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Voting by committees under constraints

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Abstract

We consider social choice problems where a society must choose a subset from a set of objects. Specifically, we characterize the families of strategy-proof voting procedures when not all possible subsets of objects are feasible, and voters' preferences are separable or additively representable.

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1. Introduction

Many problems of social choice take the following form. There are n voters and a set $K = \{1, \ldots, k\}$ of objects. These objects may be bills considered by a legislature, candidates to some set of positions, or the collection of characteristics which distinguish a social alternative from another. The voters must choose a subset of the set of objects.

Sometimes, any combination of objects is feasible: for example, if we consider the election of candidates to join a club which is ready to admit as many of them as the voters choose, or if we are modelling the global results of a legislature, which may pass or reject any number of bills. It is for these cases that Barberà et al. [7] provided characterizations of

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all voting procedures which are strategy-proof and respect voter's sovereignty (all subsets of objects may be chosen) when voters' preferences are additively representable, and also when these are separable. For both of these restricted domains, voting by committees turns out to be the family of all rules satisfying the above requirements. Rules in this class are defined by a collection of families of winning coalitions, one for each object; agents vote for sets of objects; to be elected, an object must get the vote of all members of some coalition among those that are winning for that object.

Most often, though, some combinations of objects are not feasible, while others are: if there are more candidates than positions to be filled, only sets of size less than or equal to the available number of slots are feasible; if objects are the characteristics of an alternative, some collections of characteristics may be mutually incompatible, and others not. Our purpose in this paper is to characterize the families of strategy-proof voting procedures when not all possible subsets of objects are feasible, and voters' preferences are separable or additively representable. As in [7], we can identify each set of objects with the value of its characteristic functions, and thus with some vertex of the k-dimensional hypercube. Our characterization tells us exactly what social choice functions will be strategy-proof and onto for each given set of vertices, to be interpreted as the family of feasible subsets from which society wants and can choose from. Our main conclusions are the following. First, rules that satisfy strategy-proofness are still voting by committees, with ballots indicating the best feasible set of objects. Second, the coalition structures for different objects must be interrelated, in precise ways which depend on what families of sets of objects are feasible. Specifically, each family of feasible subsets will admit a unique decomposition, which will dictate the exact form of the strategy-proof and onto social choice functions that can be defined on it. Third, unlike in [7], the class of strategy-proof rules when preferences are additively representable can be substantially larger that the set of rules satisfying the same requirement when voters' preferences are separable.

Our characterization result for separable preferences is quite negative: infeasibilities quickly turn any non-dictatorial rule into a manipulable one, except for very limited cases. In contrast, our characterization result for additive preferences can be interpreted as either positive or negative, because it has different consequences depending on the exact shape of the range of feasible choices. The contrast between these two characterization results is a striking conclusion of our research, because until now the results regarding strategy-proof mechanisms for these two domains had gone hand to hand, even if they are, of course, logically independent.

In order to compare our results with others in the literature, it is worth noticing that our framework, where alternatives (sets of objects) can be expressed as vectors of zeros and ones, has been extended. Barberà et al. [4] extended the analysis to cover situations where the objects of choice are Cartesian products of integer intervals, allowing for possibly more than two values on each dimension. The pioneering work of Border and Jordan [9] considered functions whose range is any Cartesian product of intervals in the real line. In there and in other contexts of multidimensional choice where the range of the social choice rule is a Cartesian product, strategy-proof rules are necessarily decomposable into rules which independently choose a value for each dimension, and are themselves strategy-proof (see [10,11] for general expressions of this important result).

In [5] (see also [6]) we considered the consequences of introducing feasibility constraints in that larger framework. The range of feasible choices is no longer a Cartesian product and this requires a more complex and careful analysis. All strategy-proof rules are still decomposable, but choices in the different dimensions must now be coordinated in order to guarantee feasibility. While these previous papers make an important step in understanding how this coordination is attained for each given shape of the range, it is marred by a strong assumption on the domain of admissible preferences. Specifically, we assume there that the bliss point of each agent is feasible. This assumption is not always realistic. Moreover, it makes the domain of admissible preferences dependent on the range of feasible choices.

Several authors (Serizawa [13] and Answal et al. [1]) have studied the consequences of specific restrictions on the range, like budget constraints or limitations on the number of objects that may be chosen. These authors only consider the case of separable preferences, not the additive case, which is the one providing some positive results. Our results apply generally and cover all types of infeasibilities within our context: ranges of all shapes are allowed.

In the present paper we come back to the question of strategy-proofness under constraints within a more limited framework, the one where only two values can be taken by each of the components of k-dimensional vectors, initially considered by Barberà et al., [7]. This is done for clarity of exposition, given that in all other respects we are going to substantially extend the previous analysis. One substantial extension consists in that we can apply our result regardless of the nature and the form of feasibility restrictions: our results apply to ranges of any shape. Budget constraints, capacity limits, lower bounds on the number of objects to be chosen are all specific cases that we cover with a single result. It is also worth noting that we tackle the case where all separable preferences (and all additive preferences) are admissible without any further restriction.

Perhaps the most important progress regarding previous results in this literature comes from the new insights we get on the need for strategy-proof rules to be decomposable. As already mentioned, when the range of the rule is a Cartesian product, strategy-proofness requires and allows to decompose global decisions into partial ones, one for each object (or for each dimension). What we prove is that the decomposition of the range as a Cartesian product is still essential in order to understand the possibility of defining strategy-proof rules. Even when a set is not a Cartesian product of k separate sets of values, one for each object, it can always be decomposed in different pieces (maybe only one, in the most degenerate cases), through what we call the minimal Cartesian decomposition. Then, strategy-proof rules must be decomposable into rules that choose in a strategy-proof manner within each of these pieces (sections), and then aggregate these choices into a feasible alternative. This structure generalizes our previous notions of decomposability, which was restricted before to one of the cases where the decomposition into Cartesian components is trivial.

The paper is organized as follows. Section 2 contains preliminary notation and definitions as well as previous results. In Section 3, we introduce specific definitions and notation, obtain preliminary results, and present our two characterizations: Theorem 1 for additive preferences and Theorem 2 for separable ones. Section 4 contains an important final remark: the Gibbard–Satterthwaite Theorem is a corollary of our results. Section 5 contains the proof of Theorem 1, omitted in Section 3.

2. Preliminaries

Agents are the elements of a finite set $N = \{1, 2, \dots, n\}$. The set of objects is $K = \{1, \dots, k\}$. We assume that n and k are at least 2. Generic elements of N will be denoted by i and j and generic elements of K will be denoted by x, y, and z. *Alternatives* are subsets of K which will be denoted by X, Y, and Z. Subsets of N will be represented by I and J. Calligraphic letters will represent families of subsets; for instance, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ will represent families of subsets of alternatives and $\mathcal{W}$, $\mathcal{I}$, and $\mathcal{J}$ families of subsets of agents (coalitions).

Preferences are binary relations on alternatives. Let $\mathbf{P}$ be the set of complete, transitive, and asymmetric preferences on 2^K. Preferences in $\mathbf{P}$ are denoted by P_i, P_j, P'_i, and P'_j. For $P_i \in \mathbf{P}$ and $\mathcal{X} \subseteq 2^K$, we denote the alternative in $\mathcal{X}$ most-preferred according to P_i as $\tau_{\mathcal{X}}(P_i)$, and we call it *the top of* P_i *on* $\mathcal{X}$. We will use $\tau(P_i)$ to denote the top of P_i on 2^K. Generic subsets of preferences will be denoted by $\hat{\mathbf{P}}$.

Preference profiles are n-tuples of preferences. They will be represented by $P = (P_1, \dots, P_n)$ or by $P = (P_i, P_{-i})$ if we want to stress the role of agents i's preference.

A *social choice function on* $\hat{\mathbf{P}}$ is a function $F\colon \hat{\mathbf{P}}^n \to 2^K$.

Definition 1. The social choice function $F\colon \hat{\mathbf{P}}^n \to 2^K$ *respects voter's sovereignty* if for every $X \in 2^K$ there exists $P \in \hat{\mathbf{P}}^n$ such that $F(P) = X$.

