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Undiscounted equilibrium payoffs of repeated games with a continuum of players

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We give a characterization of equilibrium payoffs of a repeated game in which players use the long-run average of the one-shot-game payoffs as the overall payoff of the repeated game and individual actions are not discernible by others. In contrast to the 'Anti-Folk Theorem' equilibria may exist even though the one-shot game has no equilibrium.

1. Introduction

The aim of this paper is to characterize the set of equilibrium payoffs of a repeated game having two important features. First, the game that is repeated over time has a large number of individually insignificant players (i.e. a continuum of them), and the knowledge that each player has at any point in time about the history of the game does not possess individualistic information about the others, so that it is invariant with respect to the behavior of small subsets of players (zero measure). Therefore, in this context, players cannot identify and punish individual deviators from the long-run plan. Second, players use the long-run average of the one-shot-game payoffs as the overall payoff of the repeated game. Therefore, in order to carry out a successful deviation, a player has to be able to improve his payoff at least an infinite number of times.

What has been known as the 'Folk Theorem' is the equilibrium-payoff characterization of infinitely-repeated games without the first feature; i.e., at any point in time the full history of the game is common knowledge. The

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Folk Theorem says that a payoff is a Nash equilibrium of the infinitely repeated game if and only if it is feasible and individually rational in the one-shot game [see Aumann (1981)]. Results about repeated games having the first feature (individual deviators cannot be identified) but using the discounting criterion are collectively termed the 'Anti-Folk Theorem' [see Green (1980), Kaneko (1983) and Dubey and Kaneko (1984)] which states that a payoff is a Nash equilibrium of the repeated game if and only if it is a discounted sum of one-shot Nash-equilibrium payoffs.

Massó and Rosenthal (1989) characterized the Nash-equilibrium strategies of repeated games with the same features that we are concerned with. However, their characterization only applies to repetitions of games with continuous payoffs and sequentially-compact[1] aggregate action sets. Under their assumptions a strategy is a Nash equilibrium of the infinitely repeated game if and only if every converging subsequence of aggregate actions generated by the strategy is either 'negligible' or has a one-shot Nash equilibrium as a limit point. The payoff characterization here will apply to a much larger class of games; in fact we have only to assume that the payoff functions are uniformly bounded.

The main difficulty in characterizing the set of equilibrium payoffs of infinitely-repeated nonatomic games with long-run-average criterion is that, roughly, without these continuity and compactness assumptions there may be subsequences of payoffs in the one-shot game with no cluster points or with a discontinuity in the limit; and therefore, payoffs that are not even feasible in the one-shot game may emerge as equilibria in the repeated game because they can be achieved through repetition. We overcome this difficulty by defining an artificial concept, ε-almost-equilibrium payoff (the measure of players that can improve their corresponding payoff by more than ε is itself smaller than ε) in the one-shot game, and as ε goes to zero, preserving the 'limit' payoff as a feasible one for the repeated game. Therefore, our characterization says that the set of undiscounted equilibrium payoffs of the repeated game is, roughly, the convex hull of 'limits', when ε goes to zero, of sequences of ε-almost equilibrium payoffs of the one-shot game.

In a broad class of economic applications one would think that a more appropriate model would be either the finitely-repeated game or the discounted game. However, for the particular information structure in which we are interested, equilibria of the finitely-repeated game or infinitely-repeated game with discounting may not exist if the one-shot game does not possess equilibria; with the infinite undiscounted model, however, this is not the case. In addition, models of infinitely-repeated games with long-run-average payoff criterion can generate insights about far-sighted players that are not obtainable from the other models.

[1]Every sequence has a convergent subsequence.

At this point, two remarks are in order. First, in order to define the infinitely-repeated game with long-run-average payoff criterion, it is necessary to associate a real number to every sequence of average payoffs. Standard ways of doing so are to use either the liminf or a Banach limit. Here, we use the concept of *limit medial*, introduced by Meyer (1973), because, besides being linear, it has the convenient property that the order of 'limit' and integration may be interchanged. Second, our characterization is not exact in the sense that we do not obtain an 'if and only if' result. However, as we will show, necessary and sufficient conditions differ only on non-converging sequences.

The next section of the paper is devoted to describing the game and to presenting the definitions needed to state the results, in turn presented in section 3. In an appendix at the end of the paper, the reader will find the proofs of certain lemmas.

2. The game

Let (I,S,λ) be a measure space, where $I=[0,1]$ is the set of players, S is the σ-algebra of Lebesgue measurable subsets of I, and λ is the Lebesgue measure on (I,S).

In the one-shot (stage) game denoted by G, A_i denotes the set of feasible actions for each $i\in I$. The elements $a_i\in A_i$ may be interpreted as either pure or randomized; we do not allow for additional randomizations over the elements of A_i. Assume that some σ-algebra is associated with $\bigcup_{i\in I} A_i$, and denote by $\hat{A}$ the set of feasible joint actions, defined by

$$\hat{A}=\left\{\hat{a}\colon I\rightarrow\bigcup_{i\in I}A_i \,\middle|\, \hat{a}(i)\in A_i\,\forall i\in I,\text{ and }\hat{a}\text{ is measurable}\right\}.^2$$

For $\hat{a},\hat{b}\in\hat{A}$ we say that $\hat{a}$ and $\hat{b}$ are equivalent if they differ only on a set of zero measure. Denote by A the set of equivalence classes of $\hat{A}'$. For each $i\in I$, the payoff function is $h_i\colon A_i\times A\rightarrow R$. We assume that $(h_i)_{i\in I}$ is a collection of uniformly-bounded functions such that for every $a\in A$ the function $h_i(a(i),a)$ is measurable as a function of i. We adopt the usual convention that if a statement holds for all elements of an equivalence class, then we say that it holds for the class itself; in particular, $h_i(a(i),a)$ measurable means that for all $\hat{a}$ in the class a, $h_i(\hat{a}(i),a)$ is measurable. Hence, there exists $y\in\mathbb{R}_+$ such that

[2]In defining a feasible joint action as a measurable function, one is really talking of a sort of generalized game. If each player is choosing an action from his action set independently of others, there is no a priori reason to expect joint measurability. In the absence of the measurability assumption, however, it would not be possible to use the concept of limit medial, crucial here.

for every $a \in A$, $\lambda(\{i \in I \| h_i(a(i), a)| \leqq y\}) = 1$ and therefore, given $a \in A$, $h_i(a(i), a)$ is an element of $L_\infty([0, 1])$ as a function of i. Let

$$V = \{w \in L_\infty \mid \exists a \in A \text{ s.t. } \lambda(\{i \in I: w(i) = h_i(a(i), a)\}) = 1\}$$

be the set of feasible payoffs of G.

A Nash equilibrium of G is an $a \in A$ such that

$$\lambda(\{i \in I \mid h_i(a(i), a) \geqq h_i(b_i, a) \, \forall b_i \in A_i\}) = 1.$$

Let A^* denote the set of Nash equilibria of the stage game. Schmeidler (1973) showed that A^* may be empty.

For every $\varepsilon > 0$, define $a \in A$ as an *ε-almost-equilibrium of* G if

$$\lambda\left(\left\{i \in I \,\middle|\, \sup_{b_i \in A_i} h_i(b_i, a) > h_i(a(i), a) + \varepsilon\right\}\right) \leqq \varepsilon.$$

(Note the appearance of ε in two places in the line above.)

