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A Note on Reputation: More on the Chain-Store Paradox

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This note considers the reputation phenomenon in the context of the Chain-Store Paradox. Two major aspects of the perfect information assumption are relaxed: potential entrants do not know the ordering in which they have to make their entry decisions and they do not have full knowledge of the past history of the market. It is shown that, without introducing private information or changing the nature of the conflict, there exist sequential equilibria of the game with imperfect information in which the monopolist is willing to build reputation. *Journal of Economic Literature* Classification Numbers: C72, D82, L13.

1. INTRODUCTION

The main purpose of this note is to present several examples illustrating the difficulty of modeling reputation in a game-theoretic context. It is easy to think of economic situations in which reputation (in its everyday meaning) may play an important role in explaining rational behavior: somebody, Mr. M, is willing to incur losses today to influence the future actions of somebody else by changing his beliefs about Mr. M's future actions. However, it has proven difficult to model such phenomena in finite horizon models.

This is by no means a new problem in economics. One of the best known examples in which one would think that reputation could emerge at equilibrium is the Chain-Store Paradox: A monopolist (M) faces a set of potential competitors ($E_1, \ldots, E_T$) deciding sequentially whether or not to enter the market. If a potential entrant E_n decides to stay out (O) he receives a payoff of zero and

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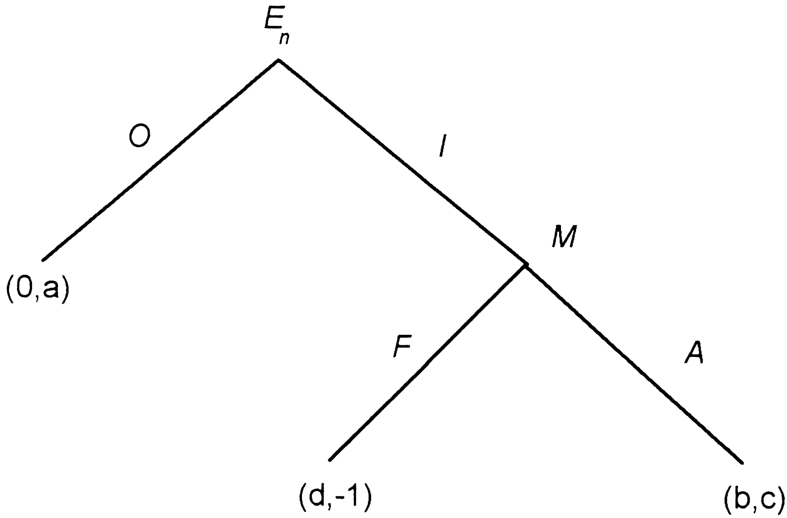

FIGURE 1

M receives a payoff of $a > 0$. If E_n decides to enter (I), payoffs depend on M's response; if M fights the entrant (F), E_n and M receive $d < 0$ and -1, respectively; if M acquiesces (A), E_n and M receive $b > 0$ and $-1 < c < a$, respectively (see Fig. 1).

Moreover, the description of the conflict and the rationality of all the firms are supposed to be common knowledge.

As Selten (1978) pointed out, even though it is a Nash equilibrium for every potential entrant to stay out and for the monopolist to fight each one of them, it is not a very sensible one[1] since the decision of the last potential entrant (E_T) to stay out can be rationalized only if he believes that the monopolist is going to carry out a non optimal decision in the last period. Hence, in any sensible equilibrium, E_T should decide to enter (independently of what happened previously) and M should play A. Knowing this, the potential entrant E_{T-1} has to decide to enter, since his action and the answer of M at period $T-1$ will not have any effect on the last period decisions and therefore, using a similar argument to that for E_T, he also knows that his decision to enter the market has to be answered with A. This argument can be carried out for all potential entrants, implying that the unique sensible (or subgame-perfect) equilibrium is the one in which all potential entrants decide to enter and the monopolist never fights. This example has been called the Chain-Store Paradox precisely because, even though it seems intuitive

[1] It is not a subgame-perfect equilibrium. See Selten (1975) and (1978).

that the monopolist can deter entrants by being prepared to build a reputation for fighting, this does not survive the perfectness criterion.

