Game Theory

Bargaining Theory

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7.1.- Bargaining Theory

- A *Bargaining problem* is a situation where a set of *agents* (players or bargainers) may cooperate to their mutual benefit.
- By bargaining, agents try to reach an agreement on how everybody will cooperate.
- In general though, agents' preferences on the set of possible agreements may differ.
- Hence, bargaining has two features:
  - cooperation (we did not consider it in the non-cooperative theory).
  - conflict.
- Examples:
  - A seller and a buyer have to agree on a price.
  - Wage setting between a firm and a union.
  - Peace conversations.
  - Etc.

7.1.- Bargaining Theory

- The important feature of the (pure) Bargaining problem is that agreements have to be reached *unanimously*.
  - There is no role for partial agreements among a subset of agents.
  - Intermediate coalitions do not play any role.
  - This feature makes Bargaining Theory an specific subclass of Cooperative Game Theory.
- If *all* agents are not able to reach an agreement, there is a disagreement outcome (*status quo*).
  - Namely, every agent has veto power over all possible agreements.
- All this literature (very active in the 90's and 00's) starts with two papers by Nash:
  - Nash, J. "The Bargaining Problem," *Econometrica* 18, 1950.
  - Nash, J. "Two-Person Cooperative Games," *Econometrica* 21, 1953.

7.1.- Bargaining Theory

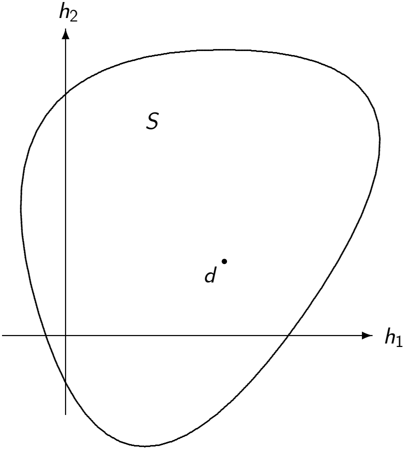
- Besides presenting the Bargaining problem and its solution (the Nash Bargaining Solution), he puts forward what has been called the *Nash Program* (on how to solve the Bargaining Problem):
  - Normative approach: axioms (values or criteria) that an arbitrator should use when solving any Bargaining problem.
  - Positive approach: specify what players can do in the bargaining process (*i.e.*, define a non-cooperative game) and look for its equilibrium.
- The Nash Program consists of studying cooperative solutions such that they are equilibria of some non-cooperative game.
- Before Nash economists though that the Bargaining problem was *indeterminate* (it had many possible solutions like the Edgeworth "contract curve"). The agents' bargaining abilities and thoughness would determine the particular chosen solution.

7.2.- The Bargaining Problem

- Set of *agents* (or players or bargainers): $N = \{1, ..., n\}$, where $n \geq 2$.
- Set of feasible *outcomes* (or *agreements*): Z.
- For $i \in N$, a vNM *utility function* $u_i : Z \longrightarrow \mathbb{R}$ representing i's preferences on Z.
  - Hence, each $i \in N$ has preferences on the set of probability distributions $\mathcal{L}(Z)$ that are represented by $h_i : \mathcal{L}(Z) \longrightarrow \mathbb{R}$ which satisfies the expected utility property.
  - Implicit hypothesis: only utilities (not the outcomes themselves) are relevant to represent and determine the solution of the Bargaining problem.
- Let $S \subset \mathbb{R}^n$ be the *set of feasible outcomes* in terms of expected utilities.

- Assumptions on S:
  - $x \in S$ if and only if there exists $p \in \mathcal{L}(Z)$ such that for every $i \in N$, $h_i(p) = x_i$.
  - S is convex (by allowing for randomizations on the set of feasible outcomes Z).
  - S is compact (for instance, if Z is finite).
- There is a *disagreement point* (or *status quo*): $d \in S$.
- Assumption: there exists $x \in S$ such that $x_i > d_i$ for all $i \in N$.
- Let $\mathcal{B}$ be the set of all pairs (S, d) with the above properties; namely, $\mathcal{B}$ is the set of all Bargaining problems.

7.2.- The Bargaining Problem



7.2.- The Bargaining Problem

Definition A *solution* of the Bargaining problem is a function $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ such that for all $(S, d) \in \mathcal{B}$, $f(S, d) \in S$.

- A solution is a rule that assigns to each bargaining problem a feasible vector of utilities.
- A solution can be interpreted as an *arbitrator* responding to a particular set of principles on how to solve the Bargaining problem.
- The following properties can be seen as desirable principles-values-axioms that a solution may satisfy.