Let $E_\varepsilon = \{a \in A \mid a$ is an ε-almost-equilibrium of $G\}$. It is straightforward to see that $A^* \subset E_\varepsilon \, \forall \varepsilon > 0$. Denote by V_ε the set of *ε-almost-equilibrium payoffs of* G defined by

$$V_\varepsilon = \{w \in V \mid \exists a \in E_\varepsilon \text{ such that } \lambda(\{i \in I \mid h_i(a(i), a) = w(i)\}) = 1\}.$$

Example 3 (at the end of the paper) shows that in general V_ε may be empty for sufficiently small ε. Therefore, in the sequel the reader should be aware of the fact that the sets defined from V_ε may also be empty.

Consider the convex hull of V_ε defined by

$$\text{co}\, V_\varepsilon = \Bigg\{ w \in L_\infty \mid \exists m, (c^1, \ldots, c^m) \in R_+^m \text{ and } (v^1, \ldots, v^m) \text{ such that:}$$

$$v^j \in V_\varepsilon \, \forall j = 1, \ldots, m, \ \sum_{j=1}^{m} c^j = 1, \text{ and } w = \sum_{j=1}^{m} c^j v^j \Bigg\}.$$

Now, define the closure of $\text{co}\, V_\varepsilon$ by

$$\overline{\text{co}\, V_\varepsilon} = \Bigg\{ w \in L_\infty \mid \exists \{m_1, m_2, \ldots\}, \{(c_n^1, \ldots, c_n^{m_n})\}_{n=1}^{\infty} \text{ and}$$

$$\{(v_n^1,\dots,v_n^{m_n})\}_{n=1}^{\infty} \text{ s.t. } \forall n \geqq 1 : (c_n^1,\dots,c_n^{m_n}) \in R_+^{m_n};$$

$$\sum_{j=1}^{m_n} c_n^j = 1; v_n^j \in V_\varepsilon \forall 1 \leqq j \leqq m_n; \text{ and } w = \lim_{n\to\infty} \sum_{j=1}^{m_n} c_n^j v_n^j \Bigg\},$$

where the limit operation is taken with respect to the L_∞ ([0, 1]) topology (a.e.-uniform convergence).[3]

In the (undiscounted) repeated game G^∞, time is indexed by t taking values in ℕ, the set of natural numbers. For each $t \in$ ℕ, the history of play through t is described by A^t, the t-fold Cartesian product of A with itself, with typical element $(a^1,\dots,a^t)$. A strategy for player i is a sequence of functions $f_i = f_i^1, f_i^2,\dots)$ satisfying

(i) $f_i^1 \in A_i$;

and $\forall t \in$ ℕ,

(ii) $f_i^{t+1} : A^t \to A_i$.

The fact that players can make contingent plans only as a function of those aspects of the history described by equivalence classes is the informational restriction which insures that individual deviators cannot be detected. Let F_i denote the set of f_i satisfying (i) and (ii), and let

$F = \{(f_i)_{i\in I} | f_i \in F_i \forall i \in I$; f_i^1 is measurable as a function of i; and $\forall t \geqq 1$ and $\forall (a^1,\dots,a^t) \in A^t$, $f_i^{t+1}(a^1,\dots,a^t)$ is measurable as a function of $i\}$.

Thus, F_i is the set of i's feasible strategies, and F is the set of joint strategies in G^∞. Given $f \in F$, the play it produces is identified as follows: Let $a^1(f) \in A$ be such that for a.e. $i \in I$ $a^1(f)(i) = f_i^1$ and, recursively, let $a^{t+1}(f) \in A$ be such that $a^{t+1}(f)(i) = f_i^{t+1}(a^1(f),\dots,a^t(f))$ for a.e. $i \in I$. For player $i \in I$, let

$$h_i^T(f) = \frac{1}{T} \sum_{t=1}^{T} h_i(f_i^t(a^1(f),\dots,a^{t-1}(f)), a^t(f)).$$

To define payoffs generally in undiscounted repeated games, one must choose some function between the extremes $\liminf_{T\to\infty} h_i^T(f)$ and $\limsup_{T\to\infty} h_i^T(f)$. For our purposes the natural choice is the concept of *limit medial* introduced by Meyer (1973). A limit medial is a linear functional H on ℓ_∞ (the set of all bounded sequences of real numbers) with the following properties:

(i) If $x \in \ell_\infty$ then $\liminf_{t\to\infty} x^t \leqq H(x) \leqq \limsup_{t\to\infty} x^t$

[3] One may alternatively define $\overline{\text{co}\, V_\varepsilon}$ simply as the L_∞-norm closure of $\text{co}\, V_\varepsilon$.

(ii) If $\{v^t\}_{t=1}^{\infty}$ is any uniformly-bounded sequence of measurable (real-valued) functions on I, then $\forall J \in S$

$$\int_J H(\{v^t(i)\}_{t=1}^{\infty})\, \mathrm{d}\lambda(i) = H\left(\left\{\int_J v^t(i)\, \mathrm{d}\lambda(i)\right\}_{t=1}^{\infty}\right).$$

Existence and measurability of such functionals follow from Theorem 2 in Meyer (1973).[4] Now, associate with each $f \in F$ the sequence $\{h_i^t(f)\}_{t=1}^{\infty} \in \ell_\infty$; pick a limit medial H on ℓ_∞; and define the payoff function $H_i(f)$ to be the limit medial evaluated at the sequence $\{h_i^t(f)\}_{t=1}^{\infty}$. Notice that every player uses the same criterion to associate real numbers to sequences of payoff averages and that $\forall f \in F$, $H_i(f)$ is measurable as a function of i. Let $V^\infty \subset L_\infty$ be the set of feasible payoffs of G^∞, i.e.

$$V^\infty = \{w \in L_\infty \mid \exists f \in F \text{ such that for a.e. } i \in I \ w(i) = H_i(f)\}.$$

Given $f \in F$ and $g_i \in F_i$, let $(f \mid g_i) \in F$ denote the joint strategy f with player i switching to g_i, defined by

$$(f \mid g_i)_j = \begin{cases} g_i & \text{if} \quad i = j \\ f_j & \text{if} \quad i \neq j \end{cases}.$$

A Nash equilibrium strategy of G^∞ is an $f \in F$ such that

$$\lambda(\{i \in I \mid H_i(f) \geqq H_i(f \mid g_i)\ \forall g_i \in F_i\}) = 1.$$

Let F^* denote the set of Nash equilibria of G^∞. Define $w \in V^\infty$ to be a *Nash equilibrium payoff of* G^∞ if there exists $f \in F^*$ s.t. $\lambda(\{i \in I \mid w(i) = H_i(f)\}) = 1$. Let $V^{\infty *} \subset V^\infty$ denote the set of Nash equilibrium payoffs of G^∞.

