Not All Majority-based Social Choice Functions Are Obviously Strategy-proof*

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Abstract: We consider two families of strategy-proof social choice functions based on the majority principle: extended majority voting rules on the universal domain of preferences over two alternatives and generalized median voter schemes on the domain of single-peaked preferences over a finite and linearly ordered set of alternatives. We characterize their respective (and substantially smaller) subclasses of obviously strategy-proof social choice functions and for each one of them we identify an extensive game form that implements it in obviously dominant strategies.

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1 Introduction

A social choice function (mapping preference profiles into alternatives) is strategy-proof if it is always in the agents' interest to reveal their preferences truthfully. This means that, in the direct revelation mechanism induced by the social choice function, the strategic problems faced by agents when submitting their preferences are not interrelated: truth-telling is an optimal decision for each agent, irrespective of the other agents' decisions. Hence, strategy-proofness is a very desirable property of a social choice function: the information that each agent has about the other agents' preferences is irrelevant.

However, the Gibbard-Satterthwaite Theorem (Gibbard (1973) and Satterthwaite (1975)) indicates the difficulties in designing non-trivial and strategy-proof social choice functions: if the set of alternatives is strictly greater than two, all unanimous and strategy-proof social choice functions on the universal domain of preferences over the set of alternatives are dictatorial. Yet, and despite this negative result, there is abundant literature studying and characterizing classes of strategy-proof social choice functions for specific settings where the Gibbard-Satterthwaite Theorem does not apply. Some consider the case where the cardinality of the set of alternatives is equal to two, when all extensions of the majority voting rule constitute the class of all strategy-proof social choice functions on the universal domain of strict preferences over two alternatives. Others question the assumption that agents may have (and submit to the mechanism) all conceivable preferences since the properties of the set of alternatives suggest that appropriate social choice functions should operate only on natural and meaningful restricted domains of preferences, those that are in agreement with the corresponding structure of the set of alternatives. There are many settings for which the class of strategy-proof social choice functions operating on a particular restricted domain is large;[1] for instance, the class of generalized median voter schemes on the domain of ordinal and single-peaked preferences over a linearly ordered set of alternatives is large.

Nevertheless, the mechanism design literature has mainly neglected two features of direct revelation mechanisms when used to implement strategy-proof social choice functions on restricted domains of preferences. The first one is related to the ease with which agents can realize that their truth-telling strategies are indeed weakly dominant (*i.e.*, how much contingent reasoning is required to do so).[2] The second one is related to the

[1] Often, we know in addition axiomatic characterizations of classes of strategy-proof social choice functions satisfying additional desirable properties.

[2] Attiyeh, Franciosi and Isaac (2000), Cason, Saijo, Sjöström and Yamato (2006), Friedman and Schenker (1998), Kawagoe and Mori (2001) and Yamamura and Kawasaki (2013) are some examples of papers dealing with this issue. Glazer and Rubinstein (1996) already argues that complexity considerations should be used to evaluate alternative mechanisms implementing a social choice function; in particular, they suggest the convenience of using extensive game forms and show that for solvable games

degree of bilateral commitment of the designer who, after collecting the revealed profile of agents' preferences, will supposedly implement the alternative that the social choice function would have chosen at the revealed profile, regardless of whether he likes it or not.[3] For example, and following Li (2017), when in a second-price sealed-bid auction the designer is simultaneously the seller of the good, he has a strong temptation to introduce an additional bid above the second submitted bid and slightly below the first one.[4] Implicitly, a vast majority of this literature has assumed that the designer can commit to not circumvent the mechanism.

Li (2017) proposes the notion of obvious strategy-proofness to deal simultaneously with both concerns (see Theorems 1 and 2 in Li (2017)). A social choice function f, on a domain $\mathcal{D}$ of profiles of n-tuples of preferences (where n is the number of agents), is obviously strategy-proof if there exists an extensive game form (or simply a *game*) Γ, whose set of outcomes is the set of alternatives, with two properties.

First, for each preference profile $P = (P_1, \ldots, P_n) \in \mathcal{D}$ one can identify a profile of truth-telling (behavioral) strategies $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n})$ with the property that if each agent i plays the game Γ according to $\sigma_i^{P_i}$, the outcome of Γ would correspond to the alternative selected by f at P; that is, Γ induces f.

Second, at Γ, agents use the two most extreme behavioral assumptions when comparing the consequences of behaving according to the truth-telling strategy with the consequences of behaving differently. In particular, for agent i with preference P_i, let σ_i' be any non-truthful strategy of agent i (*i.e.*, $\sigma_i' \neq \sigma_i^{P_i}$). Consider an earliest point of departure of $\sigma_i^{P_i}$ with σ_i'; namely, an information set I_i in Γ at which, for the first time along Γ, $\sigma_i^{P_i}$ and σ_i' are taking a different action. Then, i evaluates the consequence of choosing the action prescribed by $\sigma_i^{P_i}$ at I_i according to the *worst* possible outcome, among all outcomes that may occur as an effect of later choices made by agents along the rest of the game (fixing i's behavior to $\sigma_i^{P_i}$). In contrast, i evaluates the consequence of choosing the action prescribed by σ_i' at I_i according to the *best* possible outcome, among all outcomes that may occur, again as an effect of later choices made by agents along the rest of the game (fixing i's behavior to σ_i'). Then, $\sigma_i^{P_i}$ is obviously dominant at Γ if for any other strategy

the calculation required to obtain the set of strategies that survive iterative elimination of dominated strategies is equivalent to the calculation required to identify the backward induction outcome of a game in extensive form.

[3]Bag and Sharma (2016) is an example of a paper that considers a setting where the designer does not have commitment power at all.

[4]In the earlier wave of auctions to sell portions of the spectrum to be used for communications in New Zealand, second-price sealed-bid auctions were used. And many of them were not very successful (see MacMillan, 1994); for instance, a lot was sold for a price of NZ$6 (the second highest bid) to a bidder who placed a bit for NZ$100,000 (auctions were conducted without reserve prices!). Since 2004, New Zealand uses mostly outcry English ascending auctions.

$\sigma_i' \neq \sigma_i^{P_i}$, and from the point of view of any earliest point of departure of $\sigma_i^{P_i}$ with σ_i', the outcome of the *pessimistic* view used to evaluate $\sigma_i^{P_i}$ is at least as preferred as the outcome of the *optimistic* view used to evaluate σ_i'. If Γ induces f and, for all $P \in \mathcal{D}$ and all i, $\sigma_i^{P_i}$ is obviously dominant at Γ, then f is obviously strategy-proof.[5]

In this paper we consider two families of strategy-proof social choice functions, all based on generalizations of the majority voting procedure, and characterize their obviously strategy-proof subclasses. The notion of a committee plays a fundamental role in their description. Fix a set of agents. A *committee* is a monotonic family of subsets of agents (called winning coalitions). A winning coalition is *minimal* if it has no strict subset that is also winning.

Consider first a social choice problem with only two alternatives, x and y, and assume that agents have strict preferences over the set $\{x, y\}$. Then, a social choice function f on this domain of preferences is an *Extended Majority Voting Rule* if there exists a committee for x with the property that, for each preference profile P, x is selected by f at P if and only if the set of agents for whom x is strictly preferred to y belongs to the committee for x. It is well known that, for the case of two alternatives, a social choice function is strategy-proof if and only if it is an extended majority voting rule.

We then ask: what is the condition that a committee for x has to satisfy, so that its induced extended majority voting rule is in addition obviously strategy-proof? We identify this property, call it *Increasing Order Inclusion*, and show in Proposition 1 that it is necessary and sufficient for obvious strategy-proofness.[6] In particular, among the class of all anonymous extended majority voting rules, only those two where either x or y can be imposed by each agent unilaterally (referred to as quota 1) are obviously strategy-proof; or equivalently, only those where, to be elected, either y or x needs a unanimous support (referred to as quota n). Thus, in anonymous voting environments with two alternatives, obviously strategy-proofness has a strong bite: it requires that an alternative has to be singled out as a status quo (for instance, x) and then this alternative is always selected except when the other alternative receives unanimous support (quota 1 for x or, equivalently, quota n for y). In fact, there are voting settings where agents have veto power (or the status quo can only be changed by unanimity). For instance, the Council of the European Union (EU) has to vote unanimously on a number of matters which the state members consider to be sensitive, like common foreign and security policy,

[5] Observe three things. First, the equilibrium concept used for obviously strategy-proof implementation is obviously dominance. Second, the implementation is weak since it is not required that truth-telling be the unique obviously dominant strategy. Third, obvious strategy-proofness is a very demanding requirement.

[6] The sufficiency part of our proof shows how to construct, using the increasing order inclusion property of the extended majority voting rule f, the extensive game form that implements f in obviously dominant strategies.

citizenship (the granting of new rights to EU citizens), EU membership, harmonizations of national legislation on indirect taxation, etc.; or the jury in most of the United States' trials has to vote unanimously to convict a person.[7] We give here an additional reason for the use of unanimity, based on its strong incentive properties.

Consider now a social choice problem where the set of alternatives X is a finite and linearly ordered set.[8] For instance, when X is the set of possible levels of a public good, political parties' platforms, location of a public good in a one-dimensional space, etc. Assume that agents have strict single-peaked preferences (over X). A preference is *single-peaked* over X if it is monotonic in both sides of the best alternative: increasing at its left and decreasing at its right. There is a large literature studying this class of problems. It is well known that a social choice function is strategy-proof on the single-peaked domain of preferences over X if and only if it is a *Generalized Median Voter Scheme*.[9]

We now ask: what are the conditions that a generalized median voter scheme has to satisfy to be obviously strategy-proof? We identify the two properties that together answer this question for the general case and, given a generalized median voter scheme satisfying them, we exhibit an extensive game form that implements it in obviously dominant strategies. To give the main idea of those extensive form games, consider the anonymous median voter scheme f that selects, at each preference profile, the smallest of the best alternatives. Then f can be roughly understood as a sequence of extended majority voting rules (of quota 1) that at any generic alternative x, and starting at α, confronts two possibilities: select (using quota 1) the current alternative x as the one chosen by f or select (tentatively) the adjacent alternative $x+1$. If x is not chosen, then its adjacent alternative $x+1$ becomes the new current alternative that is confronted to $x+2$, applying again quota 1. This generalized median voter scheme f is obviously strategy-proof because whenever agent i has to decide at a node along the game, i's choices can be identified with the choice between the current alternative and its adjacent one. And i can make sure that the current alternative is the one finally selected because quota 1 is used to choose between the two alternatives. Proposition 2 generalizes this result to the class of all (not necessarily anonymous) generalized median voter schemes.

For binary allocation problems,[10] Li (2017) characterizes the monotone price mechanisms (generalizations of ascending auctions) as those that implement all obviously strategy-proof social choice functions on the domain of quasi-linear preferences. He also

[7] See for instance Buchanan and Tullock (1962), Tsebelis (2002), König and Slapin (2006), Maggi and Morelli (2006), and Bouton, Llorente-Saguer and Malherbe (2016).

[8] Without loss of generality we may assume that $X = \{\alpha, \alpha+1, \ldots, x-1, x, x+1, \ldots, \beta-1, \beta\}$.

[9] See for instance Moulin (1980) or Barberà, Gül and Stacchetti (1993). Generalized median voter schemes are non-anonymous extensions of the median voter (see Section 5 for their description).

[10] For instance, private value auctions with unit demands, procurement auctions, and the provision of a binary public good with no exclusion.

shows that, for online advertising auctions, the social choice function induced by the mechanism that selects the efficient allocation and the Vickrey-Clarke-Groves payment is obviously strategy-proof. Furthermore, he shows that the social choice function associated to the top-trading cycles algorithm in the house allocation problem of Shapley and Scarf (1974) is not obviously strategy-proof. Finally, Li (2017) reports a laboratory experiment where subjects play significantly more often their truth-telling dominant strategies when the strategy-proof mechanism they play is also obviously strategy-proof.

In addition to these specific results in Li (2017), five other papers have also asked whether well-known strategy-proof social choice functions on restricted domains of preferences are obviously strategy-proof. Ashlagi and Gonczarowski (2016) shows that the social choice function associated to the deferred acceptance algorithm is not obviously strategy-proof for the agents belonging to the offering side. They show however that this social choice function becomes obviously strategy-proof on the restricted domain of acyclic preferences introduced by Ergin (2002).

Troyan (2016) identifies a necessary and sufficient condition on the priorities (called weak acyclic, weaker than the conditions identified in Ergin (2002) and Kesten (2006)) that fully characterizes the class of obviously strategy-proof social choice functions associated to the generalizations of the top-trading cycles algorithm with priorities, introduced by Abdulkadiroğlu and Sönmez (2003).

Pycia and Troyan (2017) characterizes the family of extensive game forms that implement, in obviously dominant strategies, social choice functions for a class of ordinal problems that includes the cases of private components and voting over two alternatives. They call those games millipede because they have the property that the subgames starting at the nodes that follow nature's moves are like a centipede game (see Rosenthal (1981)), but now agents, at nodes where they have to choose along the game, may have more than one terminal choice. This characterization can be seen as a revelation principle like result, because it indicates the class of extensive game forms where to look for the implementation of social choice functions in obviously dominant strategies. They also consider, as a particular case of their model, the problem of allocating a set of objects to a set of agents when each agent only cares about the received object. They characterize, for this case, the family of obviously strategy-proof, efficient and symmetric extensive game forms as those that are equivalent to random priority rules.

Bade and Gonczarowski (2017) establishes also a revelation principle like result for obviously strategy-proofness: a social choice function is implementable in obviously dominant strategies if and only if some obviously incentive compatible gradual mechanism implements it. For the problem of assigning a set of objects to a set of agents, Bade and Gonczarowski (2017) shows that an efficient social choice function is obviously strategy-proof if and only if it can be implemented by a game with sequential barters with lurkers; this class consists of generalizations of serial dictatorships. They also show that Li (2017)'s positive result on monotone price mechanisms for binary allocation problems does not hold for more general problems with two or more goods. For the case of voting over two alternatives, Bade and Gonczarowski (2017) shows that if a social choice function is onto and obviously strategy-proof then it can be implemented by a proto-dictatorship game. Finally, for the problem of a linearly ordered set of alternatives with single-peaked preferences, Bade and Gonczarowski (2017) shows that if a social choice function is onto and obviously strategy-proof then it can be implemented by an extensive game form consisting of dictatorships with safeguards against extremisms (and arbitration via proto-dictatorships, if X is discrete).

Finally, Mackenzie (2017) contains a general revelation principle like result identifying the class of round table mechanisms: a social choice function f is obviously strategy-proof implementable if and only if f is strategy-proof implementable through a round table mechanism.[11]

We want to emphasize that, in contrast with the existing positive results described above, our characterizations are not revelation principle like results identifying a class (often very large) of extensive game forms where, without loss of generality (but not necessarily), the designer has to look for in order to implement in obviously dominant strategies a particular social choice function. But they do not identify the specific mechanism, among all in the class, that has to be used in order to implement this given social choice function; and this is important because different mechanism in the class may implement different social choice functions. Our proofs are constructive: for each obviously strategy-proof social choice function, we exhibit (and show how to construct) an extensive game form that implements the social choice function in obviously dominant strategies. Our characterizations deliver necessary and sufficient conditions for two important classes of social choice functions; given one in any of the two classes, one can easily check if it is obviously strategy-proof by using our conditions, since they are short and reasonably transparent. In the final remarks section, at the end of the paper, we relate some of the results in Pycia and Troyan (2017) and Bade and Gonczarowski (2017) with our results with more detail.

The paper is organized as follows. Section 2 contains the basic notation and definitions. Section 3 presents the notion of obvious strategy-proofness. Section 4 contains the analysis of extended majority voting rules from the point of view of obvious strategy-proofness, while Sections 5 contains the corresponding analysis of generalized median voter schemes. Section 6 concludes with final remarks. An Appendix at the end of the paper collects the proofs omitted in the main text.

[11] We have obtained our results in an independent way, before knowing the existence of Pycia and Troyan (2017), Bade and Gonczarowski (2017), and Mackenzie (2017).

2 Preliminaries

A set of *agents* $N = \{1, \ldots, n\}$, with $n \geq 2$, has to choose an alternative from a finite and given set X. Each agent $i \in N$ has a strict *preference* P_i (a linear order) over X. We denote by $t(P_i)$ the best alternative according to P_i, to which we will refer to as the *top* of P_i. We denote by R_i the weak preference over X associated to P_i; *i.e.,* for all $x, y \in X$, xR_iy if and only if either $x = y$ or xP_iy. Let $\mathcal{P}_i$ be the set of all strict preferences over X. Observe that $\mathcal{P}_i = \mathcal{P}_j$ for all $i \neq j$. A (preference) *profile* is a n-tuple $P = (P_1, \ldots, P_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n = \mathcal{P}$, an ordered list of n preferences, one for each agent. Given a profile P and an agent i, P_{-i} denotes the subprofile in $\prod_{j \in N \setminus \{i\}} \mathcal{P}_j$ obtained by deleting P_i from P. Given $i \in N$ and $x \in X$ we write $P_i^x \in \mathcal{P}_i$ to denote a generic preference such that $t(P_i^x) = x$.

Let $\mathcal{D}_i \subseteq \mathcal{P}_i$ be a generic subset of agent i's preferences over X and set $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$, which we will refer to as a *domain*.[12] A *social choice function* (SCF) on $\mathcal{D}$, $f : \mathcal{D} \to X$, selects for each preference profile $P \in \mathcal{D}$ an alternative $f(P) \in X$.

The SCF $f : \mathcal{D} \to X$ is *strategy-proof* (SP) if for all $P \in \mathcal{D}$, all $i \in N$ and all $P_i' \in \mathcal{D}_i$,

$$f(P)R_if(P_i', P_{-i}).$$

Let $f : \mathcal{D} \to X$ be a given SCF. Construct its associated normal game form $(N, \mathcal{D}, f)$, where N is the set of players, $\mathcal{D}$ is the set of strategy profiles and f is the outcome function mapping strategy profiles into alternatives. Then, f is implementable in dominant strategies (or f is SP-implementable) if the normal game form $(N, \mathcal{D}, f)$ has the property that, for all $P \in \mathcal{D}$ and all $i \in N$, P_i is a weakly dominant strategy for i in the game in normal form $(N, \mathcal{D}, f, P)$, where each $i \in N$ uses P_i to evaluate the consequences of strategy profiles. The literature refers to $(N, \mathcal{D}, f)$ as the direct revelation mechanism that SP-implements f.

We define several properties that a SCF $f : \mathcal{D} \to X$ may satisfy and that we will use in the sequel. We say that f is (i) *onto* if for all $x \in X$, there exists $P \in \mathcal{D}$ such that $f(P) = x$; (ii) *unanimous* if for all $P \in \mathcal{D}$ such that $t(P_i) = x$ for all $i \in N$, $f(P) = x$;[13] and (iii) *anonymous* if for all $P \in \mathcal{D}$ (where $\mathcal{D}_i = \mathcal{D}_j$ for all $i \neq j$) and all one-to-one $\pi : N \to N$, $f(P) = f(P^\pi)$ where for all $i \in N$, $P_i^\pi = P_{\pi(i)}$. We say that i is a *dummy* agent in f if for all P_{-i}, $f(P_i, P_{-i}) = f(P_i', P_{-i})$ for all $P_i, P_i' \in \mathcal{D}_i$.

[12] In our two applications it will hold that $\mathcal{D}_i = \mathcal{D}_j$ for all $i \neq j$.