INDEPENDENCE OF EQUIVALENT UTILITY REPRESENTATION (IEUR)

- For every $(S, d) \in \mathcal{B}$ and every $b = (b_1, ..., b_n)$ and $a = (a_1, ..., a_n)$ such that $a_i > 0$ for all $i \in N$, define $(S', d') \in \mathcal{B}$ as follows:
  - $S' = \{y \in \mathbb{R}^n \mid \text{there exists } x \in S \text{ s.t. for all } i \in N,\ y_i = b_i + a_i x_i\}$.
  - For each $i \in N$, $d'_i = b_i + a_i d_i$.

Definition A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies *Independence of Equivalent Utility Representation* if for any $(S, d) \in \mathcal{B}$,

$$f_i(S', d') = b_i + a_i f_i(S, d)$$

for all $i \in N$.

- Namely, the solution does not take into account the numerical vNM representation of the preferences (the numerical representation on lotteries is unique up to positive affine transformations). The problems (S, d) and (S', d') are equivalent and hence, the underlying outcome should be the same.

Symmetry (Sy)

- A bargaining problem $(S, d) \in \mathcal{B}$ is *symmetric* if $d_1 = \cdots = d_n$ and for any one-to-one mapping $\pi : N \longrightarrow N$ if $x = (x_1, ..., x_n) \in S$ then $y = (y_1, ..., y_n) \in S$, where $y_j = x_{\pi(j)}$ for all $j \in N$.

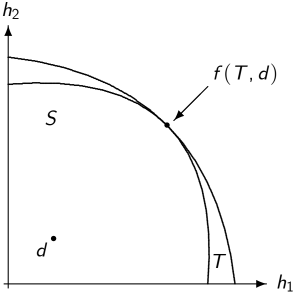
Definition A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies *Symmetry* if for every symmetric bargaining problem $(S, d) \in \mathcal{B}$,

$$f_1(S, d) = \cdots = f_n(S, d).$$

- If (S, d) is symmetric there is no information that distinguishes one player from another player. Hence, the solution should not distinguish between players.

INDEPENDENCE OF IRRELEVANT ALTERNATIVES (IIA)

Definition A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies *Independence of Irrelevant Alternatives* if for every $(S, d), (T, d) \in \mathcal{B}$ such that $S \subset T$ and $f(T, d) \in S$ then, $f(S, d) = f(T, d)$.



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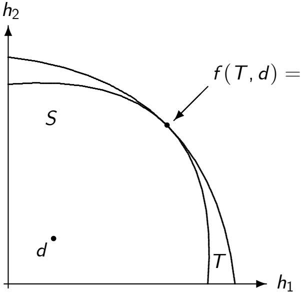


- If alternatives in $T\setminus S$ were not selected when T was available, when they are not available anymore (they are not in S), the solution should select the same outcome.
- Equivalently, when the set S is enlarged to the set T then, either the solution selects the same outcome ($f(T,d) = f(S,d)$) or else selects a *new* outcome ($f(T,d) \notin S$).

PARETO OPTIMALITY (PO)

Definition A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies *Pareto Optimality* if for every $(S, d) \in \mathcal{B}$ and every $x, y \in S$ such that $x_i > y_i$ for all $i \in N$, $f(S, d) \neq y$.

- The solution distributes all the benefits obtained from the cooperation.

7.3.- The Nash Bargaining Solution

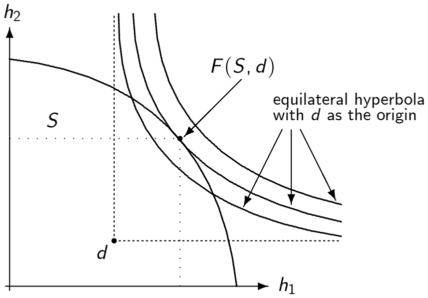
Definition The *Nash Bargaining solution* $F : \mathcal{B} \longrightarrow \mathbb{R}^n$ is defined as follows: for every $(S, d) \in \mathcal{B}$, $F(S, d) = x$ where $x \in S$ is such that $x \geq d$ and $\prod_{i=1}^{n}(x_i - d_i) > \prod_{i=1}^{n}(y_i - d_i)$ for all $y \in S \setminus \{x\}$ and $y \geq d$.

Theorem

(Nash, 1950). A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies (IEUR), $(\mathrm{S{\scriptsize Y}})$, (IIA), and (PO) if and only if $f = F$.

- $\prod_{i=1}^{n}(x_i - d_i)$ is called the Nash Product.

7.3.- The Nash Bargaining Solution



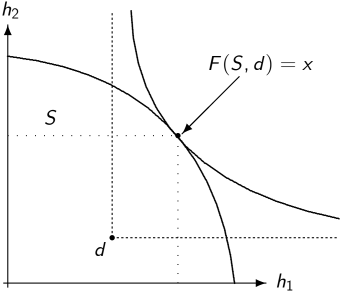
7.3.- The Nash Bargaining Solution: Summary of the proof

- It is easy to show that the Nash Bargaining solution F satisfies (IEUR), (Sy), (IIA), and (PO).
- To show that any bargaining solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfying the four properties is indeed the Nash Bargaining solution F is done in three steps:
  - Step 1: (IEUR) allows to treat every bargaining problem as a symmetric one.
  - Step 2: By (Sy) and (PO) f and F should coincide on any symmetric bargaining game since there is a unique Pareto optimal outcome whose components are all equal.
  - Step 3: By (IIA) f and F coincide on the original problem.