For every $\varepsilon > 0$ define

$$HV_\varepsilon = \left\{ w \in V^\infty \mid \exists \{v^t\}_{t=1}^{\infty} \text{ s.t. } v^t \in V_\varepsilon\ \forall t \geqq 1 \text{ and} \right.$$

$$\left. \text{a.e. } i \in I: H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} v^t(i)\right\}\right) = w(i) \right\};$$

[4]While the existence of Banach limits follows from the Hahn–Banach Theorem assuming the axiom of choice, existence of limit medial follows from a more general result (Meyer's Theorem 1) which assumes the continuum hypothesis. (We are indebted to J.F. Mertens for pointing out to us that the limit medial of a sequence of averages is a Banach limit of the initial sequence.)

that is, the set of payoffs of G^∞ that are also the long-run average of a sequence of ε-almost-equilibrium payoffs of G. It is an easy verification to see that $\forall\varepsilon>0\ \overline{\mathrm{co}V_\varepsilon} \subset HV_\varepsilon$.

Using the general structure of the strategy space of G^∞ we can also define the finitely-repeated game G^T and the discounted infinitely-repeated game G^β, respectively (as well as their respective equilibrium strategies), where T is the number of times that G is repeated and $\beta\in(0,1)$ is the discount factor. Given a strategy $f\in F$ the payoff for player $i\in I$ in G^T and G^β are defined, respectively, as

$$H_i^T(f)\equiv h_i^T(f) \quad \text{and} \quad H_i^\beta(f)\equiv(1-\beta)\sum_{t=1}^{\infty}\beta^{t-1}h_i(a^t(f)(i),a^t(f)).$$

3. Results

Before stating and proving the main results about $V^{\infty*}$ we give the well known characterization of the equilibrium strategies of G^T and G^β (the 'Anti-Folk' Theorem in strategy space) in the form of Proposition 1 below.[5] The idea behind it is simple: if the payoff criterion of the repeated game has the property that increasing the payoff in any stage game raises the overall payoff, then, because of the fact that individual actions are not discernible by the others, only sequences of one-shot equilibria are possible in an equilibrium of the repeated game.

Proposition 1. *$f\in F$ is a Nash equilibrium of G^β (resp. G^T) if and only if $\forall t\geqq 1$ (resp. $t\leqq T$)$a^t(f)\in A^*$.*

Now, we will first show that $V^{\infty*}$ (the set of equilibrium payoffs of G^∞) coincides with the set of those feasible payoffs of G^∞ having the property that for every $\varepsilon>0$ and for a.e. $i\in I$ the 'proportion of times' that player i can improve his payoff by more than ε is itself smaller than ε.

Formally, let $\{v^t\}_{t=1}^\infty$ be a sequence in V; i.e., $\forall t\geqq 1\ \exists a^t\in A$ s.t. for a.e. $i\in I$ $v^t(i)=h_i(a^t(i),a^t)$. Let $\{\hat{a}^t\}_{t=1}^\infty$ be an particular sequence of functions where $\forall t\geqq 1$, $\hat{a}^t$ belongs to the equivalence class a^t. For every $\varepsilon>0$ and for every $t\geqq 1$ define

$$I_\varepsilon^t=\left\{i\in I \;\middle|\; \sup_{b_i\in A_i} h_i(b_i,a^t)-h_i(\hat{a}^t(i),a^t)>\varepsilon\right\}.$$

[5]For the proof of the discounting case, see Green (1980); the proof of the finitely-repeated case is analogous.

Notice that if $\lambda(I_\varepsilon^t) \leqq \varepsilon$ then $v^t \in V_\varepsilon$. Now, $\forall i \in I$ define $\hat{B}_\varepsilon^i = \{t \geqq 1 | i \in I_\varepsilon^t\}$ as the subset of $\mathbb{N}$ in which player i has a one-shot deviation that gives him more than ε. For every $T \geqq 1$ let $\hat{B}_\varepsilon^i(T) = \hat{B}_\varepsilon^i \cap \{1, \ldots, T\}$ and let $\# \hat{B}_\varepsilon^i(T)$ be its cardinality. Given $T \geqq 1$, $\# \hat{B}_\varepsilon^i(T)$ is measurable as a function of i (it may be also expressed as a sum of characteristic functions, i.e., $\forall i \in I \; \# \hat{B}_\varepsilon^i(T) = \sum_{t=1}^T \chi_{I_\varepsilon^t}(i)$). Therefore, as previously, let $\# B_\varepsilon^i(T)$ denote its equivalence class.

Consider the following subset of V^∞:

$$Y^\infty = \left\{ v \in V^\infty \,|\, \exists \{v^t\}_{t=1}^\infty \text{ s.t. } v^t \in V \; \forall t \geqq 1 \text{ and for a.e. } i \in I, \right.$$

$$\left. H\left(\left\{ \frac{1}{T} \sum_{t=1}^T v^t(i) \right\}_{T=1}^\infty \right) = v(i) \text{ and } \forall \varepsilon > 0,\; H\left(\left\{ \frac{1}{T} \# B_\varepsilon^i(T) \right\}_{T=1}^\infty \right) \leqq \varepsilon \right\}.$$

Proposition 2. $V^{\infty *} = Y^\infty$.

Proof. (1) $V^{\infty *} \subset Y^\infty$. Let $v \in V^{\infty *}$; this means that $\exists f \in F^*$ s.t. for a.e. $i \in I$, $H(\{h_i^T(f)\}) = v(i)$. Next, $\forall t \geqq 1$ define $v^t \in V$ by $v^t(i) = h_i(a^t(f)(i), a^t(f))$ a.e. $i \in I$. We have to show that $\forall \varepsilon > 0$, $H(\{(1/T) \# B_\varepsilon^i(T)\}) \leqq \varepsilon$ a.e. $i \in I$. Suppose not, $\exists \varepsilon > 0$ s.t. $\exists J \in S$ with positive measure s.t. a.e. $i \in J$, $H(\{(1/T) \# B_\varepsilon^i(T)\}) > \varepsilon$. In order to contradict the fact that f is an equilibrium strategy of G^∞, we will show that a.e. $i \in J$ has available another strategy that yields him a higher payoff. For every $i \in J$ define $g_i \in F_i$ as follows:

$$g_i^t(\cdot) = \begin{cases} a^{t\,\hat{}}(f)(i) & \text{if} \quad t \notin \hat{B}_\varepsilon^i \\ b_i^t & \text{if} \quad t \in \hat{B}_\varepsilon^i, \end{cases}$$

where b_i^t is s.t. $h_i(b_i^t, a^t(f)) - h_i(a^{t\,\hat{}}(f)(i), a^t(f)) > \varepsilon/2$ and $\forall t \geqq 1$, $a^{t\,\hat{}}(f)$ is any function belonging to $a^t(f)$. For a.e. $i \in J$,

$$\begin{aligned} H_i(f | g_i) - H_i(f) &= H(\{h_i^T(f | g_i)\}) - H(\{h_i^T(f)\}) \\ &= H(\{h_i^T(f | g_i) - h_i^T(f)\}) \\ &= H\left(\left\{ \frac{1}{T} \sum_{t=1}^T [h_i(g_i^t(a^1(f | g_i), \ldots, a^{t-1}(f | g_i)), a^t(f | g_i)) \right.\right. \\ &\qquad \left.\left. - h_i(a^t(f)(i), a^t(f))] \right\} \right) \end{aligned}$$

$$=H\left(\left\{\frac{1}{T}\sum_{t\in B_\varepsilon^i(T)}^{T}[h_i(b_i^t,a^t(f))-h_i(a^t(f)(i),a^t(f))]\right\}\right)$$

$$\geqq H\left(\left\{\frac{1}{T}\#B_\varepsilon^i(T)(\varepsilon/2)\right\}\right)=H\left(\left\{\frac{1}{T}\#B_\varepsilon^i(T)\right\}\right)(\varepsilon/2)$$

$$>\varepsilon^2/2.$$

Hence, $\exists J\in S$ with positive measure such that a.e. $i\in J$: $\exists g_i\in F_i$ s.t. $H_i(f|g_i)-H_i(f)>0$, contradicting the fact that $f\in F^*$.

