All Sequential Allotment Rules Are Obviously Strategy-proof*

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October 30, 2020

Abstract: For division problems with single-peaked preferences (Sprumont, 1991) we show that all sequential allotment rules, identified by Barberà, Jackson and Neme (1997) as the class of strategy-proof, efficient and replacement monotonic rules, are also obviously strategy-proof. Although obvious strategy-proofness is in general more restrictive than strategy-proofness, this is not the case in this setting.

Keywords: Obvious Strategy-proofness; Sequential Allotment Rules; Division Problems; Single-peaked Preferences. JEL *Classification*: D71.

1 Introduction

We consider the class of division problems with single-peaked preferences where k indivisible units of a good have to be allotted among a set of agents. Each agent has an ideal amount (the top assignment) —the less, the worse; the more, the worse— inducing single-peaked preferences over the set of agent's assignments. Monetary transfers are not possible.

Different real-world problems can be framed within this model. Situations where a set of agents must share a good, a bad or a task like the surplus of a joint venture, the cost of a public good, the division of a job, or rationed goods traded at fixed prices. For example, the agents could be investors, with different risk-preferences and wealths, and the units of the good could be shares in a risky project. Agents' risk attitudes and wealths induce single-peaked preferences over their assigned shares. The agents could also be workers who

*We are grateful to Bart Lipman and three referees for helpful comments and suggestions, and to participants of seminars and conferences. We thank various financial supports. Arribillaga and Neme from the Universidad Nacional de San Luis, through grant 032016, from the Consejo Nacional de Investigaciones Científicas y Técnicas, through grant PIP112-200801-00655, and from the Agencia Nacional de Promoción de la Investigación el Desarrollo Tecnológico y la Innovación, through grant PICT2017-2355. Massó and Neme from the Spanish Ministry of Economy and Competitiveness, through grant ECO2017-83534-P. Massó from the Spanish Ministry of Science and Innovation, through the Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000915-S), and from the Generalitat de Catalunya, through grant SGR2017-711.

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have collectively agreed to complete a project requiring a fixed number of hours paid at a fixed wage. Agents' quasi-concave preferences over work and leisure induce single-peaked preferences over their assigned number of working hours. Finally, the good might be a plot of land that needs to be divided among hobby gardeners each of whom wishes to cultivate some land, but not necessarily all.[1]

The question is: How should division problems be solved? What properties should a solution have? A solution to division problems is a rule that chooses an allotment for each profile of single-peaked preferences over $\{0, \ldots, k\}$. But preferences are agents' private information and they have to be elicited. A rule is strategy-proof if, for each agent, truth-telling is always optimal, regardless of the preferences declared by the other agents. A rule is efficient if the chosen allotment is Pareto optimal at each profile of single-peaked preferences. A rule is replacement monotonic if it satisfies a weak solidarity principle requiring that if an agent obtains a different assignment by changing the revealed preference, then all other agents' assignments should change in the same direction.

Barberà, Jackson and Neme (1997) consider the class of division problems where agents might begin with natural claims to minimal or maximal assignments, or might be treated with different priorities, due for example to their seniorities, and these initial entitlements should be attended as far as possible. They characterize the class of strategy-proof, efficient and replacement monotonic rules on the domain of single-peaked preferences as the family of sequential allotment rules.

A sequential allotment rule may be thought of as starting from two reference allotments: The scarcity guaranteed allotment, to be used whenever the sum of agents' tops is larger than k, and the excess guaranteed allotment, to be used whenever the sum of agents' tops is smaller than k. If the corresponding guaranteed allotment is not efficient, the rule corrects it to select an efficient allotment. Rules within this class differ on the two pre-selected guaranteed allotments and on how the efficient correction takes place (the correction has to be monotonous for the rule to satisfy replacement monotonicity). A rule is individually rational with respect to an allotment if each agent's assignment is always at least as good as his/her assignment at this allotment. Barberà, Jackson and Neme (1997) also show that an individually rational sequential allotment rule with respect to an allotment has the property that the two reference allotments are equal to this allotment.

In this paper we ask: How might efficient allotments be implemented while, at the same time, promoting solidarity among agents who may have problems with contingent reasoning? Specifically, what would happen if we demanded that the rule be obviously strategy-proof rather than just strategy-proof? Li (2017) proposes the stronger incentive notion of obvious strategy-proofness under which agents, in order to identify that truth-telling is an optimal decision, do not need to reason contingently about other agents' decisions. This notion

[1]Division problems have been studied intensively; see for instance Thomson (1994a, 1994b and 1997), Barberà (2011)'s survey on strategy-proofness and, more recently, Moulin (2015), Wakayama (2017) or Juarez and You (2019). The continuous version of this model was first studied by Sprumont (1991). The discrete version considered here was first studied by Herrero and Martínez (2011). In the final remarks section we explain the difficulties of performing our analysis in the continuous version.

requires that, given a rule, agents' preferences should be revealed sequentially and through an extensive game form induces the rule and where truth-telling is obviously dominant and. Obvious dominance means that whenever an agent has to take a decision in a node (in the game in extensive form) he/she evaluates the consequences of truth-telling in a pessimistic way (thinking that the worst possible assignment will follow) and the consequences of not truth-telling in an optimistic way (thinking that the best possible assignment will follow); moreover, the pessimistic assignment associated to truth-telling is at least as good as the optimistic assignment associated to not truth-telling. Hence, the decision prescribed by truth-telling at that node appears as unmistakably optimal; *i.e.* obviously dominant.

The difficulty of answering the above question is that obvious strategy-proofness of a rule requires, at each profile of preferences, truthful revelation throughout an extensive game form that results in the allotment chosen by the rule at that profile. But the sequential mechanism is not given by a general revelation principle as it is for strategy-proofness in the form of the direct revelation mechanism. The main difficulty lies then in identifying, for each rule, the extensive game form that implements the rule in obviously dominant strategies.[2]

The result of this paper is the following: Any efficient and replacement monotonic rule that can be implemented in dominant strategies can moreover be done so in obviously dominant strategies. That is, in the implementation we can accommodate agents who have trouble with contingent reasoning because obvious strategy-proofness is no more restrictive than strategy-proofness. Namely, we show that all sequential allotment rules (a quite large class of rules) are obviously strategy-proof. Moreover, our proof is constructive: For each sequential allotment rule we explicitly show how to construct such extensive game form.

Our construction of the class of extensive game forms has two tiers. We first propose general traits of an algorithm along which agents sequentially face some choice sets consisting of at most three adjacent assignments, one of them guaranteed (the one in the middle, if the set has three choices): If the agent chooses it, he/she does not play again and receives it as his/her final assignment. At each step, the algorithm partially leaves open the selection of the agent that has to choose and the guaranteed assignment determining his/her choice set. When the agent chooses for the first time, he/she faces three choices: either leave with the guaranteed assignment, or ask for more and aim to get more, or ask for less and aim to get less. If the agent asks for more, then his/her guaranteed assignment may increase further, one unit at a time. If the agent asks for less, then his/her guaranteed assignment may decrease further, one unit at a time. If the procedure terminates before the agent leaves, then the agent gets his/her guaranteed assignment. Moreover, the guaranteed assignments evolve throughout the algorithm by assuring that all proposed warranties are feasible. Since the paths of guaranteed assignments throughout the algorithm are monotonous and agent

[2] Ashlagi and Gonczarowski (2018), Bade and Gonczarowski (2017), Mackenzie (2020) and Pycia and Troyan (2020) contain results identifying general features of extensive game forms that could be used to implement rules in obviously dominant strategies in different environments. We will follow Ashlagi and Gonczarowski (2018) and Mackenzie (2020) to restrict ourselves to extensive game forms with perfect information. See Mackenzie (2020) for a detailed description and discussion of the differences, similarities and nuances between the proposals of those four papers.

specific, we refer to it as the *Monotonous and Individualized Algorithm* (MIA).

The MIA defines a family of extensive game forms, each named as a *Monotonous and Individualized Game* (MIG). Truthful reporting is an obviously dominant strategy for an agent playing a MIG. Fix a node in the tree of a MIG (that corresponds to one step of the MIA) and consider the agent with a single-peaked preference that plays at this node. If the agent's top is his/her guaranteed assignment (which is always an available choice), to choose it is obviously dominant since the worst that might happen is to be assigned to his/her top. If the agent's top is strictly above his/her guaranteed assignment, the worst that might happen if he/she asks for more (*i.e.*, truth-tells) is to receive his/her guaranteed assignment. This is because the agent might still be able to select larger assignments, up to his/her top, along the monotonic path of guaranteed assignments towards his/her top. In contrast, if the agent does not ask for more (*i.e.*, does not truth-tell), the best that might happen is to receive either the guaranteed allotment or strictly less, all worse than the assignment obtained by truth-telling. Symmetrically, if agent's top is strictly below the guaranteed assignment. The key feature of the MIG is that, given the top and the initial guaranteed assignment, the agent can choose either his/her top or to push forward the guaranteed assignment towards his/her top, without exceeding it, by asking for more (if the top is above) or asking for less (if the top is below). And single-peakedness guarantees that truth-telling is obviously dominant.

The second tier consists in tailoring the MIA to each sequential allotment rule. Fix a sequential allotment rule. At each step of the MIA, the agent that has to play and his/her guaranteed assignment are selected in such a way that the corresponding MIG implements in obviously dominant strategies precisely the given sequential allotment rule.

Assume first that the sequential allotment rule is individually rational with respect to an allotment (comprising agents' initial guaranteed assignments). At Stage A of the MIA, each agent is asked, sequentially and in any order, whether he/she would like more than, less than, or exactly his/her guaranteed assignment. Agents who want exactly the guaranteed assignment leave the game and receive it for sure. Then, at each step in Stage B of the MIA, select an agent who wants more and an agent who wants less, and the later transfers one unit of the good to the former. These two new assignments together with the previous guaranteed assignments of the other agents become the new guaranteed allotment. Keep making Pareto-improving transfers until no more are available. We show that, by using the given sequential allocation rule to select the pair of agents in each Pareto improving transfer, the final guaranteed allotment is, for any profile of single-peaked preferences, the allotment that the rule would have chosen if agents had reported truthfully.

Assume now that the sequential allotment rule is not individually rational. This case is more involved because agents cannot be offered an initial guaranteed allotment as we did in Stage A of the individually rational case. However, we can modify Stage A as follows. First, use as initial warranties the scarcity and the excess guaranteed allotments, those selected by the sequential allotment rule when agents either all ask for k or all ask for zero, respectively. Second, keep updating these two allotments by evaluating the sequential allotment rule at less extreme profiles of tops until both converge to a unique guaranteed allotment.[3] Hence, Stage A produces a unique reference allotment that behaves as if it were guaranteed at the subset of single-peaked preferences that are consistent with all choices which led to it. Then, trade from this point as in Stage B of the individually rational case.

In light of the extreme behavioral criterion used to evaluate truth-telling, it is not surprising that the literature has already identified settings for which just a few and very special strategy-proof rules satisfy the stronger requirement. Li (2017) already shows that the rule associated to the top-trading cycles algorithm in the house allocation problem of Shapley and Scarf (1974) is not obviously strategy-proof, and Troyan (2019) identifies a domain of acyclic preferences that is necessary and sufficient for that rule to be obviously strategy-proof. Ashlagi and Gonczarowski (2018) show that the rule associated to the deferred acceptance algorithm is not obviously strategy-proof for the agents belonging to the offering side, but it is on the domain of acyclic preferences defined by Ergin (2002). However, some earlier possibility results can already be found in Li (2017). He characterizes the monotone price mechanisms (generalizations of ascending auctions) as those that implement all obviously strategy-proof rules on the domain of quasi-linear preferences. He also shows that, for online advertising auctions, the rule induced by the mechanism that selects the efficient allocation and the Vickrey-Clarke-Groves payment is obviously strategy-proof.[4]

Our paper contributes to the possibility strand of this literature by showing that, despite the fact that in many settings obvious strategy-proofness becomes significantly more restrictive than just strategy-proofness, for the division problems with single-peaked preferences each sequential allotment rule (*i.e.*, each strategy-proof, efficient and replacement monotonic rule) is indeed obviously strategy-proof. And we show it by exhibiting the extensive game form that implements each sequential allotment rule in obviously dominant strategies.

The paper is organized as follows. Section 2 contains the preliminaries. Section 3 presents the notion of obvious strategy-proofness adapted to our setting. Section 4 contains the description of the MIA and the statement and proof that, in any extensive game form defined by the MIA, truth-telling is obviously dominant. Section 5 defines, for each sequential allotment rule, the extensive game form that implements the rule in obviously dominant strategies. Sections 6 contains the main result and an example. Section 7 contains five final remarks. The Appendix at the end of the paper collects omitted proofs.

2 Preliminaries

Agents are the elements of a finite set $N = \{1, \ldots, n\}$, where $n \geq 2$. They have to share k indivisible units of a good, where $k \geq 2$ is a positive integer. An *allotment* is a vector

[3]Lemmata 1, 2 and 3, used in the proof of the Theorem 1, state that such convergences occur.

[4]For other partially positive or revelation principle like results see also Arribillaga, Massó and Neme (2020), Bade and Gonczarowski (2017), Pycia and Troyan (2020) and Troyan (2019). Note that although the first two papers also consider single-peaked preferences, they do so in a context of a public good (*i.e.*, voting), while here the context is of private goods, and the two models are completely different.

$x = (x_1, \ldots, x_n) \in \{0, \ldots, k\}^N$ such that $\sum_{i=1}^n x_i = k$. We refer to $x_i \in \{0, \ldots, k\}$ as agent i's *assignment*. Let X be the set of allotments. Each agent $i \in N$ has a (weak) *preference* R_i over $\{0, \ldots, k\}$, the set of i's possible assignments. Let P_i be the strict preference associated with R_i. The preference R_i is *single-peaked* if (i) it has a unique most-preferred assignment $\tau(R_i)$, the *top* of R_i, such that for all $x_i \in \{0, \ldots, k\} \setminus \{\tau(R_i)\}$, $\tau(R_i)\ P_i\ x_i$, and (ii) for any pair $x_i, y_i \in \{0, \ldots, k\}$, $y_i < x_i < \tau(R_i)$ or $\tau(R_i) < x_i < y_i$ implies $x_i\ P_i\ y_i$. We assume that agents have single-peaked preferences. Often, only $\tau(R_i)$ about R_i will be relevant and if R_i is obvious, we will refer to its top as τ_i. We denote by $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{k}$ the vectors $(0, \ldots, 0), (1, \ldots, 1), (k, \ldots, k) \in \{0, \ldots, k\}^N$ and, given $S \subset N$, by $\mathbf{0}_S$, $\mathbf{1}_S$ and $\mathbf{k}_S$ the corresponding subvectors where all agents in S receive the assignment 0, 1 or k, respectively. Given $x = (x_1, \ldots, x_n)$, we denote $(x_i)_{i \in S}$ as x_S and $(x_i - 1)_{i \in S}$ as $(x - \mathbf{1})_S$.

Let $\mathcal{R}$ be the set of all single-peaked preferences. *Profiles*, denoted by $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$, are n-tuples of single-peaked preferences. To stress the role of agent i's or agents in S, we will represent a profile R by (R_i, R_{-i}) or by (R_S, R_{-S}), respectively.

A (discrete) *division problem* is a pair (k, N), where k is the number of units of the good that have to be allotted among the agents in N with single-peaked preferences.

A solution of the division problem (k, N) is a *rule* $\Phi : \mathcal{R}^N \to X$ that selects, for each profile $R \in \mathcal{R}^N$, an allotment $\Phi(R) \in X$.

A desirable requirement on rules is *efficiency*. A rule $\Phi : \mathcal{R}^N \to X$ is *efficient* if, for each $R \in \mathcal{R}^N$, there is no $y \in X$ such that $y_i\ P_i\ \Phi_i(R)$ for all $i \in N$ and $y_j\ P_j\ \Phi_j(R)$ for at least one $j \in N$. It is easy to check that when coupled with single-peakedness, efficiency is equivalent to same-sidedness which requires that all agents are rationed in the same side of the top: Below the tops when there is scarcity and above them when there is excess. Namely, a rule $\Phi : \mathcal{R}^N \to X$ satisfies *same-sidedness* if for all $R \in \mathcal{R}^N$,

$$\textstyle\sum_{j \in N} \tau(R_j) \geq k \text{ implies } \Phi_i(R) \leq \tau(R_i) \text{ for all } i \in N, \tag{1}$$

$$\textstyle\sum_{j \in N} \tau(R_j) \leq k \text{ implies } \Phi_i(R) \geq \tau(R_i) \text{ for all } i \in N. \tag{2}$$

Rules require agents to report single-peaked preferences. A rule is strategy-proof if it is always in the best interest of agents to truthfully reveal their preferences; namely, truth-telling is a weakly dominant strategy in the game in normal form obtained from the rule at each profile. A rule $\Phi : \mathcal{R}^N \to X$ is *strategy-proof* if for all $R \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$,

$$\Phi_i(R_i, R_{-i})\ R_i\ \Phi_i(R'_i, R_{-i}).$$

If $\Phi_i(R'_i, R_{-i})\ P_i\ \Phi_i(R_i, R_{-i})$ we say that i manipulates $\Phi : \mathcal{R}^N \to X$ at $R \in \mathcal{R}^N$ via $R'_i \in \mathcal{R}$. Clearly, $\Phi : \mathcal{R}^N \to X$ is strategy-proof if no agent can manipulate it.

Replacement monotonicity is a weak solidarity property (see Thomson (2016)). It requires that if an agent obtains a different assignment by changing the revealed preference, then all other agents' assignments should change in the same direction. A rule $\Phi : \mathcal{R}^N \to X$ is *replacement monotonic* if for all $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$,[5]

$$\Phi_i(R_i, R_{-i}) \leq \Phi_i(R'_i, R_{-i}) \text{ implies } \Phi_j(R_i, R_{-i}) \geq \Phi_j(R'_i, R_{-i}) \text{ for all } j \neq i.$$

[5] As Barberà, Jackson and Neme (1997) argue, the normative justification for this property relies on

Individual rationality with respect to an allotment $q \in X$ guarantees that each agent i receives an assignment that is weakly preferred to q_i. A rule $\Phi : \mathcal{R}^N \to X$ is *individually rational* with respect to an allotment $q \in X$ if for all $R \in \mathcal{R}^N$ and $i \in N$,

$$\Phi_i(R)\ R_i\ q_i.$$

A rule $\Phi : \mathcal{R}^N \to X$ is *tops-only* if for all $R, R' \in \mathcal{R}^N$ such that $\tau(R_i) = \tau(R'_i)$ for all $i \in N$, $\Phi(R) = \Phi(R')$. Abusing notation, a tops-only rule $\Phi : \mathcal{R}^N \to X$ can be written as $\Phi : \{0, \ldots, k\}^N \to X$, and so we will often interchange $\Phi(\tau_1, \ldots, \tau_n)$ and $\Phi(R_1, \ldots, R_n)$.