[13] Although ontoness is weaker than unanimity, it is easy to see that among the class of all strategy-proof SCFs, the classes of unanimous and onto SCFs coincide.

3 Obviously strategy-proof SCFs

3.1 Definition

Adapting Li (2017) to our ordinal voting setting with no uncertainty, an *extensive game form with consequences in X* consists of:

1. A set of agents $N = \{1, \ldots, n\}$.

2. A set of alternatives X.

3. A rooted tree $(Z, \prec)$, where:

   (a) Z is the set of nodes;

   (b) $\prec$ is an irreflexive and transitive binary relation over Z;

   (c) $z_0 \in Z$ is the root of $(Z, \prec)$;[14]

   (d) Z can be partitioned into two sets, the set of terminal nodes $Z_T = \{z \in Z \mid \text{there is no } z' \in Z \text{ such that } z \prec z'\}$ and the set of non-terminal nodes $Z_{NT} = \{z \in Z \mid \text{there is } z' \in Z \text{ such that } z \prec z'\}$.

4. A mapping $\mathcal{N} : Z_{NT} \to N$ that assigns to each non-terminal node z an agent $\mathcal{N}(z)$.[15]

5. For each $i \in N$, a partition of Z_i into information sets. Denote by $\mathcal{I}_i$ this partition and by I_i one of its generic elements.[16]

6. A set of actions A and a function $\mathcal{A} : Z_{NT} \to 2^A \backslash \{\emptyset\}$ where, for each $z \in Z_{NT}$, $\mathcal{A}(z)$ is the non-empty set of actions available to player $\mathcal{N}(z)$ at z.[17]

7. An outcome function $g : Z_T \to X$ that assigns an alternative $g(z) \in X$ to each terminal node $z \in Z_T$.

[14] Namely, z_0 is the unique node that has the property that $z_0 \prec z$ for all $z \in Z \backslash \{z_0\}$.

[15] Hence, we can partition the set of non-terminal nodes Z_{NT} into n disjoint sets $Z_1, \ldots, Z_n$, where $Z_i = \{z \in Z_{NT} \mid \mathcal{N}(z) = i\}$ is the set of non-terminal nodes assigned to i by $\mathcal{N}$. To deal in the sequel with dummy agents, we admit the possibility that $\mathcal{N}$ be not onto, and so $Z_i = \emptyset$ for some $i \in N$.

[16] If $z, z' \in I_i$, then agent i cannot distinguish whether the game has reached node z_i or node z_i'.

[17] Of course, $\mathcal{A}$ has to be measurable in the sense that for any pair $z, z' \in I_i$, $\mathcal{A}(z) = \mathcal{A}(z')$. Moreover, for each $z \in Z_{NT}$, there should be a one-to-one identification between $\mathcal{A}(z)$ and the set of immediate followers of z defined as $IF(z) = \{z' \in Z \mid z \prec z' \text{ and there is no } z'' \in Z_{NT} \text{ such that } z \prec z'' \prec z'\}$. Set $\mathcal{I} = (\mathcal{I}_i)_{i \in N}$. We assume that $\mathcal{I}$ has the usual property to ensure that agents have perfect recall.

An extensive game form with consequences in X (or simply, a *game*) is a seven-tuple $\Gamma = (N, X, (Z, \prec), \mathcal{N}, \mathcal{I}, \mathcal{A}, g)$ with the above properties.[18] Since N and X will be fixed throughout the paper, let $\mathcal{G}$ be the class of all games with consequences in X and set of agents N.

Fix a game $\Gamma \in \mathcal{G}$ and an agent $i \in N$. A (behavioral and pure) *strategy* of i in Γ is a function $\sigma_i : Z_i \to A$ such that for each $z \in Z_i$, $\sigma_i(z) \in \mathcal{A}(z)$; namely, σ_i selects at each node where i has to play one of i's available actions. Moreover, σ_i is $\mathcal{I}_i$-measurable: for any $I_i \in \mathcal{I}_i$ and any pair $z, z' \in I_i$, $\sigma_i(z) = \sigma_i(z')$. Let Σ_i be the set of i's strategies in Γ. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_1 \times \cdots \times \Sigma_n = \Sigma$ is an ordered list of strategies, one for each agent. For $z \in Z\backslash\{z_0\}$, define the set of immediate predecessors of z as $IP(z) = \{z' \in Z \mid z \in IF(z')\}$.[19] A history h (of length t) is a sequence $z_0, z_1, \ldots, z_t$ of $t+1$ nodes, starting at z_0, such that for all $m = 1, \ldots, t$, $\{z_{m-1}\} = IP(z_m)$. Each history $h = z_0, \ldots, z_t$ can be uniquely identified with the node z_t and each node z can be uniquely identified with the history $h = z_0, \ldots, z$.

For a distinct pair $\sigma_i, \sigma_i' \in \Sigma_i$, the family of earliest points of departure for σ_i and σ_i' is the family of information sets where σ_i and σ_i' have made identical decisions at all previous information sets, but they are making a different decision at those information sets. Namely,

Definition 1 Let $\sigma_i, \sigma_i' \in \Sigma_i$. An information set $I_i \in \alpha(\sigma_i, \sigma_i')$ is an *earliest point of departure* for σ_i and σ_i' if for all $z \in I_i$:
1. $\sigma_i(z) \neq \sigma_i'(z)$.
2. $\sigma_i(z') = \sigma_i'(z')$ for all $z' \prec z$ such that $z' \in Z_i$.

Given a pair $\sigma_i, \sigma_i' \in \Sigma_i$, denote the set of earliest points of departure for σ_i and σ_i' by $\alpha(\sigma_i, \sigma_i')$. Given $\widehat{X} \subseteq X$ and $P_i \in \mathcal{D}_i$, we denote by $\min_{P_i} \widehat{X}$ the alternative $x \in \widehat{X}$ such that for all $y \in \widehat{X}$, yR_ix, and by $\max_{P_i} \widehat{X}$ the alternative $x \in \widehat{X}$ such that for all $y \in \widehat{X}$, xR_iy. Let $z^\Gamma(z, \sigma)$ be the terminal node that results in Γ when agents start playing at z according to σ. We are now ready the define obviously dominant strategies.

Definition 2 Let $\Gamma \in \mathcal{G}$ be a game and $P_i \in \mathcal{D}_i$ be a preference for agent $i \in N$. We say that σ_i is *obviously dominant* in Γ for i with P_i if for all $\sigma_i' \neq \sigma_i$ and all $I_i \in \alpha(\sigma_i, \sigma_i')$,

$$\begin{aligned} &\min_{P_i}\{x \in X \mid \text{there exist } \sigma_{-i} \in \Sigma_{-i} \text{ and } z \in I_i \text{ such that } x = g(z^\Gamma(z, (\sigma_i, \sigma_{-i})))\} \\ R_i &\max_{P_i}\{x \in X \mid \text{there exist } \sigma_{-i} \in \Sigma_{-i} \text{ and } z \in I_i \text{ such that } x = g(z^\Gamma(z, (\sigma_i', \sigma_{-i})))\}. \end{aligned}$$

[18] Note that the set of actions A is embeded in the definition of $\mathcal{A}$. Moreover, Γ is not yet a game in extensive form because agents' preferences over alternatives are still missing. But given a game Γ and a preference profile P over X, the pair (Γ, P) defines a game in extensive form where each agent i uses P_i to evaluate the alternatives, associated to all terminal nodes, induced by strategy profiles (defined below).

[19] Since $(Z, \prec)$ is a rooted tree, it has the property that, for all $z \in Z\backslash\{z_0\}$, $|IP(z)| = 1$ (namely, the tree is a graph with no curls).

Definition 3 The SCF $f : \mathcal{D} \to X$ is *obviously strategy-proof* (OSP, or OSP-*implementable*) if there exists $\Gamma \in \mathcal{G}$ such that (i) for each $P \in \mathcal{D}$, there exists a strategy profile $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$ such that $f(P) = g(z^\Gamma(z_0, \sigma^P))$ and (ii) for all $i \in N$ and all $P_i \in \mathcal{D}_i$, $\sigma_i^{P_i}$ is obviously dominant in Γ for i with P_i.

When (i) holds we say that $(\Gamma, \{\sigma^P\}_{P\in\mathcal{D}})$ *induces* f. When (i) and (ii) hold we say that $(\Gamma, \{\sigma^P\}_{P\in\mathcal{D}})$ OSP-*implements* f and refer to the strategy $\sigma_i^{P_i}$ played by i with P_i in Γ as the *truth-telling* strategy.[20] When $\{\sigma^P\}_{P\in\mathcal{D}}$ is obvious from the context we will just say respectively that Γ induces f and Γ OSP-implements f.

Obvious strategy-proofness entails an extreme behavioral hypothesis: agents are pessimistic when evaluating the consequences of truth-telling while they are optimistic when evaluating non-truthfulness.

It is easy to verify that similarly to what happens with SP-implementability, OSP-implementability is a hereditary property of SCFs in the following sense.[21]

Remark 1 If $f : \mathcal{D} \to X$ is OSP-implementable, then the subfunction $f : \widetilde{\mathcal{D}} \to X$ is OSP-implementable, where $\widetilde{\mathcal{D}}_i \subseteq \mathcal{D}_i$ for all $i \in N$.

3.2 The Pruning Principle

To show that a SCF $f : \mathcal{D} \to X$ is OSP, it is sufficient to exhibit a game $\Gamma \in \mathcal{G}$ that induces f, and that, for all $P \in \mathcal{D}$ and all $i \in N$, $\sigma_i^{P_i}$ is an obviously dominant strategy. Apparently, to show that f is not OSP, it would be necessary to check that for each Γ that induces f, there are i and P_i for which $\sigma_i^{P_i}$ is not obviously dominant in Γ. And this may be very difficult, indeed. The Pruning Principle facilitates this task. The idea is as follows. Let Γ be a game that induces a SCF $f : \mathcal{D} \to X$. Now, prune Γ by just keeping (from the tree used to define Γ) the plays consistent with the truth-telling strategies $\{\sigma^P\}_{P\in\mathcal{D}}$. Namely, histories that are not realized for any profile of preferences are deleted. Denote this pruned game by $\widetilde{\Gamma}$. Then, it holds that if Γ OSP-implements f, then $\widetilde{\Gamma}$ also OSP-implements f. Therefore, to show that f is not OSP it is sufficient to show that no "pruned" game OSP-implements f, and this seems much easier.

We now, following Li (2017), state the Pruning Principle formally. Assume $\Gamma \in \mathcal{G}$ induces $f : \mathcal{D} \to X$ and consider the set of strategy profiles $\{\sigma^P\}_{P\in\mathcal{D}}$. The extensive

[20] To better understand the meaning of $\sigma_i^{P_i}$ it may be useful to use the Bayesian interpretation of a strategy in an incomplete information game: each player i, before knowing his type $P_i \in \mathcal{D}_i$, chooses a strategy to play Γ, contingent on his realized type. Hence, $\sigma_i^{P_i}$ is the strategy played by i, when i's type is P_i, in the game Γ. Observe that, since whether or not $\sigma_i^{P_i}$ is obviously dominant is independent of $(P_j)_{j\in N\setminus\{i\}} \in \prod_{j\in N\setminus\{i\}} \mathcal{D}_j$, $\sigma_i^{P_i}$ can also be interpreted as i's play with type P_i in any game in extensive form $(\Gamma, (P_i, (P_j)_{j\in N\setminus\{i\}}))$. Since Γ will induce $f : \mathcal{D} \to X$, $\sigma_i^{P_i}$ will become meaningful.

[21] The proof of Proposition 5 in Li (2017) contains this observation.

game form $\widetilde{\Gamma} = (N, X, (\widetilde{Z}, \prec), \widetilde{\mathcal{N}}, \widetilde{\mathcal{I}}, \widetilde{\mathcal{A}}, \widetilde{g}) \in \mathcal{G}$ with consequences in X, called the *pruning* of Γ with respect to $\{\sigma^P\}_{P\in\mathcal{D}}$, is defined as follows:
(i) $\widetilde{Z} = \{z \in Z \mid \text{there is } P \in \mathcal{D} \text{ such that } z \preceq z^\Gamma(z_0, \sigma^P)\}$.
(ii) For all i, if $I_i \in \mathcal{I}_i$ then $I_i \cap \widetilde{Z} \in \widetilde{\mathcal{I}}_i$.
(iii) $(\prec, \widetilde{\mathcal{N}}, \widetilde{\mathcal{I}}, \widetilde{\mathcal{A}}, \widetilde{g})$ are restricted to $\widetilde{Z}$.

The Pruning Principle (Proposition 2 in Li (2017)) Assume $\Gamma \in \mathcal{G}$ induces $f : \mathcal{D} \to X$ and let $\widetilde{\Gamma}$ be the pruning of Γ with respect to $\{\sigma^P\}_{P\in\mathcal{D}}$. Denote by $\{\widetilde{\sigma}^P\}_{P\in\mathcal{D}}$ the restriction of $\{\sigma^P\}_{P\in\mathcal{D}}$ on $\widetilde{\Gamma}$. If $(\Gamma, \{\sigma^P\}_{P\in\mathcal{D}})$ OSP-implements $f : \mathcal{D} \to X$, then $(\widetilde{\Gamma}, \{\widetilde{\sigma}^P\}_{P\in\mathcal{D}})$ OSP-implements $f : \mathcal{D} \to X$.

4 Extended majority voting rules

Consider the simplest social choice problem where $X = \{x, y\}$. To define the family of extended majority voting rules on $\{x, y\}$, fix $w \in \{x, y\}$. A family $\mathcal{L}_w \subset 2^N$ of subsets of N is a *committee* for w if it satisfies the following monotonicity property: $S \in \mathcal{L}_w$ and $S \subsetneq T$ imply $T \in \mathcal{L}_w$. A monotonic $\mathcal{L}_w$ that is either empty ($\mathcal{L}_w = \{\emptyset\}$) or contains the empty set ($\{\emptyset\} \in \mathcal{L}_w$) is called a trivial committee.[22]

Definition 3 A SCF $f : \mathcal{P} \to \{x, y\}$ is an *extended majority voting rule* (EMVR) if there exists a committee $\mathcal{L}_w$ for $w \in \{x, y\}$ with the property that for all $P \in \mathcal{P}$,

$$f(P) = w \text{ if and only if } \{i \in N \mid t(P_i) = w\} \in \mathcal{L}_w. \tag{1}$$

In this case we say that $\mathcal{L}_w$ is the committee associated to f. Observe that if the EMVR is onto, then its associated committee (for w) $\mathcal{L}_w$ is not trivial (*i.e.*, $\{\emptyset\} \notin \mathcal{L}_w \neq \{\emptyset\}$). However, if the EMVR is not onto, and so it is constant, then $\{\emptyset\} \in \mathcal{L}_w$ if it is the constant w and $\mathcal{L}_w = \{\emptyset\}$ if it is the constant $w' \neq w$. Since constant SCFs are trivially OSP, from now on we will assume that all committees under consideration are not trivial.

The following remark says that if an EMVR can be simultaneously represented by a committee for x and a committee for y, then the two committees have to satisfy a consistency property, stated as condition (2) below.

Remark 2 Let $f : \mathcal{P} \to \{x, y\}$ be an EMVR. Let $\mathcal{L}_x$ be its associated committee for x (*i.e.*, condition (1) holds for $w = x$) and let $\mathcal{L}_y$ be a committee for y with the property that

$$S \in \mathcal{L}_y \text{ if and only if } S \cap S' \neq \emptyset \text{ for all } S' \in \mathcal{L}_x. \tag{2}$$

Then, condition (1) holds for $w = y$ as well; namely,

$$f(P) = y \text{ if and only if } \{i \in N \mid t(P_i) = y\} \in \mathcal{L}_y.$$

[22] A non-trivial committee can be seen as a monotonic simple TU-game (N, v) in which, in addition to $v(\emptyset) = 0$ and $v(N) = 1$, a coalition $S \subseteq N$ belongs to the committee if and only if $v(S) = 1$.

That is, an EMVR f can be associated indistinctly to its committee for x, $\mathcal{L}_x$, or to its committee for y, $\mathcal{L}_y$, whenever (2) holds.

Given $\mathcal{L}_x$ we denote by $\mathcal{L}_x^m$ the family of minimal winning coalitions of $\mathcal{L}_x$; that is, $S \in \mathcal{L}_x^m$ if and only if $S \in \mathcal{L}_x$ and $S' \notin \mathcal{L}_x$ for all $S' \subsetneq S$. Agent $i \in N$ is a *dummy* in $\mathcal{L}_x$ if $i \notin \cup_{S \in \mathcal{L}_x^m} S$. Obviously, agent i is a dummy in the EMVR $f : \mathcal{P} \to \{x, y\}$ if and only if i is a dummy in $\mathcal{L}_x$, where $\mathcal{L}_x$ is the committee associated to f. Agent i is *decisive* in $\mathcal{L}_x$ if $\{i\} \in \mathcal{L}_x$ and a *vetoer* in $\mathcal{L}_x$ if $i \in \cap_{S \in \mathcal{L}_x} S$.

4.1 Anonymous extended majority voting rules

Before considering the general case, we focus on the anonymous subfamily of EMVRs, those for which agents' identities do not play any role, and so their associated committees have the property that either all coalitions with the same cardinality belong to the committee or they do not.

A committee $\mathcal{L}_x$ is *voting by quota* $q \in \{1, \ldots, n\}$ if the following holds: $S \in \mathcal{L}_x$ if and only if $|S| \geq q$ (or equivalently, $\mathcal{L}_x^m = \{S \in \mathcal{L}_x \mid |S| = q\}$).

The following remark states two useful characterizations of strategy-proof SCFs in this setting with two alternatives.

Remark 3

(3.1) A SCF $f : \mathcal{P} \to \{x, y\}$ is strategy-proof if and only if f is an EMVR.

(3.2) A SCF $f : \mathcal{P} \to \{x, y\}$ is strategy-proof and anonymous if and only if the associated committee of f is voting by quota.

Proposition 0 *A* SCF $f : \mathcal{P} \to \{x, y\}$ *is anonymous and* OSP *if and only if* f *is an* EMVR *whose associated committee* $\mathcal{L}_x$ *is either voting by quota* 1 *or voting by quota* n.[23]

Proof Let f be an EMVR whose associated committee $\mathcal{L}_x$ is voting by quota 1, and so f is anonymous. We want to show that f is OSP. Without loss of generality, take the order $1, \ldots, n$ of the set of agents, and consider the game depicted in Figure 1, denoted by $\Gamma(x, y; \mathcal{L}_x)$, played from left to right, where $z_0 \equiv z_1$, and for all $i \in N$, $Z_i = \{z_i\}$, $\mathcal{N}(z_i) = i$ and $\mathcal{A}(z_i) = \{x, y\}$. First, observe that each agent only plays once and $\Gamma(x, y; \mathcal{L}_x) \in \mathcal{G}$. Second, fix an arbitrary $P \in \mathcal{P}$ and consider $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$ such that for all $i \in N$, $\sigma_i^{P_i}(z_i) = x$ if and only if $t(P_i) = x$; then $\Gamma(x, y; \mathcal{L}_x)$ induces f (voting by quota 1) since $f(P) = g(z^{\Gamma}(z_0, \sigma^P)) = x$ if and only if there exists i such that $\sigma_i^{P_i}(z_i) = x$.