(2) $Y^\infty\subset V^{\infty *}$. Let $v\in Y^\infty$, and let $\{v^t\}$ be s.t. $H(\{(1/T)\sum_{t=1}^T v^t(i)\})=v(i)$ a.e. $i\in I$. This means $\exists\{a^t\}_{t=1}^\infty$ from A and a corresponding sequence $\{\hat{a}^t\}_{t=1}^\infty$ with $\hat{a}^t\in a^t$, $\forall t\geqq 1$ s.t. $v^t(i)=h_i(\hat{a}^t(i),a^t)$ a.e. $i\in I$, $\forall t\geqq 1$. Define $f\in F$ as follows: $\forall t\geqq 1$ and $i\in I$, $f_i^t(\cdot)=\hat{a}^t(i)$. It will be sufficient to show that $f\in F^*$.

Let $\varepsilon>0$ be arbitrary and let $g_i\in F_i$ be any strategy for player i. Then, for a.e. $i\in I$

$$H_i(f|g_i)-H_i(f)\leqq H\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\sup_{b_i\in A_i}h_i(b_i,a^t(f))-h_i(a^t(f)(i),a^t(f))\right]\right\}$$

$$\leqq H\left(\left\{\frac{1}{T}\#B_\varepsilon^i(T)\right\}\right)2y+H\left(\left\{\frac{1}{T}(T-\#B_\varepsilon^i(T))\right\}\right)\varepsilon$$

$$\leqq 2y\varepsilon+\varepsilon.$$

Since this is true $\forall\varepsilon>0$, for a.e. $i\in I$, and for arbitrary $g_i\in F_i$ we conclude that $\lambda(\{i\in I|H_i(f|g_i)-H_i(f)\leqq 0,\ \forall g_i\in F_i\})=1$. Hence $f\in F^*$. Q.E.D.

The following example, adapted from one in Schmeidler (1973), illustrates what is going to be our next result: a feasible payoff of G^∞, obtained by a sequence of ε_t-almost-equilibrium payoffs, belongs to $V^{\infty *}$ provided that $\{\varepsilon_t\}_{t=1}^\infty$ goes to zero. Therefore, in contrast with G^T and G^β, we may have games G^∞ in which the equilibrium set of G^∞ is non-empty even when G has no equilibria.

Example 1. In the stage game G, the action set for each $i\in I$ is $A_i=\{0,1\}$ and the payoff function is

$$h_i(a_i,a)=\begin{cases}\left|a_i-\dfrac{1}{i}\displaystyle\int_{[0,i]}a\,\mathrm{d}\lambda\right| & \text{if}\quad i\neq 0\\ 0 & \text{if}\quad i=0\end{cases}.$$

As Schmeidler (1973) shows, this game has no equilibria. Consider the following strategy combination $f \in F$ in G^{∞}. At each t, partition the player set into 2^t half-open subintervals of equal size. Almost all players in the odd subintervals play 1 while almost all those in the even subintervals play 0, all independently of the history. It is easy to see that the sequence of payoffs converges a.e. $i \in I$ to $\frac{1}{2}$ and that f is an equilibrium of G^{∞}; also, $\forall t \geqq 1$, $a^t(f) \in E_{1/t}$, therefore f generates a sequence of $(1/t)$-almost-equilibrium payoffs.

Before showing in general that this is a sufficient condition for equilibrium, define

$$Z^{\infty} = \left\{ w \in V^{\infty} \,\middle|\, \exists \{v^t\}_{t=1}^{\infty} \text{ and } \{\varepsilon_t\}_{t=1}^{\infty} \text{ with } \lim_{t \to \infty} \varepsilon_t = 0 \text{ s.t.} \right.$$

$$\left. \forall t \geqq 1,\ v^t \in V_{\varepsilon_t}, \text{ and } w(i) = H\left(\left\{\frac{1}{T} \sum_{t=1}^{T} v^t(i)\right\}\right) \text{ for a.e. } i \in I \right\}.$$

Proposition 3. $Z^{\infty} \subset V^{\infty *}$.

Proof. Let $v \in Z^{\infty}$, and suppose $v \notin V^{\infty *}$. From Proposition 2, this means that $\exists \varepsilon > 0$ and $J \in S$ with $\lambda(J) = \delta > 0$ s.t. a.e. $i \in J$, $H(\{(1/T)\ \#\, B^i_{\varepsilon}(T)\}) \geqq \varepsilon$. Therefore,

$$\limsup_{T \to \infty} \int_J \frac{1}{T}\ \#\, B^i_{\varepsilon}(T)\, \mathrm{d}\lambda \geqq H\left(\left\{\int_J \frac{1}{T}\ \#\, B^i_{\varepsilon}(T)\, \mathrm{d}\lambda\right\}\right)$$

$$= \int_J H\left(\left\{\frac{1}{T}\ \#\, B^i_{\varepsilon}(T)\right\}\right) \mathrm{d}\lambda \geqq \varepsilon\delta.$$

It is an easy verification to see that in general $\forall w \in V$, $w \in V_1$ and the set $\{\tau \in [0,1] \,|\, w \in V_{\tau}\}$ is convex. Since $\lim_{t \to \infty} \varepsilon_t = 0$, $\exists \bar{T}$ s.t. $\forall t > \bar{T}$, $v^t \in V_{\varepsilon\delta/2}$; i.e.,

$$\lambda\left(\left\{ i \in I \,\middle|\, \sup_{b_i \in A_i} h_i(b_i, a^t) - v^t(i) > \varepsilon\delta/2 \right\}\right) \leqq \varepsilon\delta/2$$

implying that

$$\lambda\left(\left\{i\in I \,\middle|\, \sup_{b_i\in A_i} h_i(b_i,a^t)-v^t(i)>\varepsilon\right\}\right)\leqq \varepsilon\delta/2.$$

Now, for every $T>\bar{T}$ let $k=T-\bar{T}$. We will show by induction on k that $\forall T>\bar{T}$,

$$\int_J \frac{1}{T}\ \#B_\varepsilon^i(T)\,\mathrm{d}\lambda \leqq \frac{\bar{T}+k\varepsilon\delta/2}{\bar{T}+k}.$$

When $k=1$, since, for a.e. $i\in J$ $\#B_\varepsilon^i(\bar{T})\leqq \bar{T}$, we have

$$\int_J \frac{1}{T}\ \#B_\varepsilon^i(T)\,\mathrm{d}\lambda \leqq \frac{\bar{T}+\varepsilon\delta/2}{\bar{T}+1}.$$