Although the argument above ruling out any reputation equilibrium (a $T+1$-tuple of strategies such that there is at least one period in which the monopolist is willing to respond to an entry by fighting) seems robust to alternative specifications of the game (at least as long as the number of potential entrants is kept finite), Kreps and Wilson (1982a) and Milgrom and Roberts (1982) showed in related papers that this is not so. They argue, and I think in a sensible way, that the complete information assumption in the Chain-Store Paradox is more a modeling artifact than a good representation of real situations. It is easy to think that potential entrants are in fact uncertain about the monopolist's payoffs. Paraphrasing Kreps and Wilson (1982a), suppose that for whatever reason, potential entrants assess some positive probability δ that the monopolist's payoffs after an "in" decision by the entrant are not as in Fig. 1 (weak monopolist), but rather they are -1 if the monopolist's answer is A, or 0 if the monopolist's answer is F, reflecting a short-term benefit from a fighting response. In the latter case we say that the monopolist is strong. Using Harsanyi's (1967–1968) way of transforming a game of incomplete information into a game of imperfect information, they show that for all values of $b \in (0, 1)$ and $\delta \in (0, 1)$ there is a sequential equilibrium (see Kreps and Wilson, 1982b) in which the strong monopolist always fights and the weak monopolist fights with positive probability at early stages of the game, i.e., the weak monopolist acquiesces in only a limited number of final periods. Therefore, even though the information may be almost complete, i.e., δ may be very small, the reputation effect comes alive. Moreover, they also show that in general ($\delta \neq b^n$ for $0 < b < 1$ and $1 \leq n \leq T$) the equilibrium is unique if one restricts the beliefs of the entrants to satisfy a natural and intuitive restriction.[2]

One might be tempted to conclude that the existence of the slightest uncertainty about payoffs allows us to explain reputation as an equilibrium phenomenon and therefore that this resolves the paradox. Nevertheless, as Kreps and Wilson themselves suspected, "By cleverly choosing the nature of that small uncertainty (precisely—its support), one can get out of a game-theoretic analysis whatever one wishes."[3] Furthermore, one might have some doubts about considering the game of incomplete information as a game somehow "close" to the original game even for small amounts of uncertainty (δ close to zero).

Here I would like to argue that there are other basic features of the perfect information assumption in the model of the Chain-Store Paradox which preclude

[2] The beliefs of the entrants are called plausible if given two histories h_t and h'_t of play up to stage t, if the monopolist was more aggressive in history h_t than in history h'_t, that is, if some plays of F in h_t become A plays in h'_t, then the revised probability that the monopolist is strong after h_t can not be smaller than after h'_t.

[3] See Fudenberg and Maskin (1986) for a related statement and proof of these suspicions.

(by the backwards induction argument) sensible reputation equilibria and, furthermore, that these features may not be a good representation of real situations. I am thinking of two interrelated aspects of the perfect-information assumption. The first aspect is that before making his decision, a potential entrant E_n ($n > 1$) is assumed to have full knowledge of the history of the game up to period $n-1$; in particular, he is assumed to be able to observe that, for instance, entrant E_{n-1} has decided to stay out of the market. The second aspect is that not only do the potential entrants know how many firms are considering the possibility of entering the market, but also the order in which they will make their decisions. Let us suppose for a moment that potential entrants are uncertain about the ordering in which they have to make the "enter" or "out" decision and moreover, they are unable to observe the others' "out" decisions. Then, when a potential entrant is considering his decision after observing that so far no firm has entered the market, he may still be unsure whether he is the first one considering the possibility of entering or perhaps he is the last one but everybody else has decided to stay out. I think that this description of the game, with imperfect—rather than incomplete—information prior to the entry decision may represent more accurately the context of real situations. This approach requires neither incomplete information nor changing the structure of the game in a radical way. Moreover, there is something unnatural about assuming that agents can distinguish between those firms that have not yet decided whether to enter the market or not and those that have *already* decided *not* to enter.

Notice that, to achieve uncertainty in the environment, these two aspects must be present together, because uncertainty on the ordering but full knowledge of the past history would generate a situation in which every potential entrant E_n knows with probability one that he is in fact the tth potential entrant. In other words, if the potential entrant knows the full history h, he can infer, by just looking the length of h, say $t-1$, that he is the tth potential entrant. Then the backwards induction argument made at the beginning of this section would still apply, upsetting any possibility of observing reputation at equilibrium. Also, Appendix B in Milgrom and Roberts (1982) shows an example illustrating their claim regarding the role of lack of common knowledge in generating predation. They say that "as soon as the complete information assumption on the game is relaxed, so that the common knowledge condition no longer obtains, the logic of the backward induction breaks down."

In this note, I want to investigate whether these imperfect information modifications of the Chain-Store Paradox (potential entrants do not know the ordering in which they have to make the entry decision and they do not have full knowledge of the past history of the market) may explain the reputation phenomenon as a sequential equilibrium outcome. Before moving on, a word of caution is in order: perhaps unsurprisingly, I have not found a clear and sharp answer to the problem. Rather, as the work of Kreps and Wilson, Milgrom and Roberts, and Fudenberg and Maskin may suggest, it seems the existence of reputation equilibria is very sensitive to different specifications of the game. In particular, here, given the uncertainty on the ordering of play, the quality of the information about the history of the market held by potential entrants prior to their decisions appears to be a determinant of whether or not there exists a sequential equilibrium with reputation. This is because the information plays a double role; it may be seen by the monopolist as a way to build up his reputation, but at the same time, it may also be used by a potential entrant to compute posterior probabilities of where he is in the ordering.