For continuous division problems, Barberà, Jackson and Neme (1997) characterize the class of all strategy-proof, efficient and replacement monotonic rules as the set of all sequential allotment rules. The proof of their characterization can be adapted to discrete division problems. In discrete division problems it also holds that if Φ is strategy-proof and efficient, then no agent can affect his/her own assignment by changing to a new preference with the same top. If, in addition, Φ is non-bossy, then none of the assignments are affected. Hence, Φ is tops-only and then, the proof of the characterization for discrete division problems proceeds as in the continuous case. For further reference, we state this characterization (and the one adding individual rationality) as Proposition 1.

Proposition 1 (Barberà, Jackson and Neme, 1997) *Let (k, N) be a division problem. A rule $\Phi : \mathcal{R}^N \to X$ is strategy-proof, efficient and replacement monotonic if and only if Φ is a sequential allotment rule. Moreover, a rule $\Phi : \mathcal{R}^N \to X$ is strategy-proof, efficient, replacement monotonic and individually rational with respect to q if and only if Φ is a sequential allotment rule such that $\Phi(\mathbf{0}) = \Phi(\mathbf{k}) = q$.*

Sequential allotment rules allot the k units sequentially, using guaranteed allotments for the agents that evolve throughout the process and that are compared to their tops. We describe the general procedure that any sequential allotment rule follows.[6] The rule has to specify two initial guaranteed allotments for the agents. The *scarcity* allotment $\overline{q} \in X$, to be used when the sum of the tops is strictly larger than k, and the *excess* allotment $\underline{q} \in X$, to be used when the sum of the tops is strictly smaller than k.

To define a sequential allotment rule Φ, let $\underline{q}$ and $\overline{q}$ be respectively its excess and scarcity guaranteed allotments, and let $\tau = (\tau_1, \ldots, \tau_n) \in \{0, \ldots, k\}^N$ be an arbitrary vector of tops.

Suppose $\sum_{i=1}^n \tau_i = k$. Then, since τ is the unique efficient allotment at τ, $\Phi(\tau) = \tau$.

Suppose $\sum_{i=1}^n \tau_i > k$ (the case $\sum_{i=1}^n \tau_i < k$ is symmetric, using $\underline{q}$ instead of $\overline{q}$). If $\tau_j \geq \overline{q}_j$ for all j, then $\Phi(\tau) = \overline{q}$. Otherwise, each j with $\tau_j \leq \overline{q}_j$ receives τ_j and leaves the process with τ_j units, while the other agents remain. The guaranteed assignments of the remaining

efficiency and single-peakedness. The condition has a clear solidarity-based normative content and it is equivalent to a weakening of the welfare version called one-sided welfare-domination under preferences replacement (Thomson, 1997). It is a form of non-bossiness: An agent, without affecting his/her assignment, cannot transfer units among the other agents. A rule $\Phi : \mathcal{R}^N \to X$ is *non-bossy* if for all $R \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$, $\Phi_i(R) = \Phi_i(R'_i, R_{-i})$ implies $\Phi(R) = \Phi(R'_i, R_{-i})$. Replacement monotonicity implies non-bossiness.

[6] For a formal definition of a sequential allotment rule see Barberà, Jackson and Neme (1997). Our results will be based on the properties characterizing the class, without explicitly using this definition.

agents are weakly increased by distributing among them the not yet allotted units. Agents with a top smaller than or equal to the new guaranteed assignment receive the top and leave the process, while the others remain. The process proceeds this way until all units have been already allotted, with the remaining agents receiving their last guaranteed assignment.

At the end of the process, each agent i receives either τ_i or i's final guaranteed assignment which has been moving towards τ_i throughout the process. Hence, by single-peakedness, at all profiles with scarcity, each agent is at least as well-off as at the scarcity guaranteed assignment, and the analogous statement holds for the excess guaranteed assignment. Note that $\Phi(\mathbf{0}) = \underline{q}$ and $\Phi(\mathbf{k}) = \overline{q}$. If $q := \underline{q} = \overline{q}$ then, for every τ and j, $\Phi_j(\tau)$ lies between τ_j and q_j and, by single-peakedness, Φ is individually rational with respect to q. The process ends with an efficient allotment because, under scarcity, all agents receive less than their tops and, under excess, all receive more. Replacement monotonicity requires that the guaranteed assignments evolve monotonically. Since the sequential procedure depends on the profile of tops, strategy-proofness imposes some restrictions on the process; for instance, if the guaranteed assignment of an agent is smaller than his/her top, then it should remain the same with an even larger announced top. The differences in guaranteed assignments allow the rule to treat agents differently according to asymmetries that one wishes to respect.

The next example describes a sequential allotment rule Φ by specifying the two guaranteed allotments and how they evolve relative to two profiles of tops, one with scarcity and the other with excess.

Example 1 Let $N = \{1,2,3,4\}$, $k = 7$, $\underline{q} = (4,0,2,1)$ and $\overline{q} = (0,1,1,5)$.

Consider $\tau = (5,1,2,2)$. Since $\sum_{i=1}^{4} \tau_i > 7$, the rule Φ first uses $\overline{q} = (0,1,1,5)$ as guaranteed assignments, represented in Figure 1.a by the four large circles. In Figures 1 and 2 we represent in the horizontal axes the tops and guaranteed assignments of each of the four agents, who are represented in the vertical axes. Since $\tau_2 = 1 = \overline{q}_2$ and $\tau_4 = 2 < 5 = \overline{q}_4$, $\Phi_2(\tau) = 1$ and $\Phi_4(\tau) = 2$, and agents 2 and 4 leave with their tops. The amount not allotted yet is $\widehat{k} = 4$. Suppose that the adjusted guaranteed assignments are $q_1 = 1$ and $q_3 = 3$, represented in Figure 1.b by the two large circles. Since $\tau_3 = 2 < 3 = q_3$ and $q_1 = 1 < 5 = \tau_1$, $\Phi_3(\tau) = 2$ and agent 3 leaves with τ_3. Since only agent 1 remains and two units have not been allotted yet, the new guaranteed assignment for agent 1 has to be equal to 2 (strictly smaller than τ_1 because $\sum_{i=1}^{4} \tau_i > 7$ and all other agents have received their tops). Hence, $\Phi_1(\tau) = 2$. Therefore, $\Phi(\tau) = (2,1,2,2)$.

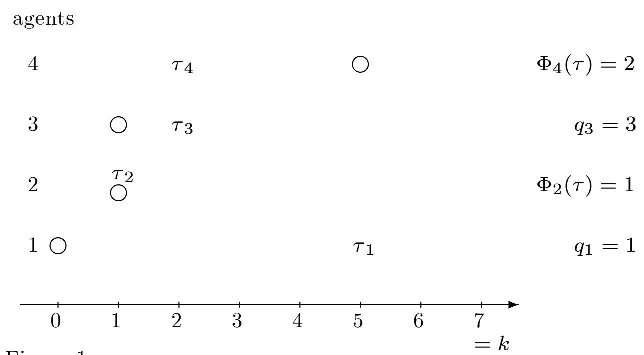

Figure 1.a

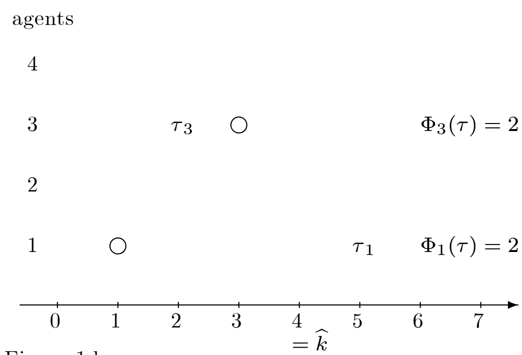

Figure 1.b

Consider $\tau' = (1, 1, 3, 0)$. Since $\sum_{i=1}^{4} \tau'_i < 7$, the rule Φ first uses $\underline{q} = (4, 0, 2, 1)$ as guaranteed assignments, represented in Figure 2.a by the small four circles. Since $\underline{q}_2 = 0 < 1 = \tau'_2$ and $\underline{q}_3 = 2 < 3 = \tau'_3$, $\Phi_2(\tau') = 1$ and $\Phi_3(\tau') = 3$, and agents 2 and 3 leave with their tops. The amount not allotted yet is $\widehat{k}' = 3$. Suppose that the adjusted guaranteed assignments are $q'_1 = 2$ and $q'_4 = 1$, represented in Figure 2.b by two small circles. Since $\tau'_1 = 1 < 2 = q'_1$ and $\tau'_4 = 0 < 1 = q'_4$, $\Phi_1(\tau') = 2$ and $\Phi_4(\tau') = 1$. Therefore, $\Phi(\tau') = (2, 1, 3, 1)$.

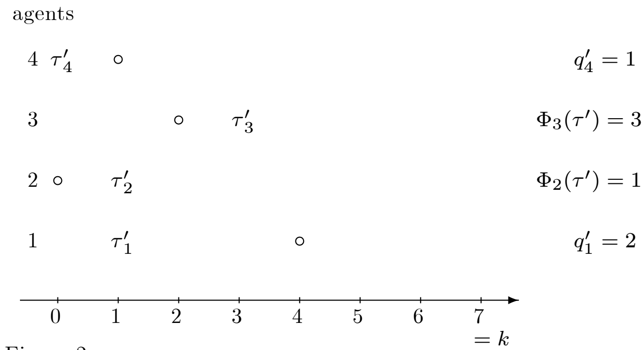

Figure 2.a

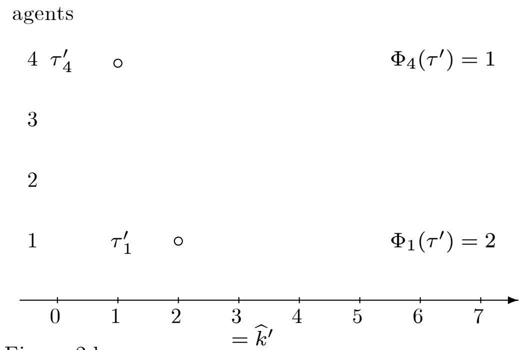

Figure 2.b

□

3 Obviously strategy-proof implementation

We briefly describe the notion of obvious strategy-proofness. Li (2017) proposes this notion with the aim of reducing the contingent reasoning that agents have to carry out to identify that, given a rule, truth-telling is always a weakly dominant strategy. A rule Φ is obviously strategy-proof if there exists an extensive game form with two properties. First, for each profile $R \in \mathcal{R}^N$ one can identify a behavioral strategy profile, associated to truth-telling, such that if agents play according to such strategy the outcome is $\Phi(R)$, the allotment selected by the rule Φ at R; that is, the extensive game form induces Φ. Second, whenever agent i with preferences R_i has to play, i evaluates the consequence of choosing the action prescribed by i's truth-telling strategy according to the *worst* possible outcome among all outcomes that may occur as an effect of later actions made by agents throughout the rest of the game. In contrast, i evaluates the consequence of choosing an action different from the one prescribed by i's truth-telling strategy according to the *best* possible outcome among all outcomes that may occur again as an effect of later actions throughout the rest of the game. Then, i's truth-telling strategy is obviously dominant in the game in extensive form if, whenever i has to play, its pessimistic outcome is at least as preferred as the optimistic outcome associated to any other strategy. If the extensive game form induces Φ and for each agent truth-telling is obviously dominant, then Φ is obviously strategy-proof.

For our context, two important simplifications related to obvious strategy-proofness have been identified in the literature that follows Li (2017). First, without loss of generality we can assume that the extensive game form that induces the rule has perfect information (see Ashlagi and Gonczarowski (2018) and Mackenzie (2020)). Second, the new notion of obvious strategy-proofness can be fully captured by the classical notion of strategy-proofness applied to games in extensive form with perfect information. This last observation follows from the fact that the best possible outcome obtained when agent i chooses an action different from the one prescribed by i's truth-telling strategy and the worst possible outcome obtained when agent i chooses the action prescribed by i's truth-telling strategy are both obtained with only one behavioral strategy profile of the other agents, because the perfect information implies that all information sets are singleton sets (and each one belongs either to the subgame that follows the truth-telling choice or else to the subgame that follows the alternative choice).[7] Then, for easy presentation and following this literature, we will say that a rule is obviously strategy-proof if it is implemented by an extensive game form with perfect information for which truth-telling is a weakly dominant strategy.

Our approach is based on a general algorithm that, if tailored to a sequential allotment rule, defines an extensive game form. Then, the algorithm gives precise instructions on how to identify at each step the agent that plays and the set of actions (associated to non-terminal nodes of the tree), and when to stop (associated to terminal nodes of the tree). We omit here the formal definition of an extensive game form.

Fix a division problem given by the integer k and the set of agents N. Let $\mathcal{G}$ be the class of all (finite) extensive game forms associated to (k, N) whose results associated to terminal nodes are allotments in X. Fix an extensive game form $\Gamma \in \mathcal{G}$ and an agent $i \in N$. A (behavioral and pure) *strategy* of i in Γ is a function σ_i that selects at each node where i has to play one of i's available actions at that node. Let Σ_i be the set of i's strategies in Γ. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_1 \times \cdots \times \Sigma_n = \Sigma$ is an ordered list of strategies, one for each agent. Given $i \in N$, $\sigma \in \Sigma$ and $\sigma'_i \in \Sigma_i$ we often write (σ'_i, σ_{-i}) to denote the strategy profile where σ_i is replaced in σ by σ'_i. Let $g : \Sigma \to X$ be the outcome function of Γ. Hence, $g(\sigma)$ is the allotment assigned to the terminal node that results when agents play Γ according to $\sigma \in \Sigma$; in particular, $\sum_{i=1}^{n} g_i(\sigma) = k$ for all $\sigma \in \Sigma$.

Fix an extensive game form $\Gamma \in \mathcal{G}$ and a preference profile $R \in \mathcal{R}^N$. Let (Γ, R) denote the game in extensive form where each agent $i \in N$ evaluates strategy profiles in Γ according to R_i. A strategy σ_i is *weakly dominant* in (Γ, R) if, for all σ_{-i} and all σ'_i,

$$g_i(\sigma) \; R_i \; g_i(\sigma'_i, \sigma_{-i}).$$

We are now ready to define obvious strategy-proofness in the context of division problems.

Definition 1 Let (k, N) be given. A rule $\Phi : \mathcal{R}^N \to X$ is *obviously strategy-proof* if there is an extensive game form $\Gamma \in \mathcal{G}$ associated to (k, N) such that, for each $i \in N$ and $R_i \in \mathcal{R}$,
(i) there exists $\sigma_i^{R_i} \in \Sigma_i$ such that $\Phi(R) = g(\sigma^R)$, where $R = (R_1, \ldots, R_n)$ and $\sigma^R = (\sigma_1^{R_1}, \ldots, \sigma_n^{R_n})$, and
(ii) $\sigma_i^{R_i}$ is weakly dominant in (Γ, R).[8]

[7] Mackenzie (2020) proves this for a class of extensive form games with perfect information, called round table mechanisms, but the proof can be adapted to any extensive game form with perfect information.

[8] Recall that by Mackenzie (2020), requiring weak dominance is equivalent to requiring obvious dominance.

When (i) holds we say that Γ *induces* Φ. When (i) and (ii) hold we say that Γ OSP-*implements* Φ and refer to $\sigma_i^{R_i}$ as i's *truth-telling* strategy.

Our main result states that all sequential allotment rules are obviously strategy-proof. Namely, in the two statements of Proposition 1, strategy-proofness can be replaced by obvious strategy-proofness. The proof of our result is constructive, and based on the Monotonous and Individualized Algorithm (MIA) that we describe in the next section.

4 The Monotonous and Individualized Algorithm (MIA)

Our aim here is to define, for the division problem (k, N), a family of extensive game forms in $\mathcal{G}$, which we will refer to as Monotonous and Individualized Games ($\mathcal{MIG}$), with the properties that (i) in each $\Gamma \in \mathcal{MIG}$, truth-telling is always weakly dominant and (ii) for each sequential allotment rule Φ, one can identify a $\Gamma \in \mathcal{MIG}$ that OSP-implements Φ. We define the family through the Monotonous and Individualized Algorithm (MIA).

At every step of the MIA an agent i and his/her guaranteed assignment β_i are selected. Agent i can either leave with his/her assignment β_i or stay, waiting for more or less. The first time that agent i is called to play has three possible actions: β_i, more than β_i or less than β_i. If agent i chooses the guaranteed assignment β_i, then i enters the set of agents that want to *stop* (N_s) and i will not be called to play anymore and will receive β_i. If agent i asks for more than β_i, then i enters the set of agents that want to go *up* (N_u) and i might be called to play later in the game, in which case i's guaranteed assignment will be equal to $\beta_i + 1$, and the choice of asking for less will not be available anymore. If agent i asks for less than β_i, then i enters the set of agents that want to go *down* (N_d) and i might be called to play later in the game, in which case i's guaranteed assignment will be equal to $\beta_i - 1$, and the choice of asking for more will not be available anymore. The guaranteed assignments evolve throughout the algorithm, by making sure that all proposed warranties are feasible.

The steps of the MIA are grouped into two stages. **Stage A** aims to identify an allotment $q \in X$ and to partition N among the sets of agents who would like more than q_i (N_u), less than q_i (N_d), or exactly q_i (N_s). If, at the end of **Stage A**, any of the sets N_u or N_d is empty, then the algorithm stops with allotment q. If the two sets N_u and N_d are non-empty, the algorithm moves to **Stage B** using as input the output of **Stage A** (the partition $N = N_u \cup N_d \cup N_s$ and the allotment q, the vector of guaranteed assignments). At each step of **Stage B** a Pareto improvement from q is carried out by identifying agents $j \in N_u$ and $r \in N_d$ and transferring one unit from r to j; these two new assignments, together with the previous guaranteed ones of the other agents, become the vector of new guaranteed assignments. Hence, all steps of the MIA use individualized guaranteed assignments that evolve monotonically, increasing for agents in N_u and decreasing for agents in N_d.