[23] Observe that by Remark 2, if $\mathcal{L}_x$ is voting by quota 1 then $\mathcal{L}_y$ is voting by quota n, and if $\mathcal{L}_x$ is voting by quota n then $\mathcal{L}_y$ is voting by quota 1. Moreover, in this binary setting non-onto SCFs are constant which correspond to the two cases where $\mathcal{L}_x$ is trivial and, as we have already said, Proposition 0 refers to onto SCF.

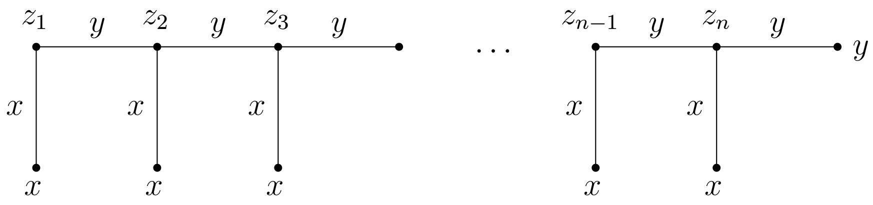

Figure 1

We want to show that, for each i, $\sigma_i^{P_i}$ is obviously dominant in $\Gamma(x, y; \mathcal{L}_x)$ for i with P_i.

Fix $i \in N$, and let $\sigma_i' \neq \sigma_i^{P_i}$ (*i.e.*, $\sigma_i'(z_i) \neq t(P_i)$). Observe that $\{z_i\} = \alpha(\sigma_i^{P_i}, \sigma_i')$ is the earliest point of departure for $\sigma_i^{P_i}$ and σ_i'. Let $i = n$ and assume $t(P_n) = x$. Then, n has to play at node z_n, reached after the sequence $\underbrace{(y, \ldots, y)}_{(n-1)\text{-times}}$ is played. Hence,

$$\min_{P_n}\{w \in X \mid w = g(z^\Gamma(z_n, (\sigma_n^{P_n}, \sigma_{-n})) \text{ for some } \sigma_{-n}\} = x \tag{3}$$

and

$$\max_{P_n}\{w \in X \mid w = g(z^\Gamma(z_n, (\sigma_n', \sigma_{-n})) \text{ for some } \sigma_{-n}\} = y, \tag{4}$$

because $\sigma_n^{P_n}(z_n) = t(P_n) = x$ and $\sigma_n'(z_n) = y$, the set in (3) is the singleton $\{x\}$ and the set in (4) is the singleton $\{y\}$. Since xP_ny, $\sigma_n^{P_n}$ is obviously dominant. Symmetrically if $t(P_n) = y$. Let $i < n$ and let $\sigma_i' \neq \sigma_i^{P_i}$ (*i.e.*, $\sigma_i'(z_i) \neq t(P_i)$). Observe that $\{z_i\} = \alpha(\sigma_i^{P_i}, \sigma_i')$ is the earliest point of departure for $\sigma_i^{P_i}$ and σ_i'. Assume $t(P_i) = y$. Then, i has to play at z_i which is either z_0 (if $i = 1$), reached after the empty history, or else $z_i \neq z_0$, (if $1 < i$), which is reached after the sequence $\underbrace{(y, \ldots, y)}_{(i-1)\text{-times}}$ is played. Hence,

$$\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i^{P_i}, \sigma_{-i})) \text{ for some } \sigma_{-i}\} = \{x, y\},$$

since there is at least one $\widehat{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i^{P_i}, \widehat{\sigma}_{-i})) = x$ and at least another $\overline{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i^{P_i}, \overline{\sigma}_{-i})) = y$. But then $\min_{P_i}\{x, y\} = x$ because $t(P_i) = y$. On the other hand,

$$\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i', \sigma_{-i})) \text{ for some } \sigma_{-i}\} = \{x\},$$

because $\sigma_i'(z_i) = x$. Since $\min_{P_i}\{x, y\} = xR_ix = \max_{P_i}\{x\}$, $\sigma_i^{P_i}$ is obviously dominant. Assume now that $t(P_i) = x$. Then,

$$\min_{P_i}\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i^{P_i}, \sigma_{-i})) \text{ for some } \sigma_{-i}\} = x.$$

and

$$\max_{P_i}\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i', \sigma_{-i})) \text{ for some } \sigma_{-i}\} = x,$$

where $\sigma' \neq \sigma_i^{P_i}$ and hence, $\sigma_i'(z_i) = y$. To see that, observe that there is at least one $\widehat{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i', \widehat{\sigma}_{-i})) = x$ and at least another one $\overline{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i', \overline{\sigma}_{-i})) =$

y, and $\max_{P_i}\{x, y\} = x$ because $t(P_i) = x$. Hence, $\sigma_i^{P_i}$ is obviously dominant in $\Gamma(x, y; \mathcal{L}_x)$ for i with P_i. Since this holds for all $i \in N$ and any arbitrary P, f is OSP.

Assume now that the associated committee for x is voting by quota n. By Remark 2, we can construct a symmetric game $\Gamma(y, x; \mathcal{L}_y)$, whose associated committee $\mathcal{L}_y$ is voting by quota 1, and proceed as we did for $\Gamma(x, y; \mathcal{L}_x)$, replacing the roles of x and y.

To prove that the converse holds, let $f : \mathcal{P} \to \{x, y\}$ be an OSP and anonymous SCF. Hence, f is SP-implementable and by condition (3.2) in Remark 3, f is voting by quota q. We now show that either $q = 1$ or $q = n$. Assume otherwise, *i.e.*, $1 < q < n$. We proceed by distinguishing between the case $n = 3$ and $n > 3$.

Assume first that $n = 3$, and so $q = 2$. We proceed by contradiction; *i.e.*, assume f is OSP and let $\Gamma \in \mathcal{G}$ be a pruned game that OSP-implements f. Since Γ induces f (voting by quota 2) there exists at least one information set at which one agent has available two actions. Let i be the first agent in Γ with this property, and denote by I_i such information set. Hence, $I_i = \{z\}$ where $z \in Z_i$. Fix a profile $(P_1, P_2, P_3) \in \mathcal{P}$. Without loss of generality, assume $t(P_i) = x$. Since Γ induces f,

$$\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{x, y\},$$

because $q = 2$ and the way i was selected. Let $\sigma_i' \in \Sigma_i$ be such that $\sigma_i' = \sigma_i^{P_i^y}$. Hence, $\sigma_i^{P_i}(z) \neq \sigma_i'(z)$, because Γ is pruned. Then, as i is the agent who first has an information set I_i with two available actions, $I_i \in \alpha(\sigma_i^{P_i}, \sigma_i')$. Consider any subprofile $\sigma_{-i}^{P'_{-i}}$ such that $\left|\{j \in N\backslash\{i\} \mid t(P_j') = x\}\right| = 1$ (and hence $\left|\{j \in N\backslash\{i\} \mid t(P_j') = y\}\right| = 1$). Since $q = 2$ and Γ is pruned and induces f, $g(z^\Gamma(z, (\sigma_i^{P_i}, \sigma_{-i}^{P'_{-i}}))) = x$ and $g(z^\Gamma(z, (\sigma_i', \sigma_{-i}^{P'_{-i}}))) = y$. Furthermore, since Γ induces f,

$$\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{x, y\},$$

because $q = 2$ and the way i was selected. Hence, since $\max_{P_i}\{x, y\} = xP_iy = \min_{P_i}\{x, y\}$, $\sigma_i^{P_i}$ is not obviously dominant, a contradiction.

Assume now that $n > 3$ and $1 < q < n$. By Remark 1, to obtain a contradiction it is sufficient to exhibit a subdomain $\widetilde{\mathcal{P}}$ of f where $f : \widetilde{\mathcal{P}} \to X$ is not OSP. Anonymity allows us to consider the particular subdomain

$$\widetilde{\mathcal{P}} = \underbrace{\{P_1^x, P_1^y\} \times \{P_2^x, P_2^y\} \times \{P_3^x, P_3^y\}}_{3 \text{ agents}} \times \underbrace{\{P_4^x\} \times \cdots \times \{P_{q+2}^x\}}_{q-2 \text{ agents}} \times \underbrace{\{P_{q+3}^y\} \times \cdots \times \{P_n^y\}}_{n-q-1 \text{ agents}}.$$

Let $\widetilde{f}$ be the restriction of f on $\widetilde{\mathcal{P}}$. Assume that $\widetilde{f}$ is OSP and let $\Gamma \in \mathcal{G}$ be a pruned game that OSP-implements $\widetilde{f}$. Since $\widetilde{f}$ is not constant and Γ induces $\widetilde{f}$, there exists an information set at which a player has available at least two actions. Let $i \in N$ be the first player who first faces this situation, and I_i be this information set. Obviously, $i \in \{1, 2, 3\}$. Agents $1, 2$ and 3 face a situation which is equivalent to the situation where $n = 3$ and

$q = 2$; *i.e.*, given the fixed preferences of the remaining $n-3$ agents, to be selected both x and y require only two additional agents to support them as top alternatives. Thus, we can also reach the conclusion that $\widetilde{f}$ is not OSP, a contradiction. ■

4.2 The general case

Let $\mathcal{L}_x$ be a committee for x and $k \in \{1, \ldots, n\}$. Denote by $\mathcal{L}_x^k = \{S \in \mathcal{L}_x^m \mid |S| = k\}$ the family of minimal winning coalitions of $\mathcal{L}_x$ with cardinality k.[24]

We present the property of a committee that plays a key role in this section as well as in Section 5. In words, a committee satisfies the increasing order inclusion property if there exists an order of distinct agents for which any minimal winning coalition of cardinality $k \geq 2$ contains the first $k-1$ agents in the order.[25]

Definition 5 A committee $\mathcal{L}_x$ for x satisfies the *increasing order inclusion* (IOI) property if there exists an order of distinct agents $i_1, \ldots, i_K$ such that for all $k > 1$,

$$\text{if } S \in \mathcal{L}_x^k \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S.$$

Example 1 illustrates the IOI property.

Example 1 The committee $\mathcal{L}_x^m = \{\{1\}, \{2,3\}, \{2,4\}, \{2,5,6,7,8\}, \{2,5,6,7,9\}\}$ satisfies IOI by the order $2,5,6,7$ or by the orders $2,5,7,6$; $2,6,5,7$; $2,6,7,5$; $2,7,5,6$ or $2,7,6,5$. On the other hand, the committee $\widehat{\mathcal{L}}_x^m = \{\{1\}, \{2,3\}, \{2,4\}, \{5,6,7,8\}, \{5,6,7,9\}\}$ does not satisfy IOI because agent 2 has to be first in any possible order since $\widehat{\mathcal{L}}_x^2 = \{\{2,3\}, \{2,4\}\}$ but $2 \notin \{5,6,7,8\} \in \widehat{\mathcal{L}}_x^4$. □

Before proceeding, several remarks about IOI are appropriate. First, there are committees that satisfy IOI trivially. For instance if $\mathcal{L}_x$ is voting by quota 1 ($\mathcal{L}_x^k = \emptyset$ for all $k > 1$) or quota n ($\mathcal{L}_x^k = \emptyset$ for all $k < n$ and $\mathcal{L}_x^n = \{N\}$), then $\mathcal{L}_x$ satisfies IOI for any order of the set of agents. Second, there may be some connected parts of the order of agents for which the specific ordering is important and some other parts for which the specific ordering is irrelevant. For instance, in any order for which the committee $\mathcal{L}_x$ in Example 1 satisfies IOI, agent 2 should be first, followed by agents 5, 6, and 7, in any ordering. Along the play of any game that could be use to show that the EMVR associated to $\mathcal{L}_x$ is OSP, the role of agent 2 will be different from the roles of agents 5, 6, 7; in particular, agent 2 will have to play earlier. And third, by its definition, if $\mathcal{L}_x$ satisfies IOI, then decisive and dummy agents do not belong to any of its associated orders, although they play a very different role in $\mathcal{L}_x$.

[24] In the notation $\mathcal{L}_x^m$, the letter m will always refer to the word 'minimal', and never to an integer.

[25] As it will become clear later, this order may not be unique.

We are now ready to state the result characterizing all SCFs that are OSP in this setting with two alternatives.

Proposition 1 *A SCF $f : \mathcal{P} \to \{x, y\}$ is* OSP *if and only if f is an* EMVR *whose committee $\mathcal{L}_x$ satisfies* IOI.

Proof See the Appendix in subsection 7.1.[26] ■

Example 1 (continued) Assume $n = 9$ and consider again the committee $\mathcal{L}_x$ for x where $\mathcal{L}_x^m = \{\{1\}, \{2, 3\}, \{2, 4\}, \{2, 5, 6, 7, 8\}, \{2, 5, 6, 7, 9\}\}$, which satisfies IOI by the order $2, 5, 6, 7$; that is, $i_1 = 2$, $i_2 = 5$, $i_3 = 6$ and $i_4 = 7$. Define the game $\Gamma(x, y; \mathcal{L}_x)$ that OSP-implements the EMVR associated to $\mathcal{L}_x$, depicted in Figure 2, as follows. Players play sequentially from left to right, $z_0 \equiv z_1$, and for all $i \in N$, $Z_i = \{z_i\}$, $\mathcal{N}(z_i) = i$ and $\mathcal{A}(z_i) = \{x, y\}$.

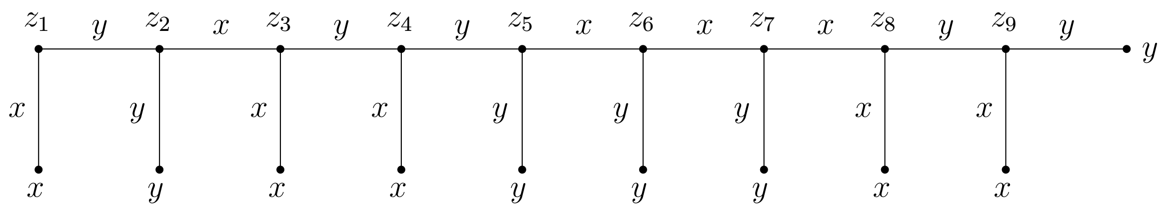

Figure 2

It is worthwhile to point out by means of this example two general properties of any such game $\Gamma(x, y; \mathcal{L}_x)$. First, although the roles of agents 1 and 2 are very different in $\mathcal{L}_x$, along the game $\Gamma(x, y; \mathcal{L}_x)$ they are somehow similar. After agent 1 ('decisive' for x) chooses y, agent 2 becomes 'decisive' for y. Also, for instance, at node z_7, agent 7 becomes 'decisive' for y while, at node z_8, agent 8 becomes 'decisive' for x. Hence, whenever an agent has to play, truth-telling is an obvious optimal choice, regardless of any consideration about the other agents' future behavior.

Second, the game depicted in Figure 2 could also be the game obtained if instead we would have used the committee $\mathcal{L}_y$ for y, the one obtained by means of Remark 2, associated to the same EMVR. By Remark 2,

$$\mathcal{L}_y^m = \{\{1, 2\}, \{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 3, 4, 7\}, \{1, 3, 4, 8, 9\}\}.$$

It is easy to see two things. First, $\mathcal{L}_y$ satisfies IOI by the order $1, 3, 4, 8$ (for instance); that is, $i_1 = 1$, $i_2 = 3$, $i_3 = 4$ and $i_4 = 8$. Second, the corresponding game $\Gamma(y, x; \mathcal{L}_y)$ coincides with $\Gamma(x, y; \mathcal{L}_x)$. Finally, the fact that $\{1, 2\} \in \mathcal{L}_y^m$ explains why the two agents have similar power, although this was not apparent in $\mathcal{L}_x$.[27] □

[26] The proof of sufficiency is constructive: it exhibits an extensive game form that OSP-implements a given EMVR whose committee $\mathcal{L}_x$ satisfies IOI.

[27] The game in Figure 2 is a proto-dictatorship game according to the terminology used by Bade and

5 Generalized median voter schemes

Consider a social choice problem where the set of *alternatives* $X = \{\alpha, \ldots, \beta\}$ is a finite and linearly ordered set. Without loss of generality we may assume that X is a finite subset of integers between α and β, where $\alpha, \beta \in \mathbb{Z}$. Moreover, we may also assume that $|X| > 2$; otherwise, we are back to the setting of the previous section.

There is a rich literature studying this class of problems for the case where, given this structure of the set of alternatives, agents' preferences are assumed to be single-peaked relative to the order over X. Agent i's preference P_i is *single-peaked* over X if for all $x, y \in X$, $x < y \leq t(P_i)$ or $t(P_i) \leq y < x$ implies yP_ix. Let $\mathcal{SP}_i$ be the set of all agent i's single-peaked preferences over X. Define $\mathcal{SP} = \mathcal{SP}_1 \times \cdots \times \mathcal{SP}_n$.

We define now a class of SCFs known as generalized median voter schemes. One description is based on the notion of *left coalition system* on X, which is a family of non-trivial committees $\{\mathcal{L}_x\}_{x \in X}$ with the additional monotonicity property that, for all $x < \beta$, $S \in \mathcal{L}_x$ implies $S \in \mathcal{L}_{x+1}$, and the boundary condition that $\mathcal{L}_\beta = 2^N \backslash \{\emptyset\}$. If $S \in \mathcal{L}_x$ we say that S is a left-winning coalition at x.

Definition 6 A SCF $f : \mathcal{SP} \to X$ is a *generalized median voter scheme* (GMVS) if there exists a left coalition system $\{\mathcal{L}_w\}_{w \in X}$ such that, for all $P \in \mathcal{SP}$,

$$f(P) = x \text{ if and only if } \quad \begin{array}{l} \text{(i) } \{i \in N \mid t(P_i) \leq x\} \in \mathcal{L}_x \text{ and} \\ \text{(ii) for all } x' < x,\ \{i \in N \mid t(P_i) \leq x'\} \notin \mathcal{L}_{x'}. \end{array}$$

Namely, the alternative x selected by the GMVS f at P is the smallest one for which the top alternatives of all agents of a left-winning coalition at x are smaller than or equal to x.

A similar description can be provided through the symmetric concept of *right coalition system* on X, which is a family of non-trivial committees $\{\mathcal{R}_x\}_{x \in X}$ with the additional monotonicity property that, for all $\alpha < x$, $S \in \mathcal{R}_x$ implies $S \in \mathcal{R}_{x-1}$, and the boundary condition that $\mathcal{R}_\alpha = 2^N \backslash \{\emptyset\}$. If $S \in \mathcal{R}_x$ we say that S is a right-winning coalition at x.

Definition 6' A SCF $f : \mathcal{SP} \to X$ is a *generalized median voter scheme* (GMVS) if there exists a right coalition system $\{\mathcal{R}_w\}_{w \in X}$ such that, for all $P \in \mathcal{SP}$,

$$f(P) = x \text{ if and only if } \quad \begin{array}{l} \text{(i) } \{i \in N \mid t(P_i) \geq x\} \in \mathcal{R}_x \text{ and} \\ \text{(ii) for all } x' > x,\ \{i \in N \mid t(P_i) \geq x'\} \notin \mathcal{R}_{x'}. \end{array}$$

Gonczarowski (2017). Their Theorem 4.1 states that a mechanism (an extensive game form) is OSP if and only if it is a proto-dictatorship. In contrast, our characterization in Proposition 1 identifies by means of the IOI property those EMVRs that are OSP. See the Final Remarks section for a comment relating our characterization result in Proposition 1 and their Theorem 4.1.