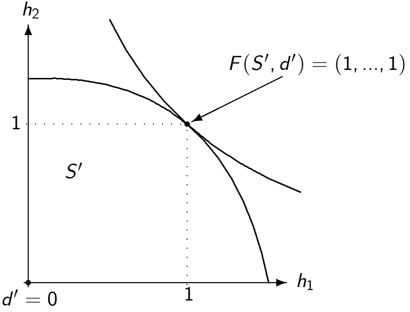
- Let $(S, d) \in \mathcal{B}$ be arbitrary and let $F(S, d) = x$.
- Observe that by assumption and (PO), $x_i > d_i$ for all $i \in N$.
- Define $(S', d') \in \mathcal{B}$ as the following positive affine transformation of (S, d): for all $i \in N$, and all $y \in S$,

$$\lambda_i(y_i) = \frac{-d_i}{x_i - d_i} + \frac{1}{x_i - d_i} y_i.$$

- Notice that $\lambda_i(x_i) = 1$ and $\lambda_i(d_i) = 0$.
- By (IEUR), $F(S', d') = (1, ..., 1)$.



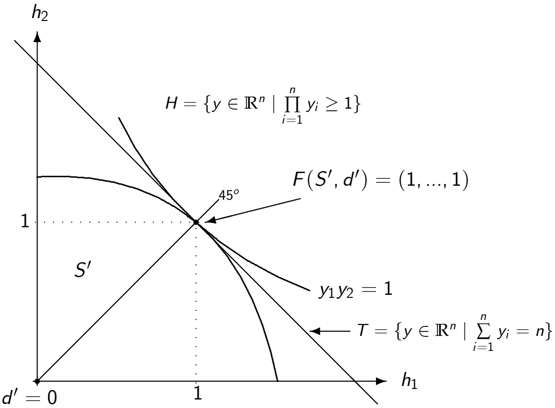
by (IEUR)



7.3.- The Nash Bargaining Solution: Proof

- Hence, $(1, ..., 1)$ is the maximizer of the Nash product on S'.
- Thus, $x' = (1, ..., 1)$ *is* the *unique* point in the intersection of S' and the convex set
$$H = \{y \in \mathbb{R}^n \mid \prod_{i=1}^{n} y_i \geq 1\}.$$
- Since $\prod_{i=1}^{n} y_i \geq 1$ is differentiable, the hyperplane
$$T = \{y \in \mathbb{R}^n \mid \sum_{i=1}^{n} y_i = n\}$$
is the unique hyperplane that is tangent to H and goes through $x' = (1, ..., 1)$.

7.3.- The Nash Bargaining Solution: Proof



7.3.- The Nash Bargaining Solution: Proof

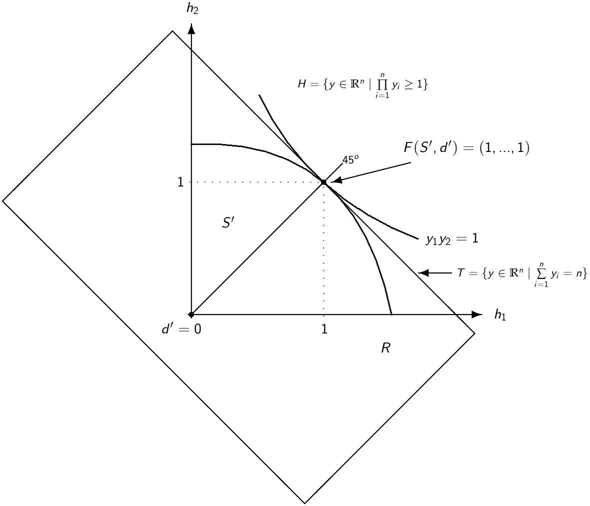
- Since H and S' are convex sets, by the separating hyperplane theorem,

$$S' \subseteq \{y \in \mathbb{R}^n \mid \sum_{i=1}^{n} y_i \leq n\}.$$

- Hence, and since S' is compact, there exists a symmetric set R such that $S' \subseteq R$ and $P(R) \subseteq T$,
  - where $P(R)$ is the set of Pareto optimal points of R; *i.e.*,

$$P(R) = \{y \in R \mid \nexists x \in R \text{ s.t. } x_i > y_i \text{ for all } i \in N\}.$$