The range of a social choice function $F\colon \hat{\mathbf{P}}^n \to 2^K$ is denoted by $\mathcal{R}_F$*;* that is,

$$\mathcal{R}_F = \left\{ X \in 2^K \mid \text{there exists } P \in \hat{\mathbf{P}}^n \text{ such that } F(P) = X \right\}.$$

Denoteby R_F the set of chosen objects; namely,

$$R_F = \{x \in K \mid x \in X \text{ for some } X \in \mathcal{R}_F\}.$$

Definition 2. A social choice function $F\colon \hat{\mathbf{P}}^n \to 2^K$ is *manipulable* if there exist $P = (P_1, \dots, P_n) \in \hat{\mathbf{P}}^n$, $i \in N$, and $P'_i \in \hat{\mathbf{P}}$ such that $F\left(P'_i, P_{-i}\right) P_i F(P)$. A social choice function on $\hat{\mathbf{P}}$ is *strategy-proof* if it is not manipulable.

Definition 3. A social choice function $F\colon \hat{\mathbf{P}}^n \to 2^K$ is *dictatorial* if there exists $i \in N$ such that $F(P) = \tau_{\mathcal{R}_F}(P_i)$ for all $P \in \hat{\mathbf{P}}^n$.

The Gibbard–Satterthwaite Theorem states that any strategy-proof social choice function on $\mathbf{P}$ will be either dictatorial or its range will have only two elements. It would apply directly if any individual preference over the sets of objects were in the domain. However, there are many situations where agents' preferences have specific structure due to the nature of the set of objects, and this structure may impose meaningful restrictions on the way agents rank subsets of objects. We will be interested in two natural domains of preferences: those that are separable and those that are additive.

Definition 4. A preference P_i on 2^K is *additive* if there exists a function $u_i : K \to \mathbb{R}$ such that for all $X, Y \subseteq K$

$$X P_i Y \text{ if and only if } \sum_{x \in X} u_i(x) > \sum_{y \in Y} u_i(y).$$

The set of additive preferences will be denoted by **A**.

An agent i has separable preferences P_i if the division between *good* objects ($\{x\} P_i \{\emptyset\}$) and *bad* objects ($\{\emptyset\} P_i \{x\}$) guides the ordering of subsets in the sense that adding a good object leads to a *better* set, while adding a bad object leads to a *worse* set. Formally,

Definition 5. A preference P_i on 2^K is *separable* if for all $X \subseteq K$ and all $y \notin X$

$$X \cup \{y\} P_i X \text{ if and only if } \{y\} P_i \{\emptyset\}.$$

Let **S** be the set of all separable preferences on 2^K. We can give a geometric interpretation to this set by identifying each object with a coordinate and each set X of objects with a vertex of a k-dimensional cube; i.e., with the k-dimensional vector of zeros and ones, where x belongs to X if and only if that vector has a one in x's coordinate. Sometimes we will make use of this geometric interpretation. For instance, given $X, Y \subseteq K$ the *minimal box on X and Y* is the smallest subcube containing the vectors corresponding to X and Y; namely,

$$MB(X, Y) = \left\{ Z \in 2^K \mid (X \cap Y) \subseteq Z \subseteq (X \cup Y) \right\}.$$

Following with this interpretation, it is easy to see that a preference P_i is separable if for all Z and $Y \in MB(\tau(P_i), Z) \setminus Z$, $Y P_i Z$.

Remark that additivity implies separability but the converse is false with more than two objects. To see that, let $K = \{x, y, z\}$ be the set of objects and consider the separable preference

$$\{x, y, z\} P_i \{y, z\} P_i \{x, z\} P_i \{x, y\} P_i \{x\} P_i \{y\} P_i \{z\} P_i \{\emptyset\},$$

which is not additive since $\{x\} P_i \{y\}$ and $\{y, z\} P_i \{x, z\}$. Geometrically, additivity imposes the condition that the orderings of all vertices on each parallel face of the hypercube coincide while separability admits the possibility that some vertices of two parallel faces have different orderings. This geometric interpretation will become very useful to understand the differences of our two characterizations.

To define voting by committees as in [7] we need the concept of a coalition structure.

Definition 6. A *coalition structure* $\mathcal{W}$ is a nonempty family of nonempty coalitions of N, which satisfies coalition monotonicity: if $I \in \mathcal{W}$ and $I \subseteq J$, then $J \in \mathcal{W}$. Coalitions in $\mathcal{W}$ are called *winning*. A coalition $I \in \mathcal{W}$ is a *minimal winning coalition* if for all $J \subsetneq I$ we have that $J \notin \mathcal{W}$.

Given a coalition structure $\mathcal{W}$, we will denote by $\mathcal{W}^m$ the set of its minimal winning coalitions. A coalition structure $\mathcal{W}$ is *dictatorial* if there exists $i \in N$ such that $\mathcal{W}^m =$

$\{\{i\}\}$. Associated to each family of coalition structures (one for each object) we can define a special type of social choice functions.

Definition 7. A social choice function $F\colon\ \hat{\mathbf{P}}^n \to 2^K$ is *voting by committees*, if for each $x \in K$, there exists a coalition structure $\mathcal{W}_x$ such that for all $P = (P_1, \ldots, P_n) \in \hat{\mathbf{P}}^n$,

$$x \in F(P) \text{ if and only if } \{i \in N \mid x \in \tau_{\mathcal{R}_F}(P_i)\} \in \mathcal{W}_x.$$

A social choice function F is called *Voting by quota q* $(1 \leqslant q \leqslant n)$ if for all x the coalition structure $\mathcal{W}_x$ is equal to the family of coalitions with cardinality equal or larger than q.

We state, as Proposition 1, Barberà et al. [7]'s characterization of voting by committees as the class of strategy-proof social choice functions on **S**, as well as on **A**, satisfying voter's sovereignty.

Proposition 1. *A social choice function* $F\colon\ \mathbf{S}^n \to 2^K$ (*or*, $F\colon\ \mathbf{A}^n \to 2^K$) *is strategy-proof and satisfies voter's sovereignty if and only if it is voting by committees.*

To cover social choice problems with constraints we have to drop the voter's sovereignty condition of Proposition 1. But a result in [5] tells us that the only strategy-proof rules in this case must still be of the same form: this is stated in Proposition 2.

Proposition 2. *Assume* $F\colon\ \mathbf{S}^n \to 2^K$ (*or*, $F\colon\ \mathbf{A}^n \to 2^K$) *is strategy-proof. Then,* F *is voting by committees.* [1]

3. Two characterization results

3.1. The need to coordinate: two examples and an outline

Because of feasibility constraints, not all voting by committees can be guaranteed to always select a feasible alternative. The exact nature of the constraints, i.e., the shape of the range, will determine which combinations of coalition structures can constitute a proper social choice function for this range. Example 1 illustrates this fact. Moreover, under the presence of infeasibilities, there are voting by committees that, although respecting feasibility, are not strategy-proof. Example 2 illustrates this possibility.

Example 1. Let $K = \{x, y\}$ be the set of objects and $N = \{1, 2, 3\}$ the set of agents. Assume that $\{\emptyset\}$, $\{x\}$, and $\{y\}$ are feasible but $\{x, y\}$ is not. Voting by quota 1 does not respect feasibility because for any preference profile P, with the property that $\tau(P_1) = \tau(P_2) = \{x\}$ and $\tau(P_3) = \{y\}$, both x and y should be elected, which is infeasible. However, voting by quota 2 does respect feasibility because x and y cannot get simultaneously two votes (remember, agents cannot vote for infeasible outcomes) since the complementary coalition of each winning coalition for x is not winning for y, and vice versa.

[1] It is easy to check that the proof of Proposition 2 in [5] which covers the case of separable preferences also applies to the smaller domain of additive preferences.

This idea will play an important role in our characterization with additive preferences. As suggested by our example, when defining a social choice function by means of coalition structures, we must guarantee that if all agents vote for a feasible alternative, then the result must also be a feasible alternative. This was the role played by the intersection property in [5]. Here we shall ensure it by a combination of conditions, one of which will be the choice of mutually exclusive coalition structures under certain situations. Mutually exclusive coalition structures, following the hint provided in the previous example, are formally defined as follows.

Definition 8. We say that two coalition structures $\mathcal{W}$ and $\mathcal{W}'$ are *mutually exclusive* if $D \in \mathcal{W}$ implies $N \backslash D \notin \mathcal{W}'$ and $D \in \mathcal{W}'$ implies $N \backslash D \notin \mathcal{W}$.

The interested reader may check that our characterization results (Theorems 1 and 2) guarantee that the intersection property in [5] will be satisfied by the rules we define in each case.