Symmetrically, the alternative x selected by the GMVS f at P is the largest one for which the top alternatives of all agents of a right-winning coalition at x are larger than or equal to x.

The left or the right coalition system can be taken indistinctly as the primitive concept for the definition of a GMVS. But yet, a precise relationship between a left coalition system and a right coalition system has to hold if they have to generate the same GMVS. We state this relationship in Remark 4, which generalizes Remark 2 for the case with more than two alternatives.[28]

Remark 4 A left coalition system $\{\mathcal{L}_x\}_{x\in X}$ and a right coalition system $\{\mathcal{R}_x\}_{x\in X}$ define the same GMVS $f : \mathcal{SP} \to X$ if and only if, for all $x > \alpha$,

$$T \in \mathcal{R}_x \text{ if and only if } T \cap S \neq \varnothing \text{ for all } S \in \mathcal{L}_{x-1}.$$

In this case we will say that $\{\mathcal{L}_x\}_{x\in X}$ is the left coalition system associated to the GMVS f and $\{\mathcal{R}_x\}_{x\in X}$ is the right coalition system associated to the GMVS f.

Alternatively, and more metaphorically, a GMVS might be understood as a force that, starting at the lowest alternative, pushes up towards the highest possible alternative. However, the left coalition system distributes the power to stop this force in such a way that all members of a left-winning coalition at x can make sure that, by declaring that their top alternative is smaller than or equal to x, the pushing force of f will not overcome x.

It is well known that a SCF $f : \mathcal{SP} \to X$ is strategy-proof if and only if f is a GMVS.[29]

The smallest alternative for which its left-committee contains a singleton set will play a relevant role in this section. Given the left coalition system $\{\mathcal{L}_w\}_{w\in X}$ and $x \in X$, let $De^L_x = \{i \in N \mid \{i\} \in \mathcal{L}_x\}$ be the set of left-decisive agents at x. Define $x_1 = \min\{x \in X \mid De^L_x \neq \emptyset\}$. Observe that x_1 is well defined since $De^L_\beta = N$. Similarly, given the right coalition system $\{\mathcal{R}_w\}_{w\in X}$ and $x \in X$, let $De^R_x = \{i \in N \mid \{i\} \in \mathcal{R}_x\}$ be the set of right-decisive agents at x. Let $i^1 \in De^L_{x_1}$ be one of the agents for which $\{i^1\} \in \mathcal{L}_{x_1}$.

We now present a strengthening of IOI that will play a crucial role in the characterization of the class of SCFs that are OSP on the domain of single-peaked preferences.

[28] See Barberà, Massó and Neme (1997) for a proof of Remark 4.

[29] See Barberà, Gul and Stacchetti (1993). Sprumont (1995) shows that the tops-only property in Moulin (1980)'s characterization is not required. If the social choice function is not onto, define a new and smaller set of alternatives by deleting the subset of alternatives that have not been chosen, and restrict then the set of single-peaked preferences and the social choice function to this new set. Then, strict single-peaked preferences remain single-peaked over the restricted set of alternatives, and the restricted social choice function is onto. Unic-top single-peaked preferences admitting indifferences may no longer be unic-top single-peaked over the restricted set of alternatives. See Barberà and Jackson (1994) to deal with this later (and much more involved) case. The characterization just stated refers to this restricted (onto) function.

Definition 7 A left (right) committee $\mathcal{L}_x$ ($\mathcal{R}_x$) for x satisfies the *increasing order inclusion* (IOI) *property with respect to* $i^x \in N$ if there exists an order of distinct agents $i_1, \ldots, i_K$ such that for all $k > 1$,

$$\text{if } S \in \mathcal{L}^k(x) \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S \text{ and } i_1 = i^x$$

$$(\text{if } S \in \mathcal{R}^k(x) \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S \text{ and } i_1 = i^x).$$

That is, a committee satisfies IOI with respect to an agent if the committee satisfies IOI relative to an order where this agent goes first.

Proposition 2 below characterizes the class of all SCFs that are OSP on the domain of single-peaked preferences.

Proposition 2 *A* SCF $f : \mathcal{SP} \to X$ *is* OSP *if and only if* f *is a* GMVS *whose associated left and right coalition systems,* $\{\mathcal{L}_x\}_{x \in X}$ *and* $\{\mathcal{R}_x\}_{x \in X}$*, satisfy the following two properties:*

(L-IOI) *For every* $\beta > x \geq x_1 - 1$*, there exists* $i^x \in N$ *such that* $\mathcal{L}_x$ *satisfies* IOI *with respect to* i^x *and* $\{i^x\} \in \mathcal{L}_{x+1}$.

(R-IOI) *For every* $\alpha < x \leq x_1 + 1$*, there exists* $i^x \in N$ *such that* $\mathcal{R}_x$ *satisfies* IOI *with respect to* i^x *and* $\{i^x\} \in \mathcal{R}_{x-1}$.[30]

Proof See the Appendix in subsection 7.2. ■

As a consequence of Proposition 2 we obtain Corollary 1 characterizing the class of all OSP and anonymous SCFs on the domain of single-peaked preferences.

Corollary 1 *A* SCF $f : \mathcal{SP} \to X$ *is anonymous and* OSP *if and only if* f *is a* GMVS *whose associated left coalition system* $\{\mathcal{L}_x\}_{x \in X}$ *has the property that there exists* $x_1 \in \{\alpha, \ldots, \beta\}$ *such that* (i) $\mathcal{L}_x = \{N\}$ *for all* $x < x_1$ *and* (ii) $\mathcal{L}_x^m = \{\{1\}, \ldots, \{n\}\}$ *for all* $x \geq x_1$.

Observe that the two SCFs associated to $x_1 = \alpha$ and $x_1 = \beta$ correspond respectively to the one that, at each profile, selects the smallest and largest peak. Corollary 1 holds for the following reasons. Let $\{\mathcal{L}_x\}_{x \in X}$ be a left coalition system satisfying the necessary and sufficient condition in Corollary 1. We check that (L-IOI) and (R-IOI) in Proposition 2 hold. First, $\{\mathcal{L}_x\}_{x \in X}$ satisfies (L-IOI): for all $x \geq x_1 - 1$, $\mathcal{L}_x$ satisfies IOI with respect to any $i \in N$ and $\{i\} \in \mathcal{L}_{x+1}$. Second, by Remark 4, the right coalition system $\{\mathcal{R}_x\}_{x \in X}$ associated to f satisfies (i) $\mathcal{R}_x^m = \{\{1\}, \ldots, \{n\}\}$ for all $x \leq x_1$ and (ii) $\mathcal{R}_x = \{N\}$ for all $x > x_1$. And indeed, the right coalition system $\{\mathcal{R}_x\}_{x \in X}$ satisfies (R-IOI): for all $x \leq x_1 + 1$, $\mathcal{R}_x$ satisfies IOI with respect to any $i \in N$ and $\{i\} \in \mathcal{R}_{x-1}$.

[30] In these two statements, $x_1 - 1$ or $x_1 + 1$ could be read as α or β if $x_1 = \alpha$ or $x_1 = \beta$, respectively.

Figure 3 illustrates Corollary 1, for the case $X = \{\alpha, \alpha+1, x_1 - 1, x_1, x_1 + 1, \beta\}$, by simultaneously describing the anonymous GMVS by means of its left and right coalition system.

	α	$\alpha+1$	$x_1 - 1$	x_1	$x_1 + 1$	β
$\mathcal{R}_x^m$	$\{i\}_{i\in N}$	$\{i\}_{i\in N}$	$\{i\}_{i\in N}$	$\{i\}_{i\in N}$	N	N
$\mathcal{L}_x^m$	N	N	N	$\{i\}_{i\in N}$	$\{i\}_{i\in N}$	$\{i\}_{i\in N}$

Figure 3

Assume $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$ are the left and the right coalition systems associated to the same GMVS f. If $x_1 \in \{\alpha, \beta\}$, Remark 4 gives the relationship between them and one can directly check whether or not $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$ respectively satisfy (L-IOI) and (R-IOI). But if $\alpha < x_1 < \beta$, (L-IOI) and (R-IOI) in Proposition 2 only impose conditions on $\{\mathcal{L}_{x_1-1}, \mathcal{L}_{x_1}, \ldots, \mathcal{L}_{\beta-1}\}$ and $\{\mathcal{R}_{\alpha+1}, \ldots, \mathcal{R}_{x_1}, \mathcal{R}_{x_1+1}\}$, respectively. In the Appendix, subsection 7.3, we answer the following natural question: can we fully describe f as a GMVS only through either $\{\mathcal{L}_x\}_{x\in X}$ or $\{\mathcal{R}_x\}_{x\in X}$? In particular, Proposition 3 identifies the property on the left coalition system, that together with (L-IOI), characterizes all SCFs that are OSP on the domain of single-peaked preferences. Although one may see the characterization in Proposition 3 as being more transparent and natural, we give more prominence to Proposition 2 because the extensive game form that OSP-implements a given GMVS is identified along the proof of Proposition 2.

We finish this section with two examples illustrating the statements of Propositions 2 and 3. The statement and proof of Proposition 3 is in the Appendix, subsection 7.3.[31]

Example 2 Assume $X = \{\alpha, x, \beta\}$, $n = 5$ and consider the left coalition system $\{\mathcal{L}_w\}_{w\in X}$ where:

$$\begin{aligned} \mathcal{L}_\alpha^m &= \{\{1\}, \{2,3,4\}, \{2,3,5\}\} \\ \mathcal{L}_x^m &= \{\{1\}, \{2\}, \{3\}, \{4,5\}\} \\ \mathcal{L}_\beta^m &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}. \end{aligned}$$

The committees $\mathcal{L}_\alpha$ and $\mathcal{L}_x$ satisfy (L-IOI) by the orders $2, 3$ and 4, and with respect to the agents $i^\alpha = 2$ and $i^x = 4$, respectively. Observe that $x_1 = \alpha$ and $i^1 = 1$. Define the game Γ, depicted in Figure 4, that OSP-implements the GMVS associated to $\{\mathcal{L}_w\}_{w\in X}$ as follows.

[31] Example 2 illustrates Case 1 in the proof of Proposition 2 and Example 3 illustrates Case 3 in the proof of Proposition 2 as well as Proposition 3.

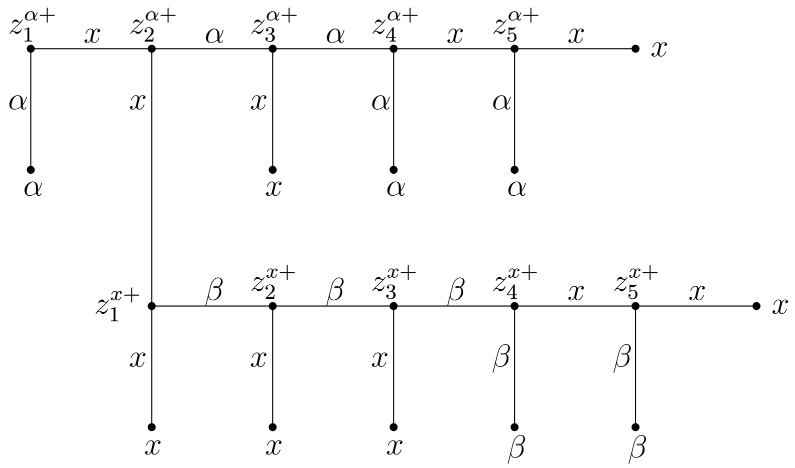

Figure 4

Players play sequentially from left to right, $z_0 = z_1^{\alpha+}$, the subscript in any of the other nodes indicates the agent that has to play at that node by choosing between α and x, if the superscript is $\alpha+$, or between x and β, if the superscript is $x+$; for instance, (i) $z_4^{\alpha+} \in Z_4$ and agent 4 has to choose at $z_4^{\alpha+}$ one action from the set $\{\alpha, x\}$ and (ii) $z_3^{x+} \in Z_3$ and agent 3 has to choose at z_3^{x+} one action from the set $\{x, \beta\}$.[32] $\square$

Example 3 Assume $X = \{\alpha, x, x_1-1, x_1, x_1+1, \beta\}$, $n = 9$ and consider the left coalition system $\{\mathcal{L}_w\}_{w \in X}$ where:

$$\begin{aligned}
\mathcal{L}_\alpha^m &= \{\{1,2,3\}\} \\
\mathcal{L}_x^m &= \{\{1,2,3\}\} \\
\mathcal{L}_{x_1-1}^m &= \{\{1,2,3\},\{1,2,4\}\} \\
\mathcal{L}_{x_1}^m &= \{\{1\},\{2,3\},\{2,4\},\{2,5,6,7,8\},\{2,5,6,7,9\}\} \\
\mathcal{L}_{x_1+1}^m &= \{\{1\},\{2\},\{3,4\}\} \\
\mathcal{L}_\beta^m &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\}\}.
\end{aligned}$$

Then, by Remark 4, the right coalition system $\{\mathcal{R}_w\}_{w \in X}$ that defines the same GMVS is:

$$\begin{aligned}
\mathcal{R}_\beta^m &= \{\{1,2,3\},\{1,2,4\}\} \\
\mathcal{R}_{x_1+1}^m &= \{\{1,2\},\{1,3,4,5\},\{1,3,4,6\},\{1,3,4,7\},\{1,3,4,8,9\}\} \\
\mathcal{R}_{x_1}^m &= \{\{1\},\{2\},\{3,4\}\} \\
\mathcal{R}_{x_1-1}^m &= \{\{1\},\{2\},\{3\}\} \\
\mathcal{R}_x^m &= \{\{1\},\{2\},\{3\}\} \\
\mathcal{R}_\alpha^m &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\}\}.
\end{aligned}$$

[32] The game in Figure 4 will be used in the final section to compare our characterization result in Proposition 2 with the characterization results in Theorems 2 and 5.1 in Pycia and Troyan (2017) and Bade and Gonczarowski (2017), respectively.

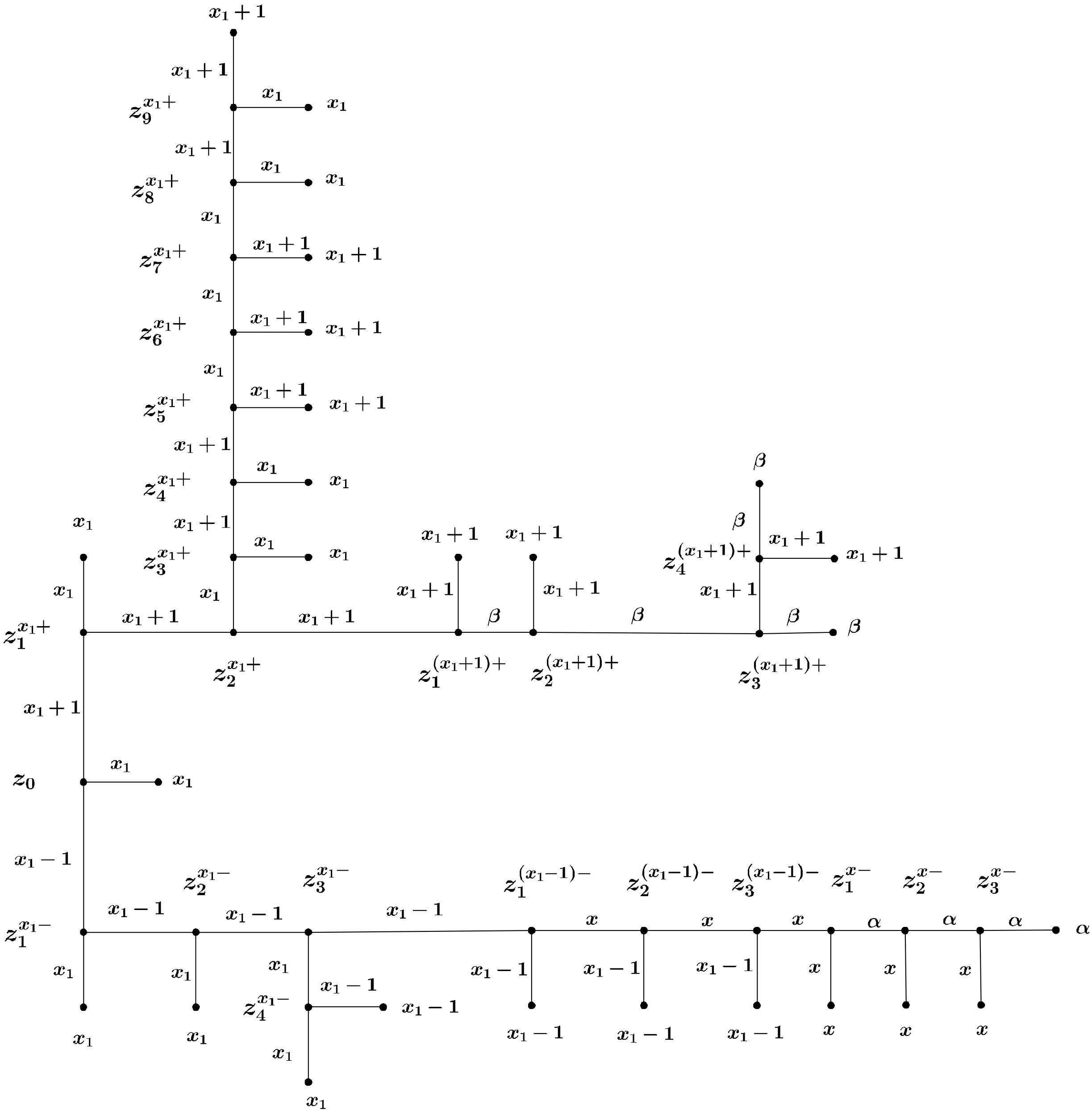

Figure 5

Figure 5 depicts a game Γ that OSP-implements the GMVS associated to $\{\mathcal{L}_w\}_{w\in X}$ and $\{\mathcal{R}_w\}_{w\in X}$, where $z_0 \in Z_1$, the subscript in any of the other nodes indicates the agent that has to play at that node by choosing between y and $y+1$, if the superscript is $y+$, or between y and $y-1$, if the superscript is $y-$, where y is a generic alternative in the set $X\backslash\{\alpha,\beta\}$; for instance, (i) $z_4^{(x_1+1)+} \in Z_4$ and agent 4 has to choose at $z_4^{(x_1+1)+}$ one action from the set $\{x_1+1,\beta\}$ and (ii) $z_3^{(x_1-1)-} \in Z_3$ and agent 3 has to choose at $z_3^{(x_1-1)-}$ one action from the set $\{x_1-1,x\}$. Indeed, x_1 is the smallest alternative for which there exists $i \in N$ such that $\{i\} \in \mathcal{L}_{x_1}$, and $\alpha < x_1 < \beta$, so Case 3 is the relevant one in the proof of Proposition 2. Note that $i^1 = 1$. We first check that $\{\mathcal{L}_w\}_{w\in X}$ satisfies (L-IOI). First,

$\mathcal{L}_{x_1-1}$ satisfies IOI by the order $1,2$ with respect to $i^{x_1-1}=1$, $\mathcal{L}_{x_1}$ satisfies IOI by the order $2,5,6,7$ with respect to $i^{x_1}=2$ and $\mathcal{L}_{x_1+1}$ satisfies IOI by the order 3 with respect to $i^{x_1+1}=3$; hence $\{\mathcal{L}_w\}_{w\in X}$ satisfies (L-IOI).[33] We now check that $\{\mathcal{R}_w\}_{w\in X}$ satisfies (R-IOI). First, $\mathcal{R}_{x_1+1}$ satisfies IOI by the order $1,3,4,8$ with respect to $i^{x_1+1}=1$, $\mathcal{R}_{x_1}$ satisfies IOI by the order 3 with respect to $i^{x_1}=3$, and $\mathcal{R}_{x_1-1}$, and $\mathcal{R}_{x_1}$ satisfy IOI by any order with respect to any agent; hence, $\{\mathcal{R}_w\}_{w\in X}$ satisfies (R-IOI). □

6 Final remarks

We first relate our results to those in Pycia and Troyan (2017) and Bade and Gonczarowski (2017). As we have already said in the Introduction, these two papers contain revelation principle like results identifying classes of games inside which one can restrict attention when searching for a game that OSP-implements some families of SCFs. However, there are two important differences between these two results and the revelation principle for SP-implementation. First, given a SCF $f:\mathcal{D}\to X$, the revelation principle for strategy-proofness identifies a unique *normal* game form $(N,\mathcal{D},f)$ for which truth-telling has to be a weakly dominant strategy for each agent in N. In contrast, the classes of games identified in Pycia and Troyan (2017) and Bade and Gonczarowski (2017) are large and many games in those classes would not OSP-implement the given f but another SCF. Hence, the question of which game has to be used to OSP-implement a particular SCF f remains open, although their results may help because they delimit the class of games within which to look for. Second, if f is not OSP (but this is still unknown to the designer) one ought to check that each game in their respective class does not OSP-implement f, and this may not be easy. In addition, Pycia and Troyan (2017) and Bade and Gonczarowski (2017) results say respectively that if a game OSP-implements a SCF, then there exists a multipede game or a gradual mechanism (or a proto-dictatorship game for the case of two alternatives) that does as well. Our characterizations in Propositions 1, 2 and 3 however give necessary and sufficient conditions on the SCFs that are OSP-implementable, and those conditions can be checked directly on the SCF under consideration and not on the game. Moreover, our sufficiency proofs of Propositions 1 and 2 are constructive; that is, they give a procedure to construct an extensive game form that OSP-implements the SCF. Examples 1 and 2, and their respective Figures 2 and 4, illustrate these points.