Suppose it is true for k, then for $k+1$:

$$\int_J \frac{1}{T}\ \#B_\varepsilon^i(T)\,\mathrm{d}\lambda \leqq \frac{\bar{T}+k\varepsilon\delta/2+\varepsilon\delta/2}{\bar{T}+k+1},$$

because, by the induction hypothesis,

$$\int_J \#B_\varepsilon^i(T-1)\,\mathrm{d}\lambda \leqq \bar{T}+k\varepsilon\delta/2.$$

Therefore,

$$\varepsilon\delta \leqq \limsup_{T\to\infty}\int_J \frac{1}{T}\ \#B_\varepsilon^i(T)\,\mathrm{d}\lambda \leqq \limsup_{k\to\infty}\frac{\bar{T}+k\varepsilon\delta/2}{\bar{T}+k}$$

$$\leqq \limsup_{k\to\infty}\frac{\bar{T}}{\bar{T}+k}+\limsup_{k\to\infty}\frac{k\varepsilon\delta/2}{\bar{T}+k}=\varepsilon\delta/2,$$

which is a contradiction. Q.E.D.

Theorem 1. $\bigcap_{\varepsilon>0}\overline{\mathrm{co}V_\varepsilon}\subset V^{\infty *}$.

Proof. Let $v\in\bigcap_{\varepsilon>0}\overline{\mathrm{co}V_\varepsilon}$. By Proposition 3, it will be sufficient to show that there exists $\bar{f}\in F$ s.t. $\lambda(\{i\in I \mid v(i)=H_i(\bar{f})\})=1$ and $v\in Z^\infty$. By assumption $v\in\overline{\mathrm{co}V_{1/n}}$ for every $n\geqq 1$. Because of the fact that the set of rational convex combinations is a dense subset of the convex hull and therefore of its closure, there exist $r_n^1,\ldots,r_n^{m_n}$, q_n positive integers and $v_n^1,\ldots,v_n^{m_n}$ in $V_{1/n}$ such that $\sum_{j=1}^{m_n} r_n^j=q_n$ and for a.e. $i\in I$:

$$\left|\sum_{j=1}^{m_n} (r_n^j/q_n)v_n^j(i) - v(i)\right| < \frac{1}{n}. \tag{1}$$

Without loss of generality we may assume that $1 \leqq q_1 < \cdots < q_n < q_{n+1} < \cdots$. (If $q_n \geqq q_{n+1}$ multiply $q_{n+1}, r_{n+1}^1, \ldots, r_{n+1}^{m_n}$ by q_n and redefine them as $\bar{q}_{n+1}$, $\bar{r}_{n+1}^1, \ldots, \bar{r}_{n+1}^{m_n}$.) For every $n \geqq 1$, define $s_n^0 = 0$, and for every $1 \leqq j \leqq m_n$, $s_n^j = \sum_{k=1}^j r_n^k$. Define the sequence $\{T_k\}_{k=0}^\infty$ as follows:

$$T_0 = 0,$$

$$T_1 = q_1 q_2,$$

$$,\ldots,$$

$$T_{k+1} = T_k + q_{k+1} q_{k+2},$$

$$,\ldots.$$

As $v_n^j \in V_{1/n}$ for every $n \geqq 1$ and every $1 \leqq j \leqq m_n$, we may associate to each v_n^j an $a_n^j \in A$ s.t. $a_n^j \in E_{1/n}$. Now, we will construct a strategy for G^∞ using the following procedure: For the first block of r_1^1 periods, players play according to the aggregate action a_1^1. For the next r_1^2 periods they play according to $a_1^2, \ldots$, up to the period $s_1^{m_1}$. This takes care of the first q_1 periods. This sequence of joint actions will next be repeated q_2 times, so that the first $q_1 q_2$ periods are taken care of. Similarly, after T_{k-1} periods, we can carry out the above construction for the succeeding $q_k q_{k+1}$ periods where: for the next block of r_k^1 periods players play according to a_k^1, for the following r_k^2 periods they play according to $a_k^2, \ldots$, up to the period $T_{k-1} + s_k^{m_k}$, therefore exhausting the first q_k periods. After T_{k-1} this sequence of length q_k is repeated q_{k+1} times. Therefore, we have described the strategy for the first $T_k = T_{k-1} + q_k q_{k+1}$ periods. Formally, define $\bar{f} \in F$ as follows: for $T_{k-1} < t \leqq T_k$, let $\bar{f}_i^t(\cdot) = a_k^j(i)$ for a.e. $i \in I$, where j is such that

$$T_{k-1} + s_k^{j-1} \leqq T_{k-1} + (t - T_{k-1}) (\operatorname{mod} q_k) < T_{k-1} + s_k^j.$$

Lemma 1. $\lambda(\{i \in I \mid H_i(\bar{f}) = v(i)\}) = 1.$

Proof. See appendix.

Therefore, the strategy $\bar{f}$ generates a sequence of payoffs $\{v^t\}_{t=1}^\infty$ where $v^t(i) = h_i(a^t(\bar{f})(i), a^t(\bar{f}))$ a.e. $i \in I$, such that for $T_{k-1} < t \leqq T_k$, $v^t \in V_{1/k}$ (call

$\varepsilon_t = 1/k$, then $\lim_{t\to\infty} \varepsilon_t = 0$). Therefore, $v \in Z^\infty$. Thus, by Proposition 3, $v \in V^{\infty *}$. Q.E.D.

The next example illustrates our next result in which we give a necessary condition for equilibrium.

Example 2. In the stage game G, the action set for each $i \in I$ is $A_i = \{0, 1\}$ and the payoff function is

$$h_i(a_i, a) = \begin{cases} 1 & \text{if} \quad a_i = 1 \text{ and } \int a < 1 \\ 0 & \text{if} \quad a_i = 0 \text{ and } \int a > 0. \\ -1 & \text{otherwise} \end{cases}$$

It is easy to see that $A^* = \emptyset$. Given $\varepsilon > 0$, consider an aggregate action a in which all players play action 1 except a subset of measure less than ε.

Clearly $a \in E_\varepsilon$ and hence $V_\varepsilon \neq \emptyset$. Consider $v \in V^{\infty *}$ defined by $v(i) = 1$ a.e. $i \in I$. A straightforward argument shows that $v \in HV_\varepsilon$ for every $\varepsilon > 0$.

The following theorem shows that this is general; i.e., an equilibrium payoff has to belong to HV_ε for every $\varepsilon > 0$.

Theorem 2. $V^{\infty *} \subset \cap_{\varepsilon > 0} HV_\varepsilon$.

Proof. Let v be an equilibrium payoff of G^∞. There exists $f \in F^*$ such that $\lambda(\{i \in I \mid H_i(f) = v(i)\}) = 1$. Let $\varepsilon > 0$ be arbitrary; we have to show that $v \in HV_\varepsilon$. Consider the sequence $\{a^t(f)\}_{t=1}^\infty$ of elements of A generated by f. Define $B = \{t \in \mathbb{R} \mid a^t(f) \notin E_\varepsilon\}$ and for every $T \geqq 1$ let $\# B(T)$ be the cardinality of $B \cap \{1, 2, \dots, T\}$.