7.3.- The Nash Bargaining Solution: Proof



- Thus, by (SY) and (PO), $f(R, d') = (1, ..., 1)$.
- By (IIA), $f(S', d') = (1, ..., 1)$.
- By (IEUR), $f(S, d) = x = F(S, d)$. ■

7.3.- The Nash Bargaining Solution: Comments

- Nash proposes an *artificial* non-cooperative game (in normal form) where players make demands; *i.e.*, for all $i \in N$,
  - $S_i = \{y_i \in \mathbb{R} \mid d_i \leq y_i \leq \max\{x_i \mid x \in S \text{ and } x \geq d\}\}$,
  - if $y \in S$ then y is implemented,
  - if $y \notin S$ then d is the solution, *and*
  - the payoff functions are (artificially) made continuous and smooth with respect to the above outcome function.
  - The Nash equilibrium of the game is the Nash bargaining solution F.
- The four axioms in the Theorem are independent (Problem set).

7.3.- The Nash Bargaining Solution: Individual Rationality

- The Theorem does not explicitly require that the Bargaining solution satisfies individual rationality.
- However, the Nash Bargaining solution is individually rational.

INDIVIDUAL RATIONALITY (IR)

Definition A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies *Individual Rationality* if for every $(S, d) \in \mathcal{B}$,

$$f_i(S, d) \geq d_i$$

for all $i \in N$.

- (IR) is an axiom related to individual choice. Surprisingly, (PO) (related with collective choice) can "almost" be replaced in the Nash's Theorem by (IR).

7.3.- The Nash Bargaining Solution: Individual Rationality

- Roth, A. "Individual Rationality and Nash's Solution to the Bargaining Problem," *Mathematics of Operations Research* 2, 1977.

Definition The *Disagreement* solution $D : \mathcal{B} \longrightarrow \mathbb{R}^n$ always selects d; *i.e.*, $D(S, d) = d$ for all $(S, d) \in \mathcal{B}$.

Theorem

(Roth, 1977) A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies (IEUR), $(\mathrm{S{\scriptsize Y}})$, (IIA), *and* (IR) *if and only if either $f = F$ or $f = D$.*

7.3.- The Nash Bargaining Solution: Individual Rationality

STRONG INDIVIDUAL RATIONALITY (SIR)

Definition A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies *Strong Individual Rationality* if for every $(S, d) \in \mathcal{B}$,

$$f_i(S, d) > d_i$$

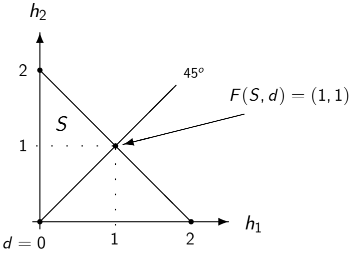
for all $i \in N$.

Theorem

(Roth, 1979) A solution $f : \mathcal{B} \longrightarrow \mathbb{R}^n$ satisfies (IEUR), (SY), (IIA), *and* (SIR) *if and only if $f = F$.*

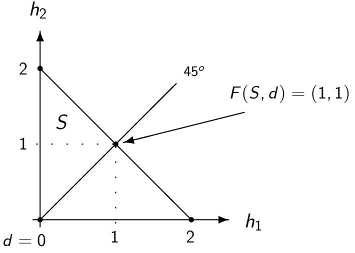
7.4.- The Kalai-Smorodinsky Solution

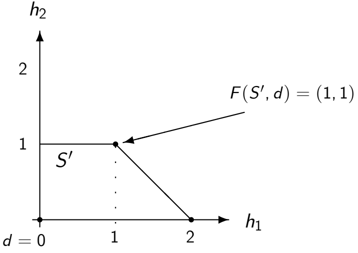
- Kalai, E. and M. Smorodinsky. "Other Solutions to Nash's Bargaining Problem," *Econometrica* 45, 1975.
- Assume from now on that $n = 2$.
  - Denote by $\mathcal{B}_2$ the family of bargaining problems with $n = 2$.
- The most controversial axiom in the axiomatic characterization of the Nash Bargaining solution is (IIA).
- The outcome of the bargaining problem may depend on opportunities that although they are not chosen (the atomic bomb) they give strength (give *bargaining power*) to a player.
- Consider the following example $(S, d) \in \mathcal{B}_2$:
  $S = Co\{(0,0),(2,0),(0,2)\}$ and $d = (0,0)$.



- Since (S, d) is symmetric, the Nash Bargaining solution chooses $F(S, d) = (1, 1)$.
- Consider now the set $S' = S \cap \{x \in \mathbb{R}^2 \mid x_2 \leq 1\}$.

7.4.- The Kalai-Smorodinsky Solution





- Since (S, d) is symmetric, the Nash Bargaining solution chooses $F(S, d) = (1, 1)$.
- Consider now the set $S' = S \cap \{x \in \mathbb{R}^2 \mid x_2 \leq 1\}$.
- Now, the bargaining problem (S', d) is very asymmetric but, by (IIA), F has to choose also $(1, 1)$.