Since the main goal of this note is to investigate the role of the perfect information assumptions in preventing the existence of sequential equilibria with reputation, the multiplicity issue is not analyzed; I restrict myself to the particular class of sequential equilibria in which consistent beliefs possesses a good deal of symmetry and behavioral strategies are always in pure actions. However, in contrast with Kreps and Wilson's work, when reputation equilibria are found, they are never unique since the constant strategy "enter" for every potential entrant and the constant strategy "acquiescence" for the monopolist (the unique subgame-perfect equilibrium of the perfect information game) are always an equilibrium. Since the unique rational justification of a fight is the possible effect that it may have on future potential entrants' decisions, if nobody pays attention to what has happened in the past (i.e., reputation has no role) the game becomes a sequence of independent games whose overall equilibrium is nothing else that the sequence of unique equilibria of every independent game. I do not see this as a negative result; rather, any model pretending to explain the reputation phenomenon as it is understood here (M is willing to incur losses today, because in doing so, he may influence potential entrants' actions) would have to allow for nonreputation behavior to be an equilibrium of the model.

In the next section, various models are described, and results are presented for different examples. Section 3 contains general comments and conclusions. An Appendix at the end of the paper contains complete proofs, or their most important arguments, of some of the results that are more illustrative in terms of the techniques that one needs to use.

2. EXAMPLES AND RESULTS

In this section I analyze different versions of the Chain-Store game. Before doing so we need a bit of notation.

The set $N \cup M$ represents the set of players where M is the monopolist and $N = \{1, 2, \ldots, T\}$ is the set of potential entrants ($n \in N$ will denote a typical element of N). Let H_t be the set of all possible histories of play up to period t ($1 \leq t \leq T-1$), i.e., $h_t \in H_t$ is a sequence of t different elements ($h_t = \{h_t^\tau\}_{\tau=1}^t$), where each element h_t^τ is drawn from the set $\{(\hat{O}_n)_{n\in N}, (\hat{F}_n)_{n\in N}, (\hat{A}_n)_{n\in N}\}$, where $h_t^\tau = \hat{O}_n$

means that at period τ $(1 \leq \tau \leq t)$ the potential entrant n decided to stay out, $h_t^\tau = \hat{F}_n$ (resp. $\hat{A}_n$) means that n decided to enter and the monopolist answered with fighting (resp. acquiescing). As a convention, set $H_0 = \emptyset$. I also assume that each entrant plays just once, that is, for any $Z, W \in \{\hat{O}, \hat{F}, \hat{A}\}$, any $n \in N$, and any ν and τ such that $1 \leq \nu < \tau \leq t \leq T$: $h_t^\nu = Z_n \Rightarrow h_t^\tau \neq W_n$. Therefore, the set of all possible histories before a potential entrant decides whether to enter or to stay out is

$$H = \cup_{t=1}^T H_{t-1}.$$

Given $h_t \in H_t$ and $h_\tau \in H_\tau$ $(t \geq \tau)$ we say that h_t includes h_τ (represented by $h_t \supset h_\tau$) if $h_t^\nu = h_\tau^\nu$, $\forall 1 \leq \nu \leq \tau$. Given $h \in H$, let $H|h = \{h' \in H \colon h' \supset h\}$.

The first important modification of the Chain-Store Paradox is that potential entrants do not know the order in which they have to make their decisions. Assume that a prior distribution p on the set of all possible orderings of N (i.e., with $T!$ points in its support) is given and it is common knowledge. From it, it is possible to compute the probability that potential entrant $n \in N$ has to make his decision at period t $(1 \leq t \leq T)$ which will be denoted by $p(n, t)$. I will first consider the situation in which at *every* period, one and only one of the potential entrants will decide on either I or O. Therefore, it is easily seen that for every $1 \leq t \leq T$ and respectively, every $n \in N$, $p(\cdot, t)$ and $p(n, \cdot)$ are indeed probability distributions, i.e.,

$$\sum_{n=1}^{T} p(n, t) = 1 \text{ for every } 1 \leq t \leq T$$

and

$$\sum_{t=1}^{T} p(n, t) = 1 \text{ for every } n \in N.$$

Let us assume that the prior is such that for every $1 \leq t \leq T$ and every $n \in N$, $p(n, t) > 0$.