Lemma 2. $f \in F^* \Rightarrow H(\{(1/T)\ \# B(T)\}_{T=1}^\infty) = 0$.

Proof. See appendix.

For every $t \geqq 1$, define $w^t(i) = h_i(a^t(f)(i), a^t(f))$ a.e. $i \in I$. Then for a.e. $i \in I$,

$$v(i) = H_i(f) = H\left(\left\{\frac{1}{T} \sum_{t=1}^{T} w^t(i)\right\}\right)$$

$$= H\left(\left\{\frac{1}{T} \sum_{t \in \sim B(T)} w^t(i)\right\}\right) + H\left(\left\{\frac{1}{T} \sum_{t \in B(T)} w^t(i)\right\}\right)$$

by linearity of H. Now, using again linearity of H, uniform-boundedness of payoffs, and Lemma 2, it is easy to see that

$$H\left(\left\{\frac{1}{T}\sum_{t\in B(T)} w^t(i)\right\}\right)=0.$$

Therefore,

$$v(i)=H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} w^t(i)\right\}\right)=H\left(\left\{\frac{1}{T}\sum_{t\in\sim B(T)} w^t(i)\right\}\right). \tag{2}$$

For every $T\geqq 1$, define $C[T]=\{t\in\sim B \mid t\leqq \bar{T}$ where $\bar{T}$ is s.t. $T=\#\sim B(\bar{T})\}$.

Lemma 3. For a.e. $i\in I$,

$$H\left(\left\{\frac{1}{T}\sum_{t\in\sim B(T)} w^t(i)\right\}\right)=H\left(\left\{\frac{1}{T}\sum_{t\in C[T]} w^t(i)\right\}\right).$$

Proof. See appendix.

Rewrite $\sim B=(t_1,t_2,\dots\}$ and define the sequence $\{v^n\}_{n=1}^{\infty}$ by $v^n=w^{t_n}$. Notice that $\forall n\geqq 1$, $v^n\in V_\varepsilon$, since $t_n\in\sim B$ implies that $w^{t_n}\in V_\varepsilon$. Clearly, for a.e. $i\in I$:

$$H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} v^t(i)\right\}\right)=H\left(\left\{\frac{1}{T}\sum_{t\in C[T]} w^t(i)\right\}\right)$$

$$=H\left(\left\{\frac{1}{T}\sum_{t\in\sim B(T)} w^t(i)\right\}\right) \quad \text{(by Lemma 3)}$$

$$=v(i) \quad \text{[by (2)].}$$

Therefore, $\exists\{v^t\}_{t=1}^{\infty}$ s.t. $\forall t\geqq 1$, $v^t\in V_\varepsilon$ and for a.e. $i\in I$,

$$H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} v^t(i)\right\}\right)=v(i).$$

Hence $v\in HV_\varepsilon$. Q.E.D.

The following example shows that the set V_ε may be empty (for sufficiently small ε). Therefore, by Theorem 2, it also shows a case in which $V^{\infty *}$ is empty.

Example 3. In the stage game G, the action set for each $i \in I$ is $A_i = \{0, 1\}$ and the payoff function is

$$h_i(0, a) = \begin{cases} 1 & \text{if } \int a \geqq 1/2, \\ -1 & \text{otherwise,} \end{cases}$$

$$h_i(1, a) = 0.$$

In order to see that $\forall \varepsilon \in [0, 1/2)$ the set V_ε is empty, consider an arbitrary $a \in A$. Either $\int a \geqq 1/2$, in which case

$$\lambda(\{i \in I \mid a(i) = 1\})$$

$$= \lambda\left(\left\{i \in I \,\middle|\, \sup_{b_i \in A_i} h_i(b_i, a) = h_i(0, a) = 1 > \varepsilon = h_i(1, a) + \varepsilon\right\}\right) > 1/2 > \varepsilon;$$

or else $\int a < 1/2$, in which case

$$\lambda(\{i \in I \mid a(i) = 0\})$$

$$= \lambda\left(\left\{i \in I \,\middle|\, \sup_{b_i \in A_i} h_i(b_i, a) = h_i(1, a) = 0 > -1 + \varepsilon = h_i(0, a) + \varepsilon\right\}\right) > 1/2 > \varepsilon.$$

Hence, the set V_ε is empty for every $\varepsilon \in [0, 1/2)$.

Before finishing this section, a few comments are in order.

First, about the characterization itself. It should be noted that Theorems 1 and 2 do not give us an 'if and only if' type of characterization of $V^{\infty *}$. Rather, they say

$$\bigcap_{\varepsilon > 0} \overline{\text{co} V_\varepsilon} \subset V^{\infty *} \subset \bigcap_{\varepsilon > 0} H V_\varepsilon. \tag{3}$$

The sets $\bigcap_{\varepsilon > 0} \text{co} V_\varepsilon$ and $\bigcap_{\varepsilon > 0} H V_\varepsilon$ differ, however, only on non-converging sequences. To see this, for every $\varepsilon > 0$ define

$$\overline{HV_\varepsilon}=\left\{v\in V^\infty \,\middle|\, \exists\{v^t\}\text{ s.t. } v^t\in V_\varepsilon \forall t\geqq 1 \text{ and a.e. } i\in I \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} v^t(i)=v(i)\right\}$$

as the set of payoffs obtained by sequences of ε-almost-equilibrium payoffs whose limit of averages converges almost everywhere. Obviously $\overline{HV_\varepsilon}\subset HV_\varepsilon$. It is a straightforward verification to see that $\overline{HV_\varepsilon}\subset\overline{\mathrm{co}V_\varepsilon}$. Therefore $\bigcap_{\varepsilon>0}\overline{HV_\varepsilon}\subset\bigcap_{\varepsilon>0}\overline{\mathrm{co}V_\varepsilon}$.

Second, about the relationship between the Nash equilibrium payoffs of the stage game (V_0) and the items in (3). It is easy to show that $V_0=\bigcap_{\varepsilon>0}V_\varepsilon$; therefore, it follows that $\overline{\mathrm{co}V_0}=\overline{\mathrm{co}(\bigcap_{\varepsilon>0}V_\varepsilon)}\subseteq\bigcap_{\varepsilon>0}\overline{\mathrm{co}V_\varepsilon}$ and $HV_0=H(\bigcap_{\varepsilon>0}V_\varepsilon)\subseteq\bigcap_{\varepsilon>0}HV_\varepsilon$. Thus, the set of Nash equilibrium payoffs of the repeated game, $V^{\infty *}$, contains elements which do not belong to $\overline{\mathrm{co}V_0}$ (the closed convex hull of the Nash equilibrium of the stage game) because one cannot interchange the order of intersection and convex hull, i.e. $\overline{\mathrm{co}(\bigcap_{\varepsilon>0}V_\varepsilon)}$ may be a strict subset of $\bigcap_{\varepsilon>0}\overline{\mathrm{co}V_\varepsilon}$.

Third, about the use of the limit medial for evaluating non-convergent sequences of payoff averages. Some of our results rely strongly on the fact that we may interchange the order of integration with the 'limit' operation [property (ii) of H resembles the Lebesgue Convergence Theorem]. For example, in the proof of Lemma 2 this property allows us to show that if for a.e. $i\in I$ the 'proportion of times' that player i belongs to the set of players that can improve their payoff (when a payoff is not an ε-almost-equilibrium one) by more than ε is equal to zero, then the 'proportion of times' in which the payoffs are not ε-almost-equilibria is also equal to zero. It is also important to notice that generally the liminf and limsup do not satisfy this property; we do not know whether or not Banach limits do in general.