A collection of selections of pairs (i, β_i), one for each step of the MIA, defines an extensive game form $\Gamma \in \mathcal{MIG}$, and a strategy profile σ in Γ determines a run of the MIA that delivers the allotment $g(\sigma)$, where g is the outcome function of Γ.[9]

[9]For each sequential allotment rule Φ, we will later describe how to identify all pairs (i, β_i) in the MIA

When the guaranteed assignment is equal to 0 or k, the agent can neither receive strictly less than 0 nor more than k, respectively. For this reason the following notation will be useful. Let $j \in N$ and $\beta_j \in \{0, \ldots, k\}$. Define $\beta_j^- = \max\{\beta_j - 1, 0\}$ and $\beta_j^+ = \min\{\beta_j + 1, k\}$. Namely, $\beta_j^- = 0$ if $\beta_j = 0$ and otherwise $\beta_j^- = \beta_j - 1$; symmetrically, $\beta_j^+ = k$ if $\beta_j = k$ and otherwise $\beta_j^+ = \beta_j + 1$. Therefore, when $\beta_j \neq k$ or $\beta_j \neq 0$, the choice β_j^+ or β_j^- can be interpreted as j asking for more or less than β_j, respectively.

Throughout the steps of the MIA we identify with the symbol $^{(*)}$ some properties of β_i, the guaranteed assignment to agent i who plays at the step. In Subsection 4.2 we will explain those properties and argue why they are indeed satisfied.

4.1 The MIA

Stage A. Step A.t ($\mathbf{t} \geq 1$)**.**

Input: N_u, N_d, N_s, N_p (with $N_p = N_u \cup N_d \cup N_s$) and $(q_i)_{i \in N_p}$, output of **Step A.t-1** if $\mathbf{t} > 1$, or $N_u = N_d = N_s = N_p = \emptyset$ if $\mathbf{t} = 1$.

Choose $j \notin N_s$ and $\beta_j \in \{0, \ldots, k\}$ such that

(i) $\sum_{i \in N_p \setminus \{j\}} q_i + \beta_j \leq k$ and $\sum_{i \in N_p \setminus \{j\}} q_i + \beta_j = k$ if $N_p \cup \{j\} = N$,$^{(*)}$

(ii) $\beta_j = q_j + 1$ if $j \in N_u$,$^{(*)}$

(iii) $\beta_j = q_j - 1$ if $j \in N_d$.$^{(*)}$

Agent j has to choose an action a_j from the set

$$A_j = \begin{cases} \{\beta_j^-, \beta_j, \beta_j^+\} & \text{if } j \notin N_p \\ \{\beta_j, \beta_j^+\} & \text{if } j \in N_u \\ \{\beta_j^-, \beta_j\} & \text{if } j \in N_d. \end{cases}$$

Set

$$N_u := \begin{cases} N_u \cup \{j\} & \text{if } a_j = \beta_j + 1 \\ N_u \setminus \{j\} & \text{if } a_j = \beta_j \\ N_u & \text{if } a_j = \beta_j - 1, \end{cases} \qquad N_d := \begin{cases} N_d \cup \{j\} & \text{if } a_j = \beta_j - 1 \\ N_d \setminus \{j\} & \text{if } a_j = \beta_j \\ N_d & \text{if } a_j = \beta_j + 1, \end{cases}$$

$$N_s := \begin{cases} N_s \cup \{j\} & \text{if } a_j = \beta_j \\ N_s & \text{otherwise,} \end{cases} \qquad N_p := N_u \cup N_d \cup N_s,$$

and $q_j := \beta_j$.

Output: Subsets N_u, N_d, N_s, N_p and $q = (q_i)_{i \in N_p}$.

If $N_p \neq N$, go to **Step A.t+1**.

If $N_p = N$, stop.

The output of **Stage A** is the partition N_u, N_d, N_s and $q = (q_i)_{i \in N}$.

If $N_u \neq \emptyset$ and $N_d \neq \emptyset$, go to **Stage B**, with input N_u, N_d, N_s and q.

If $N_u = \emptyset$ or $N_d = \emptyset$, stop, and the outcome of the MIA is the allotment q.

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that define the extensive game form $\Gamma^\Phi \in \mathcal{MIG}$ that OSP-implements Φ.

Stage B. Step B.t (t ≥ 1).

Input: Partition N_u, N_d, N_s and q, output of **Stage A** if $\mathbf{t} = 1$, or **Stage B.t-1** if $\mathbf{t} > 1$.

Choose agents $j \in N_u$ and $r \in N_d$.

Set $\beta_j = q_j + 1$ and $\beta_r = q_r - 1$.$^{(*)}$

Step B.t.a. Agent $j \in N_u$ has to choose an action a_j from the set $A_j = \{\beta_j, \beta_j^+\}$.

Step B.t.b. Agent $r \in N_d$ has to choose an action a_r from the set $A_r = \{\beta_r^-, \beta_r\}$.

Set

$$N_u := \begin{cases} N_u \setminus \{j\} & \text{if } a_j = \beta_j \\ N_u & \text{if } a_j = \beta_j + 1, \end{cases} \qquad N_d := \begin{cases} N_d \setminus \{r\} & \text{if } a_r = \beta_r \\ N_d & \text{if } a_j = \beta_r - 1, \end{cases}$$

$$N_s := \begin{cases} N_s \cup \{j\} & \text{if } a_j = \beta_j \text{ and } a_r = \beta_r - 1 \\ N_s \cup \{r\} & \text{if } a_j = \beta_j + 1 \text{ and } a_r = \beta_r \\ N_s \cup \{j, r\} & \text{if } a_j = \beta_j \text{ and } a_r = \beta_r \\ N_s & \text{if } a_j = \beta_j + 1 \text{ and } a_r = \beta_r - 1, \end{cases}$$

$q_j := \beta_j$ and $q_r := \beta_r$.

Output: The partition N_u, N_d, N_s and $q = (q_i)_{i \in N}$.

If $N_u \neq \emptyset$ and $N_d \neq \emptyset$, go to **Step B.t+1**.

If $N_u = \emptyset$ or $N_d = \emptyset$, stop, and the outcome of the MIA is the allotment q.

Denote by $\mathcal{MIG}$ the family of all extensive game forms defined by the MIA once, at each step, a pair (i, β_i) is selected out of all those satisfying the constraints imposed by the MIA. Let $\Gamma \in \mathcal{MIG}$ and let σ be a strategy in Γ. We will refer to the partition N_u, N_d, N_s and $q = (q_i)_{i \in N}$, output of either **Stage A** or **B**, as the output of the run of the MIA when agents play Γ according to σ. Observe that $q = g(\sigma)$, where g is the outcome function of Γ.

4.2 General remarks on the MIA

1. On the properties of β_j, the guaranteed assignment to agent j who plays at some step of the MIA.

   (1.i) Condition (i) in **Step A.t** says that the guaranteed assignment β_j is feasible together with $(q_i)_{i \in N_p \setminus \{j\}}$, the assignments assigned provisionally to agents other than j who have already played at earlier steps. And in particular, if j is the only agent that has not played yet, β_j is equal to the remaining units to be allotted.

   (1.ii) If $j \in N_u$ in **Step A.t** or **B.t**, then $\beta_j = q_j + 1$. We argue that $\beta_j \leq k$. Note first that $j \in N_u$ means that j has played at some earlier step. Let $\widehat{A}_j \subseteq \{\widehat{\beta}_j^-, \widehat{\beta}_j, \widehat{\beta}_j^+\}$ be the set of actions available to j the last time j was called to play. As $j \in N_u$, j chose $\widehat{a}_j = \widehat{\beta}_j^+$, where $\widehat{\beta}_j^+ = \widehat{\beta}_j + 1 \leq k$. Now, by the definition of the MIA, $\widehat{q}_j = \widehat{\beta}_j = q_j$. Hence, $q_j + 1 \leq k$.

(1.iii) If $i \in N_d$ in **Step A.t** or **B.t**, then $\beta_i = q_i - 1$. We argue that $\beta_i \geq 0$. Note first that $i \in N_d$ means that i has played at some earlier step. Let $\widehat{A}_i \subseteq \{\widehat{\beta}_i^-, \widehat{\beta}_i, \widehat{\beta}_i^+\}$ tbe he set of actions available to i the last time i was called to play. As $i \in N_d$, i chose $\widehat{a}_i = \widehat{\beta}_i^-$, where $\widehat{\beta}_i^- = \widehat{\beta}_i - 1 \geq 0$. Now, by the definition of the MIA, $\widehat{q}_i = \widehat{\beta}_i = q_i$. Hence, $q_i - 1 \geq 0$.

2. Whenever agent i has to choose an action throughout the MIA, each choice can be identified with a subset of $\mathcal{R}$: action β_i with $\{R_i \in \mathcal{R} \mid \tau(R_i) = \beta_i\}$, action β_i^+ with $\{R_i \in \mathcal{R} \mid \tau(R_i) > \beta_i\}$ and action β_i^- with $\{R_i \in \mathcal{R} \mid \tau(R_i) < \beta_i\}$.

   (2.i) If $i \notin N_p$ (at some **Step A.t**), A_i can be seen as a partition of $\mathcal{R}$.

   (2.ii) If $i \in N_p$, A_i can be seen as a partition of the subset of preferences induced by i's last previous choice.[10]

3. The evolution of the subsets N_u, N_d and N_s throughout the algorithm is as follows.

   (3.i) Once agent i enters the subset N_s at some step, i remains in N_s at all further steps and, accordingly, i is not called to play again.

   (3.ii) Once agent i enters the subset N_u at some step, i can only move to N_s or remain in N_u at further steps.

   (3.iii) Once agent i enters the subset N_d at some step, i can only move to N_s or remain in N_d at further steps.

4.3 Truth-telling is weakly dominant

Let $\Gamma \in \mathcal{MIG}$ be the extensive game form defined by the MIA once, in each of its steps, the required agent j and his/her guaranteed assignment β_j are specified. Recall that a strategy σ_i for agent i in Γ selects a choice a_i in each of the action sets A_i that i may have the chance to choose from in Γ. For $i \in N$ and $R_i \in \mathcal{R}$, the *truth-telling* strategy $\sigma_i^{R_i}$ (relative to R_i) is the strategy where, whenever agent i is called to play, i chooses the best action in A_i according to R_i. Denote this choice by $\max_{R_i} A_i$. By single-peakedness, i selects β_i if $\tau(R_i) = \beta_i$, β_i^+ if $\tau(R_i) > \beta_i$ and $\beta_i^+ \in A_i$, and β_i^- if $\tau(R_i) < \beta_i$ and $\beta_i^- \in A_i$.[11]

Remark 1 Let N_u, N_d, N_s and q be the output of the run of the MIA when agents play the game in extensive form (Γ, R) according to σ^R.

[10] As a consequence of (2.i) and (2.ii), each $\Gamma \in \mathcal{MIG}$ is a *round table mechanism* (see Mackenzie (2020)) because its sets of actions are non-empty subsets of preferences and (a) the set of actions at any node are disjoint subsets of preferences, (b) when a player has to play for the first time the set of actions is a partition of $\mathcal{R}$, and (c) later, at a node ν, the union of actions is the intersection of the actions taken by the agent assigned to ν at all predecessor nodes that lead to ν.

[11] Each $\Gamma \in \mathcal{MIG}$ is a *menu mechanism* (see Mackenzie and Zhou (2020)) because agents select from a menu of possible assignments (identified with a corresponding set of actions) and truth-telling requires to choose the most-preferred one.

(R1.1) If $i \in N_s$, then $\tau(R_i) = q_i$. To see that (R1.1) holds, let $i \in N_s$. First, by the definition of N_s, the last time that i was called to play i has chosen the guaranteed assignment $\beta_i \in A_i$ and entered the set N_s. By (3.i) in Subsection 4.2, i remains in N_s until the end of the MIA and, accordingly, i is not called to play anymore and $q_i = \beta_i$. Second, by definition of $\sigma_i^{R_i}$, $\tau(R_i) = \beta_i$. Hence, $\tau(R_i) = q_i$.

(R1.2) If $i \in N_u$, then $\tau(R_i) > q_i$. To see that (R1.2) holds, let $i \in N_u$. First, by the definition of N_u, the last time that i was called to play i has chosen $\beta_i + 1 \in A_i$ and $q_i = \beta_i$. Second, by definition of $\sigma_i^{R_i}$, $\tau(R_i) \geq \beta_i + 1$. Hence, $\tau(R_i) > q_i$.

(R1.3) If $i \in N_d$, then $\tau(R_i) < q_i$. To see that (R1.3) holds, let $i \in N_d$. First, by the definition of N_d, the last time that i was called to play i has chosen $\beta_i - 1 \in A_i$ and $q_i = \beta_i$. Second, by definition of $\sigma_i^{R_i}$, $\tau(R_i) \leq \beta_i - 1$. Hence, $\tau(R_i) < q_i$. □

Fix $\Gamma \in \mathcal{MIG}$ and $R \in \mathcal{R}^N$. We now argue why, according to Li (2017)'s original definition, truth-telling is obviously dominant in (Γ, R). Consider agent i with guaranteed assignment β_i who plays in (Γ, R) and has to chose an action from the set $A_i \subseteq \left\{\beta_i^-, \beta_i, \beta_i^+\right\}$, where $\beta_i \in A_i$. If $\tau(R_i) = \beta_i$, the truth-telling choice β_i in A_i is obviously dominant since in this case i's worst assignment will be $\tau(R_i)$, the unique possible one. If $\tau(R_i) > \beta_i$, the set of i's possible assignments induced by the truth-telling choice β_i^+ is $\{\beta_i, \ldots, \tau(R_i)\}$ (i will choose $\tau(R_i)$ whenever $\tau(R_i)$ becomes i's guaranteed assignment) and, by single-peakedness, β_i is the worst one according to R_i. However, the best possible assignment of a non truth-telling choice (β_i or β_i^-) is β_i. Hence, truth-telling is obviously dominant in this case. Symmetrically if $\tau(R_i) < \beta_i$. We now enunciate and formally prove that truth-telling is weakly dominant in (Γ, R).[12]

Proposition 2 *Let $\Gamma \in \mathcal{MIG}$ be an extensive game form defined by the MIA and let $R \in \mathcal{R}^N$ be a profile. Then, for each agent i, the strategy $\sigma_i^{R_i}$ is weakly dominant in the game in extensive form (Γ, R).*

Proof Let Γ be defined by the MIA. Fix arbitrary $i \in N$, $R_i \in \mathcal{R}$ and σ_{-i}, and consider any $\sigma_i' \neq \sigma_i^{R_i}$. Let N_u, N_d, N_s and $(q_i)_{i \in N}$ be the output of the run of the MIA when agents play according to $(\sigma_i^{R_i}, \sigma_{-i})$ and let N_u', N_d', N_s' and $(q_i')_{i \in N}$ be the output of the run of the MIA when agents play according to (σ_i', σ_{-i}). We verify that $q_i \; R_i \; q_i'$. Assume first that $i \in N_s$. Then, by (R1.1) in Remark 1, $\tau(R_i) = q_i$ and, accordingly, $q_i \; R_i \; q_i'$. Assume now that $q_i \neq q_i'$. There exists a step at which for the first time $\sigma_i^{R_i}$ and σ_i' select different actions, say a_i and a_i', and q_i follows after a_i and q_i' after a_i'. We distinguish between two symmetric cases.

<u>Case 1</u>: $i \in N_u$. Then, by (R1.2) in Remark 1, $\tau(R_i) > q_i$. By the definition of $\sigma_i^{R_i}$, $a_i = \max_{R_i} A_i \leq \tau(R_i)$. Since $i \in N_u$, the guaranteed assignment has weakly increased from $a_i - 1$ (the guaranteed assignment at the step where i could choose a_i' as well) to q_i until the end of the MIA. Hence, $a_i - 1 \leq q_i$ and

$$\max\nolimits_{R_i} A_i - 1 \leq q_i < \tau(R_i). \tag{3}$$

[12]Recall again that, according to Mackenzie (2020), being obviously dominant in (Γ, R) is equivalent to being weakly dominant in (Γ, R).

Similarly, and as $a_i \neq a'_i$,

$$a'_i \leq \max_{R_i} A_i - 1. \tag{4}$$

By (4), $i \in N'_d \cup N'_s$, and the guaranteed assignment has weakly decreased from a'_i to q'_i until the end of the MIA. Hence, $q'_i \leq a'_i$ and, together with (3) and (4), $q'_i \leq q_i < \tau(R_i)$. By single-peakedness, $q_i \; R_i \; q'_i$.

Case 2: $i \in N_d$. Then, by (R1.3) in Remark 1, $\tau(R_i) < q_i$. By the definition of $\sigma_i^{R_i}$, $\tau(R_i) \leq a_i = \max_{R_i} A_i$. Since $i \in N_d$, the guaranteed assignment has weakly decreased from $a_i + 1$ (the guaranteed assignment at the step where i could choose a'_i as well) to q_i until the end of the MIA. Hence, $q_i \leq a_i + 1$ and

$$\tau(R_i) < q_i \leq \max_{R_i} A_i + 1. \tag{5}$$

Similarly, and as $a_i \neq a'_i$,

$$\max_{R_i} A_i + 1 \leq a'_i. \tag{6}$$

By (6), $i \in N'_u \cup N'_s$, and the guaranteed assignment has weakly increased from a'_i to q'_i until the end of the MIA. Hence, $a'_i \leq q'_i$ and, together with (5) and (6), $\tau(R_i) < q_i \leq q'_i$. By single-peakedness, $q_i \; R_i \; q'_i$.

Hence, for all σ_{-i} and σ'_i, $g_i(\sigma_i^{R_i}, \sigma_{-i}) \; R_i \; g_i(\sigma'_i, \sigma_{-i})$, which means that $\sigma_i^{R_i}$ is weakly dominant in (Γ, R). ■

5 From a sequential allotment rule to the extensive game form defined by the MIA

Our objective here is to exhibit, for each sequential allotment rule Φ, an extensive game form $\Gamma^\Phi \in \mathcal{MIG}$ that OSP-implements Φ. The extensive game form Γ^Φ defined by the MIA uses Φ to select the agents who have to move and their guaranteed assignments. In what follows we apply the MIA, but we only specify the required selections of agents and guaranteed assignments (everything else is as it has been specified in the general definition of the MIA in Subsection 4.1).