Before defining behavioral strategies in the game, notice that from the point of view a potential entrant, knowing the full history of the game would tell him where he is in the ordering. Therefore in order not to remove all the uncertainty at this point, assume that there is a function $f \colon H \to X$, where X is a given set of signals. Then, f generates a partition on H as follows: $h, h' \in H$ belong to the same set of the partition iff $f(h) = f(h')$. Prior to his decision a potential entrant only knows $x \in X$ (which is equivalent to knowing the set $\{h \in H \colon f(h) = x\}$); hence, from the potential entrant's point of view, the set of periods at which he may be making his decision is denoted by $U_{f(h)}$ (i.e., a subset of $\{1, 2, \ldots, T\}$). To illustrate this, suppose there are only two possible signals,

x and x', depending, respectively, on whether or not the monopolist has failed to fight; thus, f maps any history in which the monopolist has never acquiesced into x and any history in which the monopolist has acquiesced at least once into x' (the partition of H possesses only two elements). Then, $U_x = \{1, 2, \ldots, T\}$ and $U_{x'} = \{2, 3, \ldots, T\}$. Now, in general, a strategy for $n \in N$ is a function $r_n\colon X \to \{I, O\}$. Given r_n for every $1 \leq n \leq T$, denote $r = (r_1, \ldots, r_T)$. Given $h \in H_t$ let $N(h) = \{n \in N\colon \forall 1 \leq \tau \leq t, h_t^\tau \neq W_n$ for any $W \in \{\hat{O}, \hat{F}, \hat{A}\}\}$ be the subset of potential entrants who have not yet decided. A strategy for the monopolist is a function s such that, for every $0 \leq t < T, h \in H_t$, and $n \in N(h)$, $s(h, n) \in \{A, F\}$ specifies the monopolist's action at period $t+1$ after observing a history h and the potential entrant n entering at $t + 1$.[4] Then, given (s, r), a potential entrant $n \in N$ may use $x \in X$, $p(n, \cdot)$ and Bayes Rule to compute (when possible) the posterior probability $p(n, t \mid x)$ that he is the tth potential entrant in the market.

Through this note I am going to use sequential equilibrium as the solution concept because it captures the idea of Selten's perfectness criterion in the context of imperfect information and, moreover, it has been the equilibrium concept used in the attempts to solve the paradox of the Chain Store by looking at it as a game of incomplete information (see Kreps and Wilson, 1982a; Milgrom and Roberts, 1982). In our context, given $f\colon H \to X$ and p, a sequential equilibrium consists of a $T + 1$-tuple (s, r) and a set of beliefs $B = (B_1, \ldots, B_T, B_M)$ (for every player and for each of his information sets, a probability distribution ("belief") on the set of nodes belonging to the information set) such that:

(i) for every player, and at each of his information sets, the moves prescribed by his strategy are optimal (given his beliefs) for the remainder of the game against everybody else's future moves according to their strategies and,

(ii) there exists a sequence of completely mixed strategies converging to (s, r) such that the generated sequence of conditional probability distributions over the nodes at each information set converges to the set of beliefs B.[5]

The rest of the section is devoted to showing through examples that the existence of sequential equilibria with reputation is very sensitive to changes in the information structure $f\colon H \to X$.

The first three examples consider an information structure in which there exists a sequential equilibrium with the property that the monopolist is willing to build reputation.

EXAMPLE 1. Suppose potential entrants know before their entering decision one and only one of the following three different things: (a) nobody has decided

[4] Here it is assumed that the monopolist knows the full history of the game. In Example 2 and at the end of Examples 3 and 5 this assumption is relaxed.

[5] Kreps and Wilson (1982b) give a formal definition of it as well as its existence and relation with Selten's "trembling-hand" perfection. Condition (i) says that (s, r) is sequentially rational given the set of beliefs B, and condition (ii) says that given (s, r) the set of beliefs B are consistent.

to enter yet ($\tilde{O}$); (b) at least one agent decided to enter and faced an A response ($\tilde{A}$); and (c) everybody who decided to enter was faced with an F response ($\tilde{F}$). That is, $X = \{\tilde{O}, \tilde{A}, \tilde{F}\}$ and $f\colon H \to X$, where,

$$f(h) = \begin{cases} \tilde{O} & \text{if } h \in H_0 \text{ or } h \in H_t \text{ is such that } \forall 1 \le \tau \le t,\, h_t^\tau = \hat{O}_n \\ & \text{for some } n \in N \\ \tilde{A} & \text{if } h \in H_t \text{ is such that } \exists 1 \le \tau \le t \text{ such that } h_t^\tau = \hat{A}_n \\ & \text{for some } n \in N \\ \tilde{F} & \text{if } h \in H_t \text{ is such that } \forall 1 \le \tau \le t,\, h_t^\tau \in \{(\hat{O}_n)_{n\in N}, (\hat{F}_n)_{n\in N}\} \\ & \text{and } \exists \tau \text{ such that } h_t^\tau = \hat{F}_n \text{ for some } n \in N. \end{cases}$$

In this case, if the monopolist has a reputation for being strong ($\tilde{F}$) and he plays A just once, his reputation thereafter is a weak one ($\tilde{A}$), i.e., $\tilde{A}$ is an absorbing state and the only way to maintain $\tilde{F}$ is to fight all the entrants.