Example 2. Let $K = \{x, y\}$ be the set of objects and $N = \{1, 2, 3\}$ the set of agents. Assume that $\{\emptyset\}$, $\{x\}$, and $\{y\}$ are feasible but $\{x, y\}$ is not. Consider the social choice function F defined by voting by quota 3 (which respects feasibility) and let P be any additive (as well as separable) preference profile such that $\tau(P_2) = \tau(P_3) = \{y\}$ and $\{x, y\}P_1\{x\}P_1\{y\}P_1\{\emptyset\}$. Since $\tau_{2^K \backslash \{x,y\}}(P_1) = \{x\}$, y receives two votes and x one; therefore, $F(P) = \{\emptyset\}$. However, if agent 1 declares the preference P_1' where $\{y\}P_1'\{x, y\}P_1'\{\emptyset\}P_1'\{x\}$, then y receives three votes and x none; that is, $F(P_1', P_2, P_3) = \{y\}P_1\{\emptyset\} = F(P_1, P_2, P_3)$. Hence, F is not strategy-proof.

The purpose of our two characterizations is to identify exactly the subfamilies of coalition structures that simultaneously respect feasibility and are strategy-proof for the domains of additive and separable preferences.

We begin with some intuition about the nature of our results. For that, we first remind the reader about the essential features of voting by committees when there are no constraints, as in [7]. There, the choice of a set can be decomposed into a family of binary choices, one for each object. In each case, society decides whether the object should or should not be retained, and the union of selected objects amounts to the social alternative. If the methods used to decide upon each object are each strategy-proof, then so is the method resulting from combining them into a global decision, as long as the agent's preferences are additive or separable. Agents should be asked to express their best set, and under the expressed domain restrictions this is equivalent to expressing those objects that they would prefer to be included in the social decision, rather than not.

In our case, a first difference is that the choice of sets may not be decomposable to the extreme of allowing for independent decisions on each object. Our results tell us precisely about the extent to which global decisions can be decomposed, and say how to coordinate the decisions within groups of objects that require joint treatment. Indeed, in the presence of infeasibilities, the decision on what objects to choose, and which ones not to, can no longer be decomposed into object-by-object binary decisions. For example, choosing x might only be possible if y is not chosen: then the choices regarding x and y must be joint. Similarly,

z might only be chosen if w is, and again decisions involving these two objects need to be coordinated. Yet, if all feasible choices of x and y, when coupled with any feasible choice for z and w, turn out to be feasible, there is still room for decomposition of the choices in two blocks of objects. If, on the contrary, further restrictions must be taken into account, whereby certain feasible choices from x and y become incompatible with some feasible choices from z and w, then decomposition is not possible. The paper provides a precise statement about the extent to which decisions on what sets to choose can be decomposed into partial decisions involving subsets (we call each part of the decomposition a section), in the presence of feasibility constraints. Moreover, we discuss the characteristics of the coalition structures that must be used in order to coordinate the choices of objects within each of the sections.

3.2. The minimal Cartesian decomposition of a family of subsets

In this subsection, we shall describe the way in which any family of subsets can be decomposed uniquely into what we call a minimal Cartesian decomposition. This will be exactly the decomposition that will allow us to make our previous statements precise, as expressed in Theorems 1 and 2, to be found in Sections 3.3 and 3.4. As we proceed, and in order to help the reader through the new definitions, we introduce an example to illustrate the new concepts.

Example 3. Let $K = \{a, b, z, w, t\}$ be the set of objects and assume that the set of feasible alternatives $\mathcal{M}$ is

$$\{\{b\}, \{b, t\}, \{b, z\}, \{b, z, t\}, \{b, z, w\}, \{b, z, w, t\}\}.$$

Notice that (1) a is never chosen, (2) b is always chosen, (3) w is only chosen if z is, and (4) t can be chosen or not, whatever happens.

Given a social choice function $F\colon\ \hat{\mathbf{P}}^n \to 2^K$ and a subset B of R_F define the *active components of B in the range* as

$$\mathcal{AC}(B) = \{Y \cap B \mid Y \in \mathcal{R}_F\}.$$

Active components of B are subsets of B whose union with some subset in $R_F \backslash B$ is part of the range.

Example 3 (Continued)**.** The active components of the sets $\{z\}$, $\{z, w\}$ and $\{t\}$ are $\mathcal{AC}(\{z\}) = \{\{\emptyset\}, \{z\}\}$, $\mathcal{AC}(\{z, w\}) = \{\{\emptyset\}, \{z\}, \{z, w\}\}$, and $\mathcal{AC}(\{t\}) = \{\{\emptyset\}, \{t\}\}$, respectively.

Now, given $B' \subseteq B \subseteq R_F$ define the *range complement of* B' *relative to B* as

$$\mathcal{C}_F^B\left(B'\right) = \left\{C \subseteq R_F \backslash B \mid B' \cup C \in \mathcal{R}_F\right\}.$$

The range complement of a subset B' of B is the collection of sets in $R_F \backslash B$ whose union with B' is in the range. Notice that $\mathcal{AC}(B)$ can also be written as $\{X \subseteq B \mid X \cup Y \in \mathcal{R}_F.$ for some $Y \in \mathcal{C}_F^B(X)\}$.

Example 3 (Continued)**.** The range complement of the subsets $\{\emptyset\}$, $\{z\}$, and $\{z, w\}$ relative to $\{z, w\}$ coincide and they are all equal to $\{b\} + \{\{\emptyset\}, \{t\}\}$.[2]

A section is a group of objects with the property that the decision among their active components can be made without paying attention to the infeasibilities involving objects on its complement.

Definition 9. A subset of objects $B \subseteq K$ is a *section* of R_F if for all active components $B', B'' \in \mathcal{AC}(B)$ we have $\mathcal{C}_F^B(B') = \mathcal{C}_F^B(B'')$.

Example 3 (Continued)**.** The set $\{z, w\}$ is a section of R_F because $\mathcal{AC}(\{z, w\}) = \{\{\emptyset\}, \{z\}, \{z, w\}\}$ (notice that the subset $\{w\}$ is not an active component of $\{z, w\}$) and, as we have already seen, $\mathcal{C}_F^{\{z,w\}}(\{\emptyset\}) = \mathcal{C}_F^{\{z,w\}}(\{z\}) = \mathcal{C}_F^{\{z,w\}}(\{z, w\}) = \{b\} + \{\{\emptyset\}, \{t\}\}$.

Remark 1. $B = R_F$ is a section of R_F because $\mathcal{C}_F^B(X) = \{\emptyset\}$ for all active components $X \in \mathcal{AC}(R_F) = \mathcal{R}_F$.

Remark 2. B is a section of R_F if and only if, for all $B' \in \mathcal{AC}(B)$,

$$\mathcal{R}_F = \mathcal{AC}(B) + \mathcal{C}_F^B(B').$$

Lemma 1. *Let B be a section of R_F and let B_1 and B_2 be such that $B = B_1 \cup B_2$, $B_1 \cap B_2 = \{\emptyset\}$, and B_1 is a section of R_F. Then, B_2 is also a section of R_F.*

Proof. To show that B_2 is a section of R_F, let $X_2, Y_2 \in \mathcal{AC}(B_2)$ be arbitrary. We must show

$$\mathcal{C}_F^{B_2}(X_2) = \mathcal{C}_F^{B_2}(Y_2). \tag{1}$$

By definition of active component of B_2, we can find $X, Y \in \mathcal{R}_F$ such that

$$X_2 = X \cap B_2 \in \mathcal{AC}(B_2) \tag{2}$$

and

$$Y_2 = Y \cap B_2 \in \mathcal{AC}(B_2).$$

Moreover, by definition of range complement of X_2 and Y_2 relative to B_2,

$$X \cap B_2^c \in \mathcal{C}_F^{B_2}(X_2)$$

and

$$Y \cap B_2^c \in \mathcal{C}_F^{B_2}(Y_2),$$

[2] Given two families of subsets of objects $\mathcal{X}$ and $\mathcal{Y}$ we denote by $\mathcal{X} + \mathcal{Y}$ the sum of the two; namely,

$$\mathcal{X} + \mathcal{Y} = \{X \cup Y \in 2^K \mid X \in \mathcal{X} \text{ and } Y \in \mathcal{Y}\}.$$

where, given a set $Z \subseteq K$, $Z^c \equiv K \backslash Z$. Notice, that to show that (1) holds, it is sufficient to show that $Y \cap B_2^c \in \mathcal{C}_F^{B_2}(X_2)$; that is,

$$X_2 \cup (Y \cap B_2^c) \in \mathcal{R}_F.$$

By (2), and since $B_2^c = B_1 \cup B^c$,

$$\begin{aligned} X_2 \cup (Y \cap B_2^c) &= (X \cap B_2) \cup (Y \cap B_2^c) \\ &= (X \cap B_2) \cup (Y \cap B_1) \cup (Y \cap B^c). \end{aligned}$$

Claim 1. $(X \cap B_1) \cup (X \cap B_2) \cup (Y \cap B^c) \in \mathcal{R}_F$.