The excess and scarcity allotments play an important role on determining those selections. For a given sequential allotment rule Φ, they are

$$\underline{q} := \Phi(\mathbf{0}) \text{ and } \overline{q} := \Phi(\mathbf{k}).$$

We distinguish between two cases, depending on whether $\underline{q} = \overline{q}$ or $\underline{q} \neq \overline{q}$. The first case corresponds to the subclass of individually rational sequential allotment rules.

5.1 The individually rational case $\underline{q} = \overline{q}$

Stage A. Step A.t ($\mathbf{t} \geq 1$), choose $j = t$ and $\beta_j = \underline{q}_j$.

Observe that **Stage A** finishes at **Step A.n**, when $N_p = N$ with $q := \underline{q} = \overline{q}$.

Stage B. Step B.t ($\mathbf{t} \geq 1$), choose agents $j \in N_u$ and $r \in N_d$ among those for which

$$\Phi_j(\mathbf{k}_{N_u}, (q-\mathbf{1})_{N_d}, q_{N_s}) \geq q_j + 1 \text{ and } \Phi_r(q_j+1, \mathbf{0}_{N_d}, q_{-(N_d \cup \{j\})}) \leq q_r - 1.$$

This extensive game form defined by the MIA obtained from Φ is denoted by Γ^{Φ}.

Throughout **Stage A**, all agents are classified into subsets according to whether each j prefers to receive the guaranteed assignment q_j (N_s), strictly more (N_u) or strictly less (N_d). If the algorithm goes into **Stage B** is because $N_u \neq \emptyset$ and $N_d \neq \emptyset$ in the output of **Stage A**. But this means that q is not efficient. At each **Step B.t**, agents $j \in N_u$ and $r \in N_d$ are chosen to carry out a Pareto improvement upon q, input of this step, by increasing j's guaranteed assignment by one unit and decreasing r's guaranteed assignment by one unit. Agents j and r are sequentially identified by looking at the image of Φ at two somehow extreme profiles, both with all agents in N_s having their top at q_{N_s}. First, j is one of the agents in N_u whose assignment is larger or equal to $q_j + 1$ at profile $(\mathbf{k}_{N_u}, (q-\mathbf{1})_{N_d}, q_{N_s})$, whose sum of the components is larger or equal to k. Therefore, by (1) in the definition of same-sidedness, agents in $N_d \cup N_s$ receive at most their tops and, by feasibility of q, one agent j in N_u has to receive at least $q_j + 1$. Once j is identified, r is one of the agents in N_d whose assignment is smaller or equal to $q_j - 1$ at profile $(q_j + 1, \mathbf{0}_{N_d}, q_{-(N_d \cup \{j\})})$, whose sum of the components is smaller or equal to k. Therefore, by (2) in the definition of same-sidedness, agents in $N_u \cup N_s$ receive at least their tops and, by feasibility of q, one agent r in N_d has to receive at most $q_r - 1$. We now enunciate and formally prove that Γ^{Φ} is well defined.

Proposition 3 *For each individually rational sequential allotment rule Φ, Γ^{Φ} is well defined.*

Proof We only have to show that at each **Step B.t**, agents $j \in N_u$ and $r \in N_d$ are well defined. Let N_u, N_d, N_s and q be the input of **Step B.t**, which means that $N_u \neq \emptyset$ and $N_d \neq \emptyset$.

By (1.iii) in Subsection 4.2, $i \in N_d$ implies $0 < q_i$. Therefore, the profile $x = (\mathbf{k}_{N_u}, (q-\mathbf{1})_{N_d}, q_{N_s})$ is well defined and $\sum_{i \in N} x_i \geq k$. Hence, by (1) in the definition of same-sidedness,

$$\sum_{i \notin N_u} \Phi_i(\mathbf{k}_{N_u}, (q-\mathbf{1})_{N_d}, q_{N_s}) \leq \sum_{i \in N_d} (q_i - 1) + \sum_{i \in N_s} q_i < \sum_{i \notin N_u} q_i.$$

By feasibility of q,

$$\sum_{i \in N_u} \Phi_i(\mathbf{k}_{N_u}, (q-\mathbf{1})_{N_d}, q_{N_s}) > \sum_{i \in N_u} q_i.$$

Hence, there exists $j \in N_u$ such that $\Phi_j(\mathbf{k}_{N_u}, (q-\mathbf{1})_{N_d}, q_{N_s}) \geq q_j + 1$.

By (1.ii) in Subsection 4.2, $j' \in N_u$ implies $q_{j'} < k$. Therefore, the profile $y = (q_j + 1, \mathbf{0}_{N_d}, q_{-(N_d \cup \{j\})})$, where j is the agent identified just above and the one selected to play, is well defined and $\sum_{i \in N} y_i \leq k$. Hence, by (2) in the definition of same-sidedness,

$$\sum_{i \notin N_d} \Phi_i(q_j + 1, \mathbf{0}_{N_d}, q_{-(N_d \cup \{j\})}) \geq q_j + 1 + \sum_{i \notin N_d} q_i > \sum_{i \notin N_d} q_i.$$

By feasibility of q,

$$\sum_{i\in N_d}\Phi_i(q_j+1,\mathbf{0}_{N_d},q_{-(N_d\cup\{j\})})<\sum_{i\in N_d}q_i.$$

Hence, there exists $r\in N_d$ such that $\Phi_r(q_j+1,\mathbf{0}_{N_d},q_{-(N_d\cup\{j\})})\leq q_r-1$. ■

5.2 The non-individually rational case $\underline{q}\neq\overline{q}$

The biggest difficulty in this case is that the rule does not provide a unique allotment q, whose components could be sequentially offered to agents as guaranteed assignments in order to classify them according to whether each i wants exactly q_i, more than q_i or less than q_i. To overcome it, we construct a sequence of pairs, each consisting of an agent and his/her provisionally guaranteed assignment, that ends in an allotment q and a partition of the set of agents classifying them according to their wills with respect to q.

In the individually rational case (*i.e.*, when $\underline{q}:=\Phi(\mathbf{0})=\Phi(\mathbf{k})$) the sequence of pairs is trivially $(1,\underline{q}_1),(2,\underline{q}_2),\ldots,(n,\underline{q}_n)$, where (i) each agent j appears only once in the sequence and (ii) j's guaranteed assignment $\underline{q}_j$ does not depend on the choices made by the previous agents in the sequence. These two properties greatly simplify **Stage A** of the individually rational case. To deal with the general case, agents could appear now more than once in the sequence and their guaranteed assignments could depend on the choices made by the previous agents.

The selection of the pair (j,β_j) at each step of **Stage A** is as follows. Specifically, at **Step A.1** start computing the scarcity and excess allotments $\overline{q}=\Phi(\mathbf{k})$ and $\underline{q}=\Phi(\mathbf{0})$. Select agent j among those for whom $\underline{q}_j<\overline{q}_j$ (note that $\overline{q},\underline{q}\in X$ and $\overline{q}\neq\underline{q}$ assure that such j does exist) and select j's guaranteed assignment as $\beta_j=\underline{q}_j$. The pair $(j,\underline{q}_j)$ is the first element of the sequence.

The choice of agent j playing at **Step A.t** with $\mathbf{t}>1$ and input N_u,N_d,N_s,N_p and $(q_i)_{i\in N_p}$ is more involved. Define adjusted scarcity and excess allotments by setting

$$\underline{q}:=\Phi(\mathbf{0}_{-(N_s\cup N_u)},q_{N_s\cup N_u})\quad\text{and}\quad\overline{q}:=\Phi(\mathbf{k}_{-(N_s\cup N_d)},q_{N_s\cup N_d}).$$

In the two profiles of tops agents in N_s ask for their guaranteed assignments q_{N_s}. In the profile of tops used to obtain the adjusted scarcity allotment agents in N_d also ask for their provisionally guaranteed assignments q_{N_d} (upper bounds of their finally guaranteed assignments), while agents in N_u and those that have not played yet are asking for all. Symmetrically, in the profile of tops used to obtain the adjusted excess allotment agents in N_u also ask for their provisionally guaranteed assignments q_{N_u} (lower bounds of their finally guaranteed assignments), while agents in N_d and those that have not played yet are asking for nothing at all.[13] We distinguish between two cases. In Case 1, when $\underline{q}\neq\overline{q}$, proceed as in **Step A.1** by selecting j as one agent among those for whom $\underline{q}_j<\overline{q}_j$ and setting $\beta_j=\underline{q}_j$. In Case 2, when $\underline{q}=\overline{q}$, proceed as in the individually rational case as if the set of agents were $N\setminus N_p$, those that have not played yet; that is, each agent $j\in N\setminus N_p$ plays only once

[13] Lemma 1 in the Appendix states that (i) the sum of the components of $(q_i)_{i\in N_p}$ is smaller or equal to k, (ii) $q_i=\underline{q}_i=\overline{q}_i$ for all $i\in N_s$, (iii) $q_i=\underline{q}_i\leq\overline{q}_i$ for all $i\in N_u$, and (iv) $\underline{q}_i\leq\overline{q}_i=q_i$ for all $i\in N_d$.

(in the remaining steps of **Stage A**) in any order and j's guaranteed assignment is $\underline{q}_j$. The need to distinguish between the two cases above (under which agent j is selected to play throughout **Stage A**) is due to the fact that the algorithm has reached two situations that have to be treated differently.[14]

Finally, any **Step B.t** proceeds as in the individually rational case.

We now formally define how the MIA is tailored to a sequential allotment rule Φ that is not individually rational.

Stage A. Step A.t ($\mathbf{t} \geq 1$), with input N_d, N_u, N_s (and $N_p = N_d \cup N_u \cup N_s \neq N$) and $(q_i)_{i \in N_p}$, j and β_j are chosen by looking at the allotments

$$\underline{q} := \Phi(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u}) \quad \text{and} \quad \overline{q} := \Phi(\mathbf{k}_{-(N_s \cup N_d)}, q_{N_s \cup N_d}).$$

We distinguish between two mutually exclusive cases.

Case 1: There exists $i \notin N_s$ such that $\underline{q}_i < \overline{q}_i$. One of such agents is the chosen j. If $j \notin N_p$, choose $\beta_j = \underline{q}_j$. If $j \in N_p$, choose β_j according to either (ii) or (iii) in the general definition of the MIA.

Case 2: For all $i \notin N_s$, $\underline{q}_i \geq \overline{q}_i$. Among those, choose $j \in N \setminus N_p$ and $\beta_j = \underline{q}_j$.

Stage B. Step B.t. Agents $j \in N_u$ and $r \in N_d$ are identified as in the individually rational case (in Subsection 5.1).

The extensive game form defined by the MIA obtained from Φ is denoted by Γ^Φ and, as stated by Proposition 4, it is well defined.

Proposition 4 *For each sequential allotment rule Φ, Γ^Φ is well defined.*

To prove that Γ^Φ is well defined we only have to show that the definition of the guaranteed assignment β_j at each **Step A.t** satisfies the feasibility condition (i). This proof is relegated to the Appendix and it uses Lemmata 2 and 3, whose formal statements and proofs can also be found in the Appendix.

To gain insight as to why Γ^Φ induces Φ when agents truth-tell, consider any **Step A.t** or **Step B.t** and denote its input by N_u, N_d, N_s, N_p and $(q_i)_{i \in N_p}$. Let j be the agent that is called to play at this step and let β_j be his/her proposed guaranteed assignment. Fix a preference profile τ. Since agents have been truth-telling up to this step, we have that (i) $\tau_i = q_i$ for all $i \in N_s$, (ii) $\tau_i < q_i$ for all $i \in N_d$, and (iii) $\tau_i > q_i$ for all $i \in N_u$. A preliminary key property is that β_j is the outcome of Φ for agent j when $\tau_j = \beta_j$; *i.e.*,

$$\text{if } \tau_j = \beta_j \text{ then } \Phi_j(\tau) = \beta_j. \tag{7}$$

The whole proof of (7) proceeds by induction on $\mathbf{t}$. It requires a precise and detailed argumentation collected in the lemmata that can be found in the Appendix. However, we

[14] Although, to keep the formal description of the algorithm compact, the allotments $\underline{q}$ and $\overline{q}$ are still computed once Case 2 takes over, but then $\underline{q} = \overline{q}$ holds thereafter. In particular, Claim 4 in the proof of Lemma 1 shows that the two updated guaranteed allotments are equal, and conditions (27) and (29) ensure that they remain constant.

present here the main intuition for the first step in the induction; *i.e.*, the first time that j is called to play (so at some **Step A.t**). Assume $\tau_j = \beta_j$. By definition, $\beta_j = \underline{q}_j = \Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u})$ and, by strategy-proofness, $\beta_j = \Phi_j(\mathbf{0}_{-(N s \cup N_u \cup \{j\})}, q_{N_s \cup N_u}, \tau_j)$. Since all agents have been truth-telling according to τ until **Step A.t**, $(\mathbf{0}_{-(N_s \cup N_u \cup \{j\})}, q_{N_s \cup N_u}) \leq \tau_{-j}$. Then, by strategy-proofness and replacement monotonicity, $\Phi_j(\tau) \leq \beta_j$. On the other hand, $\beta_j = \underline{q}_j \leq \overline{q}_j = \Phi_j(\mathbf{k}_{-(N_s \cup N_d)}, q_{N_s \cup N_d})$.[15] Then, by strategy-proofness and single peakedness, $\beta_j \leq \Phi_j(\mathbf{k}_{-(N_s \cup N_d \cup \{j\})}, q_{N_s \cup N_d}, \tau_j)$. Since all agents have been truth-telling according to τ until **Step A.t**, $\tau_{-j} \leq (\mathbf{k}_{-(N_s \cup N_d \cup \{j\})}, q_{N_s \cup N_d})$. Then, by strategy-proofness and replacement monotonicity, $\beta_j \leq \Phi_j(\tau)$. Hence, (7) holds.[16] Now, let N_u, N_d, N_s and q be the output of the MIA obtained from Φ when agents truth-tell according to τ. We argue that $\Phi(\tau) = q$. As N_u, N_d, N_s is part of the output of the MIA, $N_u = \emptyset$ or $N_d = \emptyset$. Assume that $N_u = \emptyset$ (if $N_d = \emptyset$, use a symmetric argument). Consider an arbitrary agent $i \in N_s \cup N_d = N$, and let **Step A.t** or **Step B.t** be the last step where i is called to play, with β_i as his/her proposed guaranteed assignment. Since $i \in N_s \cup N_d$, and i has been truth-telling according to τ_i, $\tau_i \leq \beta_i$. Then, by (7) and strategy-proofness, $\Phi_i(\tau) \leq \beta_i = q_i$. Since i was arbitrarily chosen and q is feasible, $\Phi(\tau) = q$.

6 Main result and example

Theorem 1 *All sequential allotment rules are obviously strategy-proof.*

We have already proved some building blocks of the proof of Theorem 1. Namely, for all Φ, Propositions 3 and 4 state that Γ^Φ is well defined and Proposition 2 states that, for all $R \in \mathcal{R}^N$, truth-telling is weakly dominant in (Γ^Φ, R). The Appendix contains the rest, those blocks of a more technical nature. As we have already mentioned, Lemma 1 says that the sequences of q's, $\underline{q}$'s and $\overline{q}$'s generated by a non-individually rational Φ in **Stage A** are monotonous (in the right direction) and the sum of the components of each $(q_i)_{i \in N_p}$ in the sequence is smaller or equal to k. Lemmata 2 and 3 say respectively that, at the end of **Stage A**, $q = \underline{q} = \overline{q}$ and $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = \Phi(\mathbf{k}_{N_u}, q_{N_s}, q_{N_d}) = q$ hold. Both imply that q is a feasible allotment and the former says that the process somehow converges, while the latter is an intermediate result for the proof of Lemma 4 about **Stage B**. In turn, Lemma 4 is required to prove Lemma 5, a result used (and explicitly stated) along the proof of Theorem 1, presented below.

Proof of Theorem 1 Let Φ be a sequential allotment rule. Consider the extensive game form Γ^Φ defined by the MIA obtained from Φ and let $R \in \mathcal{R}^N$ be a profile. By Proposition 2, σ^R is a weakly dominant strategy in the game in extensive form (Γ^Φ, R). We now prove that Γ^Φ induces Φ. Let N_u, N_d, N_s and q be the output of the run of the MIA when agents

[15] We have already argued that $\underline{q}_j \leq \overline{q}_j$, regardless of whether j has been selected under Case 1 or Case 2.

[16] If j has already played before, a similar argument can be used to show (7). If this happened in **Step A.t**, appealing now to Lemma 1, if **t** is not the last step of **Stage A**, and to Lemma 2, otherwise. If this happened in **Step B.t**, appealing now to Lemma 4.

play Γ^{Φ} according to σ^R. This means that $q = g(\sigma^R)$. We show that $\Phi(R) = q$ holds by distinguishing between two cases.

Case 1: $N_u = \emptyset$. Statement (L5.1) in Lemma 5 (whose enunciate and proof can be found in the Appendix) says that $\Phi(\mathbf{0}_{N_d}, q_{N_s}) = q$. By (R1.1) in Remark 1, and with the abuse of notation of mixing a profile of preferences and a profile of tops, $\Phi(\mathbf{0}_{N_d}, R_{N_s}) = q$. Let $i \in N_d$. By (R1.3) in Remark 1, $\tau(R_i) < q_i$. By strategy-proofness and single-peakedness, $\Phi_i(\mathbf{0}_{N_d \setminus \{i\}}, R_{N_s \cup \{i\}}) = q_i = \Phi_i(\mathbf{0}_{N_d}, R_{N_s})$. Since Φ is replacement monotonic, $\Phi(\mathbf{0}_{N_d \setminus \{i\}}, R_{N_s \cup \{i\}}) = q = \Phi(\mathbf{0}_{N_d}, R_{N_s})$. Successively using the same argument for the remaining agents in $N_d \setminus \{i\}$, we obtain that $\Phi(R) = q = \Phi(R_{N_d}, R_{N_s})$.