7.4.- The Kalai-Smorodinsky Solution

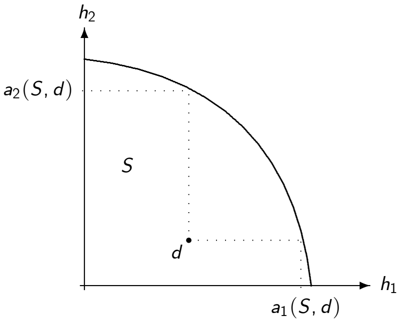
- Let $S \subset \mathbb{R}^2$ be a compact and convex set. The *comprehensive set of* S is

$$Comp(S) = \{y \in \mathbb{R}^2 \mid \exists x \in S \text{ s.t. } y \leq x\}.$$

- Interpretation: there is free disposal of utility.

Definition Given $(S, d) \in \mathcal{B}_2$ and $i = 1, 2$, define i's *ideal point* as

$$a_i(S, d) = \max \{x_i \in \mathbb{R} \mid x \in S \text{ and } x_{3-i} \geq d_{3-i}\}.$$


h_2
$a_2(S, d)$
S
d
h_1
$a_1(S, d)$

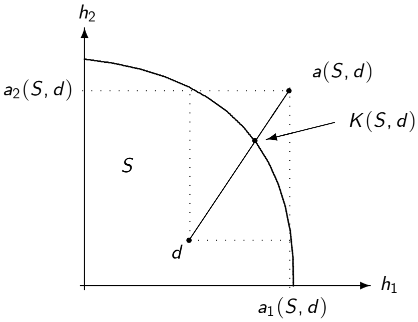
Definition The *Kalai-Smorodinsky Bargaining solution* $K : \mathcal{B}_2 \longrightarrow \mathbb{R}^n$ is defined by letting $K(S,d)$ to be equal to the maximal point of S on the segment connecting d and $a(S,d)$. Namely, for every $(S,d) \in \mathcal{B}_2$, $K(S,d) = x$ where $x \in S$ is such that

$$\frac{x_1 - d_1}{x_2 - d_2} = \frac{a_1(S,d) - d_1}{a_2(S,d) - d_2}$$

and $x \geq y$ for all $y \in S$ such that

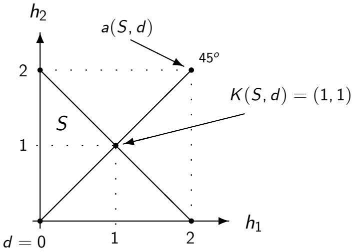
$$\frac{y_1 - d_1}{y_2 - d_2} = \frac{a_1(S,d) - d_1}{a_2(S,d) - d_2}.$$

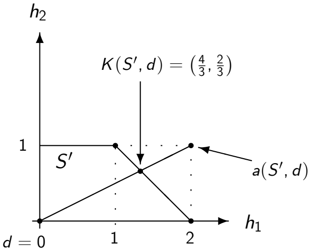
7.4.- The Kalai-Smorodinsky Solution

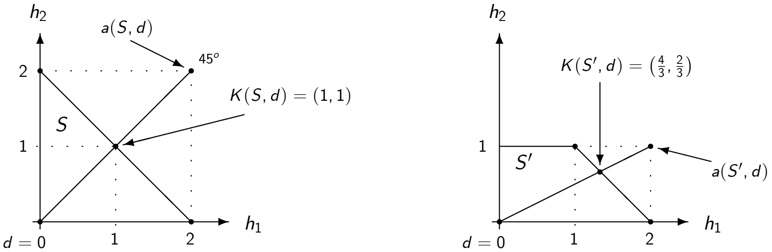


- The following example shows that the Kalai-Smorodinsky solution does not satisfy (IIA).
- **Example** Let
  - $S = CoComp\{(2,0),(0,2)\} \cap \mathbb{R}^2_+$,
  - $S' = CoComp\{(1,1),(2,0)\} \cap \mathbb{R}^2_+$ and
  - $d = (0,0)$.
  - Obviously, $S' \subset S$.
  - $K(S,d) = (1,1) \in S'$.
  - Hence,
    - (IIA) would require that the solution also chooses $(1,1)$ in the problem (S',d),
    - but $K(S',d) = (\frac{4}{3},\frac{2}{3})$.

7.4.- The Kalai-Smorodinsky Solution







- Player 2 has less bargaining power in (S', d) than in (S, d), and the Kalai-Smorodinsky solution reflects that.

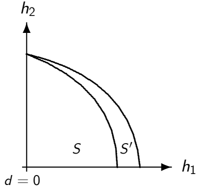
- The agreement $(1, 1)$ seems reasonable for (S, d) since the problem is symmetric, but it does not for the problem (S', d).

- The Kalai-Smorodinsky solution violates (IIA), but it does it in the intuitively good direction.