Proof. Since $Y \in \mathcal{R}_F$, $(Y \cap B) \cup (Y \cap B^c) \in \mathcal{R}_F$. Therefore, $Y \cap B^c \in \mathcal{C}_F^B(\bar{B})$ for some $\bar{B} \in \mathcal{AC}(B)$. Moreover, since B is a section and $X \cap B \in \mathcal{AC}(B)$, Remark 2 implies that $(X \cap B) \cup (Y \cap B^c) \in \mathcal{R}_F$. Hence, $(X \cap B_1) \cup (X \cap B_2) \cup (Y \cap B^c) \in \mathcal{R}_F$, which is the statement of the claim.

Therefore, by Claim 1 and the hypothesis that B_1 is a section,

$$(X \cap B_2) \cup (Y \cap B^c) \in \mathcal{C}_F^{B_1}(B_1')$$

for all $B_1' \in \mathcal{AC}(B_1)$. Because $(Y \cap B_1) \in \mathcal{AC}(B_1)$ we have, by Remark 2, $(X \cap B_2) \cup (Y \cap B_1) \cup (Y \cap B^c) \in \mathcal{R}_F$. Hence, $(Y \cap B_2^c) \in \mathcal{C}_F^{B_2}(X_2)$. □

Definition 10. A partition $\{B_1, \ldots, B_q\}$ of R_F is a *Cartesian decomposition of* R_F if for all $p = 1, \ldots, q$, B_p is a section of R_F. A Cartesian decomposition is called *minimal* if there is no finer Cartesian decomposition of R_F.

Example 3 (Continued)**.** The partition $\{\{b\}, \{z, w\}, \{t\}\}$ of R_F is the minimal Cartesian decomposition of R_F, since one can check that all of its elements are minimal sections. The section $\{z, w\}$ is minimal since neither $\{z\}$ nor $\{w\}$ are sections because, for instance, $\mathcal{AC}(\{w\}) = \{\{\emptyset\}, \{w\}\}$ but

$$\mathcal{C}_F^{\{w\}}(\{\emptyset\}) = \{b\} + \{\{\emptyset\}, \{z\}\} + \{\{\emptyset\}, \{t\}\}$$

and

$$\mathcal{C}_F^{\{w\}}(\{w\}) = \{b\} + \{z\} + \{\{\emptyset\}, \{t\}\},$$

and hence, $\mathcal{C}_F^{\{w\}}(\{\emptyset\}) \neq \mathcal{C}_F^{\{w\}}(\{w\})$.

The proof that all other components of the decomposition are also minimal sections is similar and left to the reader.

Remark 3. Let $\{B_1, \ldots, B_q\}$ be a partition of R_F. Then, $\{B_1, \ldots, B_q\}$ is a Cartesian decomposition of R_F if and only if

$$\mathcal{R}_F = \mathcal{AC}(B_1) + \cdots + \mathcal{AC}(B_q).$$

We want to show (Proposition 3 below) that, given any social choice function F, its corresponding set R_F has a unique minimal Cartesian decomposition. In the proof of Proposition 3 we will use the following Lemma.

Lemma 2. *Let B_1 and B_2 be two sections of R_F. Then $B = B_1 \cup B_2$ is also a section of R_F.*

Proof. Let $B = B_1 \cup B_2$ and assume that B_1 and B_2 are sections of R_F. Let $X, Y \in \mathcal{R}_F$ be arbitrary. They can also be written as

$$X = (X \cap B) \cup (X \cap B^c)$$

and

$$Y = (Y \cap B) \cup (Y \cap B^c).$$

To show that B is a section, it is sufficient to show that $(X \cap B) \cup (Y \cap B^c) \in \mathcal{R}_F$. Rewrite X and Y as

$$X = (X \cap (B_1 \backslash B_2)) \cup (X \cap (B_2 \backslash B_1)) \cup (X \cap (B_1 \cap B_2)) \cup (X \cap B^c)$$

and

$$Y = (Y \cap (B_1 \backslash B_2)) \cup (Y \cap (B_2 \backslash B_1)) \cup (Y \cap (B_1 \cap B_2)) \cup (Y \cap B^c).$$

Since B_1 is a section, $(Y \cap (B_1 \backslash B_2)) \cup (Y \cap (B_1 \cap B_2))$ and $(X \cap (B_1 \backslash B_2)) \cup (X \cap (B_1 \cap B_2))$ belong to $\mathcal{AC}(B_1)$, and $(Y \cap (B_2 \backslash B_1)) \cup (Y \cap B^c) \in \mathcal{C}_F^{B_1}((Y \cap (B_1 \backslash B_2)) \cap (Y \cap (B_1 \cap B_2)))$. Therefore,

$$(X \cap (B_1 \backslash B_2)) \cup (X \cap (B_1 \cap B_2)) \cup (Y \cap (B_2 \backslash B_1)) \cup (Y \cap B^c) \in \mathcal{R}_F.$$

By definition of the range complement of $(Y \cap (B_2 \backslash B_1)) \cup (X \cap (B_1 \cap B_2))$ relative to B_2,

$$(X \cap (B_1 \backslash B_2)) \cup (Y \cap B^c) \in \mathcal{C}_F^{B_2}((Y \cap (B_2 \backslash B_1)) \cup (X \cap (B_1 \cap B_2))). \quad (3)$$

Also, since X and Y belong to $\mathcal{R}_F$ and B_2 is a section,

$$(X \cap B_2) \cup (Y \cap B_2^c) \in \mathcal{R}_F. \quad (4)$$

Rewriting (4), we have

$$(Y \cap (B_1 \backslash B_2)) \cup (X \cap (B_2 \backslash B_1)) \cup (X \cap (B_1 \cap B_2)) \cup (Y \cap B^c) \in \mathcal{R}_F.$$

Therefore,

$$(X \cap (B_2 \backslash B_1)) \cup (X \cap (B_1 \cap B_2)) \in \mathcal{AC}(B_2). \quad (5)$$

Then, by (3) and (5), the fact again that B_2 is a section, and Remark 2,

$$(X \cap (B_2 \backslash B_1)) \cup (X \cap (B_1 \cap B_2)) \cup (X \cap (B_1 \backslash B_2)) \cup (Y \cap B^c) \in \mathcal{R}_F.$$

This implies that $(X \cap B) \cup (Y \cap B^c) \in \mathcal{R}_{F.}$ Hence, B is a section of R_F. □

Proposition 3. *R_F has a unique minimal Cartesian decomposition.*

Proof. Assume not. Let $\{B_1^1, \ldots, B_{q_1}^1\}$ and $\{B_1^2, \ldots, B_{q_2}^2\}$ be two distinct minimal Cartesian decompositions of R_F. There exists at least one pair such that $B_{p_1}^1 \cap B_{p_2}^2 \neq \{\emptyset\}$ and $B_{p_1}^1 \neq B_{p_2}^2$. By Lemma 2, $B_{p_1}^1 \cup B_{p_2}^2$ is a section of R_F. By Lemma 1, $B_{p_1}^1 \backslash B_{p_2}^2$ is also a section of R_F implying, again by Lemma 1, that $\{B_1^1, \ldots, B_{q_1}^1\}$ was not minimal. □

3.3. Additive preferences

We can now state our first characterization.

Theorem 1. *A social choice function $F\colon \mathbf{A}^n \to 2^K$ is strategy-proof if and only if it is voting by committees with the following properties*:

(1.1) *$\mathcal{W}_x$ and $\mathcal{W}_y$ are equal for all x and y in the same active component of any section with two active components in R_F's minimal Cartesian decomposition,*

(1.2) *$\mathcal{W}_x$ and $\mathcal{W}_y$ are mutually exclusive for all x and y in different active components of the same section in R_F's minimal Cartesian decomposition, when there are only two active components in this section, and*

(1.3) *$\mathcal{W}_x$ is dictatorial and equal for all x's in the same section in R_F's minimal Cartesian decomposition, when this section has more than two active components.*

The proof of Theorem 1 is in the Appendix at the end of the paper.

Our Theorem refers to the set R_F, and is thus stated as if we started from a given function F and then described the necessary and sufficient conditions for this F to be strategy-proof. We can take another point of view, which is also compatible with our purposes. Start from any family $\mathcal{M}$ of subsets of K. Interpret $\mathcal{M}$ as the set of feasible outcomes. We can then re-read Theorem 1 as telling us everything about the strategy-proof social choice functions which can be defined onto $\mathcal{M}$ (which will then be the range of these functions). True, there may also exist other strategy-proof functions which start with a feasible set $\mathcal{M}$ and end up having a subset of $\mathcal{M}$ as the range. But then, if there are alternatives that the designer is willing to give up as possible outcomes, we might as well reinterpret them and include these outcomes among those which we consider unfeasible, for practical purposes. Example 4 below illustrates the statement of Theorem 1.