Case 2: $N_u \neq \emptyset$. Statement (L5.2) in Lemma 5 (whose enunciate and proof can be found in the Appendix) says that $N_d = \emptyset$ and $\Phi(\mathbf{k}_{N_u}, q_{N_s}) = q$. By (R1.1) in Remark 1, and again with an abuse of notation, $\Phi(\mathbf{k}_{N_u}, R_{N_s}) = q$. Let $i \in N_u$. By (R1.2) in Remark 1, $\tau(R_i) > q_i$. Therefore, by an argument symmetric to the one already used in Case 1 applied now to agents in N_u instead of N_d, we obtain that $\Phi(R) = q = \Phi(R_{N_u}, R_{N_s})$. ■

We return to Example 1 to describe, given a sequential allotment rule Φ and two profiles of tops, τ with scarcity and τ' with excess, two runs of the MIA obtained from Φ when agents play the extensive game form Γ^{Φ} according to the truth-telling strategies σ^{τ} and $\sigma^{\tau'}$, respectively. Note that each path of the extensive game form Γ^{Φ} can be obtained by letting players to play Γ^{Φ} according to a profile of behavioral strategies σ.

Example 1 (continued) Let $N = \{1,2,3,4\}$, $k = 7$, $\Phi(0,0,0,0) = \underline{q} = (4,0,2,1)$ and $\Phi(7,7,7,7) = \overline{q} = (0,1,1,5)$. Let Φ be the non-individually rational sequential allotment rule, partially studied in Example 1 and described in Table 1 below. Observe that Table 1, which will be used in what follows, is consistent with the existence of a rule satisfying strategy-proofness, efficiency and replacement monotonicity, and with the description of a sequential allotment rule made in Section 2.

$\sum_{i=1}^4 \tau_i < 7$	$\sum_{i=1}^4 \tau_i \geq 7$
$\Phi(0,0,0,0) = (4,0,2,1)$	$\Phi(7,7,7,7) = (0,1,1,5)$
$\Phi(0,1,0,0) = (3,1,2,1)$	$\Phi(7,1,7,7) = (0,1,1,5)$
$\Phi(0,1,0,1) = (3,1,2,1)$	$\Phi(7,1,7,2) = (2,1,2,2)$
$\Phi(0,1,0,2) = (2,1,2,2)$	$\Phi(7,1,7,1) = (3,1,2,1)$
$\Phi(2,1,0,2) = (2,1,2,2)$	$\Phi(3,1,7,1) = (3,1,2,1)$
$\Phi(0,1,3,0) = (2,1,3,1)$	$\Phi(2,1,7,0) = (2,1,4,0)$
$\Phi(1,1,3,0) = (2,1,3,1)$	$\Phi(5,1,2,2) = (2,1,2,2)$

Table 1

Consider first the profile of tops $\tau = (5,1,2,2)$. We run the MIA obtained from Φ when agents play Γ^{Φ} according to the truth-tell strategy σ^{τ}. We represent below the six steps of this run of the MIA in Figures 3.a, 3.b and 3.c (using similar conventions to those already used in Figures 1 and 2 in Example 1) and the path of Γ^{Φ} when agents play it according to σ^{τ} in Figure 4.

Stage A: Set $\mathbf{t} = 1$ and go to **Step A.1**.

Step A.1: *Input:* Subsets of agents $N_u = N_d = N_s = N_p = \emptyset$. Set

$$\underline{q} = \Phi(0,0,0,0) = (4,0,2,1) \quad \text{and} \quad \overline{q} = \Phi(7,7,7,7) = (0,1,1,5)\,. \tag{8}$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\} = \{2,4\}$.[17] Choose $j = 2$ and, since $2 \notin N_p$, set $\beta_2 = \underline{q}_2 = 0$ and $A_2 = \{0,1\}$. Agent 2 chooses $a_2 = 1$ because $\tau_2 = 1$. *Output:* $N_u = N_p = \{2\}$, $N_d = N_s = \emptyset$ and $q_2 = 0$. Go to **Step A.2**.

Step A.2: *Input:* The output of **Step A.1**. Set

$$\underline{q} = \Phi(0,0,0,0) = (4,0,2,1) \quad \text{and} \quad \overline{q} = \Phi(7,7,7,7) = (0,1,1,5)\,.$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\} = \{2,4\}$. Choose $j = 2$ and, since $2 \in N_u$, set $\beta_2 = q_2 + 1 = 1$ and $A_2 = \{1,2\}$. Agent 2 chooses $a_2 = 1$ because $\tau_2 = 1$. *Output:* $N_u = N_d = \emptyset$, $N_s = N_p = \{2\}$ and $q_2 = 1$. Go to **Step A.3**.

Step A.3: *Input:* The output of **Step A.2**. Set

$$\underline{q} = \Phi(0,1,0,0) = (3,1,2,1) \quad \text{and} \quad \overline{q} = \Phi(7,1,7,7) = (0,1,1,5)\,.$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\} = \{4\}$. Choose $j = 4$ and, since $4 \notin N_p$, set $\beta_4 = \underline{q}_4 = 1$ and and $A_4 = \{0,1,2\}$. Agent 4 chooses $a_4 = 2$ because $\tau_4 = 2$. *Output:* $N_u = \{4\}$, $N_d = \emptyset$, $N_s = \{2\}$, $N_p = \{2,4\}$, $q_2 = 1$ and $q_4 = 1$. Go to **Step A.4**.

Step A.4: *Input:* The output of **Step A.3**. Set

$$\underline{q} = \Phi(0,1,0,1) = (3,1,2,1) \quad \text{and} \quad \overline{q} = \Phi(7,1,7,7) = (0,1,1,5)\,.$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\} = \{4\}$. Choose $j = 4$ and, since $4 \in N_u$, set $\beta_4 = q_4 + 1 = 2$ and $A_4 = \{2,3\}$. Agent 4 chooses $a_4 = 2$ because $\tau_4 = 2$. *Output:* $N_u = N_d = \emptyset$, $N_s = N_p = \{2,4\}$, $q_2 = 1$ and $q_4 = 2$. Go to **Step A.5**.

Step A.5: *Input:* The output of **Step A.4**. Set

$$\underline{q} = \Phi(0,1,0,2) = (2,1,2,2) \quad \text{and} \quad \overline{q} = \Phi(7,1,7,2) = (2,1,2,2).$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\}$ is empty (indeed, $\underline{q} = \overline{q}$ and j is selected under Case 2). Since $N \setminus N_p = \{1,3\}$, choose $j = 1$ and set $\beta_1 = \underline{q}_1 = 2$ and $A_1 = \{1,2,3\}$. Agent 1 chooses $a_1 = 3$ because $\tau_1 = 5$. *Output:* $N_u = \{1\}$, $N_d = \emptyset$, $N_s = \{2,4\}$, $N_p = \{1,2,4\}$, $q_1 = 2$, $q_2 = 1$ and $q_4 = 2$.

Step A.6: *Input:* The output of **Step A.5**. Set

$$\underline{q} = \Phi(2,1,0,2) = (2,1,2,2) \quad \text{and} \quad \overline{q} = \Phi(7,1,7,2) = (2,1,2,2).$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\}$ is empty (indeed, $\underline{q} = \overline{q}$ and j is selected under Case 2). Since $N \setminus N_p = \{3\}$, choose $j = 3$ and set $\beta_3 = \underline{q}_3 = 2$ and $A_3 = \{1,2,3\}$. Agent 3 chooses $a_3 = 2$

[17] Whenever there is more than one player that can be chosen to play, we select the player with the smallest index. The outcome is always independent of this choice.

because $\tau_3 = 2$. *Output:* $N_u = \{1\}$, $N_d = \emptyset$, $N_s = \{2,3,4\}$, $N_p = N$, $q_1 = 2$, $q_2 = 1$, $q_3 = 2$ and $q_4 = 2$.

Since $N_p = N$ and $N_d = \emptyset$ stop. The allotment $q = (2,1,2,2)$ is the outcome of the extensive game form $\Gamma^\Phi \in \mathcal{MIG}$ defined by the MIA obtained from Φ when agents play it according to the truth-telling strategy profile σ^τ.

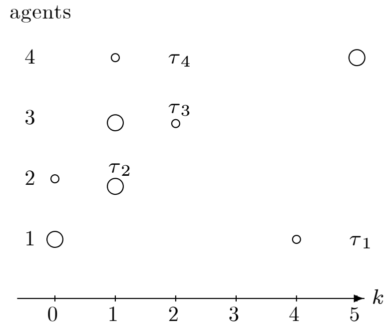

Figure 3.a: **Steps A.1** and **A.2**

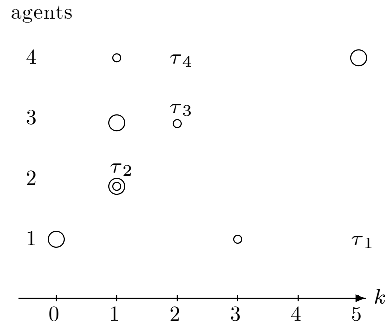

Figure 3.b: **Steps A.3** and **A.4**

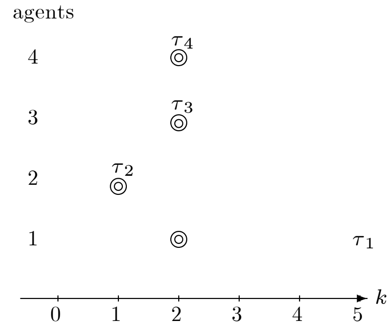

Figure 3.c: **Steps A.5** and **A.6**

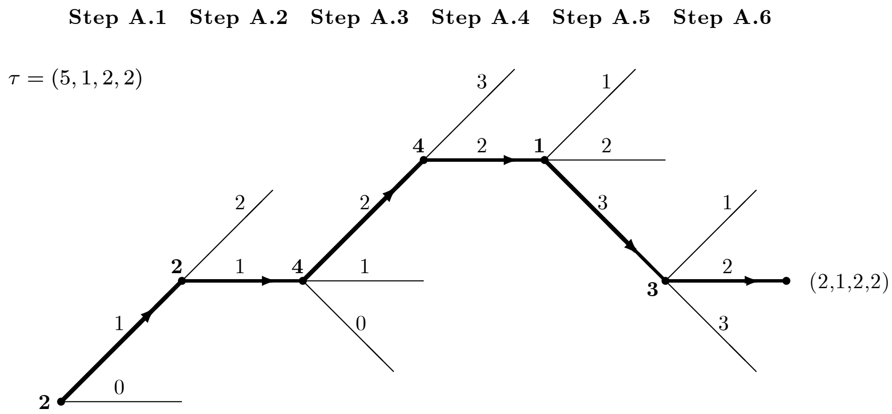

Figure 4: The path of Γ^Φ when agents play it according to σ^τ, where agents are in bold numbers

Consider now the profile of tops $\tau' = (1,1,3,0)$. We run the MIA obtained from Φ when agents play Γ^Φ according to the truth-tell strategy $\sigma^{\tau'}$. We represent below the last four steps of this run of the MIA in Figures 5.a, 5.b and 5.c and the path of Γ^Φ when agents play it according to $\sigma^{\tau'}$ in Figure 6.

Step A.1 and **Step A.2** are as in previous case. *Output:* $N_u = N_d = \emptyset$, $N_s = N_p = \{2\}$ and $q_2 = 1$. Go to **Step A.3**.

Step A.3: *Input:* The output of **Step A.2**. Set

$$\underline{q} = \Phi(0,1,0,0) = (3,1,2,1) \quad \text{and} \quad \overline{q} = \Phi(7,1,7,7) = (0,1,1,5)\,.$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\} = \{4\}$. Choose $j = 4$ and, since $4 \notin N_p$, set $\beta_4 = \underline{q}_4 = 1$ and $A_4 = \{0,1,2\}$. Agent 4 chooses $a_4 = 0$ because $\tau'_4 = 0$. *Output:* $N_u = \emptyset$, $N_d = \{4\}$, $N_s = \{2\}$, $N_p = \{2,4\}$, $q_2 = 1$ and $q_4 = 1$. Go to **Step A.4**.

Step A.4: *Input:* The output of **Step A.3**. Set

$$\underline{q} = \Phi(0,1,0,0) = (3,1,2,1) \quad \text{and} \quad \overline{q} = \Phi(7,1,7,1) = (3,1,2,1).$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\}$ is empty (indeed, $\underline{q} = \overline{q}$ and j is selected under Case 2). Since $N \setminus N_p = \{1,3\}$, choose $j = 1$ and set $\beta_1 = \underline{q}_1 = 3$ and $A_1 = \{2,3,4\}$. Agent 1 chooses $a_1 = 2$ because $\tau'_1 = 1$. *Output:* $N_u = \emptyset$, $N_d = \{1,4\}$, $N_s = \{2\}$, $N_p = \{1,2,4\}$, $q_1 = 3$, $q_2 = 1$ and $q_4 = 1$.

Step A.5: *Input:* The output of **Step A.4**. Set

$$\underline{q} = \Phi(0,1,0,0) = (3,1,2,1) \quad \text{and} \quad \overline{q} = \Phi(3,1,7,1) = (3,1,2,1).$$

Then, $\{i \notin N_s \mid \underline{q}_i < \overline{q}_i\}$ is empty (indeed, $\underline{q} = \overline{q}$ and j is selected under Case 2). Since $N \setminus N_p = \{3\}$, choose $j = 3$ and set $\beta_3 = \underline{q}_3 = 2$ and $A_3 = \{1,2,3\}$. Agent 3 chooses $a_3 = 3$ because $\tau'_3 = 3$. *Output:* $N_u = \{3\}$, $N_d = \{1,4\}$, $N_s = \{2\}$, $N_p = N$, $q_1 = 3$, $q_2 = 1$, $q_3 = 2$ and $q_4 = 1$.

Since $N_p = N$ stop, and as $N_u \neq \emptyset$ and $N_d \neq \emptyset$ go to **Stage B** with input $N_u = \{3\}$, $N_d = \{1,4\}$, $N_s = \{2\}$, and $q = (3,1,2,1)$.

Stage B: Set $\mathbf{t} = 1$ and go to **Step B.1**.

Step B.1: *Input:* The output of **Stage A**. Since $\Phi(2,1,7,0) = (2,1,4,0)$,

$$\{i \in N_u \mid \Phi_i(2,1,7,0) \geq q_i + 1\} = \{3\}$$

and $j = 3 \in N_u$. Since $\Phi(0,1,3,0) = (2,1,3,1)$,

$$\{i \in N_d \mid \Phi_i(0,1,3,0) \leq q_i - 1\} = \{1\}$$

and $r = 1 \in N_d$. Therefore, set $\beta_3 = q_3 + 1 = 3$ and $\beta_1 = q_1 - 1 = 2$. In **Step B.1.a**, agent 3 chooses $a_3 = 3 \in A_3 = \{3,4\}$ because $\tau'_1 = 3$. In **Step B.1.b**, agent 1 chooses $a_1 = 1 \in A_1 = \{1,2\}$ because $\tau'_1 = 1$. *Output:* $N_u = \emptyset$, $N_d = \{1,4\}$, $N_s = \{2,3\}$ and $q = (2,1,3,1)$. Since $N_u = \emptyset$, stop. The allotment $q = (2,1,3,1)$ is the outcome of the extensive game form $\Gamma^\Phi \in \mathcal{MIG}$ defined by the MIA obtained from Φ when agents play it according to the truth-telling strategy profile $\sigma^{\tau'}$.

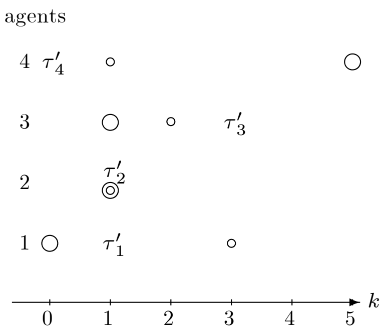

Figure 5.a: **Step A.3**

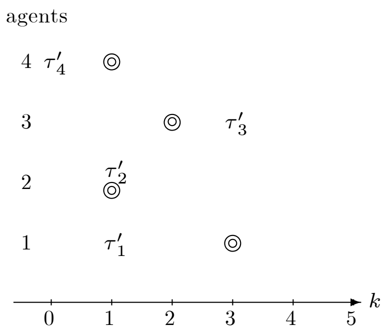

Figure 5.b: **Steps A.4** and **A.5**

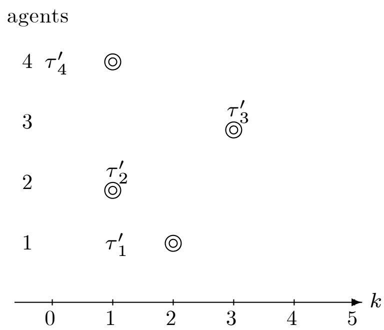

Figure 5.c: **Steps B.1.a** and **B.1.b**

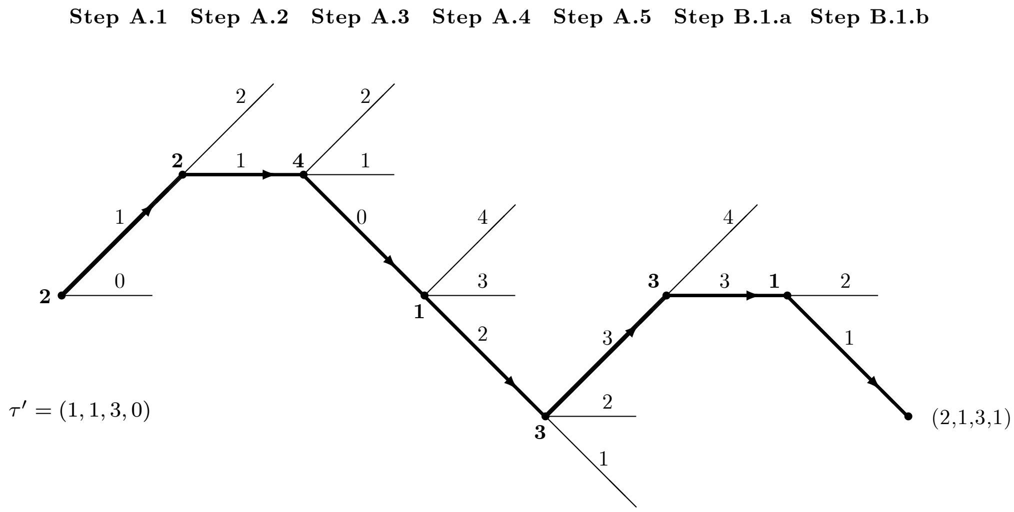

Figure 6: The path of Γ^{Φ} when agents play it according to $\sigma^{\tau'}$, where agents are in bold numbers $\square$

7 Final Remarks

We finish the paper with five remarks.

First, our implementation result requires that the rule be replacement monotonic. Example 2 contains a division problem where there is a strategy-proof, efficient and non-replacement monotonic rule that is not obviously strategy-proof.