7.4.- The Kalai-Smorodinsky Solution: Individual Monotonicity

INDIVIDUAL MONOTONICITY (IM)

Definition A solution $f : \mathcal{B}_2 \longrightarrow \mathbb{R}^2$ satisfies *Individual Monotonicity* if for all $(S, d) \in \mathcal{B}_2$ and all $i = 1, 2$, if $S \subset S'$ and $a_{3-i}(S, d) = a_{3-i}(S', d)$ then, $f_i(S', d) \geq f_i(S, d)$.

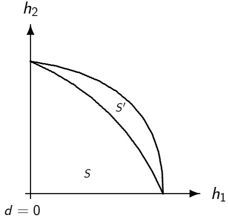


- If two problems have the property that for every level of utility of one player the other player obtains more utility in the first problem than in the second one, then he has to get at least as much utility in the first problem.

7.4.- The Kalai-Smorodinsky Solution: Restricted Monotonicity

RESTRICTED MONOTONICITY (RM)

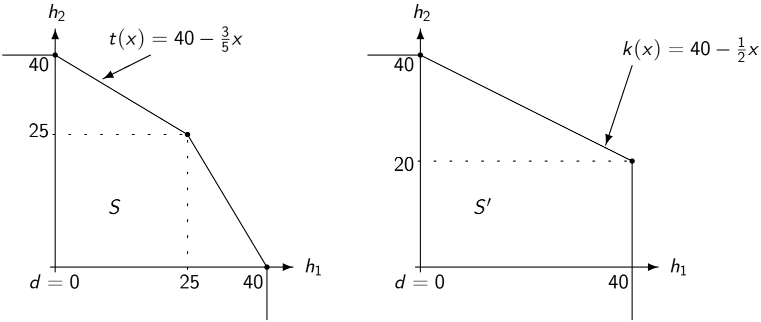
Definition A solution $f : \mathcal{B}_2 \longrightarrow \mathbb{R}^2$ satisfies *Restricted Monotonicity* if for all $(S, d) \in \mathcal{B}_2$, $S \subset S'$ and $a(S, d) = a(S', d)$ imply $f(S', d) \geq f(S, d)$.



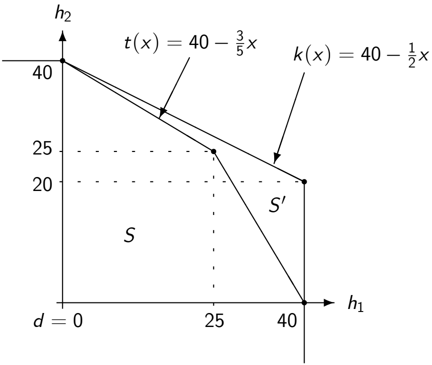
7.4.- The Kalai-Smorodinsky Solution: Monotonicity

- It is easy to see that if a solution satisfies (IM) then it satisfies (RM).
- The Kalai-Smorodinsky solution satisfies (IM) and therefore it satisfies (RM) as well.
- These properties are not easily generalized to bargaining problems with $n > 2$.
- The Nash Bargaining solution does not satisfy (RM) (and consequently it does not satisfy (IM)) as the following example shows.

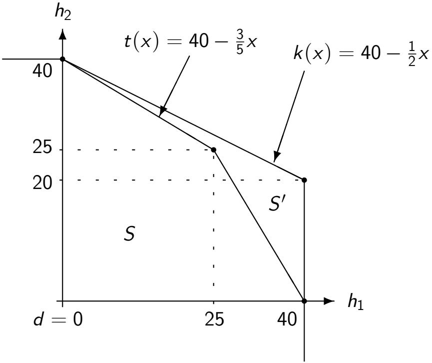
- **Example** Let
  - $S = CoComp\{(40,0),(25,25),(0,40)\} \cap \mathbb{R}^2_+$,
  - $S' = CoComp\{(40,20),(0,40)\} \cap \mathbb{R}^2_+$ and
  - $d = (0,0)$.



7.4.- The Kalai-Smorodinsky Solution: Monotonicity

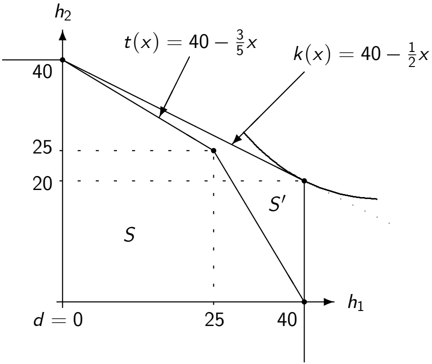


7.4.- The Kalai-Smorodinsky Solution: Monotonicity



- By symmetry, $F(S, d) = (25, 25)$.