Appendix

Lemma 1. $\lambda(\{i\in I \mid H_i(\bar{f})=v(i)\})=1$.

Proof. In order to shorten the notation let $h_i^t\equiv h_i(a^t(\bar{f})(i),a^t(\bar{f}))$. Assume $T_K\leqq T<T_{K+1}$ and let

$$\phi_{pk}=\{t\geqq 1 \mid T_{k-1}+s_k^{p-1}\leqq T_{k-1}+(t-T_{k-1})(\mathrm{mod}\, q_k)<T_{k-1}+s_k^p\}.$$

For a.e. $i\in I$,

$$\left|\frac{1}{T}\sum_{t=1}^{T}h_i^t-\frac{1}{T}\sum_{t=1}^{T}v(i)\right|$$

$$=\left|\frac{1}{T}\left[\sum_{k=1}^{K}\sum_{T_{k-1}<t\leqq T_k}(h_i^t-v(i))+\sum_{T_K<t\leqq T}(h_i^t-v(i))\right]\right|$$

$$=\left|\frac{1}{T}\left[\sum_{k=1}^{K}\sum_{p=1}^{m_k}\sum_{t\in\phi_{pk}}(h_i^t-v(i))+\sum_{T_K<t\leqq T}(h_i^t-v(i))\right]\right|$$

$$\leqq\frac{1}{T}\left|\sum_{k=1}^{K}\sum_{p=1}^{m_k}\sum_{t\in\phi_{pk}}(v_k^p(i)-v(i))\right|+\frac{1}{T}\left|\sum_{T_K<t\leqq T}(h_i^t-v(i))\right|$$

by definition of $\tilde{f}$.

The first term of this last expression is equal to

$$\frac{1}{T}\left|\sum_{k=1}^{K}q_{k+1}\sum_{p=1}^{m_k}r_k^p(v_k^p(i)-v(i))\right|$$

$$=\frac{1}{T}\left|\sum_{k=1}^{K}q_{k+1}\left(\sum_{p=1}^{m_k}r_k^p v_k^p(i)-q_k v(i)\right)\right|$$

$$=\frac{1}{T}\left|\sum_{k=1}^{K}\left[\frac{T_k-T_{k-1}}{q_k}\right]\left[\sum_{p=1}^{m_k}r_k^p v_k^p(i)-q_k(v(i)\right]\right|$$

(because $T_k=T_{k-1}+q_kq_{k+1}$)

$$\leqq\frac{1}{T}\sum_{k=1}^{K}(T_k-T_{k-1})\left|\sum_{p=1}^{m_k}\left(\frac{r_k^p}{q_k}\right)v_k^p(i)-v(i)\right|$$

$$<\frac{1}{T}\sum_{k=1}^{K}\frac{T_k-T_{k-1}}{k}\quad\text{[by eq. (1)]}$$

$$\leqq\frac{1}{T_k}\sum_{k=1}^{K}\frac{T_k-T_{k-1}}{k}\quad\text{(because } T_K\leqq T\text{)}.$$

Now, in order to find an upper bound for the second term $(1/T)$ $\left|\sum_{T_K<t\leqq T}(h_i^t-v(i))\right|$, notice that $\exists C,\ D\in\mathbb{R}$ s.t. $T-T_K=Cq_{K+1}+D$, $0\leqq D<q_{K+1}$ and $0\leqq C<q_{K+2}$. Therefore this second term is bounded above by

$$\frac{1}{T}\left|\sum_{T_K<t\leqq Cq_{K+1}+T_K}(h_i^t-v(i))\right|+\frac{1}{T}\left|\sum_{T_K+Cq_{K+1}<t\leqq T_K+Cq_{K+1}+D}(h_i^t-v(i))\right|$$

$$\leqq\frac{1}{T}\left|\sum_{p=1}^{m_K}\sum_{t\in\phi_{pK+1}}(h_i^t-v(i))\right|+\frac{1}{T}\,q_{K+1}\,y$$

$$\leqq\frac{1}{T}\,C\left|\sum_{p=1}^{m_K}r_{K+1}^p v_{K+1}^p(i)-q_{K+1}v(i)\right|+\frac{1}{T}\,q_{K+1}\,y$$

$$=\frac{1}{T}\,Cq_{K+1}\left|\sum_{p=1}^{m_K}\left(\frac{r_{K+1}^p}{q_{K+1}}\right)v_{K+1}^p(i)-v(i)\right|+\frac{1}{T}\,q_{K+1}\,y$$

$$<\frac{1}{T}\,\frac{Cq_{K+1}}{(K+1)}+\frac{1}{T}\,q_{K+1}\,y$$

$$\leqq\frac{1}{T}\,\frac{(T-T_K)}{(K+1)}+\frac{1}{T_K}\,q_{K+1}\,y$$

$$\leqq\frac{1}{K+1}+\frac{y}{q_K}\quad\text{(because } T_K\geqq q_K q_{K+1}\text{)}.$$

Therefore, for a.e. $i\in I$ and $T_K\leqq T<T_{K+1}$,

$$\left|\frac{1}{T}\sum_{t=1}^{T}h_i^t-v(i)\right|<\frac{1}{T_K}\sum_{k=1}^{K}\frac{T_k-T_{k-1}}{k}+\frac{1}{K+1}+\frac{y}{q_K}.$$

Hence,

$$\lim_{T\to\infty}\left|\frac{1}{T}\sum_{t=1}^{T}h_i^t-v(i)\right|<\lim_{K\to\infty}\frac{1}{T_K}\sum_{k=1}^{K}\left(\frac{T_k-T_{k-1}}{k}\right).$$

Fix K_1 and let $K>K_1>1$, then

$$\frac{1}{T_K}\sum_{k=1}^{K}\left(\frac{T_k-T_{k-1}}{k}\right)=\frac{1}{T_K}\sum_{k=1}^{K_1-1}\left(\frac{T_k-T_{k-1}}{k}\right)+\frac{1}{T_K}\sum_{k=K_1}^{K}\left(\frac{T_k-T_{k-1}}{k}\right)$$

$$\leqq\frac{1}{T_K}\sum_{k=1}^{K_1-1}\left(\frac{T_k-T_{k-1}}{k}\right)+\frac{1}{T_K}\sum_{k=K_1}^{K}\left(\frac{T_k-T_{k-1}}{K_1}\right)$$

$$\leqq \frac{1}{T_K} \sum_{k=1}^{K_1-1} \left(\frac{T_k - T_{k-1}}{k}\right) + \frac{1}{T_K}\left(\frac{T_K - T_{K-1}}{K_1}\right)$$

$$\leqq \frac{1}{T_K} \sum_{k=1}^{K_1-1} \left(\frac{T_k - T_{k-1}}{k}\right) + \frac{1}{K_1}.$$