Example 4. Let $K = \{a, b, x, y, z, w, r, s, q, t\}$ be the set of objects and assume that the set of feasible alternatives is

$$\mathcal{M} = \{\{b\}\} + \{\{\emptyset\}, \{x\}, \{y\}\} + \{\{\emptyset\}, \{z\}, \{z, w\}\} + \{\{r\}, \{s, q\}\} + \{\{\emptyset\}, \{t\}\}.$$

Any voting by committees $F\colon \mathbf{A}^n \to 2^K$ will be strategy-proof and will have $\mathcal{R}_F = \mathcal{M}$ as long as it satisfies the following properties: (a) by (1.3) of Theorem 1, $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i_1\}\}$ and $\mathcal{W}_z^m = \mathcal{W}_w^m = \{\{i_2\}\}$ for some $i_1, i_2 \in N$; (b) by (1.1) of Theorem 1, $\mathcal{W}_s^m = \mathcal{W}_q^m$; and (c) by (1.2) of Theorem 1, $\mathcal{W}_r$ and $\mathcal{W}_s$ are mutually exclusive. To illustrate these conditions, let $N = \{1, 2\}$ be the set of agents and consider the voting by committees F where $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{1\}\}$, $\mathcal{W}_z^m = \mathcal{W}_w^m = \{\{2\}\}$, and $\mathcal{W}_a^m = \mathcal{W}_b^m = \mathcal{W}_r^m = \mathcal{W}_s^m =$

$\mathcal{W}_q^m = \mathcal{W}_t^m = \{\{1, 2\}\}$. Observe that F satisfies properties (a), (b), and (c), and hence, by Theorem 1, it is strategy-proof on the domain of additive preferences and $\mathcal{R}_F = \mathcal{M}$.

3.4. Separable preferences

In contrast with the unconstrained case, our results for separable preferences are quite different (and much more negative) than for additive preferences. Essentially, this is because in the presence of infeasibilities, agents are not asked to vote for their preferred sets, but rather for their preferred feasible sets. Hence, they may end up voting for their second best, their third best, etc. Now, some of the individual objects they vote for may be retained, and others not. Likewise, some objects they do not vote for can obtain. What matters for strategy proofness is whether the best set for each agent among those that contain some externally fixed objects (those that are chosen in spite of the agent's negative vote) and do not contain some others (those that are not chosen even if the agent supports them) is the set that contains, in addition to those, as many elements from the agent's preferred feasible set. This is the case for additive preferences in all cases. It is also the case for separable preferences if the first best for the agent is feasible, but not necessarily otherwise. That is why, in the presence of infeasibilities, declaring the best feasible set may not be a dominant strategy for some voters, even when voting by committees are used (except if the first best is always feasible, a situation studied in [5]). Whereas it is always a dominant strategy for additive preferences. This accounts for the differences in results under these two different domains. To further illustrate this general point, we can go back to Example 3.

Example 3 (Continued)**.** Let $F\colon\ \mathbf{S}^2 \to 2^K$ be defined by the coalition structures $\mathcal{W}_z^m = \mathcal{W}_w^m = \{\{1\}\}$, and $\mathcal{W}_a^m = \mathcal{W}_b^m = \mathcal{W}_t^m = \{\{1, 2\}\}$. To see that F is manipulable on the domain of separable preferences, consider any separable preference P_1 with the following properties:

(1) $\tau(P_1) = \{b, w, t\}$ and $\tau_{\mathcal{R}_F}(P_1) = \{b, z, w, t\}$.

(2) $\{b, z, w, t\}\ P_1\ \{b, t\}$ and $\{b\}\ P_1\ \{b, z, w\}$.

Observe that P_1 is not additive because adding t to $\{b, z, w\}$ and to $\{b\}$ inverts its ordering. Take any separable profile of preferences $\left(P_1', P_2\right)$ with the properties that $\tau\left(P_1'\right) = \{b\}$ and $\tau(P_2) = \{b, z, w\}$. Then,

$$F\left(P_1', P_2\right) = \{b\}\ P_1\ \{b, z, w\} = F(P_1, P_2),$$

implying that F is manipulable by agent 1 at profile (P_1, P_2) with the preference P_1'.

Theorem 2 below characterizes the family of strategy-proof social choice functions when voters' preferences are separable. Our result shows that the class of strategy-proof social choice functions under additive representable preferences identified in Theorem 1 is drastically reduced as a consequence of this enlargement of the domain of preferences. This is an important novelty with respect to the situation without constraints. Now, only social choice functions with Cartesian product ranges (up to constant and/or omitted objects,) are strategy-proof. Namely, the range of F has to be a subcube: all sections of the minimal Cartesian decomposition of R_F (the set of not omitted objects) are singletons, either with the object itself as the unique active component (constant object) or else with the object itself *and* the empty set as the two active components. Formally,

Theorem 2. *A social choice function $F: \mathbf{S}^n \to 2^K$ is strategy-proof if and only if it is voting by committees with the following property*:

(2.1) *If $\#\mathcal{R}_F \geqslant 3$ then either F is dictatorial or all sections of the minimal Cartesian decomposition of R_F are singletons*.

Proof. Let $F: \mathbf{S}^n \to 2^K$ be a voting by committees satisfying property (2.1). If F is dictatorial then it is obviously strategy-proof. If $\#\mathcal{R}_F \geqslant 3$ and all sections of the minimal Cartesian decomposition of R_F are singletons, then the set of active components in the range of each object x of this Cartesian decomposition of R_F is either $\{\{\emptyset\}, \{x\}\}$ or $\{\{x\}\}$. When the set of active components is of the form $\{\{x\}\}$, this means that object x is always chosen. When the set of active components is of the form $\{\{\emptyset\}, \{x\}\}$, then voters have a choice between including x and not doing it. Leaving aside the constant elements, which have no consequence for strategy-proofness, the remaining choices between the objects with active components of the form $\{\{\emptyset\}, \{x\}\}$ are of the type contemplated by Barberà et al. [7]. Hence, since we assume voting by committees, then F is strategy-proof.

For the converse, assume that F is strategy-proof. By Proposition 2, F is voting by committees. To show that F satisfies property (2.1) assume $\#\mathcal{R}_F \geqslant 3$. Since all additive preferences are separable, Theorem 1 applies to the subdomain of additive preferences. Therefore, the coalition structures associated to F satisfy properties (1.1), (1.2), and (1.3) of Theorem 1. Assume F is non-dictatorial. Then, property (1.3) implies that the minimal Cartesian decomposition of R_F cannot consist of just one section with strictly more than two active components. Therefore, and since $\#\mathcal{R}_F \geqslant 3$, the minimal Cartesian decomposition of R_F contains at least two sections. Now, notice that when preferences are separable but not additively representable, the active components of a section can be ordered differently among themselves, depending on which objects are present in another section. That is, for each pair of sections B_1 and B_2 of the minimal Cartesian decomposition of R_F there exist at least one separable preference $P_i \in \mathbf{S}$, $X_1, Y_1 \in \mathcal{AC}(B_1)$, $X_2, Y_2 \in \mathcal{AC}(B_2)$, and $Z \subseteq R_F \setminus (B_1 \cup B_2)$ such that

$$(X_1 \cup X_2 \cup Z)\ P_i\ (X_1 \cup Y_2 \cup Z) \ \text{ and } \ (Y_1 \cup Y_2 \cup Z)\ P_i\ (Y_1 \cup X_2 \cup Z)\,. \tag{6}$$

This can now be used to show that we cannot have a section with more than two active components together with another section having more than one active component. To prove it, it is enough to construct profiles where the presence of an object affects the ordering of the active components in another section. Assume that a section B_1 has the property that $\#\mathcal{AC}(B_1) \geqslant 3$. Then, by property (1.3) of Theorem 1, for all $x \in B_1$, $\mathcal{W}_x$ is dictatorial (i.e., $\mathcal{W}_x^m = \{\{i\}\}$ for some $i \in N$). Also assume that there exists another section B_2 such that $\#\mathcal{AC}(B_2) \geqslant 2$. Then, for all $y \in B_2$, $\mathcal{W}_y^m = \{\{i\}\}$, since there exists a separable preference P_i satisfying (6). By applying the same argument we could prove that dictatorship extends to all objects belonging to sections with more than two active components. Therefore, all sections have either only one active component (the objects that are always selected) or they have just two active components. Following a similar argument to the one already used to establish (6) it is immediate to see that if a section has two active components they are of the form $\{\{\emptyset\}, \{x\}\}$. Hence, all sections in the minimal Cartesian decomposition of R_F are singletons. □

4. Final remark

Until now, we have taken the dimension of our problems (i.e., the number of objects), as well as the feasibility constraints, as given data. Our analysis admits another reading without any formal change, except for its interpretation.