The game in Figure 2 is indeed a multipede and proto-dictatorship game. However, consider the game in Figure 6, which is obtained from Figure 2 by exchanging at node z_3 the y and the x choices of agent 3, because now the y choice finishes the game with outcome y while the x choice moves the game to node z_4, and everything

[33] Observe that $\mathcal{L}_\alpha$ and $\mathcal{L}_x$ satisfy IOI both by the order $1,2$ and $\{\mathcal{L}_w\}_{w\in X}$ satisfies (L2-IOI) by setting $i_{x_1-1}=3$ and $i_x=1$, defined in the Appendix.

else remains the same. The new game is also multipede and proto-dictatorship but now it OSP-implements another SCF, the one whose associated committee is $\widehat{\mathcal{L}}_x = \{\{1\}, \{2,3,4\}, \{2,3,5,6,7,8\}, \{2,3,5,6,7,9\}\}$, which satisfies also the IOI property (by the order $2, 3, 5, 6, 7$). Our proof tells us, given a committee associated to an EMVR, how to construct the (multipede and proto-dictatorship) game that OSP-implements it.

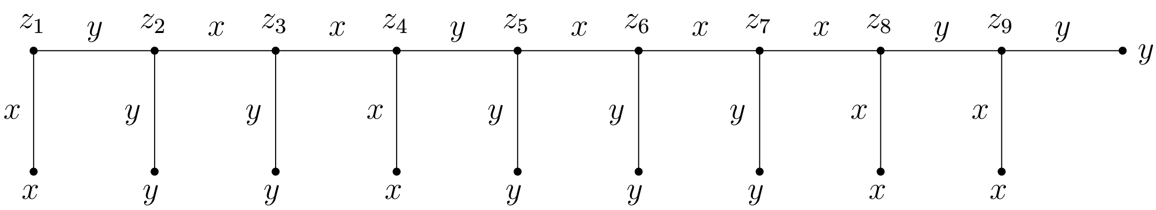

Figure 6

The game in Figure 4 is neither multipede nor gradual. Theorem 2 in Pycia and Troyan (2017) characterizing multipede games does not apply to the problem of a linearly ordered set with single-peaked preferences because the general conditions on preferences in their general model imply the universal domain of preferences, when the cardinality of the set of alternatives is larger than or equal to three.[34] It is not gradual because at node $z_2^{\alpha+}$ agent 2, by playing the strategy $\sigma_2(z_2^{\alpha+}) = \sigma_2(z_2^{x+}) = x$, can force the outcome to be x, regardless of the other agents' strategies. But agent 2 does not have a choice at $z_2^{\alpha+}$ inducing immediately outcome x. Our design of the game in Figure 4 comes from the description of the GMVS as a sequence of EMVRs satisfying (in this case) the (L-IOI) property. However, the game can be modified into a gradual one, as the one depicted in Figure 7. This game is a dictatorship with safeguards against extremism with proto-dictatorship games added to the non-terminal nodes z_4^1 and z_4^2 (this is what Bade and Gonczarowski (2017) call "arbitration"). Together, and only together, agents 4 and 5 are also playing a safeguard role against the extremes α (by sharing the power to induce jointly x at the subgame starting at z_4^1) and β (by sharing the power to induce jointly x at the subgame starting at z_4^2). The general message that Bade and Gonczarowski (2017) tries to convey is that in the limit, with a continuum of possible alternatives, dictatorships with safeguards against extremism (without arbitration via proto-dictatorships) are the unique OSP and onto SCFs. Our Proposition 2 characterizes, for a discrete and finite set of linearly ordered alternatives, the subclass of SCFs on the single-peaked domain that are OSP; and the characterization is based on the description of the SCFs as GMVSs whose associated left and right coalition systems satisfy respectively the (L-IOI) and (R-IOI) properties. Of course, the proofs of Theorem 5.1 in Bade and Gonczarowski (2017) and

[34] Hence, the game depicted in Figure 4 illustrates why the general conditions (a) and (b) on preferences in Pycia and Troyan (2017) can not be dispensed for their Theorem 2 to hold.

our Proposition 2 are very different because they are based on two alternative descriptions of the SCFs under consideration.

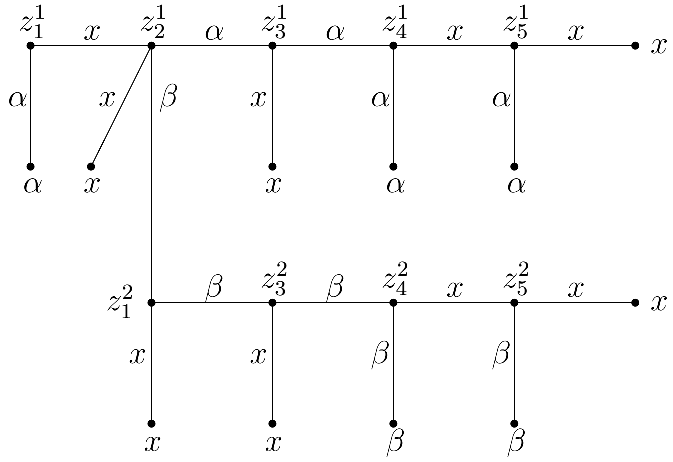

Figure 7

We want to emphasize that our statements in Propositions 1 and 2 do not refer explicitly to either ontoness or unanimity, although the SCF under consideration has to be onto, perhaps relative to a subset of alternatives and corresponding subdomains of preferences obtained from the range of the original and non-onto SCF that we are interested in (see footnotes 23 and 29).

We want to finish by referring to two other settings where the class of all strategy-proof and onto social choice functions are based on the majority principle. The first one is the multidimensional extension of the single-peaked model studied by Barberà, Gul and Stacchetti (1993). In this case, the family of multidimensional generalized median voter schemes coincides with the class of strategy-proof and onto social choice functions on the domain of multidimensional single-peaked preferences. The second setting is the one where voting by committees (studied in Barberà, Sonnenschein and Zhou (1991)) are used to collectively select a subset, from a given set of objects K. The family of voting by committees constitute the class of all strategy-proof and onto social choice functions, mapping profiles of separable preferences (over 2^K) into the family 2^K. They are also based on the extension of the majority principle, applied to each object in order to decide whether or not the object belongs to the chosen subset, at a given preference profile. However, neither multidimensional generalized median voter schemes nor voting by committees are efficient and hence, they are not weak group strategy-proof. Then, by Proposition 1 in Li (2017), which states that obviously strategy-proofness implies weak group strategy-proofness, they are not obviously strategy-proof.

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7 Appendix

7.1 Proof of Proposition 1

Proposition 1 *A* SCF $f : \mathcal{P} \to \{x, y\}$ *is* OSP *if and only if* f *is an* EMVR *whose committee* $\mathcal{L}_x$ *satisfies* IOI.

Proof of Proposition 1 To prove necessity, assume that f is OSP, and hence f is SP. By (3.1) in Remark 3, f is an EMVR. Let $\mathcal{L}_x$ be its associated committee for x and fix $k \in \{1, \ldots, n\}$. Denote by S_k the intersection of minimal winning coalitions of cardinality k; namely,

$$S_k = \bigcap_{S \in \mathcal{L}_x^k} S.$$

We start with a recursive definition and two key results, stated in Lemma 1 below. Define first $r_1 = \min\{|S| \mid S \in \mathcal{L}_x^m \text{ and } |S| > 1\}$ and for $t \in \{2, \ldots, T\}$, given r_{t-1}, define recursively $r_t = \min\{|S| \mid S \in \mathcal{L}_x^m \text{ and } |S| > r_{t-1}\}$.[35]

Lemma 1 *Let* f *be* OSP *and let* $\mathcal{L}_x$ *be its associated committee for* x. *Then, for all* $t \in \{1, \ldots, T\}$, *the following two statements hold.*

(1.1) *If* $|\mathcal{L}_x^{r_t}| \geq 2$, *then* $|S_{r_t}| = r_t - 1$ *and* $S_{r_t} \subseteq S_{r_{t'}}$ *for all* $t' > t$.

(1.2) *If* $|\mathcal{L}_x^{r_t}| = 1$, *then there exists* $j_t \in S_{r_t}$ *such that* $S_{r_t} \backslash \{j_t\} \subseteq S_{r_{t'}}$ *for all* $t' > t$.

Proof (1.1) Let $t \in \{1, \ldots, T\}$ be such that $|\mathcal{L}_x^{r_t}| \geq 2$ and assume $|S_{r_t}| < r_t - 1$. Then, there exist $S, S', S'' \in \mathcal{L}_x^{r_t}$ (where S' and S'' may be the same set, for instance whenever $|\mathcal{L}_x^{r_t}| = 2$) and $j', j'' \in S$ such that $j' \in S \backslash S'$ and $j'' \in S \backslash S''$. Define $S^* = S \cap S' \cap S''$ and $\overline{S} = S \cup S' \cup S''$. Note that S^* could be empty and that, since $S^* \subseteq S \backslash \{j', j''\}$, $|S^*| < r_t - 1$ and $\overline{S} \backslash S^* \neq \emptyset$. Let P_i^x and P_i^y be the two preferences such that $x P_i^x y$ and $y P_i^y x$, respectively. When agent i's preference is P_i^w, we will say that i votes for w. Define the subdomain $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_1 \times \cdots \times \widetilde{\mathcal{P}}_n$ where for all $i \in S^*$, $\widetilde{\mathcal{P}}_i = \{P_i^x\}$, for all $i \in N \backslash \overline{S}$, $\widetilde{\mathcal{P}}_i = \{P_i^y\}$ and for all $i \in \overline{S} \backslash S^*$, $\widetilde{\mathcal{P}}_i = \{P_i^x, P_i^y\}$. Assume that $f : \mathcal{P} \to \{x, y\}$ is OSP. Let $\widetilde{f}$ be the restriction of f on $\widetilde{\mathcal{P}}$. Then, by Remark 1, $\widetilde{f}$ is OSP. Let $\Gamma \in \mathcal{G}$ be a pruned game that OSP-implements $\widetilde{f}$. Since Γ induces $\widetilde{f}$ and $\widetilde{f}$ is not constant, there exists an information set at which a player has available two actions. Let $i \in N$ be the agent who first faces this situation, and let I_i be this information set. Since Γ induces $\widetilde{f}$, $i \in \overline{S} \backslash S^*$. Fix a profile $P \in \widetilde{\mathcal{P}}$ and assume $t(P_i) = x$. Since Γ induces $\widetilde{f}$,

$$\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i} \text{ and } z \in I_i\} = \{x, y\}.$$

To see that x belongs to this set, observe that there is a profile in the subdomain where all agents in $\overline{S}$ vote for x, and this is a winning coalition for x. To see that y belongs to this set, observe that there is a profile in the subdomain where only the agents in

[35] For the committee $\mathcal{L}_x$ in Example 1, $r_1 = 2$ and $r_2 = 5$.

$S^* \cup \{i\}$ vote for x, but this is not a winning coalition for x, because $S^* \cup \{i\} \subsetneq S$ or $S^* \cup \{i\} \subsetneq S'$ or $S^* \cup \{i\} \subsetneq S''$, where the strict inclusions follow from $|S^*| < r_t - 1$ and $|S| = |S'| = |S''| = r_t$.

Now, let $\sigma_i' \in \Sigma_i$ be such that $\sigma_i' = \sigma_i^{P_i^y}$. Hence, $\sigma_i^{P_i}(z) \neq \sigma_i'(z)$, because Γ is pruned. Then, as i is the agent who first has an information set I_i with two available actions, $I_i \in \alpha(\sigma_i^{P_i}, \sigma_i')$. Now, as $t(P_i') = \{y\}$ and since Γ induces $\widetilde{f}$,

$$\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i} \text{ and } z \in I_i\} = \{x, y\}.$$

Then,

$$\max_{P_i}\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i} \text{ and } z \in I_i\} = \max_{P_i}\{x, y\},$$

$$\min_{P_i}\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i} \text{ and } z \in I_i\} = \min_{P_i}\{x, y\}$$

and $\max_{P_i}\{x, y\} P_i \min_{P_i}\{x, y\}$. Thus, $\sigma_i^{P_i}$ is not obviously dominant in Γ, and so Γ does not OSP-implement $\widetilde{f}$, a contradiction.

Assume now that $t \in \{1, \ldots, T\}$ is such that $|\mathcal{L}_x^{r_t}| \geq 2$, $|S_{r_t}| = r_t - 1$ and $S_{r_t} \nsubseteq S_{r_{t'}}$ for some $t' > t$. Hence, there exists $i \in S_{r_t} \backslash S_{r_{t'}}$, implying that there exists $S' \in \mathcal{L}_x^{r_{t'}}$ such that $i \notin S'$. Let $S, S'' \in \mathcal{L}_x^{r_t}$ be two distinct coalitions, which they do exist because $|\mathcal{L}_x^{r_t}| \geq 2$. Since S and S'' are minimal winning, there exists $j \in S \backslash S''$ such that $j \neq i$ because $i \in S_{r_t}$. Define $S^* = S \cap S' \cap S''$ and $\overline{S} = S \cup S' \cup S''$, and note that, since $S^* \subseteq S \backslash \{i, j\}$, $|S^*| < r_t - 1$ and $\overline{S} \backslash S^* \neq \emptyset$. By following an argument similar to the one already used, we obtain a contradiction.

(1.2) Let $t \in \{1, \ldots, T\}$ be such that $|\mathcal{L}_x^{r_t}| = 1$. We first show that the following two claims hold.

Claim 1 *There exists $j_t \in S_{r_t}$ such that $S_{r_t} \backslash \{j_t\} \subseteq S_{r_{t+1}}$.*

Proof of Claim 1 Assume there exist $i, j \in S_{r_t}$ such that $i, j \notin S_{r_{t+1}}$. Since $\mathcal{L}_x^{r_t} = \{S_{r_t}\}$, S_{r_t} is a minimal winning coalition. Hence, there exist $S', S'' \in \mathcal{L}_x^{r_{t+1}}$ such that $i \notin S'$ and $j \notin S''$. Define $S^* = S_{r_t} \cap S' \cap S''$ and $\overline{S} = S_{r_t} \cup S' \cup S''$, and note that, since $S^* \subseteq S_{r_t} \backslash \{i, j\}$, $|S^*| < r_t - 1$ and $\overline{S} \backslash S^* \neq \emptyset$. By following an argument similar to the one already used, we obtain a contradiction. □

Claim 2 *There exists $S'' \in \mathcal{L}_x^{r_{t+1}}$ such that $j_t \notin S''$, where j_t is the agent identified in Claim 1.*

Proof of Claim 2 Assume $j_t \in S$ for all $S \in \mathcal{L}_x^{r_{t+1}}$. Then, by Claim 1, $S_{r_t} \subseteq S_{r_{t+1}}$. Hence, $S_{r_t} \subsetneq S$ for all $S \in \mathcal{L}_x^{r_{t+1}}$, which is a contradiction with $S_{r_t} \in \mathcal{L}_x^m$, which follows from $|\mathcal{L}_x^{r_t}| = 1$. □

To proceed with the proof of (1.2), assume that there exists $t' > t + 1$ such that $S_{r_t} \backslash \{j_t\} \nsubseteq S_{r_{t'}}$. Then, there exist $j \in S_{r_t} \backslash \{j_t\}$ and $S' \in \mathcal{L}_x^{r_{t'}}$ such that $j \notin S'$. By Claim 2, there exists $S'' \in \mathcal{L}_x^{r_{t+1}}$ such that $j_t \notin S''$. Define $S^* = S_{r_t} \cap S' \cap S''$ and $\overline{S} = S_{r_t} \cup S' \cup S''$, and note that, since $S^* \subseteq S_{r_t} \backslash \{j, j_t\}$, $|S^*| < r_t - 1$ and $\overline{S} \backslash S^* \neq \emptyset$. Following an argument similar to the one already used, we obtain a contradiction. And this finishes the proof of Lemma 1. □

Before proceeding with the proof of necessity, define, for each $t \in \{1, \ldots, T\}$, the set

$$Q_t = \begin{cases} S_{r_t} & \text{if } |\mathcal{L}_x^{r_t}| \geq 2 \text{ or } t = T \\ S_{r_t} \backslash \{j_t\} & \text{if } |\mathcal{L}_x^{r_t}| = 1 \text{ and } t < T, \end{cases}$$

where j_t is the agent identified in Claim 1. We first argue that

$$Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_T \tag{5}$$

holds. If $|\mathcal{L}_x^{r_{t+1}}| \geq 2$ or $t + 1 = T$, Lemma 1 directly implies $Q_t \subseteq Q_{t+1}$. Assume now that $|\mathcal{L}_x^{r_{t+1}}| = 1$ and $t + 1 < T$ hold. By Lemma 1, $Q_t \backslash \{j_{t+1}\} \subseteq Q_{t+1}$. By definition of j_{t+1}, $j_{t+1} \notin S_{r_{t+2}}$. Suppose that $j_{t+1} \in Q_t$. Then, by Lemma 1, $j_{t+1} \in S_{r_{t+2}}$ which is a contradiction. Therefore, $j_{t+1} \notin Q_t$ and $Q_t \subseteq Q_{t+1}$. It is easy to check that, also by Lemma 1,

$$Q_t \subseteq S \text{ for all } S \in \mathcal{L}_x^{r_t} \text{ and all } t \in \{1, \ldots, T\} \tag{6}$$

and

$$|Q_t| = r_t - 1 \text{ for all } t \in \{1, \ldots, T\} \tag{7}$$

hold as well.

