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NOTE

NONEXISTENCE OF STABLE THREESOME MATCHINGS

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A counterexample shows stable matchings need not exist in societies where threesomes are to form. The stability theorem breaks down, in fact, for K-some formations for all $K \geq 3$, even when preferences are restricted to be separable.

Key words: Matching; stability.

Gale and Shapley (1962) proved stability is the rule for 'marriage', i.e., matchings of twosomes. This rule says given any society with two kinds of agents to be matched in pairs, each agent having a preference ordering on agents of the opposite kind, there exists at least one stable matching. In this note we show by example that stable matchings may fail to exist in societies with three kinds of agents to be matched in threesomes. The stability theorem breaks down, in fact, for K-some formations for all $K \geq 3$, in quite an unexceptional way and even when preferences are restricted to be separable.

We consider societies with three kinds of agents, naming them 'men', 'women', and 'children', respectively. A 'family' is a triplet with one of each kind. Any set of distinct families constitutes a *matching* and is said to be *stable* if there exists no family which would block it, i.e., one which every one of its members finds superior to his/her lot in the matching.

Our story is told with three agents of each kind. We denote the men A, B, C, the women $\tilde{A}, \tilde{B}, \tilde{C}$, and the children a, b, c. An agent's preferences then amount to a ranking from 1st to 9th best of all the nine distinct pairs (s)he could join to form a family. For our example, it suffices to specify preferences partially as displayed in Fig. 1. To illustrate, family A consisting of agents A, $\tilde{A}$, and a is third best for A, first best for the other two. We will write $A = A\tilde{A}a$ and $r(A) = (3,1,1)$.

We now prove there exists no stable matching in any society fitting these specifications:

A:	a	b	c
$\tilde{A}$	3		1
$\tilde{B}$			2
$\tilde{C}$			

B:	a	b	c
$\tilde{A}$			
$\tilde{B}$		2	1
$\tilde{C}$			3

C:	a	b	c
$\tilde{A}$			
$\tilde{B}$			
$\tilde{C}$			1

$\tilde{A}$:	a	b	c
A	1		
B			
C			

$\tilde{B}$:	a	b	c
A			2
B		3	1
C			

$\tilde{C}$:	a	b	c
A			
B			1
C			2

a:	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
A	1		
B			
C			

b:	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
A			
B		1	
C			

c:	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
A		3	1
B			2
C			4

Fig. 1. Preferences of agents.

It follows from $r(A) = (3,1,1)$ that a matching in order to be stable must give agent A at least his third best family: on the other hand, any matching that contains A's first best family $F = A\tilde{A}c$ is blocked by $C = C\tilde{C}c$ and any matching that contains his second best family $D = A\tilde{B}c$ is blocked by $E = B\tilde{C}c$. We conclude all matchings except possibly those containing A are unstable.

A parallel argument establishes the same for family $B = B\tilde{B}b$: agents B and $\tilde{B}$'s first best family is $G = B\tilde{B}c$, and $\tilde{B}$'s second best family is D. Family C blocks any matching containing G, family E blocks any matching containing D, and $r(B) = (2,3,1)$.

The only possibly stable matching, therefore, is the one containing both A and B, and (residually) C. The matching (A, B, C), however, is blocked by D, that is to say, unstable as well.

Preferences in our construction are all in accordance with the highly restrictive additive separability condition. (An ordering over pairs $(a, \tilde{a})$ in $A \times \tilde{A}$ is *additively separable* if there exists a real-valued function f on $A \cup \tilde{A}$ such that $(a, \tilde{a})$ is better than $(a', \tilde{a}')$ iff $f(a) + f(\tilde{a}) > f(a') + f(\tilde{a}')$.) This can be checked by regarding, for example, the valuations $f(A) = 5, f(B) = 3, f(C) = 1, f(\tilde{A}) = 0, f(\tilde{B}) = 5, f(\tilde{C}) = 8$ for child c, and observing that c's ordering covers all others up to a permutation each. Without the separability requirement, our example can in fact be stated with considerably fewer specifications: Discard all 1st best entries for the agents A, B, $\tilde{B}$, C, $\tilde{C}$, c as well as the 4th best entry for c and increase their remaining entries by one. Instability can now be demonstrated even more simply.

Omitting details (see Alkan, 1987), we just mention in closing that our counterexample extends easily to societies with $K > 3$ kinds of agents where K-somes form, and so the various positive results pertaining to pairwise matching (see e.g., Crawford and Knoer, 1981; Demange and Gale, 1985; Roth, 1982) all resting on the stability theorem, vanish at the outset for $K \geq 3$. Building on the matching problem in Roth (1984) for one possible illustrative case, we suggest replacing 'children' in our example with hospitals, and 'men' and 'women' with male and female interns, respectively.

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