Consider a situation where society faces four alternatives, *a*, *b*, *c*, and *d*. One possibility is that each of these alternatives might be described by two characteristics, and that identifying $a = (0, 0)$, $b = (1, 0)$, $c = (0, 1)$, and $d = (1, 1)$ provides a good description of the actual choices (this particular choice would indicate that *a* and *c* are similar in the first characteristic but differ on the second, etc.). It may also be, in another extreme, that these four alternatives share nothing relevant in common. They can still be represented as vectors of zeros and ones, but now it is better to embed them in $\mathbb{R}^4$, and identify them as $a = (1, 0, 0, 0)$, $b = (0, 1, 0, 0)$, $c = (0, 0, 1, 0)$, and $d = (0, 0, 0, 1)$. There may still be intermediate cases where three characteristics are necessary and sufficient to distinguish between these four alternatives. Two examples may be given by the cases

$$a = (1, 0, 0)\,, \;\; b = (1, 1, 0)\,, \;\; c = (1, 0, 1)\,, \;\; \text{and } d = (0, 0, 0)$$

or

$$a = (1, 0, 0)\,, \;\; b = (0, 1, 0)\,, \;\; c = (0, 0, 1)\,, \;\; \text{and } d = (0, 1, 1)\,.$$

In the four- and three-dimensional cases, these four alternatives are only some of the conceivable vertices of the corresponding cubes. Other combinations of zeros and ones represent conceivable but unfeasible choices.

These examples suggest that the objects in our model (interpreted as characteristics) may be taken as partial aspects of the overall alternatives (whose role is played in our model by the feasible sets). This interpretation is not restrictive: any alternative (out of a finite set) can be described by a (finite) set of characteristics. What is restrictive is that once we identify each alternative with a set of characteristics (thus embedding it into some *l*-dimensional cube), we also determine the shape of the set of feasible alternatives, and this has consequences on the class of preferences which pass the test of additivity (or separability). [3]

In fact, thanks to the above observations, we can conclude by arguing that the Gibbard–Satterthwaite Theorem arises as a particular corollary of our Theorem 1. Indeed, take any finite set $A = \{x, y, \ldots, w\}$ of *k* alternatives ($k > 2$). Identify them with the *k* unit vectors and assume that the set of feasible alternatives $\mathcal{M}$ is $\{\{x\}, \{y\}, \ldots, \{w\}\}$. Notice that all

[3] Actually, identifying the alternatives of a social choice problem as points in a grid can give us some interesting insights. In particular, many problems can be rewritten as ones where alternatives are strings of 0 or 1 vectors. For example, the setting of Barberà et al. [7] can be viewed as defining rules to choose among the vertices of a hypercube. This point of view has been expressed and used in [2,3,8]. It is the object of recent work by Nehring and Puppe [12].

preferences over A are restrictions of some additive preference on the k-dimensional cube. Hence, we are considering the universal domain assumption of the Gibbard–Satterthwaite result. Let $F\colon \mathbf{A}^n \to 2^A$ be such that $\mathcal{R}_F = \{\{x\}, \{y\}, \dots, \{w\}\}$. The minimal Cartesian decomposition of $R_F\ (= A)$ contains only the section $B = \{x, y, \dots, w\}$, whose set of active components is $\mathcal{AC}(B) = \{\{x\}, \{y\}, \dots, \{w\}\}$. Since $\#\mathcal{AC}(B) > 2$, Property (1.3) of Theorem 1 tells us that only dictatorial rules are strategy-proof on additive preferences. This is the conclusion we wanted. [4]

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Appendix A. Proof of Theorem 1

The proof of Theorem 1 is based on a decomposition argument that applies an important result of Le Breton and Sen [10] to our context. This argument, which will be exploited in the proof of Theorem 1, is expressed as Proposition A.2 below. But before, we need the following notation.

Let P_i be an additively representable preference on 2^K and consider a subset B of K. Let P_i^B stand for the preferences on 2^B generated by the utilities which represent P_i. Let $\mathbf{A}_B$ be the set of additive preferences on 2^B. For a profile P of preferences on 2^K, P^B will denote the profile of preferences so restricted, for all $i \in N$.

Given a strategy-proof social choice function $F\colon \mathbf{A}^n \to 2^K$ and a subset B of objects, let $F^B\colon \mathbf{A}_B^n \to 2^B$ be defined so that for all $P^B \in \mathbf{A}_B^n$

$$F^B\left(P^B\right) = F(P) \cap B,$$

where P is any additive preference such that P^B is generated by the utilities which represent P.

[4] In an earlier paper [5] we had already used the same embedding or identification of alternatives with unit vectors in order to prove the Gibbard–Satterthwaite Theorem. In the earlier paper, this was a corollary of a different characterization than the one we offer here. As a result, our arguments in the present paper, which apply Theorem 1, are simpler and more direct than in the previous case.

Remark A.1. Notice that, since $F\colon \mathbf{A}^n \to 2^K$ is a strategy-proof social choice function, it is voting by committees (by Proposition 2). Hence, for any $B \subseteq K$, $F(P) \cap B = F\left(\hat{P}\right) \cap B$ for all P, $\hat{P} \in \mathbf{A}^n$ such that $P^B = \hat{P}^B$. Therefore, F^B is well-defined.

Proposition A.2. (1) *Let* $F\colon \mathbf{A}^n \to 2^K$ *be a social choice function and let* $\{B_1, \ldots, B_q\}$ *be a Cartesian decomposition of* R_F. *If F is strategy-proof then* $F^{B_1}, \ldots, F^{B_q}$ *are strategy-proof and* $F(P) = \bigcup_{p=1}^{q} F^{B_p}\left(P^{B_p}\right)$ *for all* $P \in \mathbf{A}^n$.

(2) *Conversely, let* $\{B_1, \ldots, B_q\}$ *be a partition of* $K' \subseteq K$ *and let* $\{\mathcal{B}_1, \ldots, \mathcal{B}_q\}$ *be a collection of subsets of objects, with* $\mathcal{B}_p \subseteq 2^{B_p}$ *for all* $p = 1, \ldots, q$. *Let* $F^{B_p}\colon \mathbf{A}^n_{B_p} \to \mathcal{B}_p$ *be a collection of onto social choice functions, one for each* $p = 1, \ldots, q$. *If* $F^{B_1}, \ldots, F^{B_q}$ *are strategy-proof, then the function* $F(P) = \bigcup_{p=1}^{q} F^{B_p}\left(P^{B_p}\right)$ *for all* $P \in \mathbf{A}^n$ *is strategy-proof,* $\{B_1, \ldots, B_q\}$ *is a Cartesian decomposition of* $R_F = K'$, *and* $\mathcal{R}_F = \mathcal{B}_1 + \cdots + \mathcal{B}_q$.

Proof. (1) Assume $\{B_1, \ldots, B_q\}$ is a Cartesian decomposition of R_F and let $P \in \mathbf{A}^n$. Then,

$$\begin{aligned} F(P) &= F(P) \cap R_F \quad \text{by definition of } R_F \\ &= \bigcup_{p=1}^{q} [F(P) \cap B_p] \quad \text{since } \{B_1, \ldots, B_q\} \text{ is a partition of } R_F \\ &= \bigcup_{p=1}^{q} F^{B_p}(P^{B_p}) \quad \text{by definition of } F^{B_p} \text{ and } P^{B_p}. \end{aligned}$$

To obtain a contradiction, assume that F^{B_p} is not strategy-proof; that is, there exist P^{B_p}, i, and $\hat{P}_i^{B_p}$ such that $F^{B_p}(\hat{P}_i^{B_p}, P_{-i}^{B_p}) P_i^{B_p} F^{B_p}(P^{B_p})$. Therefore, and since preferences are additive,

$$\sum_{y \in F^{B_p}(\hat{P}_i^{B_p}, P_{-i}^{B_p})} u_i^{B_p}(y) > \sum_{y \in F^{B_p}(P^{B_p})} u_i^{B_p}(y), \tag{A.1}$$

for any $u_i^{B_p} : B_p \to \mathbb{R}$ representing $P_i^{B_p}$.