We want to show that $\mathcal{L}_x$ satisfies IOI. By (5) and (7), we can write, for all $t \in \{1, \ldots, T\}$, the set Q_t as

$$Q_t = \{i_1, \ldots, i_{r_1 - 1}, i_{r_1}, \ldots i_{r_2 - 1}, i_{r_2}, \ldots, i_{r_3 - 1}, \ldots, i_{r_{t-1}}, \ldots, i_{r_t - 1}\}. \tag{8}$$

Consider the order

$$i_1, \ldots, i_{r_1 - 1}, i_{r_1}, \ldots i_{r_2 - 1}, i_{r_2}, \ldots, i_{r_3 - 1}, \ldots, i_{r_{t-1}}, \ldots, i_{r_t - 1}, \ldots, i_{r_{T-1}}, \ldots, i_{r_T - 1}, \tag{9}$$

and note that it is not necessarily unique since any reordering of the agents inside each Q_t in (9) is arbitrary and it would also allow us to follow the argument below.

Consider $S \in \mathcal{L}_x^{r_t}$ for some $t \geq 1$. Then, by (6), $Q_t \subseteq S$, implying that

$$\{i_1, \ldots, i_{r_1 - 1}, i_{r_1}, \ldots i_{r_2 - 1}, i_{r_2}, \ldots, i_{r_3 - 1}, \ldots, i_{r_{t-1}}, \ldots, i_{r_t - 1}\} \subseteq S,$$

which means that $\mathcal{L}_x$ satisfies IOI with respect to the order in (9). This finishes the proof of necessity.

To prove sufficiency, assume $\mathcal{L}_x$ satisfies IOI; namely, there exists an order of distinct agents $i_1, \ldots, i_K$ such that for all $k > 1$,

$$\text{if } S \in \mathcal{L}_x^k \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S.$$

Using the notation established in the proof of necessity and, by (9) letting $K = r_T - 1$, define the following subsets of agents:[36]

$$\begin{aligned} X_0^x &= \{i \in N \mid \{i\} \in \mathcal{L}_x\}, \\ Y_1^x &= \{i_1, \ldots, i_{r_1-1}\}, \\ X_1^x &= \{i \in N\backslash (X_0^x \cup Y_1^x) \mid \text{there exists } S \in \mathcal{L}_x^{r_1} \text{ such that } i \in S\}, \end{aligned}$$

for $1 < t < T$,

$$\begin{aligned} Y_t^x &= \{i_{r_{t-1}}, \ldots, i_{r_t-1}\}. \\ X_t^x &= \{i \in N\backslash[(\bigcup_{t'<t} X_{t'}^x) \cup (\bigcup_{t'\leq t} Y_{t'}^x)] \mid \text{there exists } S \in \mathcal{L}_x^{r_t} \text{ such that } i \in S\}, \end{aligned}$$

and

$$\begin{aligned} Y_T^x &= \{i_{r_{T-1}}, \ldots, i_{r_T-1}\}, \\ X_T^x &= \{i \in N\backslash[(\bigcup_{t'<T} X_{t'}^x) \cup (\bigcup_{t'\leq T} Y_{t'}^x)] \mid \text{there exists } S \in \mathcal{L}_x^{r_T} \text{ such that } i \in S\}. \end{aligned}$$

We now construct an extensive game form with perfect information $\Gamma(x, y; \mathcal{L}_x)$.[37] Each agent only plays once, following the ordering given by the (obvious) order of agents induced by the sequence of sets $X_0^x, Y_1^x, X_1^x, \ldots, Y_t^x, X_t^x, \ldots, Y_T^x, X_T^x$. Denote this order by $j_1, \ldots, j_n$.[38] Define the set of non-terminal nodes Z_{NT} by assigning each agent i in the order to a non-terminal node z_i, in such a way that if i goes earlier in the order than j, then $z_i \prec z_j$. At each $z_i \in Z_{NT}$, agent $i \in N$ has available the set of actions $\mathcal{A}(z_i) = \{x, y\}$. Look at any agent j_h in the order with $1 \leq h < n$. If $j_h \in X_t^x$, for $t = 0, \ldots, T$, and $\sigma_{j_h}(z_{j_h}) = x$, then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z$ is a terminal node and set $g(z) = x$. If $\sigma_{j_h}(z_{j_h}) = y$ then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z_{j_{h+1}}$ is a non-terminal node at which agent j_{h+1} plays. If $j_h \in Y_t^x$, for $t = 1, \ldots, T$, and $\sigma_{j_h}(z_{j_h}) = y$, then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z$ is a terminal node and set $g(z) = y$. If $\sigma_{j_h}(z_{j_h}) = x$ then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z_{j_{h+1}}$ is a non-terminal node at which agent j_{h+1} plays. Look now at agent j_n, the last in the order. Then, the history $(z_{j_n}, \sigma_{j_n}(z_{j_n})) = z$ is a terminal node, independently of whether $\sigma_{j_n}(z_{j_n}) = x$ (in which case set $g(z) = x$) or $\sigma_{j_n}(z_{j_n}) = y$ (in which case set $g(z) = y$). And this finishes the definition of $\Gamma(x, y; \mathcal{L}_x)$ (Figure 2, at the end of the statement of Proposition 1, depicts $\Gamma(x, y; \mathcal{L}_x)$ for the case of the committee $\mathcal{L}_x$ of Example 1).

[36] We use the superscript x in the notation of these sets because later on we will need to define the corresponding sets for the committee $\mathcal{L}_y$, for which we will use then the superscript y.

[37] Remember that there may be many such games because agents belonging to the sets X_0^x and Y_t^xs can be freely ordered. The orderings inside the sets X_t^xs are determined by the sequence $i_1, \ldots, i_K$ which also may not be unique.

[38] Without loss of generality we are assuming that no agent is dummy in $\mathcal{L}_x$; otherwise, the obtained sequence would be $j_1, \ldots, j_{n'}$, with $n' < n$, and we would proceed by setting $Z_i = \emptyset$ for any dummy i, so that i would not play at $\Gamma(x, y; \mathcal{L}_x)$.

For each $P \in \mathcal{P}$, let $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$ be the truth-telling profile of strategies in $\Gamma(x, y; \mathcal{L}_x)$; *i.e.*, for all $i \in N$, $\sigma_i^{P_i}(z_i) = x$ if and only if $t(P_i) = x$, where z_i denotes the unique node at which agent i has to play at $\Gamma(x, y; \mathcal{L}_x)$. It is easy to see that $\Gamma(x, y; \mathcal{L}_x)$ induces f since $f(P) = g(z^\Gamma(z_0, \sigma^P))$ for arbitrary $P \in \mathcal{P}$. We want to show that, for each agent i, $\sigma_i^{P_i}$ is obviously dominant in $\Gamma(x, y; \mathcal{L}_x)$. Fix $j_h \in N$, and suppose j_h is called to play. We distinguish between two cases.

Case 1: $j_h \in X_t^x$ for some $t = 0, \ldots, T$. Assume first that $t(P_{j_h}) = x$, and so $\sigma_{j_h}^{P_{j_h}}(z_{j_h}) = x$. Then, $(z_{j_h}, \sigma_{j_h}^{P_{j_h}}(z_{j_h})) = z \in Z_T$ and $g(z) = x = t(P_{j_h})$. Hence, $\sigma_{j_h}^{P_{j_h}}$ is trivially obviously dominant for j_h. Suppose now that $t(P_{j_h}) = y$, and let σ'_{j_h} be the strategy $\sigma'_{j_h}(z_{j_h}) = x$. Then, $(z_{j_h}, \sigma'_{j_h}(z_{j_h})) = z \in Z_T$ and $g(z) = x$, which is the worst alternative according to P_{j_h}. Hence, $\sigma_{j_h}^{P_{j_h}}$ is obviously dominant for j_h.

Case 2: $j_h \in Y_t^x$ for some $t = 1, \ldots, T$. Assume first that $t(P_{j_h}) = y$, and so $\sigma_{j_h}^{P_{j_h}}(z_{j_h}) = y$. Then,$(z_{j_h}, \sigma_{j_h}^{P_{j_h}}(z_{j_h})) = z \in Z_T$ and $g(z) = y = t(P_{j_h})$. Hence, $\sigma_{j_h}^{P_{j_h}}$ is trivially obviously dominant for j_h. Suppose now that $t(P_{j_h}) = x$, and let σ'_{j_h} be the strategy $\sigma'_{j_h}(z_{j_h}) = y$. Then, $(z_{j_h}, \sigma'_{j_h}(z_{j_h})) = z \in Z_T$ and $g(z) = y$, which is the worst alternative according to P_{j_h}. Hence, $\sigma_{j_h}^{P_{j_h}}$ is obviously dominant for j_h. ■

7.2 Proof of Proposition 2

Proposition 2 *A* SCF $f : \mathcal{SP} \to X$ *is* OSP *if and only if* f *is a* GMVS *whose associated left and right coalition systems,* $\{\mathcal{L}_x\}_{x \in X}$ *and* $\{\mathcal{R}_x\}_{x \in X}$*, satisfy the following two properties:*

(L-IOI) *For every* $\beta > x \geq x_1 - 1$*, there exists* $i^x \in N$ *such that* $\mathcal{L}_x$ *satisfies* IOI *with respect to* i^x *and* $\{i^x\} \in \mathcal{L}_{x+1}$.

(R-IOI) *For every* $\alpha < x \leq x_1 + 1$*, there exists* $i^x \in N$ *such that* $\mathcal{R}_x$ *satisfies* IOI *with respect to* i^x *and* $\{i^x\} \in \mathcal{R}_{x-1}$.

The proof of Proposition 2 will require, at each $x \in X$ and each $k \in \{2, \ldots, n\}$, to look at the family of minimal winning coalitions of cardinality k, as well as at their intersections. Given $x \in X$ and $k \in \{1, \ldots, n\}$, denote by

$$\mathcal{L}^k(x) = \{S \in \mathcal{L}_x^m \mid |S| = k\} \text{ and } \mathcal{R}^k(x) = \{S \in \mathcal{R}_x^m \mid |S| = k\}$$

the respective families of minimal winning coalitions with cardinality $k \in \{1, \ldots, n\}$, and let

$$S_k^L(x) = \bigcap_{S \in \mathcal{L}^k(x)} S \text{ and } S_k^R(x) = \bigcap_{S \in \mathcal{R}^k(x)} S$$

be their intersections.[39] We say that k is a non-empty left-cardinality at x, written

[39] At the beginning of Subsection 4.2, we have already defined $\mathcal{L}^k(x)$ as $\mathcal{L}_x^k$, but we now change slightly the notation to write it in the context also of many committees and right coalition systems.

$k \in NE^L(x)$, if $\mathcal{L}^k(x) \neq \emptyset$ and $k \geq 2$, and similarly, we say that k is a non-empty right-cardinality at x, written $k \in NE^R(x)$, if $\mathcal{R}^k(x) \neq \emptyset$ and $k \geq 2$.

Proof of Proposition 2 To prove necessity, assume $f : \mathcal{SP} \to X$ is OSP. To obtain a contradiction, suppose first that (L-IOI) does not hold. We distinguish between two cases.

Case 1: There exists $x \in X$ such that $x \geq x_1 - 1$ and $\mathcal{L}_x$ does not satisfy IOI. Since $\mathcal{L}_\beta$ satisfies IOI trivially, $x < \beta$. Define $\widehat{N} = \bigcup_{S \in \mathcal{L}^m_x} S$ and $\widehat{\mathcal{L}}^m_x = \mathcal{L}^m_x$. For $i \in \widehat{N}$, let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose tops are either x or $x+1$, and for $i \in N \backslash \widehat{N}$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose top is $x+1$. Define $\widetilde{\mathcal{SP}} = \widetilde{\mathcal{SP}}_1 \times \cdots \times \widetilde{\mathcal{SP}}_n$ and consider the SCF $\widetilde{f} : \widetilde{\mathcal{SP}} \to \{x, x+1\}$ which is the restriction of f in the subdomain $\widetilde{\mathcal{SP}}$. Since, by assumption, $\widehat{\mathcal{L}}^m_x$ does not satisfy IOI, Proposition 1 implies that $\widetilde{f} : \widetilde{\mathcal{SP}} \to \{x, x+1\}$ is not OSP and, by Remark 1, $f : \mathcal{SP} \to X$ is not OSP-implementable, a contradiction.

Case 2: Assume $\mathcal{L}_x$ satisfies IOI for all $x \geq x_1 - 1$, but there exists $\widehat{x} \geq x_1 - 1$ such that for all $i^{\widehat{x}}$ such that $\mathcal{L}^m_{\widehat{x}}$ satisfies IOI with respect to $i^{\widehat{x}}$, $\{i^{\widehat{x}}\} \notin \mathcal{L}_{\widehat{x}+1}$. Since for all $i \in N$, $\{i\} \in \mathcal{L}^1(\beta)$, we have that $\widehat{x} < \beta$. Furthermore, if $|S| = 1$ for all $S \in \mathcal{L}^m_{\widehat{x}}$, then there exists i such that $\{i\} \in \mathcal{L}^m_{\widehat{x}}$ which, by the monotonicity property in the definition of a left coalition system, requires that $\{i\} \in \mathcal{L}^m_{\widehat{x}+1}$. Furthermore $\mathcal{L}^m_{\widehat{x}}$ satisfies IOI with respect to i trivially, in contradiction with our initial hypothesis. Hence,

$$\{T \in \mathcal{L}^m_{\widehat{x}} \mid |T| \geq 2\} \neq \emptyset. \tag{10}$$

Denote by $F(\mathcal{L}_{\widehat{x}})$ the set of agents for whom $\mathcal{L}_{\widehat{x}}$ satisfies IOI with respect to each of them; namely, $F(\mathcal{L}_{\widehat{x}}) = \{i \in N \mid \mathcal{L}_{\widehat{x}}$ satisfies IOI with respect to $i\}$. By assumption $F(\mathcal{L}_{\widehat{x}}) \neq \emptyset$. Let $x' = \min\{x \in X \mid$there exists $i^{\widehat{x}} \in F(\mathcal{L}_{\widehat{x}})$ and $\{i^{\widehat{x}}\} \in \mathcal{L}_x\}$ and let $i^{*\widehat{x}}$ be one of the agents in $F(\mathcal{L}_{\widehat{x}})$ such that $\{i^{*\widehat{x}}\} \in \mathcal{L}_{x'}$. By definition of x', and the contradiction hypothesis, $\widehat{x}+1 < x'$ (note that x' may be equal to β) and for all $i^{\widehat{x}} \in F(\mathcal{L}_{\widehat{x}})$, $\{i^{\widehat{x}}\} \notin \mathcal{L}^m_{\widetilde{x}}$ for all $\widetilde{x} < x'$. Since $\widehat{x} \geq x_1 - 1$, and the definition of x_1, there exists i^* such that $\{i^*\} \in \mathcal{L}_{\widehat{x}+1}$ (note that $\widehat{x}+1 \geq x_1$). Since $\{i\} \notin \mathcal{L}_{\widehat{x}+1}$ for all $i \in F(\mathcal{L}_{\widehat{x}})$, we have that $i^* \notin F(\mathcal{L}_{\widehat{x}})$. Moreover, because $\mathcal{L}_{\widehat{x}}$ satisfies IOI and $i^* \notin F(\mathcal{L}_{\widehat{x}})$, by (10), there exists $S \in \{T \in \mathcal{L}^m_{\widehat{x}} \mid |T| \geq 2\}$ such that $i^* \notin S$. As $i^{*\widehat{x}} \in \bigcap_{k \in NE^L(x)} S^L_k(\widehat{x})$, $i^{*\widehat{x}} \in S$ and there exists $j \neq i^{*\widehat{x}}$ such that $j \in S$. Given $i^*, j, i^{*\widehat{x}} \in N$ and S we define a Cartesian product subset of the set of all single-peaked preference profiles as follows. Let $\widetilde{\mathcal{SP}}_{i^*}$ be i^*'s subset of single-peaked preferences whose tops are either $\widehat{x}+1$ or x'. For $i \in \{j, i^{*\widehat{x}}\}$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose tops are either $\widehat{x}$ or x'. For $i \in S \backslash \{j, i^{*\widehat{x}}\}$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose top is $\widehat{x}$. Finally, for $i \in N \backslash (S \cup \{i^*\})$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose top is x'. Define $\widetilde{\mathcal{SP}} = \widetilde{\mathcal{SP}}_1 \times \cdots \times \widetilde{\mathcal{SP}}_n$ and consider the SCF $\widetilde{f} : \widetilde{\mathcal{SP}} \to \{\widehat{x}, \widehat{x}+1, x'\}$ which is f restricted to this subdomain $\widetilde{\mathcal{SP}}$. As f is OSP, by Remark 1, $\widetilde{f}$ is OSP. Let Γ be a pruned game that OSP-implements $\widetilde{f}$. Since Γ induces $\widetilde{f}$ and $\widetilde{f}$ is not constant, there exists an information set at which a player has available two actions. It is clear that such agent belongs to the set $\{i^*, j, i^{*\widehat{x}}\}$.

Assume i^* is the agent who first has an information set I_{i^*} with at least two available actions in Γ and suppose that $t(P_{i^*}) = \widehat{x}+1$ and $x' P_{i^*} \widehat{x}$. Then, since $\{i^*\} \in \mathcal{L}^m_{\widehat{x}+1}$ and $i^* \notin S \in \mathcal{L}^m_{\widehat{x}}$,

$$\min_{P_{i^*}}\{x \in X \mid x = g(z^\Gamma(z, (\sigma_{i^*}^{P_{i^*}}, \sigma_{-i^*}))) \text{ for some } \sigma_{-i^*} \text{ and } z \in I_{i^*}\} = \min_{P_{i^*}}\{\widehat{x}, \widehat{x}+1\} = \widehat{x}. \tag{11}$$

Now, let $\sigma'_{i^*} \in \Sigma_{i^*}$ be such that $\sigma'_{i^*} = \sigma_{i^*}^{P'_{i^*}}$, where $t(P'_{i^*}) = x'$. Remember that since Γ is pruned, agent i^* only has at Γ strategies associated to single-peaked preferences whose tops are either $\widehat{x}+1$ or x'. Hence, $\sigma_{i^*}^{P_{i^*}}(z) \neq \sigma'_{i^*}(z)$ because Γ induces the tops-only SCF $\widetilde{f}$. Then, as i^* is the agent who first has an information set I_{i^*} with at least two available actions, $I_{i^*} \in \alpha(\sigma_{i^*}^{P_{i^*}}, \sigma'_{i^*})$ and by the definitions of x', P_{i^*} and $\widetilde{\mathcal{SP}}$,

$$\max_{P_{i^*}}\{x \in X \mid x = g(z^\Gamma(z, (\sigma'_{i^*}, \sigma_{-i^*}))) \text{ for some } \sigma_{-i^*} \text{ and } z \in I_{i^*}\} = \max_{P_{i^*}}\{\widehat{x}, x'\} = x'. \tag{12}$$

But again, $x' P_{i^*} \widehat{x}$ and conditions (11) and (12) imply that $\sigma_{i^*}^{P_{i^*}}$ is not obviously dominant in Γ, contradicting that Γ OSP-implements $\widetilde{f}$.