Consider now the following strategies (s, r):

$$r_n(x) = \begin{cases} O & \text{if } x \in \{\tilde{O}, \tilde{F}\} \\ I & \text{if } x = \tilde{A} \end{cases} \quad \forall n \in N$$

and

$$s(h, n) = \begin{cases} F & \text{if } h \in H_t, \text{ where } t < T-1 \text{ and} \\ & f(h) \in \{\tilde{O}, \tilde{F}\} \\ A & \text{if either } h \in H_t, \text{ where } t < T-1 \\ & \text{and } f(h) = \tilde{A}, \text{ or } h \in H_{T-1} \end{cases} \quad \forall n \in N(h).$$ [6]

RESULT 1.1. *Suppose that $-1 + a > 2c$ and p is such that for every $n \in N$,*

$$\sum_{t=1}^{T-1} p(n,t)d + p(n,T)b < 0 \quad \text{and} \quad \sum_{t=2}^{T-1} (t-1)p(n,t)d + (T-1)p(n,T)b < 0. \tag{1}$$

Then, there exists a set of beliefs B such that $((s, r), B)$ is a sequential equilibrium.

Intuitively, condition (1) is sufficient for sequential rationality of potential entrants given (s, r) and the beliefs B. In particular, the first term of the condition ensures that an entrant that observed $\tilde{O}$ will prefer to stay out, and the second term ensures that an entrant that observed $\tilde{F}$ will prefer to stay out. The weights $(t-1)$ and $(T-1)$ come from the numerator of the Bayesian updating given B. For instance, if $d = -1$ and $b = 1$, then any p close "enough" to the uniform distribution satisfies condition (1). The condition $-1 + a > 2c$ is sufficient for

[6] Notice that the monopolist is able to observe the potential entrants staying out of the market. In Example 2 this assumption will be removed.

sequential rationality of the monopolist given (s, r) and the beliefs B. One set of beliefs B that may do the job are the ones obtained (applying Bayes Rule) from a sequence of completely mixed strategies converging to (s, r) in which the probability of mistake in every point of the sequence is the same for all players. Unless otherwise noted, these are going to be the class of beliefs considered hereafter.

To describe an equilibrium play, given (s, r), define $a_t(s, r)$ as the actual action of the potential entrant n at period t according to r_n and the monopolist's planned action, contingent on entry, according to s. In this case, the equilibrium strategies (s, r) generate the following sequence of actions $\{a_t(s, r)\}_{t=1}^{T}$: $a_t(s, r) = (O, F), \forall 1 \le t < T$ and $a_T(s, r) = (O, A)$. Even though the monopolist is willing to fight all the entrants but the last one, he never has to do it since all potential entrants decide to stay out of the market.

EXAMPLE 2. Suppose now that in fact the monopolist can observe only potential entrants deciding to enter, not the ones deciding to stay out. To describe this information structure, consider $h \in H_t$ and define $M(h) = \{n \in N\colon \forall 1 \le \tau \le t, h_t^{\tau} \ne W_n$ for any $W \in \{\hat{F}, \hat{A}\}\}$ to be the set of potential entrants who have not yet entered (either because they have already decided to stay out or because they are going to consider their decision later on). Now consider the following partition of the history space H. Since M cannot distinguish between, say, history $(\hat{F}_3, \hat{F}_1, \hat{O}_2)$ and history $(\hat{F}_3, \hat{O}_2, \hat{O}_4, \hat{F}_1)$, but knows which player is facing now, let Z (with typical element z) be the set consisting of either $\tilde{O}$ (nobody got in yet) or all sequences of $\hat{F}_n$'s and $\hat{A}_n$'s (with different subindexes) of any length between 1 and $T-1$ (for instance, $\hat{F}_n\hat{F}_m$ means that so far potential entrants n and m have decided to enter and both faced a fight as monopolist's response). Define $Y = Z \times N$ and let the signal-generating function $g\colon H \to Y$ be such that

(i) $g(h_0) = (z, n) \Rightarrow z = \tilde{O}$, and
(ii) $\forall h \in H_t$, $g(h) = (z, n) \Rightarrow$ if $z \ne \tilde{O}$ then $n \in M(h)$,

where for $h \in H$, $g(h) = (z, n)$ means that the monopolist knows z about h and that, right now, the potential entrant n has decided to enter. In this case a monopolist's strategy is a function $\hat{s}\colon Y \to \{F, A\}$. Assume $T = 5$ and consider the following strategies:

$$r_n(x) = \begin{cases} O & \text{if } x \in \{\tilde{O}, \tilde{F}\} \\ I & \text{if } x = \tilde{A} \end{cases} \qquad \forall n \in N \text{ (as in the previous example)}$$

and

$$\hat{s}(y) = \begin{cases} A & \text{if } y = (z, m) \text{ is s.t. either } z \text{ is a sequence of } 4\hat{F}_n\text{'s or there is} \\ & \text{at least one } \hat{A}_n \\ F & \text{otherwise.} \end{cases}$$

RESULT 2.1. *Suppose that* $a > 6c + 5$,[7] *and* p *is the uniform distribution. Then there exists a set of beliefs* B *such that* $((\hat{s}, r), B)$ *is a sequential equilibrium.*

The equilibrium play is $a_t(\hat{s}, r) = (O, F)$ for every $1 \leq t \leq 5$.