7.4.- The Kalai-Smorodinsky Solution: Monotonicity



- By the Implicit Function Theorem, the slope of the Nash product at $(40, 20)$ is

$$\frac{\frac{\partial(x_1 \cdot x_2)}{\partial x_1}}{\frac{\partial(x_1 \cdot x_2)}{\partial x_2}} = -\frac{x_2}{x_1} \mid_{(40,20)} = -\frac{1}{2} = k'(x).$$

7.4.- The Kalai-Smorodinsky Solution: Monotonicity

- The Nash product is maximized at $(40, 20)$.
- Hence, $F(S', d) = (40, 20)$.
- Therefore, $F_2(S, d) = 25 > 20 = F_2(S', d)$.
- Observe that $S \subset S'$ and $a(S, d) = a(S', d) = (40, 40)$.
- Thus, the Nash Bargaining solution does not satisfy (RM).

7.4.- The Kalai-Smorodinsky Solution: Characterization

Theorem

(Kalai and Smorodinsky, 1975). A solution $f : \mathcal{B}_2 \longrightarrow \mathbb{R}^2$ satisfies (IEUR), (SY), (PO), and (RM) if and only if f is the Kalai-Smorodinsky solution (i.e., $f = K$).

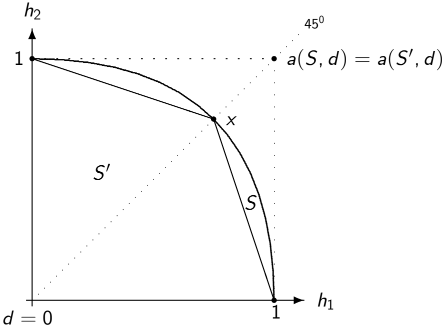
Proof

- It is easy to see that K satisfies the four properties.
- Assume that $f : \mathcal{B}_2 \longrightarrow \mathbb{R}^2$ satisfies (IEUR), (SY), (PO), and (RM).
- We want to show that for all $(S, d) \in \mathcal{B}_2$, $f(S, d) = K(S, d)$.
- Let $(S, d) \in \mathcal{B}_2$ be arbitrary.

7.4.- The Kalai-Smorodinsky Solution: Proof of the Characterization

- By (IEUR) we can assume that $d = (0, 0)$ and $a(S, d) = (1, 1)$.
- By definition of K, $K(S, d) = (\alpha, \alpha) = x$.
- By convexity of S, $\alpha \geq \frac{1}{2}$.
- Observe that $(1, 0), (0, 1) \in S$.
- Let $S' = CoComp\{(1, 0), x, (0, 1)\} \cap \mathbb{R}^2_+$.
- Since $(0, 0), (1, 0), (0, 1), x \in S$ and S is convex, $S' \subseteq S$.

7.4.- The Kalai-Smorodinsky Solution: Proof

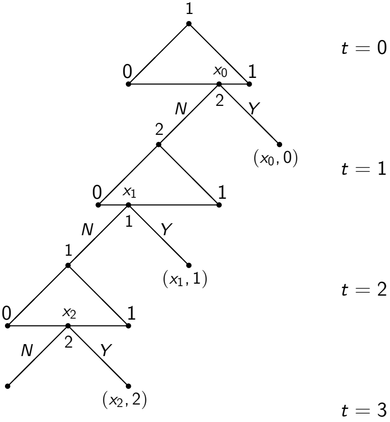


- Since (S', d) is symmetric, by (SY) and (PO), $f(S', d) = x$.
- Since $S' \subseteq S$ and $a(S, d) = a(S', d) = (1, 1)$, by (RM), $f(S, d) \geq x$.
- But since $\{x\} = S \cap \{y \in \mathbb{R}^2_+ \mid y \geq x\}$, $f(S, d) = x$, and hence, $f(S, d) = K(S, d)$. ■

7.5.- Strategic Bargaining

- Rubinstein, A. “Perfect Equilibrium in a Bargaining Model,” *Econometrica* 50, 1982.
- Two players, $I = \{1, 2\}$, are bargaining on how to divide a unit of a good (a pie): $X = [0, 1]$.
  - $x \in [0, 1]$ means that they reach an agreement and player 1 gets x and player 2 gets $1 - x$.
- Bargaining takes place over (discrete) time: $t = 0, 1, 2, \ldots$
- Rules:
  - Player 1 makes a proposal $x_0 \in [0, 1]$ at period t_0.
  - Player 2 either accepts or rejects the proposal x_0.
  - If the proposal x_0 is accepted, bargaining finishes and the outcome is $(x_0, 0)$.
  - If the proposal is rejected then player 2 makes a proposal x_1 at period $t = 1$.
  - Player 1 either accepts (outcome $(x_1, 1)$) or rejects the proposal x_1. ...