Example 2 Consider the division problem where $N = \{1, 2, 3\}$ and $k = 2$. Let $\Psi : \mathcal{R}^N \to X$ be the tops-only rule that, for every $\tau = (\tau_1, \tau_2, \tau_3) \in \{0, 1, 2\}^N$, $\Psi(\tau)$ is determined sequentially. The top of agent 1 determines the order in which agents 2 and 3 have to successively choose their most preferred assignments (among those left available by the predecessor, if any). If agent 1 chooses 0 or 1, then agent 2 moves before 3. If agent 1 chooses 2, then agent 3 moves before 2. Agent 1's assignment is equal to the remainder. Namely,

$$\Psi(\tau_1, \tau_2, \tau_3) = \begin{cases} (2 - \tau_2 - \min\{2 - \tau_2, \tau_3\}, \tau_2, \min\{2 - \tau_2, \tau_3\}) & \text{if } \tau_1 \in \{0, 1\} \\ (2 - \tau_3 - \min\{2 - \tau_3, \tau_2\}, \min\{2 - \tau_3, \tau_2\}, \tau_3) & \text{if } \tau_1 = 2. \end{cases}$$

It is easy to check that Ψ is strategy-proof and efficient. To see that Ψ is not replacement monotonic, consider $\tau = (\tau_1, \tau_2, \tau_3) = (0, 1, 2)$ and $\tau' = (\tau_1', \tau_2, \tau_3) = (2, 1, 2)$. Then, $\Psi(\tau) = (0, 1, 1)$ and $\Psi(\tau') = (0, 0, 2)$. Since $\Psi_1(\tau) = \Psi_1(\tau')$, $\Psi_2(\tau) > \Psi_2(\tau')$ and $\Psi_3(\tau) < \Psi_3(\tau')$, Ψ is not replacement monotonic.

To obtain a contradiction, assume Ψ is obviously strategy-proof. Let Γ be the extensive game form that OSP-implements Ψ. Given a profile of tops τ, let $\sigma^{\tau} = (\sigma_1^{\tau_1}, \sigma_2^{\tau_2}, \sigma_3^{\tau_3})$ be a strategy profile such that $\Psi(\tau) = g(\sigma^{\tau})$. As Γ induces Ψ, there must exists a non-terminal node ν such that (i) the agent who moves at ν has at least two available actions (denoted by a^1 and a^2) and (ii) at all nodes preceding ν (if any) the agents who play have only one available action. Suppose agent 1 is who moves at ν. Consider the two profiles of tops $\tau = (1, 0, 0)$ and $\tau' = (2, 1, 0)$. As Γ induces Ψ, $g_1(\sigma^{\tau}) = \Psi_1(\tau) = 2$ and $g_1(\sigma^{\tau'}) = \Psi_1(\tau') = 1$. Consider σ_2 and σ_3 with the properties that (i) they respectively coincide with $\sigma_2^{\tau_2}$ and $\sigma_3^{\tau_3}$

at all nodes that follow ν after agent 1 chooses a^1 and (ii) they respectively coincide with $\sigma_2^{\tau_2'}$ and $\sigma_3^{\tau_3'}$ at all nodes that follow ν after agent 1 chooses a^2. Note that by its definition, node ν is reached regardless of the strategy profile used by the agents and, since Γ has perfect information, σ_2 and σ_3 are well defined. Because there exists $R_1 \in \mathcal{R}$ such that $\tau(R_1) = 1$ and

$$g_1(\sigma_1^{\tau_1'}, \sigma_2, \sigma_3) = 1 P_1 2 = g_1(\sigma_1^{\tau_1}, \sigma_2, \sigma_3),$$

strategy $\sigma_1^{\tau_1}$ is not weakly dominant in Γ, a contradiction with the assumption that Γ OSP-implements Ψ. Suppose agent 2 is who moves at ν. Consider the two profiles of tops $\tau = (2,1,2)$ and $\tau' = (1,1,2)$. As Γ induces Ψ, $g_2(\sigma^\tau) = \Psi_2(\tau) = 0$ and $g_2(\sigma^{\tau'}) = \Psi_2(\tau') = 1$. Consider σ_1 and σ_3 with the properties that (i) they respectively coincide with $\sigma_1^{\tau_1}$ and $\sigma_3^{\tau_3}$ at all nodes that follow ν after agent 2 chooses a^1 and (ii) they respectively coincide with $\sigma_1^{\tau_1'}$ and $\sigma_3^{\tau_3'}$ at all nodes that follow ν after agent 2 chooses a^2. Note that by its definition, node ν is reached regardless of the strategy profile used by the agents and, since Γ has perfect information, σ_1 and σ_3 are well defined. Because there exists $R_2 \in \mathcal{R}$ such that $\tau(R_2) = 1$ and

$$g_2(\sigma_1, \sigma_2^{\tau_2'}, \sigma_3) = 1 P_2 0 = g_2(\sigma_1, \sigma_2^{\tau_2}, \sigma_3),$$

strategy $\sigma_2^{\tau_2}$ is not weakly dominant in Γ, a contradiction with the assumption that Γ OSP-implements Ψ. A similar argument can be used to obtain a contradiction when 3 is the agent who moves at ν. $\square$

Second, there are strategy-proof and efficient rules that are not replacement monotonic (and so, they are not sequential), but they are obviously strategy-proof. Example 3 illustrates this possibility

Example 3 Consider the division problem where $N = \{1,2,3\}$ and $k = 2$. Let $\varphi : \mathcal{R}^N \to X$ be the tops-only rule that, for every $\tau = (\tau_1, \tau_2, \tau_3) \in \{0,1,2\}^N$, $\varphi(\tau)$ is determined sequentially. Agent 1 receives his/her top. If $\tau_1 = 0$, agent 2 receives τ_2 and agent 3 receives $2 - \tau_2$. If $\tau_1 \in \{1,2\}$, agent 3 receives his/her best assignment in $[0, 2-\tau_1]$, denoted by τ_3^{rest}, and agent 2 receives $2 - \tau_1 - \tau_3^{rest}$. Namely,

$$\varphi(\tau_1,\tau_2,\tau_3) = \begin{cases} (0, \tau_2, 2-\tau_2) & \text{if } \tau_1 = 0 \\ (\tau_1, 2-\tau_1-\tau_3^{rest}, \tau_3^{rest}) & \text{if } \tau_1 \in \{1,2\}. \end{cases}$$

It is easy to check that φ is strategy-proof and efficient. To see that φ is not replacement monotonic, consider $\tau = (\tau_1,\tau_2,\tau_3) = (0,2,2)$ and $\tau' = (\tau_1', \tau_2, \tau_3) = (1,2,2)$. Then, $\varphi(\tau) = (0,2,0)$ and $\varphi(\tau') = (1,0,1)$. Since $\varphi_1(\tau) < \varphi_1(\tau')$, $\varphi_2(\tau) > \varphi_2(\tau')$ and $\varphi_3(\tau) < \varphi_3(\tau')$, φ is not replacement monotonic.

However, φ is obviously strategy-proof. The extensive game form depicted in Figure 7 OSP-implements φ, where agents are in bold numbers. Together, Examples 2 and 3 show that while replacement monotonicity is indispensable for our main result to hold, it is not necessary.

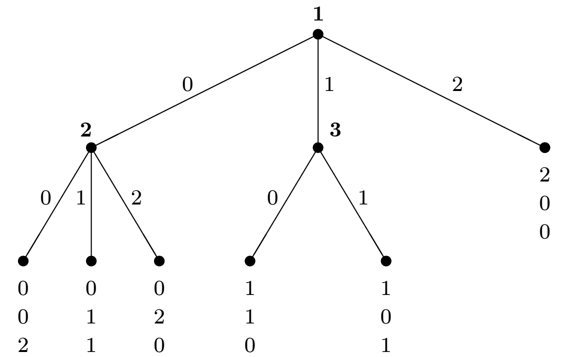

Figure 7: The extensive game form that OSP-implements φ □

Third, Pycia and Troyan (2020) propose a strengthening of obvious strategy-proofness, called strong obvious strategy-proofness (SOSP). In our context, a rule $\Phi : \mathcal{R}^N \to X$ is *strongly obviously strategy-proof* if there is an extensive game form $\Gamma \in \mathcal{G}$ associated to k and N such that Γ induces φ (*i.e.*, for each $R \in \mathcal{R}^N$ there exists a strategy profile $\sigma^R = (\sigma_1^{R_1}, \ldots, \sigma_n^{R_n}) \in \Sigma$ such that $\Phi(R) = g(\sigma^R)$) and, for all $i \in N$ and $R_i \in \mathcal{R}$, $\sigma_i^{R_i}$ is strongly obviously dominant in Γ at R_i, where the later condition requires that when comparing the worst possible outcome of the choice prescribed by $\sigma_i^{R_i}$ at an earliest point of departure ν with any other σ_i, the choices made by i at all nodes that follow $\sigma_i^{R_i}(\nu)$ do not have to be truth-telling any more since i may now choose, at a node γ that follows $\sigma_i^{R_i}(\nu)$, a different action to $\sigma_i^{R_i}(\gamma)$. Therefore, the worst possible outcome associated to the stronger OSP notion could be strictly worse than the one obtained when agent i is required to stay with the truth-telling strategy, as required by the original Li (2017)'s OSP notion. Example 4 below shows that not all sequential allotment rules are strongly obviously strategy-proof. However, the subclass of sequential dictators (that can be described as sequential allotment rules) satisfy the stronger requirement since agents play only once. In light of Theorem 5 in Pycia and Troyan (2020), the class of all efficient and strongly obviously strategy-proof rules coincides with the class of all sequential dictator rules.

Example 4 Consider the division problem where $N = \{1, 2, 3\}$ and $k = 3$. Let $\psi : \mathcal{R}^N \to X$ be any individually rational sequential allotment rule with respect to the allotment $q = (1, 1, 1)$.[18] To obtain a contradiction, assume that Γ is an extensive game form that SOSP-implements ψ. Let ν be the node in Γ at which for the first time a player has available at least two actions. Without loss of generality, let 1 be such agent. Fix an arbitrary $R_1 \in \mathcal{R}$ and let a^ℓ be the action such that $\sigma_1^{R_1}(\nu) = a^\ell$ with $\tau(R_1) = \ell$. Since ψ is tops-only, it is sufficient to distinguish between two different general cases.

<u>Case 1</u>: Assume $a^2 \neq a^3$. Since φ is efficient, $\psi(2, 1, 0) = (2, 1, 0)$ and, because Γ induces ψ, the allotment $(2, 1, 0)$ is possible after 1 chooses a^2 at ν. Since ψ is efficient, $\psi(1, 1, 1) = (1, 1, 1)$ and, by individual rationality, $\psi(3, 1, 1) = (1, 1, 1)$. Then, the allotment $(1, 1, 1)$ is

[18]Note that this implies that ψ is not a sequential dictator rule.

possible after the choice a^3. However, for all single-peaked preference $R_1 \in \mathcal{R}$ with $\tau(R_1) = 3$, $2P_1 1$. Hence, ψ is not SOSP.

Case 2: Assume $a^2 = a^3$. We refer to this action as $a^{2,3}$. We distinguish between two subcases.

Case 2.1: Assume $a^1 \neq a^{2,3}$. Then, using similar arguments to those used in Case 1, the allotment $(1,1,1)$ is possible after 1 chooses a^1 and the allotment $(3,0,0)$ is possible after 1 chooses $a^{2,3}$. However, there is a single-peaked preference $R_1 \in \mathcal{R}$ with $\tau(R_1) = 2$, for which $1P_1 3$. Hence, ψ is not SOSP.

Case 2.2: Assume $a^1 = a^{2,3}$. We refer to this action as $a^{1,2,3}$. We distinguish between two further subcases.

Case 2.2.1: Assume $a^0 \neq a^{1,2,3}$. Then, using similar arguments to those used in Case 1, the allotment $(0,3,0)$ is possible after 1 chooses a^0 while the allotment $(3,0,0)$ is possible after 1 chooses $a^{1,2,3}$. However, there is a single-peaked preference $R_1 \in \mathcal{R}$ with $\tau(R_1) = 2$, for which $0P_1 3$. Hence, ψ is not SOSP.

Case 2.2.2: Assume $a^0 = a^{1,2,3}$. But this means that agent 1 has a unique available action at ν. A contradiction. $\square$

Fourth, Barberà, Jackson and Neme (1997) observe that each sequential allotment rule is fully implementable in dominant strategies by the direct revelation mechanism. It is easy to see that our extensive game forms provide *full* OSP-implementation of all sequential allotment rules. Namely, for each sequential allotment rule, the extensive game form defined by the MIA obtained from the rule has the property that, for each preference profile, each obviously dominant strategy profile leads to the allotment specified by the rule for that preference profile. Moreover, they provide ex-post perfect and full subgame perfect implementation of the rules (see Mackenzie and Zhou (2020)).

Fifth, our extensive game forms are based on the discrete version of Sprumont (1991)'s continuous model. An OSP-implementation of any sequential allotment rule in the continuous version of the division problem should deal with the technical difficulties that may arise in games in extensive form where agents play in a continuous fashion (see for instance Alós-Ferrer and Ritzberger (2013)). For simplicity, we have decided to undertake our analysis in the discrete division problem, first studied by Herrero and Martínez (2011).

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Appendix: Proofs

We start with a remark that will be intensively used in the proofs that follow.

Remark 2 Let $\Phi : \{0, \ldots, k\}^N \to X$ be a sequential allotment rule. Then, for all $\tau \in \{0, \ldots, k\}^N$, $i \in N$ and $\tau'_i \in \{0, \ldots, k\}$, the following two statements hold.

(R2.1) If $\Phi_i(\tau) \geq \tau'_i$, then $\Phi_i(\tau) \geq \Phi_i(\tau'_i, \tau_{-i}) \geq \tau'_i$ and $\Phi_j(\tau) \leq \Phi_j(\tau'_i, \tau_{-i})$ for all $j \in N\backslash\{i\}$.

To see that (R2.1) holds, assume first that $\Phi_i(\tau) = \tau'_i$. Then, by strategy-proofness, $\Phi_i(\tau) = \Phi(\tau'_i, \tau_{-i})$ and, by replacement monotonicity, $\Phi_j(\tau) = \Phi_j(\tau'_i, \tau_{-i})$ for all $j \in N\backslash\{i\}$. Assume now that $\Phi_i(\tau) > \tau'_i$. To obtain a contradiction, suppose that either (i) $\Phi_i(\tau'_i, \tau_{-i}) > \Phi_i(\tau) > \tau'_i$ or (ii) $\Phi_i(\tau) > \tau'_i > \Phi_i(\tau'_i, \tau_{-i})$ hold. By single-peakedness, (i) contradicts that Φ is strategy-proof. Suppose (ii) holds. Then, there is $R''_i \in \mathcal{R}$ such that $\tau(R''_i) = \tau''_i = \tau'_i$ and $\Phi_i(\tau)\ P''_i\ \Phi_i(\tau'_i, \tau_{-i})$. By tops-onlyness, $\Phi_i(\tau)\ P''_i\ \Phi_i(\tau''_i, \tau_{-i})$ holds, which contradicts that Φ is strategy-proof. Hence, $\Phi_i(\tau) \geq \Phi_i(\tau'_i, \tau_{-i}) \geq \tau'_i$. By replacement monotonicity, $\Phi_j(\tau) \leq \Phi_j(\tau'_i, \tau_{-i})$ for all $j \in N \setminus \{i\}$.

(R2.2) If $\Phi_i(\tau) \leq \tau'_i$, then $\Phi_i(\tau) \leq \Phi(\tau'_i, \tau_{-i}) \leq \tau'_i$ and $\Phi_j(\tau) \geq \Phi_j(\tau'_i, \tau_{-i})$ for all $j \in N\backslash\{i\}$.

A symmetric argument to the one used in (R2.1) shows that (R2.2) holds.

Lemma 1 *Let $\Phi : \mathcal{R}^N \to X$ be a non individually rational sequential allotment rule. Let N_u, N_d, N_s, N_p and $(q_i)_{i\in N_p}$ be the input of* **Step A.t** *of the* MIA *obtained from Φ, and let*

$$\underline{q} = \Phi(\mathbf{0}_{-(N_s\cup N_u)}, q_{N_s\cup N_u}) \quad \textit{and} \quad \overline{q} = \Phi(\mathbf{k}_{-(N_s\cup N_d)}, q_{N_s\cup N_d}). \tag{9}$$

Then, the following four conditions hold.

(L1.1) $\sum_{i\in N_p} q_i \leq k$.

(L1.2) *If* $i \in N_s$ *then* $q_i = \underline{q}_i = \overline{q}_i$.

(L1.3) *If* $i \in N_u$ *then* $q_i = \underline{q}_i \leq \overline{q}_i$.

(L1.4) *If* $i \in N_d$ *then* $\underline{q}_i \leq \overline{q}_i = q_i$.

Proof We proceed by induction on $\mathbf{t}$. When $\mathbf{t} = 1$ the four statements hold trivially because $N_u = N_d = N_s = N_p = \emptyset$. Suppose $\mathbf{t} \geq 2$.

Induction Hypothesis (IH): Let N'_u, N'_d, N'_s and $(q'_i)_{i\in N'_p}$ be the input of **Step A.t-1** of the MIA obtained from Φ, and let

$$\underline{q}' = \Phi(\mathbf{0}_{-(N'_s\cup N'_u)}, q'_{N'_s\cup N'_u}) \quad \text{and} \quad \overline{q}' = \Phi(\mathbf{k}_{-(N'_s\cup N'_d)}, q'_{N'_s\cup N'_d}). \tag{10}$$

Then, the following four conditions hold.

(IH.L1.1) $\sum_{i\in N_p'} q_i' \leq k$.
(IH.L1.2) If $i \in N_s'$ then $q_i' = \underline{q}_i' = \overline{q}_i'$.
(IH.L1.3) If $i \in N_u'$ then $q_i' = \underline{q}_i' \leq \overline{q}_i'$.
(IH.L1.4) If $i \in N_d'$ then $\underline{q}_i' \leq \overline{q}_i' = q_i'$.

Let j be the agent that was called to play at **Step A.t-1** and let N_u, N_d, N_s, N_p and $(q_i)_{i\in N_p}$ be the input of **Step A.t.** By the definition of the MIA and the (IH),

$$N_p = N_p' \cup \{j\} \text{ and } q_i = q_i' \text{ for all } i \in N_p \setminus \{j\}. \tag{11}$$

We distinguish between two cases, which corresponds to the two possible ways in which player j is selected to play at **Step A.t**.