Therefore,

$$\lim_{T\to\infty} \left|\frac{1}{T} \sum_{t=1}^{T} h_i^t - v(i)\right| < \lim_{K\to\infty} \frac{1}{T_K} \sum_{k=1}^{K_1-1} \left(\frac{T_k - T_{k-1}}{k}\right) + \frac{1}{K_1} = \frac{1}{K_1}.$$

But since this is true for every $K_1 \geqq 1$, we conclude that for a.e. $i \in I$

$$v(i) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} h_i^t = H_i(\tilde{f}). \qquad \text{Q.E.D.}$$

Lemma 2. $f \in F^* \Rightarrow H(\{(1/T)\ \#\ B(T)\}) = 0.$

Proof. Suppose $H(\{(1/T)\ \#\ B(T)\}) > 0$ and consider a particular sequence $\{a^{t\,\hat{}}(f)\}_{t\in B}$ where for every $t \in B$, $a^{t\,\hat{}}(f)$ is a function belonging to the equivalence class $a^t(f)$. For $t \in B$, define

$$I_t = \left\{ i \in I \,\middle|\, \sup_{b_i \in A_i} h_i(b_i, a^t(f)) > h_i(a^{t\,\hat{}}(f)(i), a^t(f)) + \varepsilon \right\}.$$

Since $t \in B$, we know that $\lambda(I_t) > \varepsilon$. For every $i \in I$, define $B^i = \{t \in B \mid i \in I_t\}$. Let K be the smallest integer s.t. $K\varepsilon \geqq 1$. Define the sequence of measurable functions $\{Y_k : I \to \mathbb{R} \cup \{0\}\}_{k=1}^{\infty}$ as follows:

$$Y_1 = \sum_{\substack{t\in B \\ 1 \leqq t \leqq T \\ \#B(T) = K+1}} \chi_{I_t},$$

where χ_{I_t} is the characteristic function of I_t; and for $k > 1$

$$Y_k = \sum_{\substack{t\in B \\ 1 \leqq t \leqq T \\ \#B(T) = k(K+1)}} \chi_{I_t}.$$

Define $J_k = \{i \in I \mid Y_k(i) \geqq k+1\}$. First, we will show that for every $k \geqq 1$, $\lambda(J_k) > \varepsilon/(K+1)$. In order to do so, notice that

$$\varepsilon k(K+1) < \sum_{\substack{t\in B\\ 1\leqq t\leqq T\\ \# B(T)=k(K+1)}} \lambda(I_t) = \sum_{\substack{t\in B\\ 1\leqq t\leqq T\\ \# B(T)=k(K+1)}} \int_I \chi_{I_t}\,\mathrm{d}\lambda$$

$$= \int_I Y_k\,\mathrm{d}\lambda = \int_{J_k} Y_k\,\mathrm{d}\lambda + \int_{\sim J_k} Y_k\,\mathrm{d}\lambda \leqq \int_{J_k} Y_k\,\mathrm{d}\lambda + k,$$

the last inequality because $\sim J_k = \{i\in I \mid Y_k(i)\leqq k\}$ and its measure is trivially less than or equal to 1. Therefore,

$$\varepsilon k \leqq \varepsilon kK + \varepsilon k - k < \int_{J_k} Y_k\,\mathrm{d}\lambda \leqq \lambda(J_k)k(K+1)$$

since $Y_k(i)\leqq k(K+1)\ \forall i\in I$. Hence $\lambda(J_k) > \varepsilon/(K+1)$.

Now, for every $T\geqq 1$ there exists $k\geqq 1$ s.t.

$$\frac{1}{T}(k-1)(K+1) \leqq \frac{1}{T}\ \# B(T) < \frac{1}{T}\ k(K+1).$$

This implies that

$$\lambda\left(\left\{i\in I \,\middle|\, \frac{1}{T}\ \# B^i(T) \geqq \frac{1}{T}\ Y_{k-1}(i) \geqq \frac{1}{T}\ k > \frac{1}{T}\frac{\# B(T)}{(K+1)}\right\}\right) > \frac{\varepsilon}{K+1}.$$

Therefore, for every $T\geqq 1$,

$$\int_I \frac{1}{T}\ \# B^i(T)\,\mathrm{d}\lambda > \frac{1}{T}\frac{\# B(T)\varepsilon}{(K+1)^2}.$$

Now for property (ii) of the limit medial

$$\int_I H\left(\left\{\frac{1}{T}\ \# B^i(T)\right\}\right)\mathrm{d}\lambda = H\left(\left\{\int_I \frac{1}{T}\ \# B^i(T)\,\mathrm{d}\lambda\right\}\right) \geqq H\left(\left\{\frac{1}{T}\ \# B(T)\right\}\right)\frac{\varepsilon}{(K+1)^2}.$$

Since our hypothesis is that $H(\{(1/T)\ \# B(T)\})>0$, we conclude that $\int_I H(\{(1/T)\ \# B^i(T)\})\,\mathrm{d}\lambda > 0$, implying that there exists $J\in S$ with positive measure and with the property that for a.e. $i\in J$, $H(\{(1/T)\ \# B^i(T)\})>0$. Since Y^∞ can be written as

$$Y^\infty = \Bigg\{v\in V^\infty \mid \exists \{v^t\}_{t=1}^\infty \text{ s.t. } v^t\in V\ \forall t\geqq 1 \text{ and for a.e. } i\in I,$$

$$H\left(\left\{\frac{1}{T}\sum_{t=1}^{T} v^t(i)\right\}_{T=1}^{\infty}\right) = v(i) \text{ and } \forall \varepsilon>0,\ H\left(\left\{\frac{1}{T}\ \# B^i_\varepsilon(T)\right\}_{T=1}^{\infty}\right) = 0\Bigg\},$$

by Proposition 2, it would contradict the fact that $f \in F^*$. Q.E.D.

Lemma 3. *For a.e.* $i \in I$,

$$H\left(\left\{\frac{1}{T}\sum_{t\in \sim B(T)} w^t(i)\right\}\right) = H\left(\left\{\frac{1}{T}\sum_{t\in C[T]} w^t(i)\right\}\right).$$

Proof. Since $H(\{(1/T)\ \#\, B(T)\}) = 0$ we have that $H(\{(1/T)\ \#\ \sim B(T)\}) = 1$ by linearity of H. Therefore, for a.e. $i \in I$,

$$H\left(\left\{\frac{1}{T}\sum_{t\in \sim B(T)} w^t(i)\right\}\right) - H\left(\left\{\frac{1}{T}\sum_{t\in C[T]} w^t(i)\right\}\right)$$

$$= H\left(\left\{-\frac{1}{T}\sum_{t=\#\sim B(T)+1} w^t(i)\right\}\right) \qquad (*)$$

$$\geqq H\left(\left\{\frac{1}{T}(T - \# \sim B(T))\right\}\right)(-y)$$

$$= \left[1 - H\left(\left\{\frac{1}{T}\ \# \sim B(T)\right\}\right)\right](-y)$$

$$= 0.$$

Also,

$$(*) \leqq H\left(\left\{\frac{1}{T}(T - \# \sim B(T))\right\}\right) y$$

$$= \left[1 - H\left(\left\{\frac{1}{T}\ \# \sim B(T\right\}\right)\right] y$$

$$= 0. \qquad \text{Q.E.D.}$$

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