The next example shows that to get reputation in equilibrium with the information structure $f\colon H \to X$ of Examples 1 and 2, it is not crucial that the prior distribution be close to the uniform one,[8] and moreover that equilibrium play of a reputation equilibrium might imply, at least with positive probability, that the monopolist has to fight some potential entrant who decided to enter.

EXAMPLE 3. A game in which player i will probably be the last.

Consider the information structure $f\colon H \to X$ of Example 1. Suppose that $T = 5$ and that p is such that there exists $i \in N$ such that for every $j, k, l, n \neq i$, $p(jklni) = 6/240$ and $p(ijlkn) = p(jikln) = p(jkiln) = p(jklin) = 1/240$. Therefore, $p(i, 5) = 72/120$, and $p(i, t) = 12/120$ for every $1 \leq t < 5$; for every n in N, $n \neq i$, $p(n, 5) = 12/120$ and $p(n, t) = 27/120$ for every $1 \leq t < 5$ (i.e., it is likely that player i will be the last potential entrant).

Now consider the following strategies (s, r):

$$r_n(x) = \begin{cases} O & \text{if } x \in \{\tilde{O}, \tilde{F}\} \\ I & \text{if } x = \tilde{A} \end{cases} \quad \text{for } n \neq i,$$
$$r_i(x) = I \text{ for every } x \in X,$$

and, $\forall (h, n)$ such that $n \in N(h)$,

$$s(h, n) = \begin{cases} F & \text{if } h \in H_t \text{ where either } t < 3 \text{ and } f(h) \in \{\tilde{O}, \tilde{F}\}, \\ & \text{or } t = 3 \text{ and } f(h) \in \{\tilde{O}, \tilde{F}\} \text{ and } \exists 1 \leq \tau \leq 4 \text{ s.t. player } i \\ & \text{made his decision at period } \tau \\ A & \text{otherwise.}^{9} \end{cases}$$

That is, player i always enters the market, player $n \neq i$ enters the market only if the monopolist has failed to fight, and the monopolist fights if previously he has never failed to fight and either h has length smaller than 3 or if it has length 3 and player i has already made his decision.

RESULT 3.1. *Suppose that* $3d + 2b < 0$, $2d + 3b > 0$,[10] *and* $7a > 15c + 8$. *Then, there exists a set of beliefs* B *such that* $((s, r), B)$ *is a sequential equilibrium.*

[7] For example $a > 2$ and $-1 < c < -1/2$ satisfy this condition.

[8] I am indebted to Bob Rosenthal for raising this question.

[9] Notice again that the monopolist is able to observe the full history of the market, in particular the identity of the player who stays out. This assumption will be removed later in the example.

[10] For instance, if $b = -d$ (which is in the range of payoffs considered in Kreps and Wilson, 1982a).

The equilibrium play is now the following probability distribution: with probability 1/10 each of the four different plays corresponding to the period in which i makes his decision when his turn was $k = 1$, $k = 2$, $k = 3$, or $k = 4$, that is,

$$a_t(s, r) = \begin{cases} (I, F) & \text{if } t = k \\ (O, F) & \text{if } t \neq k \text{ and } t < 5. \\ (O, A) & \text{if } t = 5, \end{cases}$$

and with probability 6/10 the play corresponding to player i being the last:

$$a_t(s, r) = \begin{cases} (O, F) & \text{if } t = 1, 2, 3 \\ (O, A) & \text{if } t = 4 \\ (I, A) & \text{if } t = 5. \end{cases}$$

To remove the fact that the monopolist is able to know the full history of the market, let $f\colon H \to Y$ be the same information structure of Example 2 (i.e., the monopolist cannot discern whether a potential entrant already decided to stay out of the market or has not made his decision yet) and consider $(\hat{s}, r)$ defined by

$$r_n(x) = \begin{cases} O & \text{if } x \in \{\tilde{O}, \tilde{F}\} \\ I & \text{if } x = \tilde{A} \end{cases} \quad \forall n \in N$$

and

$$\hat{s}(y) = \begin{cases} A & \text{if } y = (z, m) \text{ is such that } z \text{ is either a sequence of } 4\hat{F}_n\text{'s} \\ & \text{or there is at least one } \hat{A}_n \\ F & \text{otherwise.} \end{cases}$$

RESULT 3.2. *Suppose that $a > 26c + 25$. Then, there exists a set of beliefs B such that $((\hat{s}, r), B)$ is a sequential equilibrium.*

In this case the equilibrium play is $a_t(\hat{s}, r) = (O, F)$ for every $1 \leq t \leq 5$.