Case 1: There exists $i \notin N_s'$ such that $\underline{q}_i' < \overline{q}_i'$. Then, j is one of such agents and

$$\underline{q}_j' < \overline{q}_j'. \tag{12}$$

First, we show that (L1.1) holds. By the (IH) and (11), $q_i \leq \overline{q}_i'$ for all $i \in N_p \setminus \{j\}$. Now we show that $q_j \leq \overline{q}_j'$. If $j \in N_d'$, by the definitions of β_j and q_j, $q_j = q_j' - 1 < q_j'$. Then, by (IH.L1.4), $q_j = q_j' - 1 < \overline{q}_j'$. If $j \in N_u'$, by the definitions of β_j and q_j, $q_j = q_j' + 1$. Then, by (IH.L1.3) and (12), $q_j = q_j' + 1 = \underline{q}_j' + 1 \leq \overline{q}_j'$. If $j \notin N_p'$, by the definitions of β_j and q_j, and (12), $q_j = \underline{q}_j' < \overline{q}_j'$. Therefore, $q_i \leq \overline{q}_i'$ for all $i \in N_p$. Then, by feasibility of $\overline{q}'$,

$$\textstyle\sum_{i\in N_p} q_i \leq \sum_{i\in N_p} \overline{q}_i' \leq k,$$

which is (L1.1). To prove that the other three statements hold, we divide Case 1 in three cases, depending on weather j belongs to N_u', N_d' or $-N_p'$. But before doing so, we state two general observations. As N_u, N_d, N_s, N_p is an input of **Step A.t**, $N_p \neq N$ and accordingly, $N_s \cup N_u \neq N$ and $N_s \cup N_d \neq N$. Furthermore, we have just shown that $\sum_{i\in N_p} q_i \leq k$ in (L1.1) holds Then, by (9), and (1) and (2) in the definition of same-sidedness,

$$q_i \leq \Phi_i(\mathbf{0}_{-(N_s\cup N_u)}, q_{N_s\cup N_u}) = \underline{q}_i \text{ for all } i \in N_s \cup N_u \tag{13}$$

and

$$\overline{q}_i = \Phi_i(\mathbf{k}_{-(N_s\cup N_d)}, q_{N_s\cup N_d}) \leq q_i \text{ for all } i \in N_s \cup N_d. \tag{14}$$

Case 1.a: $j \in N_u'$. By the definition of the MIA obtained from Φ and (IH.L1.3),

$$\begin{aligned} &q_j = q_j' + 1 = \underline{q}_j' + 1, \\ &j \in N_s \cup N_u = N_s' \cup N_u' \text{ and } j \notin N_s' \cup N_d'. \end{aligned} \tag{15}$$

Claim 1 $\overline{q}_i \geq \overline{q}_i'$ for all $i \neq j$, $\overline{q}_i = \overline{q}_i'$ for all $i \in (N_s \cup N_d) \setminus \{j\}$ and $\overline{q}_j \leq \overline{q}_j'$.

Proof of Claim 1 We distinguish between two cases.

Case C1.1: $j \in N_u$. Then, $j \notin N_s \cup N_d = N_s' \cup N_d'$ and, by (11), $(\mathbf{k}_{-(N_s\cup N_d)}, q_{N_s\cup N_d}) = (\mathbf{k}_{-(N_s'\cup N_d')}, q'_{N_s'\cup N_d'})$. Therefore, by (10) in the (IH) and (9), $\overline{q} = \overline{q}'$, which means that Claim 1 holds in this case.

Case C1.2: $j \in N_s$. By (R2.1), (9), (10) in the (IH), (11), (12) and (15),

$$\overline{q}_j = \Phi_j(\mathbf{k}_{-(N_s \cup N_d)}, q_{N_s \cup N_d}) \leq \Phi_j(\mathbf{k}_{-(N'_s \cup N'_d)}, q'_{N'_s \cup N'_d}) = \overline{q}'_j$$

and $\overline{q}_i \geq \overline{q}'_i$ for all $i \neq j$. Hence, by (IH.L1.2), (IH.L1.4), (11), and (14), $\overline{q}_i \geq \overline{q}'_i = q'_i = q_i \geq \overline{q}_i$ holds for all $i \in (N_s \cup N_d) \setminus \{j\}$, which means that Claim 1 holds in this case. $\square$

By (13) and (15),

$$\underline{q}'_j < q_j \leq \underline{q}_j. \tag{16}$$

By replacement monotonicity,

$$\underline{q}'_i \geq \underline{q}_i \text{ for all } i \in N \setminus \{j\}. \tag{17}$$

By (11), (IH.L1.2), (IH.L1.3), (IH.L1.4) and Claim 1,

$$\begin{aligned} q_i = q'_i = \overline{q}'_i = \overline{q}_i &\text{ if } i \in (N_s \cup N_d) \setminus \{j\} \text{ and} \\ q_i = q'_i \leq \overline{q}'_i \leq \overline{q}_i &\text{ if } i \in N_u \setminus \{j\}. \end{aligned}$$

By (11), (IH.L1.2), (IH.L1.3), (IH.L1.4), (17), and (13),

$$\begin{aligned} q_i = q'_i = \underline{q}'_i \geq \underline{q}_i \geq q_i &\text{ if } i \in (N_s \cup N_u) \setminus \{j\} \text{ and} \\ q_i = q'_i \geq \underline{q}'_i \geq \underline{q}_i &\text{ if } i \in N_d. \end{aligned}$$

Therefore, (L1.2), (L1.3) and (L1.4) in Lemma 1 hold for all $i \neq j$. Now, we show that they also hold for j. First we show that $\underline{q}_j = q_j$. By (15), $j \in N_s \cup N_u$. By (11) and (2) in the definition of same-sidedness, $\Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{(N_s \cup N_u) \setminus \{j\}}, q'_j) \geq q'_j$. By strategy-proofness and single-peakedness,

$$\Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{(N_s \cup N_u) \setminus \{j\}}, q'_j) + 1 \geq \Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{(N_s \cup N_u) \setminus \{j\}}, q'_j + 1), \tag{18}$$

because otherwise j would manipulate Φ at $(\mathbf{0}_{-(N_s \cup N_u)}, q_{(N_s \cup N_u) \setminus \{j\}}, q'_j + 1)$ via q'_j. By (10) (11), (15), (18) and (9),

$$\underline{q}'_j + 1 = \Phi_j(\mathbf{0}_{-(N'_s \cup N'_u)}, q_{(N'_s \cup N'_u) \setminus \{j\}}, q'_j) + 1 \geq \Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{(N_s \cup N_u) \setminus \{j\}}, q_j) = \underline{q}_j, \tag{19}$$

that together with (16) imply $\underline{q}_j = \underline{q}'_j + 1$. By (15), $q_j = q'_j + 1 = \underline{q}'_j + 1 = \underline{q}_j$, which is the equality in (L1.3) and the first one in (L1.2). In order to prove that the inequality $q_j \leq \overline{q}_j$ in (L1.3), and the equality $q_j = \overline{q}_j$ in (L1.2) also hold, we distinguish between two cases.

Case 1a.1: $j \in N_u$. As in the proof of Case C1.1 in Claim 1, we obtain that $\overline{q}_j = \overline{q}'_j$ holds. Then, by (12), (IH.L1.3) and (15), $\overline{q}_j = \overline{q}'_j \geq \underline{q}'_j + 1 = q'_j + 1 = q_j$.

Case 1a.2: $j \in N_s$. By (15), (12) and (10) in the (IH), $q_j = \underline{q}'_j + 1 \leq \overline{q}'_j = \Phi_j(\mathbf{k}_{-(N'_s \cup N'_d)}, q'_{N'_s \cup N'_d})$, which together with (9), $j \notin N'_s \cup N'_d$, (11) and (R2.1) imply

$$\overline{q}_j = \Phi_j(\mathbf{k}_{-(N_s \cup N_d)}, q_{N_s \cup N_d}) = \Phi_j(\mathbf{k}_{-(N'_s \cup N'_d \cup \{j\})}, q'_{N'_s \cup N'_d}, q_j) \geq q_j. \tag{20}$$

Conditions (14) and (20) imply $\overline{q}_j = q_j$.

Case 1.b: $j \in N'_d$. The proof that (L1.2), (L1.3) and (L1.4) hold in this case is symmetric to the one used in Case 1.a (when $j \in N'_u$), after replacing Claim 1 by Claim 2 below.

CLAIM 2 $\underline{q}_i \leq \underline{q}'_i$ for all $i \neq j$, $\underline{q}_i = \underline{q}'_i$ for all $i \in (N_s \cup N_u) \setminus \{j\}$ and $\underline{q}_j \geq \underline{q}'_j$.

Case 1.c: $j \notin N'_p$. By the definition of the MIA obtained from Φ and (10) in the (IH),

$$q_j = \underline{q}'_j = \Phi_j(\mathbf{0}_{-(N'_s \cup N'_u)}, q'_{N'_s \cup N'_u}) \text{ and } j \notin N'_u \cup N'_d \cup N'_s. \tag{21}$$

CLAIM 3 $\overline{q}_i \geq \overline{q}'_i$ for all $i \neq j$, $\overline{q}_i = \overline{q}'_i$ for all $i \in (N_s \cup N_d) \setminus \{j\}$ and $\overline{q}_j \leq \overline{q}'_j$.

PROOF OF CLAIM 3 The proof follows similar arguments to those already used in the proof of Claim 1, and therefore it is omitted. □

We now show that $\Phi(\mathbf{0}_{-(N'_s \cup N'_u)}, q'_{N'_s \cup N'_u}) = \Phi(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u})$ holds. Suppose $j \in N_d$. By (11), the equality holds. Suppose $j \in N_s \cup N_u$. By (21) and (R2.1), $\Phi_j(\mathbf{0}_{-(N'_s \cup N'_u)}, q'_{N'_s \cup N'_u}) = q_j = \underline{q}'_j \leq \Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u})$. By (2) in the definition of same-sidedness, $\Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u}) \leq q_j$. Hence, $\Phi_j(\mathbf{0}_{-(N'_s \cup N'_u)}, q'_{N'_s \cup N'_u}) = \Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u})$. By replacement monotonicity, $\Phi_i(\mathbf{0}_{-(N'_s \cup N'_u)}, q'_{N'_s \cup N'_u}) = \Phi_i(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u})$ for all $i \neq j$. Hence,

$$\underline{q}' = \Phi(\mathbf{0}_{-(N'_s \cup N'_u)}, q'_{N'_s \cup N'_u}) = \Phi(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u}) = \underline{q}. \tag{22}$$

By (11), (IH.L1.2), (IH.L1.3), (IH.L1.4) and CLAIM 3,

$$\begin{aligned} q_i &= q'_i = \overline{q}'_i = \overline{q}_i \text{ if } i \in (N_s \cup N_d) \setminus \{j\} \text{ and} \\ q_i &= q'_i \leq \overline{q}'_i \leq \overline{q}_i \text{ if } i \in N_u \setminus \{j\}. \end{aligned}$$

By (11), (IH.L1.2), (IH.L1.3), (IH.L1.4) and (22),

$$\begin{aligned} q_i &= q'_i = \underline{q}'_i = \underline{q}_i \text{ if } i \in (N_s \cup N_u) \setminus \{j\} \text{ and} \\ q_i &= q'_i \geq \underline{q}'_i = \underline{q}_i \text{ if } i \in N_d \setminus \{j\}. \end{aligned}$$

Therefore, (L1.2), (L1.3) and (L1.4) in Lemma 1 hold for all $i \neq j$. Now, we show that they also hold for j. By (21) and (22),

$$q_j = \underline{q}'_j = \underline{q}_j. \tag{23}$$

We distinguish between two possibilities, depending on the set of agents in the output of **Step A.t** to which j belongs to.

Case 1c.1: $j \in N_u$. By using a similar argument to the one used in the proof of Case C1.1 in CLAIM 1, $\overline{q}_j = \overline{q}'_j$. Moreover, by (21) and (12) $q_j = \underline{q}'_j < \overline{q}'_j = \overline{q}_j$. By (22), $q_j = \underline{q}_j < \overline{q}_j$ which implies (L1.3).

Case 1c.2: $j \in N_s \cup N_d$. By (21), (12) and (10) in the (IH), $q_j = \underline{q}'_j < \overline{q}'_j = \Phi_j(\mathbf{k}_{-(N'_s \cup N'_d)}, q'_{N'_s \cup N'_d})$. By (R2.1), (9) and the fact that $j \notin N'_s \cup N'_d$,

$$q_j \leq \Phi_j(\mathbf{k}_{-(N'_s \cup N'_d \cup \{j\})}, q'_{N'_s \cup N'_d}, q_j) \leq \Phi_j(\mathbf{k}_{-(N_s \cup N_d)}, q_{N_s \cup N_d}) = \overline{q}_j. \tag{24}$$

Then, (24) and (14) imply $q_j = \overline{q}_j$, which together with (23) imply $q_j = \underline{q}_j = \overline{q}_j$. But this is (L1.2) if $j \in N_s$ or implies (L1.4) if $j \in N_d$.

Case 2: There does not exist $i \notin N_s'$ such that $\underline{q}_i' < \overline{q}_i'$. Let $j \in N \setminus N_p$ be the agent selected to play at **Step A.t** and let $\underline{q}_j$ be j's guaranteed assignment. By the definition of the MIA obtained from Φ, (IH.L1.2), (IH.L1.3), (IH.L1.4), (10) in the (IH), (11) and the condition defining Case 2 (*i.e.*, $\underline{q}_i' \geq \overline{q}_i'$ for all $i \notin N_s'$),

$$\begin{aligned} &q_j = \underline{q}_j = \underline{q}_j' = \Phi_j(\mathbf{0}_{-(N_s' \cup N_u')}, q_{N_s' \cup N_u'}') \text{ and} \\ &q_i = q_i' = \overline{q}_i' = \underline{q}_i' \text{ for all } i \in N_p \setminus \{j\}. \end{aligned} \tag{25}$$

Therefore, $q_i = \underline{q}_i'$ for all $i \in N_p$. By feasibility of $\underline{q}'$, $\sum_{i \in N_p} q_i = \sum_{i \in N_p} \underline{q}_i' \leq k$ which is (L1.1).

Claim 4 $\overline{q}' = \underline{q}'$.

Proof of Claim 4 Assume otherwise. By the feasibility of $\overline{q}'$ and $\underline{q}'$, there exists $i \in N$ such that $\overline{q}_i' < \underline{q}_i'$. Then, by (25), there exists $i \notin N_p \setminus \{j\} = N_p'$ such that $\underline{q}_i' < \overline{q}_i'$, which by (IH.L1.2) contradicts the hypothesis of Case 2. Therefore, $\overline{q}_i' \geq \underline{q}_i'$ for all $i \in N$. Then, by feasibility of $\overline{q}'$ and $\underline{q}'$, $\overline{q}' = \underline{q}'$. □

We now show that (L1.2), (L1.3) and (L1.4) hold. By (R2.1), (R2.2), (25), (9) and (11),

$$\underline{q}_j' = \Phi_j(\mathbf{0}_{-(N_s' \cup N_u')}, q_{N_s' \cup N_u'}') = q_j = \Phi_j(\mathbf{0}_{-(N_s \cup N_u)}, q_{N_s \cup N_u}) = \underline{q}_j \tag{26}$$

and

$$\underline{q}_i' = \underline{q}_i \text{ for all } i \in N. \tag{27}$$

By (26), Claim 4, and (10), $q_j = \overline{q}_j' = \Phi_j(\mathbf{k}_{-(N_s' \cup N_d')}, q_{N_s' \cup N_d'}')$. Then, (R2.1) and (11) imply

$$q_j = \Phi_j(\mathbf{k}_{-(N_s' \cup N_d')}, q_{N_s' \cup N_d'}') = \Phi_j(\mathbf{k}_{-(N_s \cup N_d)}, q_{N_s \cup N_d}) \tag{28}$$

and

$$\overline{q}_i' = \overline{q}_i \text{ for all } i \in N. \tag{29}$$

From (25), (26), (27), (28) and (29) it follows that $q_i = \overline{q}_i = \underline{q}_i$ for all $i \in N_p$. Hence, (L1.2), (L1.3) and (L1.4) hold. ■

Lemma 2 *Let $\Phi : \mathcal{R}^N \to X$ be a non individually rational sequential allotment rule. Let N_u', N_d', N_s', N_p' and $(q_i')_{i \in N_p'}$ be the input of* **Step A.t** *of the* MIA *obtained from Φ and let N_u, N_d, N_s, N_p and $(q_i)_{i \in N_p}$ be the output of its* **Step A.t** *and* **Stage A** *(i.e., $N_p = N$). Then, $q = \underline{q}' = \overline{q}'$ and $\sum_{i \in N} q_i = k$.*

Proof Let j be the agent called to play at **Step A.t**. By hypothesis, this means that $\{j\} = N \setminus N_p'$. By the definition of the MIA obtained from Φ, $q_i = q_i'$ for all $i \neq j$. By Lemma 1,

$$\underline{q}_i' \leq q_i \leq \overline{q}_i' \text{ for all } i \neq j. \tag{30}$$

Then, $\sum_{i \in N \setminus \{j\}} \underline{q}_i' \leq \sum_{i \in N \setminus \{j\}} q_i \leq \sum_{i \in N \setminus \{j\}} \overline{q}_i'$. The feasibility of $\underline{q}'$ and $\overline{q}'$ imply that $\underline{q}_j' \geq \overline{q}_j'$. Because agent j is called to play at **Step A.t**, $\underline{q}_i' \geq \overline{q}_i'$ for all $i \notin N_s'$. Furthermore, by Lemma 1, $\underline{q}_i' = \overline{q}_i'$ for all $i \in N_s'$. Then, by the definition of the MIA obtained from Φ, $q_j = \underline{q}_j'$. The feasibility of $\underline{q}'$ and $\overline{q}'$ and (30) imply that $q = \underline{q}' = \overline{q}'$ and $\sum_{i \in N} q_i = k$. ■

Lemma 3 *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule and N_u, N_d, N_s and q be the output of* **Stage A** *of the* MIA *obtained from Φ. Then,*

$$\Phi(\mathbf{0}_{N_d}, q_{N_u}, q_{N_s}) = \Phi(\mathbf{k}_{N_u}, q_{N_d}, q_{N_s}) = q.$$

Proof We distinguish between two cases, depending on whether or not Φ is individually rational.