Assume now that agent $j' \in \{i^{*\widehat{x}}, j\}$ is the agent who first has an information set $I_{j'}$ with at least two available actions in Γ and suppose that $t(P_{j'}) = \widehat{x}$. Then, since $S \in \mathcal{L}^m_{\widehat{x}}$ implies $S \in \mathcal{L}^m_{\widehat{x}+1}$, by single-peakedness of $P_{j'}$ and the definition of $\widetilde{\mathcal{SP}}$,

$$\min_{P_{j'}}\{x \in X \mid x = g(z^\Gamma(z, (\sigma_{j'}^{P_{j'}}, \sigma_{-j'}))) \text{ for some } \sigma_{-j'} \text{ and } z \in I_{j'}\} = \min_{P_{j'}}\{\widehat{x}, \widehat{x}+1, x'\} = x'. \tag{13}$$

Now, let $\sigma'_{j'} \in \Sigma_{j'}$ be such that $\sigma'_{j'} = \sigma_{j'}^{P'_{j'}}$, where $t(P'_{j'}) = x'$. Remember that since Γ is pruned, agent j' only has at Γ strategies associated to single-peaked preferences whose tops are either $\widehat{x}$ or x'. Hence, $\sigma_{j'}^{P_{j'}}(z) \neq \sigma'_{j'}(z)$ because Γ induces the tops-only SCF $\widetilde{f}$. Then, as j' is the agent who first has an information set $I_{j'}$ with at least two available actions, $I_{j'} \in \alpha(\sigma_{j'}^{P_{j'}}, \sigma'_{j'})$ and by definitions of x', $P'_{j'}$ and $\widetilde{\mathcal{SP}}$,

$$\max_{P_{j'}}\{x \in X \mid x = g(z^\Gamma(z, (\sigma'_{j'}, \sigma_{-j'}))) \text{ for some } \sigma_{-j'} \text{ and } z \in I_{j'}\} = \max_{P_{j'}}\{\widehat{x}+1, x'\} = \widehat{x}+1. \tag{14}$$

By single-peakedness of $P_{j'}$, $\widehat{x}+1 P_{j'} x'$, which together with conditions (13) and (14) imply that $\sigma_{j'}^{P_{j'}}$ is not obviously dominant in Γ, contradicting that Γ OSP-implements $\widetilde{f}$. Hence, (L-IOI) holds.

Now we prove that (R-IOI) holds. Since (L-IOI) holds, by Lemma 2 (in Case 3 below), there exists $i \in N$ such that $\{i\} \in \mathcal{R}^m_{x_1}$. Then, the largest alternative for which the right coalition system has a decisive agent is equal to or larger than x_1. Now the proof that (R-IOI) holds follows a symmetric argument to the one used to show that (L-IOI) holds.

To prove sufficiency, assume $f : \mathcal{SP} \to X$ is a GMVS whose associated left and right coalition systems, $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$, satisfy (L-IOI) and (R-IOI), respectively. We distinguish among three cases, depending on whether $x_1 = \alpha$ (case 1), $x_1 = \beta$ (case 2) or $x_1 \notin \{\alpha, \beta\}$ (case 3). In the three cases the game constructed in the sufficiency proof of Proposition 1 will play a fundamental role, since a GMVS f may be seen as a sequence of EMVRs, each between x and $x+1$, when f is described as a left coalition system (with the associated game $\Gamma(x, x+1; \mathcal{L}_x)$), or a sequence of EMVRs, each between x and $x-1$, when f is described as a right coalition system (with the associated game $\Gamma(x, x-1; \mathcal{R}_x)$). If $x_1 \in \{\alpha, \beta\}$ only one of the two sequences will be needed in the construction of the overall Γ, while if $x_1 \notin \{\alpha, \beta\}$ we will have to consider $\Gamma(x_1, x_1+1; \mathcal{L}_{x_1}), \ldots, \Gamma(\beta-1, \beta; \mathcal{L}_{\beta-1})$ and $\Gamma(x_1, x_1-1; \mathcal{R}_{x_1}), \ldots, \Gamma(\alpha+1, \alpha; \mathcal{R}_{\alpha+1})$. The choice of whether the game Γ proceeds by following the first or the second sequence will depend on a particular agent that will simultaneously be left-decisive and right-decisive at x_1, and that we will identify in Lemma 2 (in Case 3 below).

<u>Case 1</u>: $x_1 = \alpha$. Suppose that for all $\alpha \le x < \beta$, $\mathcal{L}_x$ satisfies IOI with respect to i^x and $\{i^x\} \in \mathcal{L}_{x+1}$. We define a game Γ by considering the sequence of games $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha), \Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1}), \ldots, \Gamma(\beta-1, \beta; \mathcal{L}_{\beta-1})$ defined in the proof of Proposition 1, where for each $\alpha \le x < \beta$, the first agents to play at the game $\Gamma(x, x+1; \mathcal{L}_x)$ are the set of decisive agents at x (*i.e.*, the set X_0^x in the notation used in the proof of Proposition 1), with any ordering, but making sure that for each $\alpha < x < \beta$, agent i^x is the agent that plays immediately after the decisive agents in x (*i.e.*, $i^x \in Y_1^x$ in the notation used in the proof of Proposition 1 and among the set of agents in Y_1^x, i^x is the first agent to play). We will write $g(z^{x+}(.,.))$ instead of $g(z^{\Gamma(x,x+1;\mathcal{L}_x)}(.,.))$. We now proceed to describe the details of the steps used to define Γ.

The set of agents in $N_\alpha = \cup_{S\in\mathcal{L}_\alpha^m} S$ play the game $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$. In this game, each $i \in N_\alpha$ plays only once. Let $z_i^{\alpha+} \in Z^{\Gamma(\alpha,\alpha+1;\mathcal{L}_\alpha)}$ be the node at which i plays, where i has available the set of actions $\mathcal{A}(z_i^{\alpha+}) = \{\alpha, \alpha+1\}$. For each $i \in N_\alpha$, we denote by $a_i^{\alpha+} \in \{\alpha, \alpha+1\}$ the action chosen by i at $z_i^{\alpha+}$ in $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$ and by $a^{\alpha+} = (a_i^{\alpha+})_{i\in N_\alpha}$ the profile of actions.[40] Abusing notation, let $z_0^{\alpha+}$ be the node assigned to the first agent playing in $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$. Then, we make sure that the following three properties of Γ hold, regarding the outcome of $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$.

First, if $g(z^{\alpha+}(z_0^{\alpha+}, a^{\alpha+})) = \alpha$, then the overall game Γ ends and the outcome is α.

Second, if $g(z^{\alpha+}(z_0^{\alpha+}, a^{\alpha+})) = \alpha+1$ and $a_{i^\alpha}^{\alpha+} = \alpha$, then the overall game ends and the outcome is $\alpha+1$.

Third, if $g(z^{\alpha+}(z_0^{\alpha+}, a^{\alpha+})) = \alpha+1$ and $a_{i^\alpha}^{\alpha+} \neq \alpha$, then agents in $N_{\alpha+1} = \cup_{S\in\mathcal{L}_{\alpha+1}^m} S$ play the game $\Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1})$, whose initial node is this terminal node of $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$.

[40] We are defining the (*behavioral*) *strategies* in the full game Γ by specifying the *actions* taken by agents at each of the games induced by their corresponding EMVRs.

In this game $\Gamma(\alpha+1,\alpha+2;\mathcal{L}_{\alpha+1})$, each agent $i \in N_{\alpha+1}$ plays only once. Let $z_i^{(\alpha+1)+} \in Z^{\Gamma(\alpha+1,\alpha+2;\mathcal{L}_{\alpha+1})}$ be the node at which i plays. Then, agents in $De^L_{\alpha+1}$ play in any order and they are immediately followed by agent $i^{\alpha+1}$ (such agent exists since $\mathcal{L}_{\alpha+1}$ satisfies (L-IOI) with respect to $i^{\alpha+1}$ and $i^{\alpha+1} \in De^L_{\alpha+2}$). Each agent $i \in N_{\alpha+1}$ has available at $z_i^{(\alpha+1)+}$ the set of actions $\mathcal{A}(z_i^{(\alpha+1)+}) = \{\alpha+1, \alpha+2\}$. For each $i \in N_{\alpha+1}$, we denote by $a_i^{(\alpha+1)+} \in \{\alpha+1,\alpha+2\}$ the action chosen by i in $\Gamma(\alpha+1,\alpha+2;\mathcal{L}_{\alpha+1})$ and by $a^{(\alpha+1)+} = (a_i^{(\alpha+1)+})_{i\in N_{\alpha+1}}$ the profile of actions. Abusing notation, let $z_0^{(\alpha+1)+}$ be the node assigned to the first agent playing in $\Gamma(\alpha+1,\alpha+2;\mathcal{L}_{\alpha+1})$. Then, we make sure that the following three properties of Γ hold, regarding the outcome of $\Gamma(\alpha+1,\alpha+2;\mathcal{L}_{\alpha+1})$.

First, if $g(z^{(\alpha+1)+}(z_0^{\alpha+1}, a^{(\alpha+1)+})) = \alpha+1$, then the overall game Γ ends and the outcome is $\alpha+1$.

Second, if $g(z^{(\alpha+1)+}(z_0^{\alpha+1}, a^{(\alpha+1)+})) = \alpha+2$ and $a_{i^{\alpha+1}}^{(\alpha+1)+} = \alpha+1$, then the overall game Γ ends and the outcome is $\alpha+2$.

Third, if $g(z^{(\alpha+1)+}(z_0^{\alpha+1}, a^{(\alpha+1)+})) = \alpha+2$ and $a_{i^{\alpha+1}}^{(\alpha+1)+} \neq \alpha+1$, then agents in $N_{\alpha+2} = \cup_{S\in\mathcal{L}^m_{\alpha+2}} S$ play the game $\Gamma(\alpha+2,\alpha+3;\mathcal{L}_{\alpha+2})$, whose initial node is this terminal node of $\Gamma(\alpha+1,\alpha+2;\mathcal{L}_{\alpha+1})$.

We continue with the construction of Γ in the same way for each $x \in \{\alpha,\ldots,\beta-2\}$, if any. Let z_0^{x+} the node assigned to the first agent playing in the game $\Gamma(x,x+1;\mathcal{L}_x)$. Identify the ordering of play and the set of available actions as in the previous cases and, in particular, make sure that the following three properties of Γ hold, regarding the outcome of $\Gamma(x,x+1;\mathcal{L}_x)$.

First, if $g(z^{x+}(z_0^{x+}, a^{x+})) = x$, then the overall game Γ ends and the outcome is x.

Second, if $g(z^{x+}(z_0^{x+}, a^{x+})) = x+1$ and $a_{i^x}^{x+} = x$, then the overall game Γ ends and the outcome is $x+1$.

Third, if $g(z^{x+}(z_0^{x+}, a^{x+})) = x+1$ and $a_{i^x}^{x+} \neq x$, then agents in $N_{x+1} = \cup_{S\in\mathcal{L}^m_{x+1}} S$ play the game $\Gamma(x+1,x+2;\mathcal{L}_{x+1})$, whose initial node is this terminal node of $\Gamma(x,x+1;\mathcal{L}_x)$.

Finally, when $\beta-1$ is reached, agents in $N_{\beta-1} = \cup_{S\in\mathcal{L}^m_{\beta-1}} S$ play the game $\Gamma(\beta-1,\beta;\mathcal{L}_\beta)$ starting at $z_0^{\beta-1}$ with the feature that the following two properties hold.

First, if $g(z^{(\beta-1)+}(z_0^{\beta-1}, a^{(\beta-1)+})) = \beta-1$, then the overall game Γ ends and the outcome is $\beta-1$.

Second, if $g(z^{(\beta-1)+}(z_0^{\beta-1}, a^{(\beta-1)+})) = \beta$, then the overall game Γ ends and the outcome is β.

Let Γ be the extensive game form just constructed. Since all information sets are singletons, Γ has perfect information. Fix $x < \beta$ and let $i \in N$ be arbitrary. If $i \in N_x$, then there exists one and only one node in $\Gamma(x,x+1;\mathcal{L}_x)$ at which agent i plays. We have denoted this node by z_i^{x+}. Again, for an arbitrary $i \in N$, let $A_i = \{x \in X \mid i \in N_x\}$ be the set of such x's at which i is called to play at z_i^{x+} in $\Gamma(x,x+1;\mathcal{L}_x)$. If $A_i = \emptyset$ then i is a dummy agent in all committees (*i.e.*, for all $x < \beta$, $i \notin S$ for all $S \in \mathcal{L}^m_x$) and $Z_i = \emptyset$

in Γ. But then, i's truth-telling strategy is trivially obviously dominant. For each agent $i \in N$, a strategy $\sigma_i : Z_i \to A$ in Γ is a function that, for each z_i^{x+} with $x \in A_i$, selects an action in $\mathcal{A}(z_i^{x+}) = \{x, x+1\}$ (*i.e.*, $\sigma_i(z_i^{x+}) \in \{x, x+1\}$).

For $P \in \mathcal{SP}$, let $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$ be the profile of truth-telling strategies; namely, for all $x \in X$, all $i \in N_x$, and all $z_i^{x+} \in Z_i$, $\sigma_i^{P_i}(z_i^{x+}) = x$ if and only if $t(P_i) \leq x$ (and hence, $\sigma_i^{P_i}(z_i^{x+}) = x+1$ if and only if $t(P_i) \geq x+1$).

Let $f : \mathcal{SP} \to X$ be a GMVS whose left coalition system has the property that $x_1 = \alpha$. Then, it is easy to see that Γ induces $f : \mathcal{SP} \to X$ since for all $P \in \mathcal{SP}$, $f(P) = g(z^\Gamma(z_0, \sigma^P))$.

We want to show that, for each i, $\sigma_i^{P_i}$ is obviously dominant in Γ. Fix $i \in N$ and let σ_i' be any strategy of i with the property that $\sigma_i' \neq \sigma_i^{P_i}$. Denote by $z_i^{\bar{x}+}$ the earliest point of departure for $\sigma_i^{P_i}$ and σ_i'; *i.e.*, $\sigma_i^{P_i}(z_i^{x+}) = \sigma_i'(z_i^{x+})$ for all $x < \bar{x}$ with $x \in A_i$ and $\sigma_i^{P_i}(z_i^{\bar{x}+}) \cup \sigma_i'(z_i^{\bar{x}+}) = \{\bar{x}, \bar{x}+1\}$. We proceed by distinguishing among several cases, depending on the role of i with respect to the committee $\mathcal{L}_{\bar{x}}$.

<u>Case 1.a</u>: $i \in X_t^{\bar{x}+}$ for some $t = 0, \ldots, T$, where $X_t^{\bar{x}+}$ corresponds to the set of agents that by choosing $\bar{x}$ in the game $\Gamma(\bar{x}, \bar{x}+1, \mathcal{L}_{\bar{x}})$ it ends at $\bar{x}$ (see the sufficiency proof of Proposition 1).

<u>Case 1.a.1</u>: Assume first that $t(P_i) \leq \bar{x}$, and so $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}$. Then, the node z that follows $z_i^{\bar{x}+}$ after i plays $\bar{x}$ has the property that $z \in Z_T$ and

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}\}.$$

As $z_i^{\bar{x}+}$ is the earliest point of departure for $\sigma_i^{P_i}$ and σ_i', $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}+1$. Hence,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} \subseteq \{\bar{x}, \ldots, \beta\}.$$

Therefore, since $t(P_i) \leq \bar{x}$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

<u>Case 1.a.2</u>: Assume now that $\bar{x} < t(P_i)$, and so $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}+1$ and $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}$. By the definition of Γ, the node z that follows $z_i^{\bar{x}+}$ after i plays $\bar{x}$ has the property that $z \in Z_T$ and

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}\}.$$

The last equality follows because if $i \in X_t^{\bar{x}+}$ for some $t = 0, \ldots, T$, then i can induce $\bar{x}$ by choosing $\bar{x}$ in the game $\Gamma(\bar{x}, \bar{x}+1; \mathcal{L}_{\bar{x}})$, which means that $\bar{x}$ is the outcome of Γ as well. However,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} \subset \{\bar{x}, \ldots, t(P_i)\},$$

where the last inclusion follows because, according to the hypothesis of Case 1.a, either (i) $i \in X_0^{\bar{x}+}$ or else (ii) $i \in X_t^{\bar{x}+}$ for some $t \geq 1$. If (i) holds, $\{i\} \in \mathcal{L}_{x'}$ for all $x' \geq \bar{x}$, and thus $g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))$ will not be larger than $t(P_i)$. If (ii) holds, observe that when i

is called to play at $z_i^{\bar{x}+}$, agent $i^{\bar{x}}$ (who plays before i in $\Gamma(\bar{x}, \bar{x}+1, \mathcal{L}_{\bar{x}})$ because is the first agent in $Y_1^{\bar{x}}$) has already chosen the action $\bar{x}$ in $z_i^{\bar{x}+}$. Then, the outcome of the game is $\bar{x}$ or $\bar{x}+1$ and $\bar{x}+1 \leq t(P_i)$ in this case.

Case 1.b: $i \in Y_t^{\bar{x}+}$ for some $t = 1, \ldots, T$, where $Y_t^{\bar{x}+}$ corresponds to the set of agents that by choosing $\bar{x}+1$ in the game $\Gamma(\bar{x}, \bar{x}+1; \mathcal{L}_{\bar{x}})$ it ends at $\bar{x}+1$ (see the sufficiency proof of Proposition 1).

Case 1.b.1: Assume first that $\bar{x} < t(P_i)$. Thus, $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}+1$ and $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}$. We distinguish between two cases, depending on i's identity.

Case 1.b.1.1: $i = i^{\bar{x}}$. Then, by (L-IOI) and the monotonicity property in the definition of a left coalition system, $\{i^{\bar{x}}\} \in \mathcal{L}_{x'}^m$ for all $x' \geq \bar{x}+1$. Therefore,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}+1, \ldots, t(P_i)\}.$$

Furthermore, since $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}$,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}, \bar{x}+1\}.$$

Then, since $\bar{x} < \bar{x}+1 \leq t(P_i)$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

Case 1.b.1.2: $i \neq i^{\bar{x}}$. Then,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}+1\}$$

and

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} \subset \{\bar{x}, \bar{x}+1\}.$$

To see that the last statements hold, observe that when i is called to play at $z_i^{\bar{x}+}$, agent $i^{\bar{x}}$ (who plays before i in $\Gamma(\bar{x}, \bar{x}+1, \mathcal{L}_{\bar{x}})$) has already chosen the action $\bar{x}$ in $z_i^{\bar{x}+}$. Therefore, and since $\bar{x} < \bar{x}+1 \leq t(P_i)$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

Case 1.b.2: Assume now that $t(P_i) \leq \bar{x}$. Thus, $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}$ and $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}+1$. Hence,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} \subset \{\bar{x}+1, \ldots, \beta\}.$$

Furthermore,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} \subset \{\bar{x}, \bar{x}+1\}.$$

Therefore, since $t(P_i) \leq \bar{x} < \bar{x}+1$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

Case 2: $x_1 = \beta$. Suppose that for all $\alpha < x \leq \beta$, $\mathcal{R}_x$ satisfies IOI with respect to i^x and $\{i^x\} \in \mathcal{R}_{x-1}$. Now, the proof follows a symmetric argument to the one already used in Case 1, using instead the right coalition system $\{\mathcal{R}_x\}_{x \in X}$ and the sequence of games $\Gamma(\beta, \beta-1; \mathcal{R}_\beta), \Gamma(\beta-1, \beta-2; \mathcal{R}_{\beta-1}), ..., \Gamma(\alpha+1, \alpha; \mathcal{R}_{\alpha+1})$.