Take any $P \in \mathbf{A}^n$ generating P^{B_p} and $\hat{P}_i$ generating $\hat{P}_i^{B_p}$ with the property that

$$P_i^{B_{p'}} = \hat{P}_i^{B_{p'}} \tag{A.2}$$

for all $p' \neq p$. For each $p' \neq p$, take any $u_i^{B_{p'}}$ representing $P_i^{B_{p'}}$. Then, by (A.1),

$$\sum_{p' \neq p} \sum_{x \in F^{B_{p'}}(P^{B_{p'}})} u_i^{B_{p'}}(x) + \sum_{y \in F^{B_p}(P^{B_p})} u_i^{B_p}(y)$$

$$= \sum_{p' \neq p} \sum_{x \in F^{B_{p'}}(\hat{P}_i^{B_{p'}}, P_{-i}^{B_{p'}})} u_i^{B_{p'}}(x) + \sum_{y \in F^{B_p}(P^{B_p})} u_i^{B_p}(y)$$
$$< \sum_{p' \neq p} \sum_{x \in F^{B_{p'}}(\hat{P}_i^{B_{p'}}, P_{-i}^{B_{p'}})} u_i^{B_{p'}}(x) + \sum_{y \in F^{B_p}(\hat{P}_i^{B_p}, P_{-i}^{B_p})} u_i^{B_p}(y),$$

where the equality follows from (A.2) and the inequality follows from (A.1). Therefore, $F(\hat{P}_{-i}, P_i) P_i F(P)$; that is, F is not strategy-proof.

(2) Let $\{B_1, \ldots, B_q\}$ be a partition of $K' \subseteq K$ and consider any $P \in \mathbf{A}^n$, $i \in N$, and $\hat{P}_i \in \mathbf{A}$. Since for all $p = 1, \ldots, q$ the functions F^{B_p} are strategy-proof, we have that $F^{B_p}(P^{B_p}) R_i^{B_p} F^{B_p}(\hat{P}_i^{B_p}, P_{-i}^{B_p})$; that is, for all $p = 1, \ldots, q$,

$$\sum_{x \in F^{B_p}(P^{B_p})} u_i^{B_p}(x) \geqslant \sum_{y \in F^{B_p}(\hat{P}_i^{B_p}, P_{-i}^{B_p})} \hat{u}_i^{B_p}(y),$$

where $u_i^{B_p}$ and $\hat{u}_i^{B_p}$ are any pair of functions on B_p representing $P_i^{B_p}$ and $\hat{P}_i^{B_p}$, respectively. Therefore, adding up,

$$\sum_{p=1}^{q} \sum_{x \in F^{B_p}(P^{B_p})} u_i^{B_p}(x) \geqslant \sum_{p=1}^{q} \sum_{y \in F^{B_p}(\hat{P}_i^{B_p}, P_{-i}^{B_p})} \hat{u}_i^{B_p}(y).$$

Hence, $F(P) R_i F(\hat{P}_i, P_{-i})$; that is, F is strategy-proof. That $\{B_1, \ldots, B_q\}$ is a Cartesian decomposition of $R_F = K'$ and $\mathcal{R}_F = \mathcal{B}_1 + \cdots + \mathcal{B}_q$ follow immediately from the fact that $F(P) = \bigcup_{p=1}^{q} F^{B_p}\left(P^{B_p}\right)$ for all $P \in \mathbf{A}^n$. □

Our strategy of proof for necessity relies heavily on invoking the Gibbard–Satterthwaite Theorem for the case where there are more than three active components in a section B_p of the minimal Cartesian decomposition of the range. This is done by proving that, then, there will be three feasible outcomes which agents can rank as the three most-preferred, and in any relative order (a "free triple"). But F^{B_p} must be strategy-proof if F is (Proposition A.2). If F^{B_p} was non-dictatorial, we could use it to construct a non-dictatorial and strategy-proof social choice function over our free triple, which we know is impossible by the Gibbard–Satterthwaite Theorem. As for the case where a section has two active components only, notice that we can divide the objects of this section into two sets, such that all the elements in one of the sets obtains when those on the other do not, and vice versa. Our restriction that the coalition structures corresponding to these two sets of objects are mutually exclusive guarantees that no vote can lead to choose at the same time objects from these two active components. Otherwise, no further restriction is imposed on our coalition structures by strategy-proofness when only two outcomes arise.

Now, we state and prove that whenever a section in the minimal Cartesian decomposition of R_F contains more than two active components, then we get a dictator. This is achieved by showing that a free triple always exists in this case.

Proposition A.3. *Assume that the following properties of $\mathcal{R}_F$ hold:* (1) *the minimal Cartesian decomposition of R_F has a unique section and* (2) $\#\mathcal{R}_F \geqslant 3$. *Then, there exists $i \in N$ such that for all $x \in R_F$, $\mathcal{W}_x^m = \{\{i\}\}$.*

Proof. By properties (1) and (2) there exists $Z \in 2^K$ such that $Z \notin \mathcal{R}_F$. Without loss of generality, first assume that there exists x such that either $Z \cup \{x\} \in \mathcal{R}_F$ or $Z \backslash \{x\} \in \mathcal{R}_F$. Moreover, by rotating the hypercube to locate Z to its origin and redefining all coordinates accordingly, assume that $Z = \{\emptyset\}$ and $\{x\} \in \mathcal{R}_F$. Let $y \in R_F \backslash \{x\}$ be arbitrary. We will show that there exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i\}\}$. We will distinguish between two cases.

Case 1: There exists $D \in \mathcal{R}_F$ such that $y \in D$ and $MB(D, \{\emptyset\}) \cap \mathcal{R}_F = \{D\}$.

Subcase 1.1: Assume $MB(D \cup \{x\}, \{\emptyset\}) \neq \{\{x\}, D\}$. Since $MB(D, \{\emptyset\}) \cap \mathcal{R}_F = \{D\}$ there exists B such that $\{\emptyset\} \neq B \subseteq D$, $B \cup \{x\} \neq D$, and $B \cup \{x\} \in MB(D \cup \{x\}, \{\emptyset\}) \cap \mathcal{R}_F$.

Subcase 1.1.1: Assume $B \subsetneq D$. Without loss of generality assume that $MB(B \cup \{x\}, \{x\}) \cap \mathcal{R}_F = \{B \cup \{x\}, \{x\}\}$. Then we can generate, by an additive preference with top on $\{\emptyset\}$, the orderings $D \succ^1 \{x\} \succ^1 B \cup \{x\}$, $\{x\} \succ^2 D \succ^2 B \cup \{x\}$, and $\{x\} \succ^3 B \cup \{x\} \succ^3 D$, by an additive preference with top on B, the orderings $D \succ^4 B \cup \{x\} \succ^4 \{x\}$, $B \cup \{x\} \succ^5 \{x\} \succ^5 D$, and $B \cup \{x\} \succ^6 D \succ^6 \{x\}$. Moreover, by associating large negative values to objects outside $D \cup \{x\}$, we must be able to put these three alternatives at the tops of the individual orderings. Therefore, we have a free-triple on the elements of the range D, $\{x\}$, and $B \cup \{x\}$. Then the Gibbard–Satterthwaite Theorem implies that there exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i\}\}$.

Subcase 1.1.2: Assume $B = D$. Because $MB(D \cup \{x\}, \{\emptyset\}) \cap \mathcal{R}_F \neq \{\{x\}, D\}$ then $D \cup \{x\} \in \mathcal{R}_F$. Then $MB(D \cup \{x\}, \{x\}) \cap \mathcal{R}_F = \{\{x\}, D \cup \{x\}\}$, $MB(D \cup \{x\}, D) \cap \mathcal{R}_F = \{D, D \cup \{x\}\}$. Notice that $MB(D, \{\emptyset\}) \cap \mathcal{R}_F = \{D\}$. Therefore, using an argument similar to the one already used in the proof of Subcase 1.1.1, we have a free triple on elements of the range D, $\{x\}$ and $D \cup \{x\}$, and again, the Gibbard–Satterthwaite Theorem implies that there exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i\}\}$.

Subcase 1.2: Assume $MB(D \cup \{x\}, \{\emptyset\}) = \{\{x\}, D\}$.

Subcase 1.2.1: There exists $C \in \mathcal{R}_F$, such that $C \cap (D \cup \{x\}) \notin \{\{x\}, D\}$. Let $\overline{C} = C \cap D \cup \{x\}$ and without loss of generality assume $MB\left\{\overline{C}, C\right\} \cap \mathcal{R}_F = C$. Since $MB\left\{\overline{C}, \{x\}\right\} \cap \mathcal{R}_F = \{x\}$ and $MB\left\{\overline{C}, D\right\} \cap \mathcal{R}_F = \{D\}$ we have a free triple on elements of the range D, $\{x\}$ and C, implying that there exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i\}\}$, because $y \in D$.