Case 1: Φ is not individually rational. Let N_u, N_d, N_s and q be the output of **Step A.t**, the last step of **Stage A** of the MIA obtained from Φ. Let N'_u, N'_d, N'_s and $(q'_i)_{i\in N'_p}$ be the input of **Step A.t** and let j be the agent that is called to play at **Step A.t**. Accordingly, $N'_p = N \setminus \{j\}$. By the definition of the MIA obtained from Φ, $q'_i = q_i$ for all $i \neq j$. Then, by Lemma 2 and $j \notin N'_p$,

$$\Phi(\mathbf{0}_{-(N'_s\cup N'_u)}, q_{N'_s\cup N'_u}) = \underline{q}' = q = \overline{q}' = \Phi(\mathbf{k}_{-(N'_s\cup N'_d)}, q_{N'_s\cup N'_d}). \tag{31}$$

If $j \in N_d$ then $N'_s \cup N'_u = N_s \cup N_u$ holds and (31) imply $\Phi(\mathbf{0}_{-(N_s\cup N_u)}, q_{N_s\cup N_u}) = q$. If $j \in N_u$ then $N'_s \cup N'_d = N_s \cup N_d$ holds and (31) imply $\Phi(\mathbf{k}_{-(N_s\cup N_d)}, q_{N_s\cup N_d}) = q$. Finally, if $j \in N_s$, then by (31), (R2.1) and (R2.2), $\Phi(\mathbf{0}_{-(N_s\cup N_u)}, q_{N_s\cup N_u}) = \Phi(\mathbf{k}_{-(N_s\cup N_d)}, q_{N_s\cup N_d}) = q$. Moreover, since $N_p = N$, we have that $-(N_s \cup N_u) = N_d$ and $-(N_s \cup N_d) = N_u$. Hence, $\Phi(\mathbf{0}_{N_d}, q_{N_u}, q_{N_s}) = \Phi(\mathbf{k}_{N_u}, q_{N_d}, q_{N_s}) = q$.

Case 2: Φ is individually rational. Then, $\Phi(\mathbf{0}) = \Phi(\mathbf{k}) = q$. By iterated applications of (R2.1) and (R2.2), $\Phi(\mathbf{0}_{N_d}, q_{N_u}, q_{N_s}) = \Phi(\mathbf{k}_{N_u}, q_{N_d}, q_{N_s}) = q$. ■

Lemma 4 *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule. Let N_u, N_d, N_s and q be the output of* **Step B.t** *of the* MIA *obtained from Φ and let q' be one of its inputs. Then, the following two conditions hold.*

(L4.1) $\Phi(\mathbf{0}_{N_d}, q_{-N_d}) = q$.

(L4.2) *If $N_u \neq \emptyset$, then* $\Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) = \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases}$

Proof Let N'_u, N'_d, N'_s and q' be the input of **Step B.t** and let $j \in N'_u$ and $r \in N'_d$ be respectively the agents that are called to play at **Step B.t.a** and **Step B.t.b**. By the definition of the MIA obtained from Φ,

$$q_i = \begin{cases} q'_i + 1 & \text{if } i = j \\ q'_i - 1 & \text{if } i = r \\ q'_i & \text{if } i \in N \setminus \{j, r\}. \end{cases} \tag{32}$$

We now prove that (L4.1) and (L4.2) hold.

(L4.1) If $N_d = \emptyset$, the statement follows by the efficiency of Φ. Assume $N_d \neq \emptyset$. We proceed by induction on $\mathbf{t}$. Suppose $\mathbf{t} = 1$. Let N'_u, N'_d, N'_s and q' be the input of **Step B.1**. Then, N'_u, N'_d, N'_s and q' is the output of **Stage A**. By Lemma 3,

$$\Phi(\mathbf{0}_{N'_d}, q'_{N'_u}, q'_{N'_s}) = q'. \tag{33}$$

By (R2.2) and (33),

$$\Phi_j(\mathbf{0}_{N'_d}, q'_{N'_u\setminus\{j\}}, q'_{N'_s}, q'_j + 1) \leq q'_j + 1 \tag{34}$$

and

$$\Phi_i(\mathbf{0}_{N'_d}, q'_{N'_u\setminus\{j\}}, q'_{N'_s}, q'_j + 1) \leq q'_i \text{ for all } i \in N \setminus \{j\}. \tag{35}$$

By the definition of agent r,

$$\Phi_r(\mathbf{0}_{N'_d}, q'_{N'_u\setminus\{j\}}, q'_{N'_s}, q'_j + 1) \leq q'_r - 1. \tag{36}$$

Since q' is feasible, the inequalities in (34), (35) and (36) can be replaced by equalities. By (32), and since $r \in N'_d$, $\Phi(\mathbf{0}_{N'_d}, q_{-N'_d}) = q$. Either $N_d = N'_d$, in which case $\Phi(\mathbf{0}_{N_d}, q_{-N_d}) = q$ follows, or $N_d \neq N'_d$, in which case $r \in N_s$ and, by (R2.1), $\Phi(\mathbf{0}_{N_d}, q_{-N_d}) = q$. This finishes the proof of (L4.1) for the case $\mathbf{t} = 1$. Suppose $\mathbf{t} \geq 2$.

Induction hypothesis: Let N'_u, N'_d, N'_s and q' be the output of **Step B.t-1**. Then,

$$\Phi(\mathbf{0}_{N'_d}, q'_{-N'_d}) = q'. \tag{37}$$

Observe that in the proof for the case $\mathbf{t} = 1$, (33) can be replaced by (37) and, with the same argument used there, we can show that $\Phi(\mathbf{0}_{N_d}, q_{N\setminus N_d}) = q$. This proves (L4.1).

(L4.2) Assume $N_u \neq \emptyset$. We proceed by induction on $\mathbf{t}$. Suppose $\mathbf{t} = 1$. Let N'_u, N'_d, N'_s and q' be the input of **Step B.1**. Then, N'_u, N'_d, N'_s and q' is the output of **Stage A**. By Lemma 3,

$$\Phi(\mathbf{k}_{N'_u}, q'_{N'_d}, q'_{N'_s}) = q'. \tag{38}$$

By definition of the MIA obtained from Φ, $N'_d \neq \emptyset$. Let $i_1 \in N'_d$. By (R2.1) and (38),

$$\Phi_{i_1}(\mathbf{k}_{N'_u}, q'_{N'_d\setminus\{i_1\}}, q'_{i_1} - 1, q'_{N'_s}) \geq q'_{i_1} - 1 \text{ and}$$

$$\Phi_i(\mathbf{k}_{N'_u}, q'_{N'_d\setminus\{i_1\}}, q'_{i_1} - 1, q'_{N'_s}) \geq q'_i \text{ for all } i \in N \setminus \{i_1\}.$$

Proceeding similarly for each remaining agent in $N'_d \setminus \{i_1\}$, we obtain that

$$\Phi_i(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N'_d \\ q'_i & \text{if } i \in N'_s. \end{cases} \tag{39}$$

Furthermore, by the definition of agent $j \in N'_u$, who plays at **Step B.1.a**,

$$\Phi_j(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}) \geq q'_j + 1. \tag{40}$$

From (39) and (40), and (32),

$$\Phi_i(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q_{N'_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N'_d \\ q_i & \text{if } i \in N'_s \cup \{j, r\}. \end{cases} \tag{41}$$

We now look at the different possibilities depending on the subsets of agents to which r and j enter in this **StepB.1**.

First, $j \in N_u$ and $r \in N_d$. Then, $N_u = N'_u$, $N_d = N'_d$ and $N_s = N'_s$, and by (41),

$$\Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \textit{if } i \in N_d \\ q_i & \textit{if } i \in N_s. \end{cases}$$

Second, $j \in N_u$ and $r \notin N_d$. Then, $N_u = N'_u$, $N_d = N'_d \setminus \{r\}$ and $N_s = N'_s \cup \{r\}$, and by (32), $(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) = (\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q_{N'_s})$. Then, by (41),

$$\Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \textit{if } i \in N_d \\ q_i & \textit{if } i \in N_s. \end{cases}$$

Third, $j \notin N_u$ and $r \in N_d$. Then, $N_u = N'_u \setminus \{j\}$, $N_d = N'_d$ and $N_s = N'_s \cup \{j\}$, and by (41) and (R2.1),

$$\Phi_j(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq q_j \text{ and } \Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \textit{if } i \in N_d \\ q_i & \textit{if } i \in N_s \setminus \{j\}. \end{cases}$$

Fourth, $j \notin N_u$ and $r \notin N_d$. Then, $N_u = N'_u \setminus \{j\}$, $N_d = N'_d \setminus \{r\}$ and $N_s = N'_s \cup \{j, r\}$, and by (32), $(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N_d}, q_{N_s \setminus \{j\}}) = (\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q_{N'_s})$. By (41) and (R2.1),

$$\Phi_j(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq q_j \text{ and } \Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \textit{if } i \in N_d \\ q_i & \textit{if } i \in N_s \setminus \{j\}. \end{cases}$$

Then, in all four cases we have

$$\Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases}$$

Hence, by (1) in the definition of same-sidedness and the fact that $N_u \neq \emptyset$,

$$\Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) = \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases} \tag{42}$$

This finishes the proof of (L4.2) for the case $\mathbf{t} = 1$. Suppose $\mathbf{t} \geq 2$.

Induction hypothesis: Let N'_u, N'_d, N'_s and q' be the output of **Step B.t-1** and N''_u, N''_d, N''_s and q'' be its input; observe that $\emptyset \neq N_u \subset N'_u$ holds. Then,

$$\Phi_i(\mathbf{k}_{N'_u}, (q'' - \mathbf{1})_{N'_d}, q'_{N'_s}) = \begin{cases} q''_i - 1 & \text{if } i \in N'_d \\ q'_i & \text{if } i \in N'_s. \end{cases} \tag{43}$$

We first prove that (43) implies (39). Then, to obtain (42), the proof follows from (39) with the same argument used in the case $\mathbf{t} = 1$.

Let $j' \in N''_u$ and $r' \in N''_d$ be the agents who play at **Step B.t-1**. If $r' \notin N'_d$, then $q'_i = q''_i$ for all $i \in N'_d$. Therefore (43) implies (39) and the proof follows as in the case $\mathbf{t} = 1$. If

$r' \in N'_d$, then $q'_i = q''_i$ for all $i \in N'_d \setminus \{r'\}$ and $q'_{r'} = q''_{r'} - 1$. Then, as $q''_{r'} - 1 > q'_{r'} - 1$, by (R2.1) and (43),

$$\Phi_{r'}(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}) \geq q'_{r'} - 1, \text{ and} \tag{44}$$

$$\Phi_i(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N'_d \setminus \{r'\} \\ q'_i & \text{if } i \in N'_s. \end{cases} \tag{45}$$

Then, (44) and (45) imply (39) and the proof of (L4.2) follows as in the case $\mathbf{t} = 1$. ■

Lemma 5 *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule and N_u, N_d, N_s and q be the output of the* MIA *obtained from Φ. Then, the following two conditions hold.*

(L5.1) *If $N_u = \emptyset$ then $\Phi(\mathbf{0}_{N_d}, q_{N_s}) = q$.*

(L5.2) *If $N_u \neq \emptyset$ then $N_d = \emptyset$ and $\Phi(\mathbf{k}_{N_u}, q_{N_s}) = q$.*

Proof Suppose the output of the MIA obtained from Φ is the output of **Stage A**. Then, by Lemma 3,

$$\Phi(\mathbf{k}_{N_u}, q_{-N_u}) = \Phi(\mathbf{0}_{N_d}, q_{-N_d}) = q. \tag{46}$$

Assume $N_u = \emptyset$. Then, $-N_d = N_s$ and (L5.1) follows from (46). Assume $N_u \neq \emptyset$. Since the MIA obtained from Φ does not move to **Stage B**, $N_d = \emptyset$ and $-N_u = N_s$. Then, (L5.2) follows from (46).

Now suppose the output of the MIA obtained from Φ is the output of **Stage B**. Then, (L5.1) follows from (L4.1) since $N_u = \emptyset$ implies $N_s = -N_d$. To show (L5.2), assume $N_u \neq \emptyset$. As N_u, N_d, N_s and q is the output of the MIA obtained from Φ, $N_d = \emptyset$. By (L4.2), for all $i \in N_s$,

$$\Phi_i(\mathbf{k}_{N_u}, q_{N_s}) = q_i. \tag{47}$$

Let $j \in N_u$ be arbitrary. We first show that $\Phi_j(\mathbf{k}_{N_u}, q_{N_s}) \geq q_j$ holds by distinguishing between two cases.

Case 1: Suppose j has not played throughout **Stage B**. Let N^*_u, N^*_d, N^*_s and q^* be the output of **Stage A** in the path to the final output N_u, N_d, N_s and q of the MIA obtained from Φ. Hence, $N^*_d \neq \emptyset$, $j \in N_u \subset N^*_u$, $q^*_i = q_i$ for all $i \in N^*_s \cup \{j\}$ and $q^*_i \geq q_i$ for all $i \in N^*_d$. By Lemma 3, $\Phi(\mathbf{k}_{N^*_u}, q^*_{-N^*_u}) = q^*$. Hence, because $-N^*_u = N^*_s \cup N^*_d$, $\Phi_i(\mathbf{k}_{N^*_u}, q_{N^*_s}, q^*_{N^*_d}) = q_i$ for all $i \in N^*_s \cup \{j\}$.

Let $i \in N^*_d$. Then, $\Phi_i(\mathbf{k}_{N^*_u}, q_{N^*_s}, q^*_{N^*_d}) \geq q_i$. By (R2.1), $\Phi_i(\mathbf{k}_{N^*_u}, q_{N^*_s}, q^*_{N^*_d \setminus \{i\}}, q_i) \geq q_i$, $\Phi_j(\mathbf{k}_{N^*_u}, q_{N^*_s}, q^*_{N^*_d \setminus \{i\}}, q_i) \geq q_j$ and $\Phi_{i'}(\mathbf{k}_{N^*_u}, q_{N^*_s}, q^*_{N^*_d \setminus \{i\}}, q_i) \geq q_{i'}$ for all $i' \in N^*_d \setminus \{i\}$ (if any). By iteratively applying (R2.1) to all remaining agents in $N^*_d \setminus \{i\}$ (if any), we obtain that for the arbitrarily fixed agent $j \in N_u$,

$$\Phi_j(\mathbf{k}_{N^*_u}, q_{-N^*_u}) \geq q_j. \tag{48}$$

Let $i \in N^*_u \setminus N_u$. By strategy-proofness, $\Phi_i(\mathbf{k}_{N^*_u}, q_{-N^*_u}) \geq \Phi_i(\mathbf{k}_{N^*_u \setminus \{i\}}, q_{-N^*_u}, q_i)$. By replacement monotonicity and (48), $q_j \leq \Phi_j(\mathbf{k}_{N^*_u}, q_{-N^*_u}) \leq \Phi_j(\mathbf{k}_{N^*_u \setminus \{i\}}, q_{-N^*_u}, q_i)$. Iteratively applying the same argument to all remaining agents in $(N^*_u \setminus \{i\}) \setminus N_u$ (if any), we obtain that for the arbitrarily fixed agent $j \in N_u$, $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{-N_u})$.

Case 2: Suppose j has played throughout **Stage B**. Let **Step B.t** be last step at which agent j has played and let N_u^*, N_d^*, N_s^* and q^* be the input of **Step B.t** in the path to the final output N_u, N_d, N_s and q of the MIA obtained from Φ. By definition, $j \in N_u^*$ and

$$q_j^* + 1 \leq \Phi_j(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}), \tag{49}$$

and j's guaranteed assignment at **Step B.t** is $q_j^* + 1$. Furthermore, as agent j does not play anymore, $q_j = q_j^* + 1$. Therefore, (49) can be written as

$$q_j \leq \Phi_j(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}). \tag{50}$$

Let $\widehat{N}_u, \widehat{N}_d, \widehat{N}_s$, and $\widehat{q}$ be the output of **Step B.t**.

CLAIM $(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s})$.

PROOF OF CLAIM As $j \in N_u$, $\widehat{N}_u = N_u^*$. Let $r \in N_d^*$ be the agent that plays at **Step B.t.b**. Then, $\widehat{q}_i = q_i^*$ for all $i \in N_s^* \cup N_d^* \setminus \{r\}$ and $\widehat{q}_r = q_r^* - 1$. If $\widehat{N}_d = N_d^*$, then $\widehat{N}_s = N_s^*$ and $(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s})$ holds trivially. If $\widehat{N}_d = N_d^* \setminus \{r\}$, then $N_s^* = \widehat{N}_s \setminus \{r\}$ and $(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s \setminus \{r\}}, q_r^* - 1) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s})$.
$\square$

Since $\widehat{N}_u \neq \emptyset$, we can apply (L4.2) to obtain

$$\Phi_i(\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s}) = \begin{cases} q_i^* - 1 & \text{if } i \in \widehat{N}_d \\ \widehat{q}_i & \text{if } i \in \widehat{N}_s. \end{cases}$$

If $i \in \widehat{N}_s$, then $i \in N_s$ and $q_i = \widehat{q}_i$. If $i \in \widehat{N}_d$, then $i \in N_s$ because $N_d = \emptyset$ and i is called to play at least once at some **Step B.t′.b** with $\mathbf{t} < \mathbf{t}'$. Then, by definition of q_i and the fact that $i \in \widehat{N}_d$, $q_i \leq \widehat{q}_i - 1 \leq q_i^* - 1$.

Then, as in Case 1, by iteratively applying (R2.1) to all $i \in (\widehat{N}_s \cup \widehat{N}_d)$, and strategy-proofness to all $i \in \widehat{N}_u \setminus N_u$ and replacement monotonicity to j, we obtain that

$$\Phi_j(\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s}) \leq \Phi_j(\mathbf{k}_{\widehat{N}_u}, q_{-\widehat{N}_u}) \leq \Phi_j(\mathbf{k}_{N_u}, q_{-N_u}). \tag{51}$$

Therefore, by the claim above, (50) and (51), $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{-N_u})$.

Hence, $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{-N_u})$ holds, independently of whether or not j plays throughout **Stage B**. Since j was arbitrary and $-N_u = N_s$, for all $j \in N_u$, $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{N_s})$. Thus, by (47) and feasibility of q, $\Phi(\mathbf{k}_{N_u}, q_{N_s}) = q$. ■