EXAMPLE 4. Suppose that potential entrants are able to observe the complete past history of the market except the staying out decisions (see the Introduction for a justification of this information structure). Formally, assume that f satisfies: for every $1 \leq t < T$ and every $n \in N$ if $h_t \supset h_{t-1}$ and $h_t = (h_{t-1}, \hat{O}_n)$, then $f(h_t) = f(h_{t-1})$. Given this particular information structure f, one might ask, does there exist any sequential equilibria with reputation? The next result answers the question in a negative way even for more general information structures f. To state it, an additional definition is needed. A history h_{T-1} of length $T - 1$ is called informative if $U_{f(h_{T-1})} = \{T\}$, and recursively, a history h_t of length $0 \leq t < T - 1$ is called informative if, $\forall n \in N(h_t)$, $(h_t, \hat{A}_n)$ and $(h_t, \hat{F}_n)$ are informative and $U_{f(h_t)} = \{t + 1, \ldots, T\}$. That is, h_t is informative if f

reveals that at least t potential entrants have already made their decisions and, moreover, this property is maintained for the histories h_t followed by entries. Using a backwards induction argument it is easy to show the following result.

RESULT 4.1. *Suppose $h_\tau (\tau \leq T - 1)$ is informative; then in any sequential equilibrium:*

(1) *for every $n \in N, r_n(f(h)) = I \ \forall h \in H \mid h_\tau$, and*

(2) $s(h, n) = A \ \forall h \in H$ *such that* $f(h) = f(h')$, *where* $h' \in H \mid h_\tau$ *and* $n \in N(h)$.

[(2′) $\hat{s}(y) = A \ \forall y \in Y$ *s.t.* $\forall h \in g^{-1}(y)$ *is such that* $f(h) = f(h')$, *where* $h' \in H \mid h_\tau$.]

COROLLARY. *Suppose $h_0 = \emptyset$ is informative; then there exists only one sequential equilibrium strategy which is $s(\cdot, \cdot) = A[\hat{s}(\cdot) = A]$ and $r_n(\cdot) = I$ $\forall n \in N$.*

It is easily seen that, for the information structure that we are interested in, h_0 is informative and hence no reputation equilibria are possible. This is because in the event of every potential entrant getting in and independently of monopolist's responses the last potential entrant knows with probability 1 that he is in fact the last one, thereby producing the unraveling. Notice that the argument is independent of the prior distribution and of whether or not the monopolist is able to observe potential entrants staying out of the market. For some histories, the information structure f tells potential entrants too much about where they may be in the ordering after computing the conditional probabilities given the history and the hypothesized strategies.

So far we have considered only the situation in which sooner or later all players in N must decide at some $1 \leq t \leq T$ whether or not to enter the market. The next example considers an alternative specification and interpretation of the set of potential entrants.

EXAMPLE 5. Suppose now that N is seen as the set of firms which for some reason (technological, product related, and so on) may potentially enter the market, but perhaps some of them will never even consider the possibility of doing so. In this example, then, the number of firms considering entry is unknown. In this case, p is a probability distribution on the set of all possible orderings of nonempty subsets of N; therefore a point on the support of p is an ordering of a nonempty (not necessarily proper) subset of N. Given p, it is possible to compute $p(n, t)$ (as before, the probability that player n is going to make his decision at period t, but now $p(n, \cdot)$ is no longer a probability distribution on $\{1, 2, \ldots, T\}$) and q_t for $1 \leq t \leq T$, where q_t means the probability that t and only t members of N will in fact consider the possibility of entering the market. Consider the information structure $f\colon H \to X$ of Example 1 and assume that the monopolist is able to observe the past history of the market (later in the example this will be relaxed), but does not know how many potential entrants are in fact considering whether or not to enter the market (as a maximum he knows that there are T).

Consider now the following strategies (s, r):

$$r_n(x) = \begin{cases} O & \text{if } x \in \{\tilde{O}, \tilde{F}\} \\ I & \text{if } x = \tilde{A} \end{cases} \qquad \forall n \in N$$

and

$$s(h, n) = \begin{cases} F & \text{if } h \in H_t, \text{ where } t < T - 1 \text{ and} \\ & f(h) \in \{\tilde{O}, \tilde{F}\} \\ A & \text{if either } h \in H_t, \text{ where } t < T - 1 \\ & \text{and } f(h) = \tilde{A}, \text{ or } h \in H_{T-1} \end{cases} \qquad \forall n \in N(h).$$

