctions de choix non manipulables. Revue d'Économie Politique 106:61–81 (1996)

Barberà S (2001) An introduction to strategy-proof social choice functions. Soc Choice Welf 18:619–653

Barberà S (2010) Strategy-proof social choice. In: Arrow K, Suzumura K (eds) Handbook of social choice and welfare, vol 2. Kluwer, Dordrecht

Barberà S, Jackson M (1994) A characterization of strategy-proof social choice functions for economies with pure public goods. Soc Choice Welf 11:241–252

Barberà S, Sonnenschein H, Zhou L (1991) Voting by committees. Econometrica 59:595–609

Barberà S, Jackson M, Neme A (1997) Strategy-proof allotment rules. Games Econ Behav 18:1–21

Barberà S, Bogomolnaia A, van der Stel H (1998) Strategy-proof probabilistic rules for expected utility maximizers. Math Soc Sci 35:89–103

Bergantiños G, Massó J, Neme A (2011) The division problem with voluntary participation. Soc Choice Welf. doi:10.1007/s00355-011-0530-9

Ching S (1992) A simple characterization of the uniform rule. Econ Lett 40:57–60

Ching S (1994) An alternative characterization of the uniform rule. Soc Choice Welf 11:131–136

Dagan N (1996) A note on Thomson's characterizations of the uniform rule. J Econ Theory 69:255–261

Ehlers L (2002a) On fixed-path rationing methods. J Econ Theory 106:172–177

Ehlers L (2002b) Resource-monotonic allocation when preferences are single-peaked. Econ Theory 20: 113–131

Kibris O (2003) Constrained allocation problems with single-peaked preferences: an axiomatic analysis. Soc Choice Welf 20:353–362

Kim S (2010) The separability principle for the division problem with participation constraints. Mimeo

Manjunath V (2010) When too little is as good as nothing at all: rationing a disposable good among satiable people with acceptance thresholds. Mimeo

Moulin H (1980) On strategy-proofness and single peakedness. Public Choice 35:437–455

Moulin H (1999) Rationing a commodity along fixed paths. J Econ Theory 84:41–72

Schummer J, Thomson W (1997) Two derivations of the uniform rule and an application to bankruptcy. Econ Lett 55:333–337
Sprumont Y (1991) The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica 59:509–519
Sprumont Y (1995) Strategy-proof collective choice in economic and political environments. Can J Econ 28:68–107
Sönmez T (1994) Consistency, monotonicity, and the uniform rule. Econ Lett 46:229–235
Thomson W (1994a) Consistent solutions to the problem of fair division when preferences are single-peaked. J Econ Theory 63:219–245
Thomson W (1994b) Resource monotonic solutions to the problem of fair division when preferences are single-peaked. Soc Choice Welf 11:205–223
Thomson W (1995) Population monotonic solutions to the problem of fair division when preferences are single-peaked. Econ Theory 5:229–246
Thomson W (1997) The replacement principle in economies with single-peaked preferences. J Econ Theory 76:145–168
Thomson W (2003) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math Soc Sci 45:249–297