7.5.- Strategic Bargaining



7.5.- Strategic Bargaining

- Players' preferences on their shares can be represented by the utility functions $u_1 : [0,1] \longrightarrow \mathbb{R}$ and $u_2 : [0,1] \longrightarrow \mathbb{R}$, where for every agreement $x \in [0,1]$, $u_1(x)$ is player 1's utility of x and $u_2(1-x)$ is player 2's utility of x.

- Players are impatient: $\delta_i \in (0,1)$ is player i's discount factor.

- An agreement on x at t is the pair (x,t).

- Player i's preferences $\succeq_i$ on the set of agreements can be represented as follows: for any pair of agreements (x,t) and (y,m) :
  - $(x,t) \succeq_1 (y,m) \Longleftrightarrow \delta_1^t u_1(x) \geq \delta_1^m u_1(y)$.
  - $(x,t) \succeq_2 (y,m) \Longleftrightarrow \delta_2^t u_2(1-x) \geq \delta_2^m u_2(1-y)$.

- However, assume that $u_i(z) = z$ (risk neutral players) and $\delta_1 = \delta_2 = \delta \in (0,1)$.

- Let $\Gamma(\delta)$ be this (infinite) game in extensive form with perfect information.
- It is easy to see that for every possible agreement (x, t) there is a Nash equilibrium of this game in which the player that has to make the proposal at period t offers x and the other player accepts it. All other proposals are rejected.

Theorem

(Rubinstein, 1982) There exists a unique subgame perfect equilibrium of $\Gamma(\delta)$ *in which the player who has to make a proposal offers* $\frac{1}{1+\delta}$ *for himself and* $\frac{\delta}{1+\delta}$ *to the other player, and the player who does not have to make a proposal accepts this offer or any better offer and rejects all strictly worse offers.*

7.5.- Strategic Bargaining: Remarks

- Uniqueness.
- Stationary strategies.
- Bargaining is efficient:
  - At $t = 0$, player 1 proposes $x_0 = \frac{1}{1+\delta}$ and player 2 says Yes.
  - Unrealistic result: delay is almost always present in all real world bargaining.
  - Under incomplete information on the discount factors, (sequential) equilibria have delays.
- Since players are impatient, player 1 has advantage and gets a larger share: for all $\delta \in (0, 1)$,

$$\frac{1}{1+\delta} > \frac{\delta}{1+\delta}.$$

7.5.- Strategic Bargaining: Remarks

- When $\delta \to 1$, because either impatience or time units between offers get smaller,
$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right) \underset{\delta \to 1}{\longrightarrow} \left(\frac{1}{2}, \frac{1}{2}\right);$$
namely, strategic bargaining converges to the Nash solution.
- Rubinstein result opens an extremely large literature. His result is not robust to many things:
  - Incomplete information.
  - Existence of outside options.
  - If $n > 2$, then results depend on the particular bargaining protocol.
- Moulin proposes a bargaining game in extensive form whose unique subgame perfect equilibrium is the Kalai-Smorodinsky bargaining solution.
  - Moulin, H. "Implementing the Kalai-Smorodinsky Bargaining Solution," *Journal of Economic Theory* 33, 1984.

7.5.- Strategic Bargaining: Idea of the Proof

- Consider the following stationary strategy:
  - x: player 1's proposal (at any period when is has to make a proposal).
  - y: player 2's proposal (at any period when is has to make a proposal).
  - Player 1 accepts z if and only if $z \geq \delta x$.
  - Player 2 accepts z if and only if $(1 - z) \geq \delta(1 - y)$.
- Observe:
  - $y > \delta x$ is not equilibrium, since player 2 could propose $y - \varepsilon > \delta x$ (which is accepted by player 1) and obtain the share $1 - y + \varepsilon > 1 - y$.
  - $1 - x > \delta(1 - y)$ is not an equilibrium, since player 1 could propose $x + \varepsilon$ with $1 - x - \varepsilon > \delta(1 - y)$ (which is accepted by player 2) and obtain the share $x + \varepsilon > x$.

7.5.- Strategic Bargaining: Idea of the Proof

- Hence, to be part of an equilibrium (x, y) have to satisfy:
  - $y \leq \delta x$
  - $1 - x \leq \delta(1 - y)$.
- Moreover,
  - player 1 accepts y if and only if $y \geq \delta x$,
  - player 2 accepts x if and only if $1 - x \geq \delta(1 - y)$.
- Thus,

$$\begin{aligned} y &= \delta x \\ 1 - x &= \delta(1 - y). \end{aligned}$$

$$
\begin{aligned}
y &= \delta x \\
1 - x &= \delta(1 - y).
\end{aligned}
$$

- Substituting,
$$1 - x = \delta(1 - \delta x) = \delta - \delta^2 x.$$
- Hence,
$$1 - \delta = x(1 - \delta^2) = x(1 - \delta)(1 + \delta).$$
- Thus,
$$x = \frac{1}{1 + \delta} \text{ and } y = \frac{\delta}{1 + \delta}.$$