Case 3: $x_1 \notin \{\alpha, \beta\}$. We start by identifying an agent who is simultaneously left-decisive and right-decisive at x_1. Lemma 2 does that, but to state it we need some additional notation. Define

$$S^L(x_1 - 1) = \bigcap_{k \in NE^L(x_1-1)} S^L_k(x_1 - 1)$$

and

$$S^R(x_1 + 1) = \bigcap_{k \in NE^R(x_1+1)} S^R_k(x_1 + 1),$$

where recall that $NE^L(x) = \{k \in \{2, \ldots, n\} \mid \mathcal{L}^k(x) \neq \emptyset\}$, and the other sets needed to define $S^L(x_1 - 1)$ and $S^R(x_1 + 1)$ are $NE^R(x) = \{k \in \{2, \ldots, n\} \mid \mathcal{R}^k(x) \neq \emptyset\}$, $S^L_k(x_1 - 1) = \bigcap_{S \in \mathcal{L}^k(x_1-1)} S$ and $S^R_k(x_1 + 1) = \bigcap_{S \in \mathcal{R}^k(x_1+1)} S$.[41]

Lemma 2 *Assume $i \in S^L(x_1 - 1)$ and $\{i\} \in \mathcal{L}^m_{x_1}$. Then,*

(L2.1) $\{i\} \in \mathcal{R}^m_{x_1}$;

(L2.2) *either* (a) $i \in S^R(x_1 + 1)$ *if* $S^R(x_1 + 1) \neq \emptyset$ *or* (b) $\{i\} \in \mathcal{R}^m_{x_1+1}$; *and*

(L2.3) *if $S \in \mathcal{R}^m_x$ and $i \notin S$, then $x \leq x_1$.*

Proof of Lemma 2 Condition (L2.1) follows from $i \in S^L(x_1 - 1)$, the relationship between the families of left and right coalition systems stated in Remark 4 and the definition of x_1. To see that (L2.2) holds, observe that since $\{i\} \in \mathcal{L}^m_{x_1}$ holds, Remark 4 implies that $i \in T$ for every $T \in \mathcal{R}^m_{x_1+1}$; then, either (a) $i \in S^R(x_1 + 1)$ if $S^R(x_1 + 1) \neq \emptyset$ or (b) $\{i\} \in \mathcal{R}^m_{x_1+1}$ follow. To see that (L2.3) holds, observe that since $\{i\} \in \mathcal{L}^m_{x_1}$ holds, again by Remark 4, $i \in T$ for every $T \in \mathcal{R}^m_x$ for each $x \geq x_1 + 1$. □

Since $\mathcal{L}_{x_1-1}$ satisfies (L-IOI) and by x_1's definition, there exists $i_1 \in N$ such that $i_1 \in S^L(x_1 - 1)$ (*i.e.*, $\mathcal{L}_{x_1-1}$ satisfies IOI with respect to the i_1 and $S^L(x_1 - 1) \neq \emptyset$) and $\{i_1\} \in \mathcal{L}_{x_1}$. By Lemma 2, $\{i_1\} \in \mathcal{R}_{x_1}$ as well. To define a game Γ that OSP-implements f, agent i_1 is the first to play, at z_0 (the initial node of Γ), and has available the following three actions: $\mathcal{A}(z_0) = \{x_1 - 1, x_1, x_1 + 1\}$. To continue with the construction of Γ we describe the subgame (if any) that follows each of the three choices of i_1 at z_0.

(a) Agent i_1 selects x_1. Then, the overall game Γ ends and the outcome is x_1.

(b) Agent i_1 selects $x_1 + 1$. Then, the game Γ proceeds with the sequence of games $\Gamma(x_1, x_1 + 1; \mathcal{L}_{x_1}), \ldots, \Gamma(\beta - 1, \beta; \mathcal{L}_{\beta-1})$ as described in Case 1 starting at x_1 instead of α.

(c) Agent i_1 selects $x_1 - 1$. Then, the game Γ proceeds with the sequence of games $\Gamma(x_1, x_1 - 1; \mathcal{R}_{x_1}), \ldots, \Gamma(\alpha + 1, \alpha; \mathcal{R}_{\alpha+1})$ as described in Case 2 starting at x_1 instead of β.

Let Γ be the game described above and let $P \in \mathcal{SP}$ be arbitrary. For any agent $i \neq i_1$, the reasons why $\sigma^{P_i}_i$ (see its definition in Case 1) is obviously dominant in Γ are the same

[41] To illustrate these sets, consider the left and right committees, $\mathcal{L}_x$ and $\mathcal{R}_x$ (where $\mathcal{R}_x$ was $\mathcal{L}_y$ in the notation of Section 4), in Example 1 at the end of Section 4. Then, $NE^L(x) = \{2, 5\}$, $NE^R(x) = \{2, 4, 5\}$, $S^L(x) = \{2\}$ and $S^R(x) = \{1\}$.

to the ones already used to prove it in Cases 1 and 2, since when the game Γ proceeds into either case (b) or (c) above it follows only one of the two corresponding sequences until Γ ends. Now, consider agent i_1. We want to show that agent i_1's truth-telling strategy $\sigma_{i_1}^{P_{i_1}}$ is also obviously dominant in Γ. Any strategy of agent i_1 selects an action at z_0 and at a node in each of the games $\Gamma(x, x+1; \mathcal{L}_x)$ for $x_1 \leq x < \beta$, and $\Gamma(x, x-1; \mathcal{R}_x)$ for $\alpha < x \leq x_1$. In particular, agent i_1's truth-telling strategy $\sigma_{i_1}^{P_{i_1}}$ is defined as follows: at z_0,

$$\sigma_{i_1}^{P_{i_1}}(z_0) = \begin{cases} x_1 - 1 & \text{if } t(P_{i_1}) < x_1 \\ x_1 & \text{if } t(P_{i_1}) = x_1 \\ x_1 + 1 & \text{if } t(P_{i_1}) > x_1, \end{cases}$$

at any z_i^{x+} where $x_1 \leq x < \beta$,

$$\sigma_{i_1}^{P_{i_1}}(z_{i_1}^{x+}) = \begin{cases} x & \text{if } t(P_{i_1}) \leq x \\ x+1 & \text{if } t(P_{i_1}) > x, \end{cases}$$

and at any z_i^{x-} where $\alpha < x \leq x_1$,

$$\sigma_{i_1}^{P_{i_1}}(z_{i_1}^{x-}) = \begin{cases} x & \text{if } t(P_{i_1}) \geq x \\ x-1 & \text{if } t(P_{i_1}) < x. \end{cases}$$

To show that $\sigma_{i_1}^{P_i}$ is obviously dominant in Γ, let σ'_{i_1} be any strategy of agent i_1 with the property that $\sigma'_{i_1} \neq \sigma_{i_1}^{P_i}$. Denote by z the earliest point of departure for $\sigma_{i_1}^{P_{i_1}}$ and σ'_{i_1}. If $z \neq z_0$, then $z \in \{z_{i_1}^{x+}, z_{i_1}^{x-}\}$ for some x. As we did in Case 1 (if $z = z_{i_1}^{x+}$) and in Case 2 (if $z = z_{i_1}^{x-}$), we can show that

$$\min_{P_{i_1}} X_{+,-}^{P_{i_1}} R_{i_1} \max_{P_{i_1}} X'_{+,-}, \tag{15}$$

where $X_{+,-}^{P_{i_1}} = \{x \in X \mid x = g(z^\Gamma(z, (\sigma_{i_1}^{P_{i_1}}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\}$ and $X'_{+,-} = \{x \in X \mid x = g(z^\Gamma(z, (\sigma'_{i_1}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\}$. Assume $z = z_0$ and suppose first that $t(P_{i_1}) = x_1$, and so $\sigma_{i_1}^{P_{i_1}}(z) = x_1$. Then,

$$\{x \in X \mid x = g(z^\Gamma(z, (\sigma_{i_1}^{P_{i_1}}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\} = \{x_1\}.$$

Since $t(P_{i_1}) = x_1$,

$$x_1 R_{i_1} \max_{P_{i_1}} \{x \in X \mid x = g(z^\Gamma(z, (\sigma'_{i_1}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\}. \tag{16}$$

Suppose now that $t(P_{i_1}) < x_1$, and so $\sigma_{i_1}^{P_{i_1}}(z) = x_1 - 1$. Then,

$$X_0^{P_{i_1}} = \{x \in X \mid x = g(z^\Gamma(z, (\sigma_{i_1}^{P_{i_1}}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\} \subseteq \{t(P_{i_1}), \ldots, x_1\}. \tag{17}$$

The inclusion follows from the definition of Γ and because, by Lemma 2 and the monotonicity property in the definition of a right coalition system, $\{i_1\} \in \mathcal{R}_x$ for all $x \leq x_1$. Since $\sigma'_{i_1}(z) \neq x_1 - 1$, $\sigma'_{i_1}(z) \in \{x_1, x_1 + 1\}$. Then,

$$X'_0 = \{x \in X \mid x = g(z^{\Gamma}(z, (\sigma'_{i_1}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\} \subseteq \{x_1, \ldots, \beta\}. \tag{18}$$

The inclusion follows because $x_1 \in X'_0$ if $\sigma'_{i_1}(z) = x_1$ and because $X'_0 \subseteq \{x_1, \ldots, \beta\}$ if $\sigma'_{i_1}(z) = x_1 + 1$, where this last inclusion follows again from the definition of Γ and because $\{i_1\} \in \mathcal{L}_{x_1}$ implies that, by the monotonicity property in the definition of a left coalition system, $\{i_1\} \in \mathcal{L}_x$ for all $x_1 \leq x$. By (17) and (18), single-peakedness of P_{i_1} and $t(P_{i_1}) < x_1$,

$$\min_{P_{i_1}} X_0^{P_{i_1}} R_{i_1} \max_{P_{i_1}} X'_0. \tag{19}$$

Suppose that $t(P_{i_1}) > x_1$, and so $\sigma_{i_1}^{P_{i_1}}(z) = x_1 + 1$. Then, the proof proceeds as in the above case where $t(P_{i_1}) < x_1$. Hence, from (15), (16), and (19) (and the symmetric condition to (19) when $t(P_{i_1}) > x_1$), $\sigma_{i_1}^{P_{i_1}}$ is obviously dominant in Γ. ■

7.3 Proposition 3 and its proof

The (L2-IOI) property stated below plays a crucial role to identify the property on the left coalition system, that together with (L-IOI), characterize all GMVSs that are OSP in terms only of its associated left coalition system.[42]

(L2-IOI) For every $\alpha < x \leq x_1 - 1$, $\mathcal{L}_x$ satisfies IOI and (i) there exists $i_x \in N$ such that $\{i_x\} \cup (\bigcap_{S \in \mathcal{L}_x^m} S) \in \mathcal{L}_x^m$ and (ii) $i_x \in S$ for all $S \in \mathcal{L}_{x-1}^m$.

By the monotonicity property in the definition of a left coalition system, Remark 5 holds.

Remark 5 Assume $x \leq x_1 - 1$. If $\mathcal{L}_x^m = \{S\}$, then for all $x' \leq x$ and all $S' \in \mathcal{L}_{x'}^m$, $S \subset S'$.

Lemma 3 will be useful in the proof of Proposition 3, which is the result that contains the answer to our question. It roughly says that IOI for the left translates into IOI for the right, +1; namely, for all $\alpha < x \leq \beta$, either $\mathcal{L}_{x-1}$ and $\mathcal{R}_x$ satisfy both IOI or neither of them do.

Lemma 3 *Let $\{\mathcal{L}_w\}_{w \in X}$ and $\{\mathcal{R}_w\}_{w \in X}$ be, respectively, the left and the right coalition systems associated to the same* GMVS *f and let $\alpha < x \leq \beta$. Then, $\mathcal{R}_x$ satisfies* IOI *if and only if $\mathcal{L}_{x-1}$ satisfies* IOI.

[42] Of course, we could also state a corresponding property (R2-ISI) for the right coalition system. However, we omit this symmetric analysis.

Proof of Lemma 3 Assume $\mathcal{L}_{x-1}$ satisfies IOI. Let $\widehat{N} = \bigcup_{S\in\mathcal{L}^m_{x-1}} S$ and, for each $i \in \widehat{N}$, let $\widehat{\mathcal{P}}_i$ be the set of i's strict preferences on $\{x-1, x\}$. Let $\widehat{f} : \prod_{i\in\widehat{N}} \widehat{\mathcal{P}}_i \to \{x-1, x\}$ be the EMVR associated to the committee $\widehat{\mathcal{L}}_{x-1}$, the restriction of $\mathcal{L}_{x-1}$ into $\widehat{N}$. Observe that if $j \notin \widehat{N}$, then j is dummy at $\mathcal{L}_{x-1}$ and j is dummy at $\mathcal{R}_x$. Since $\mathcal{L}_{x-1}$ satisfies IOI, $\widehat{\mathcal{L}}_{x-1}$ does as well. By Proposition 1, $\widehat{f}$ is OSP. Then, again by Proposition 1, and a symmetric argument, $\widehat{\mathcal{R}}_x$ satisfies IOI. But then, $\mathcal{R}_x$ satisfies IOI as well. Using a symmetric argument we can show that if $\mathcal{R}_x$ satisfies IOI, then $\mathcal{L}_{x-1}$ satisfies IOI as well. $\square$

Proposition 3 *Let $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$ be, respectively, the left and the right coalition systems associated to the same* GMVS *and let $\alpha < x_1 < \beta$. Then,* (L-IOI) *and* (R-IOI) *hold if and only if* (L-IOI) *and* (L2-IOI) *hold.*

Proof of Proposition 3 Assume (L-IOI) and (R-IOI) hold. It is sufficient to show that (L2-IOI) holds. Let $\alpha < x \leq x_1 - 1$ and assume first that $|\mathcal{L}^m_x| = 1$. Let $S \neq \emptyset$ be such that $\mathcal{L}^m_x = \{S\}$ and so, for any $i \in S$, $\{i\} \cup S \in \mathcal{L}^m_x$ holds trivially and, by Remark 5, if $S' \in \mathcal{L}^m_{x-1}$, then $S \subset S'$ and $i \in S'$. Hence, (L2-IOI) holds. Assume now that $|\mathcal{L}^m_x| \geq 2$. Then, $x + 1 < x_1 + 1$ and by (R-IOI), $\mathcal{R}_{x+1}$ satisfies IOI. By Lemma 3, $\mathcal{L}_x$ satisfies IOI. Furthermore, by (R-IOI) and $x + 1 < x_1 + 1$, there exists $i^{x+1} \in N$ such that $\mathcal{R}_{x+1}$ satisfies IOI with respect to i^{x+1} and $\{i^{x+1}\} \in \mathcal{R}_x$. Since i^{x+1} is the first element in the order for which $\mathcal{R}_{x+1}$ satisfies IOI with respect to, $i^{x+1} \in S$ for all $S \in \mathcal{R}^k(x+1)$ and all $k \geq 2$. Since $(\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\})) \cap S \neq \emptyset$ for all $S \in \mathcal{R}^m_{x+1}$, by Remark 4, $\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}) \in \mathcal{L}_x$ holds. Now, we prove that $\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}) \in \mathcal{L}^m_x$. Assume there exists $S' \subsetneq \{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\})$ such that $S' \in \mathcal{L}_x$. By Remark 4, $S' \cap \{i\} \neq \emptyset$ for all i such that $\{i\} \in \mathcal{R}_{x+1}$. Hence, $S' = \bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}$. By Remark 4,

$$\bigcup_{\{i\}\in\mathcal{R}_{x+1}} \{i\} = \bigcap_{S\in\mathcal{L}^m_x} S \tag{20}$$

holds, implying that $\bigcap_{S\in\mathcal{L}^m_x} S \in \mathcal{L}^m_x$ and $|\mathcal{L}^m_x| = 1$, which contradicts that $|\mathcal{L}^m_x| \geq 2$. Therefore, $\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}) \in \mathcal{L}^m_x$. By (20), $\{i^{x+1}\} \cup (\bigcap_{S\in\mathcal{L}^m_x} S) \in \mathcal{L}^m_x$, which is (i) in (L2-IOI). Moreover, since $\{i^{x+1}\} \in \mathcal{R}_x$, by Remark 4, $i^{x+1} \in S$ for all $S \in \mathcal{L}^m_{x-1}$.

Assume (L-IOI) and (L2-IOI) hold. It is sufficient to show that (R-IOI) holds. Let $\alpha < x \leq x_1 + 1$. We proceed by considering two cases separately.

<u>Case 1</u>: $\alpha < x < x_1 + 1$. Then, $x - 1 \leq x_1 - 1$ and by (L2-IOI), $\mathcal{L}_{x-1}$ satisfies IOI. Then, by Lemma 3, $\mathcal{R}_x$ satisfies IOI. We further distinguish between two subcases.

<u>Case 1.a</u>: $\alpha = x - 1$. Then, for any $i \in N$, $\mathcal{R}_x$ satisfies trivially IOI with respect to i, since the boundary condition in the definition of a right coalition system implies that $\{i\} \in \mathcal{R}_{x-1} = \mathcal{R}_\alpha$. Hence, (R-IOI) holds in Case 1.a.

<u>Case 1.b</u>: $\alpha < x - 1$. By (L2-IOI), there exists $i_{x-1} \in N$ such that

$$\{i_{x-1}\} \cup (\bigcap_{S\in\mathcal{L}^m_{x-1}} S) \in \mathcal{L}^m_{x-1} \tag{21}$$

and

$$i_{x-1} \in S \text{ for all } S \in \mathcal{L}^m_{x-2}. \tag{22}$$

By Remark 4 and (22), $\{i_{x-1}\} \in \mathcal{R}_{x-1}$. It is sufficient to show that $\mathcal{R}_x$ satisfies IOI with respect to i_{x-1} or, equivalently, that $i_{x-1} \in S$ for all $S \in \mathcal{R}^m_x$ with $|S| \geq 2$. By Remark 4, $\bigcup_{\{i\}\in\mathcal{R}_x}\{i\} = \bigcap_{S\in\mathcal{L}^m_{x-1}} S$, and, by (21), $\{i_{x-1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_x}\{i\}) \in \mathcal{L}_{x-1}$. Consider any $S \in \mathcal{R}^m_x$ with $|S| \geq 2$ and assume that $i_{x-1} \notin S$. By the fact that $S \in \mathcal{R}^m_x$, $i \notin S$ for all i such that $\{i\} \in \mathcal{R}_x$. Therefore, $(\{i_{x-1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_x}\{i\}) \cap S) = \emptyset$ which contradicts, together with Remark 4, that $\{i_{x-1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_x}\{i\}) \in \mathcal{L}^m_{x-1}$.

Case 2: $x = x_1+1$. By (L-IOI), $\mathcal{L}_{x_1-1}$ satisfies IOI with respect to i^{x_1-1} and $\{i^{x_1-1}\} \in \mathcal{L}_{x_1}$. By definition of x_1, $i^{x_1-1} \in S^L(x_1-1) \neq \emptyset$. By (L2.1) and (L2.2) in Lemma 2, $\mathcal{R}_{x_1}$ satisfies IOI with respect to i^{x_1-1} and $\{i^{x_1-1}\} \in \mathcal{R}_{x_1-1}$. Thus, (R-IOI) follows. ■