Subcase 1.2.2: For all $C \in \mathcal{R}_F$, $C \cap D \cup \{x\} \in \{\{x\}, D\}$.

Claim 1. Assume that for all $C \in \mathcal{R}_F$ either $\{x\} \subseteq C$ or $D \subseteq C$. Then, there exists $A, B \in \mathcal{R}_F$ and $Z \in \{\{x\}, D\}$ such that:

(C.1) $MB(A, B) \cap \mathcal{R}_F = \{A, B\}$.
(C.2) $Z \subseteq A \cap B$.
(C.3) $MB\left(\overline{A}, \overline{B}\right) \cap \mathcal{R}_F = \overline{A}$,

where $\overline{A} = (A \cup (\{x\} \cup D)) \backslash Z$ and $\overline{B} = (B \cup (\{x\} \cup D)) \backslash Z$.

Proof of Claim 1. Since $\mathcal{R}_F$ has the property that its minimal Cartesian decomposition has a unique section there exits $G \in \mathcal{R}_F$ and $Z \in \{\{x\}, D\}$ such that $Z \subseteq G$ and $\overline{G} = (G \cup (\{x\} \cup D)) \setminus Z \notin \mathcal{R}_F$. Define

$$\overline{MB}(H, Z) = \left\{ \overline{E} \in 2^K \mid \overline{E} = (E \cup (\{x\} \cup D)) \setminus Z \text{ for } E \in MB\,(H, Z) \cap \mathcal{R}_F \right\}.$$

Denote $\sim Z = x$ if $Z = D$ or $\sim Z = D$ if $Z = x$. Because $G \in MB\,(G, Z) \cap \mathcal{R}_F$, then $\overline{G} \in \overline{MB}\,(G, Z)$. Since $\overline{G} \notin MB\left(\overline{G}, \sim Z\right) \cap \mathcal{R}_F$ then $\overline{MB}\,(G, Z) \nsubseteq MB\left(\overline{G}, \sim Z\right) \cap \mathcal{R}_F$. Let B be the element in the range with minimal L_1-distance to Z with the property that $\overline{MB}\,(B, Z) \nsubseteq MB\left(\overline{B}, \sim Z\right) \cap \mathcal{R}_F$. This implies that

$$\overline{MB}\,(B, Z) \setminus \overline{B} = MB\left(\overline{B}, \{\sim Z\}\right) \cap \mathcal{R}_F. \tag{A.3}$$

Let $A \in MB\,(B, Z) \setminus B$ be such that $MB\,(A, B) = \{A, B\}$. Condition (A.3) implies that $\overline{A} \in \mathcal{R}_F$ and $MB\left(\overline{A}, \overline{B}\right) \cap \mathcal{R}_F = \overline{A}$. This proves the Claim.

Let $A, B \in \mathcal{R}_F$ and $Z \in \{\{x\}, D\}$ be such that conditions (C.1)–(C.3) of Claim 1 hold. Then we can generate, by an additive preference with top on $A \cup \{\sim Z\}$, the orderings $A \succ^1 B \succ^1 \overline{A}$, $A \succ^2 \overline{A} \succ^2 B$, and $\overline{A} \succ^3 A \succ^3 B$, by an additive preference with top on $B \cup \{\sim z\}$, the orderings $B \succ^4 A \succ^4 \overline{A}$ and $B \succ^5 \overline{A} \succ^5 A$, and by an additive preference with top on $\overline{B}$, the ordering $\overline{A} \succ^6 B \succ^6 A$. Therefore, we have a free-triple on the elements of the range A, B, and $\overline{A}$, implying that here exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i\}\}$.

Case 2: Assume that for every $D \in \mathcal{R}_F$ such that $y \in D$, there exists $B \neq D$ such that $B \in MB\,(D, \{\emptyset\}) \cap \mathcal{R}_F$.

Let D be such that

$$MB\,(D, \{y\}) \cap \mathcal{R}_F = \{D\} \tag{A.4}$$

and let B be such that

$$MB\,(B, \{\emptyset\}) \cap \mathcal{R}_F = \{B\}. \tag{A.5}$$

If $y \in B$ then we are back to Case 1. Therefore, assume that $y \notin B$. For each $z \in B$ we can apply Case 1 and obtain that there exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_z^m = \{\{i\}\}$.

Subcase 2.1: Assume that $\{x, y\} \in \mathcal{R}_F$. We claim that $MB\,(\{y\}, B) \cap \mathcal{R}_F = \{B\}$. To see it, assume that there exists $C \neq B$ such that $C \in MB(\{y\}, B) \cap \mathcal{R}_F$. If $y \in C$ then $C \in MB(D, \{y\}) \cap \mathcal{R}_F$ contradicting (A.4). If $y \notin C$ then $C \subseteq B$, contradicting the fact that $C \neq B$ because $MB(B, \{\emptyset\}) \cap \mathcal{R}_F = \{B\}$. Moreover, since $MB\,(\{y\}, D) \cap \mathcal{R}_F = \{D\}$ and $MB\,(\{y\}, \{x, y\}) \cap \mathcal{R}_F = \{x, y\}$ we can generate all orderings on D, B, $\{x, y\}$ (with these three subsets on the top); therefore, there exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i\}\}$.

Subcase 2.2: Assume that $\{x, y\} \notin \mathcal{R}_F$. First suppose that $MB\,(\{y\}, B) \cap \mathcal{R}_F = \{B\}$. Since $MB\,(\{y\}, D) \cap \mathcal{R}_F = \{D\}$ and $MB\,(\{y\}, \{x\}) \cap \mathcal{R}_F = \{x\}$ (remember, by (A.4) we know that $y \in \mathcal{R}_F$) we can generate all orderings on D, B, and $\{x\}$ (with these three subsets on the top); therefore, there exists $i \in N$ such that $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i\}\}$. Suppose that $MB(\{y\}, B) \neq \{B\}$. We claim that $D = B \cup \{y\}$ and therefore $MB(\{y\}, B) = \{B, D\}$. To see it, let $C \in MB(\{y\}, B)$. If $y \in C$ then, by (A.4), $C = D$ and $C = D \cup \{y\}$. If $y \notin C$ then $C \subseteq B$ and, by (A.5), $C = B$. Now, if $MB\,(\{y\}, B) \cap \mathcal{R}_F = \{B, D\}$ we can also generate all orderings on D, B, and $\{x\}$ with two preferences: one with top on y (orderings $D \succ^1 B \succ^1 \{x\}$, $D \succ^2 \{x\} \succ^2 B$, and $\{x\} \succ^3 D \succ^3 B$) and the other with top on $\{\emptyset\}$ (orderings $\{x\} \succ^4 B \succ^4 D$, $B \succ^5 D \succ^5 \{x\}$, and $B \succ^6 \{x\} \succ^6 D$). □

Proof of Theorem 1. To prove necessity, let F: $\mathbf{A}^n \to 2^K$ be a strategy-proof social choice function and let $\{B_1, \ldots, B_q\}$ be the minimal Cartesian decomposition of R_F, which exists by Proposition 3.

(1) Assume that $x, y \in Z_1 \in \mathcal{AC}(B_p) = \{Z_1, Z_2\}$. Since $\{B_1, \ldots, B_q\}$ is minimal we have that $Z_1 \cap Z_2 = \{\emptyset\}$. Assume that $\mathcal{W}_x^m \neq \mathcal{W}_y^m$; that is, there exists $I \in \mathcal{W}_x^m$ such that $I \notin \mathcal{W}_y^m$. Consider any P such that $\tau(P_i) \cap B_p = Z_1$ for all $i \in I$ and $\tau(P_j) \cap B_p = Z_2$ for all $j \in N \backslash I$. Then, $x \in F(P)$ and $y \notin F(P)$ contradicting that x and y belong to the same active component of B_p.

(2) Assume $x \in X$, $y \in Y$, and $\mathcal{AC}(B_p) = \{X, Y\}$. To obtain a contradiction assume there exists $D \in \mathcal{W}_x^m$ and $N \backslash D \in \mathcal{W}_y^m$. It is easy to find P such that $x, y \in F(P)$ contradicting that x and y belong to different active components of B_p.

(3) Follows from (1) of Proposition A.2 and Proposition A.3.

Sufficiency follows from (2) of Proposition A.2, since it is clear that all social choice functions defined on each of the sections are onto the active components of the section and strategy-proof. □

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