RESULT 5.1. *Suppose that* $T = 4$, $a > 3c + 2$, $19d + 9b < 0$ *and* p *is the uniform distribution. Then, there exists a set of beliefs* B *such that* $((s, r), B)$ *is a sequential equilibrium.*

The equilibrium play generated by (s, r) is the following probability distribution: with probability $q_1 = 1/16$, $a_1(s, r) = (O, F)$; with probability $q_2 = 3/16$, $\{a_\tau(s, r)\}_{\tau=1}^2$, where $a_\tau(s, r) = (O, F)$ for every $1 \leq \tau \leq 2$; with probability $q_3 = 6/16$, $\{a_\tau(s, r)\}_{\tau=1}^3$, where $a_\tau(s, r) = (O, F)$ for every $1 \leq \tau \leq 3$; and with probability $q_4 = 6/16$, $\{a_\tau(s, r)\}_{\tau=1}^4$, where $a_\tau(s, r) = (O, F)$ for every $1 \leq \tau < 4$ and $a_4(s, r) = (O, A)$.

To analyze the situation in which the monopolist does not know nature's move, let $g\colon H \to Y$ be the information structure defined in Example 2. Consider the following strategies:

$$r_n(x) = \begin{cases} O & \text{if } x \in \{\tilde{O}, \tilde{F}\} \\ I & \text{if } x = \tilde{A} \end{cases} \qquad \forall n \in N$$

and

$$\hat{s}(y) = \begin{cases} A & \text{if } y = (z, m) \text{ is such that either } z \text{ is a sequence} \\ & \text{of } T - 1\ F_n\text{'s or there is at least one } \hat{A}_n \\ F & \text{otherwise.} \end{cases}$$

RESULT 5.2. *Suppose that* $T = 4$, $a > 6c + 5$, *and* p *is the uniform distribution. Then, there exists a set of beliefs* B *such that* $((\hat{s}, r), B)$ *is a sequential equilibrium.*

The equilibrium play generated by $(\hat{s}, r)$ is the following probability distribution: with probability $q_1 = 1/16$, $a_1(\hat{s}, r) = (O, F)$; with probability $q_2 = 3/16$,

$\{a_\tau(\hat{s}, r)\}_{\tau=1}^{2}$, where $a_\tau(\hat{s}, r) = (O, F)$ for every $1 \leq \tau \leq 2$; with probability $q_3 = 6/16$, $\{a_\tau(\hat{s}, r)\}_{\tau=1}^{3}$, where $a_\tau(\hat{s}, r) = (O, F)$ for every $1 \leq \tau \leq 3$; and with probability $q_4 = 6/16$, $\{a_\tau(\hat{s}, r)\}_{\tau=1}^{4}$, where $a_\tau(\hat{s}, r) = (O, F)$ for every $1 \leq \tau \leq 4$.

3. COMMENTS AND CONCLUSIONS

Before proceeding, it may be useful to briefly summarize what we have learned through the preceding examples. Examples 1, 2, and 3 have shown that if the reputation the monopolist can build upon is of the type "all or nothing" then, independently of whether or not the monopolist is fully informed about the market history, for some parameter configurations (payoffs and p) there exist sequential equilibria with reputation whose equilibrium play may involve actual fighting (that is, fighting may be a credible threat at equilibrium). In Example 4 it was shown that if the information structure (f) of the potential entrants is too informative then, independently of the prior distribution (p), there is no sequential equilibria with reputation; that is, the unique perfect-equilibrium outcome is the perfect equilibrium of the game with perfect information. Example 5 considered the case in which, in addition to the information structure of Examples 1, 2, and 3, the number of potential entrants was uncertain; it was shown that for some parameter configurations, reputation equilibrium (with symmetric mistakes) does exist.

I would like to emphasize the role that the information structure f plays as a reputation index for the potential entrants. The amount of information that f carries about determines not only the variety of reputation levels the monopolist may acquire but also the potential entrants' perception about the monopolist's willingness to fight. However, as Example 4 shows, this may have a perverse effect in the sense that a too broad set of possible reputations removes the uncertainty about the ordering needed to generate reputation at equilibrium. The "all or nothing" type of information structure in Example 1 does not have this effect because after any history only one period may be ruled out at most, and thus it leaves enough uncertainty in the environment. Additionally, the comparison of both types of information suggests that the existence of reputation equilibria is sensitive to the information structure in a very particular way: when reputation is difficult to build up and maintain (and hence, it is easy to lose) reputation equilibria seem more likely to exist. To corroborate this suggestion, consider the following alternative information structure.

Suppose $X = \{\bar{O}, \bar{A}, \bar{F}\}$, where $\bar{O}$ means that nobody decided to enter, $\bar{A}$ means that everybody who decided to enter was faced with an A response, and $\bar{F}$ means that at least one potential entrant decided to enter and faced an F

response. Define $f\colon H \to X$ as follows:

$$f(h) = \begin{cases} \bar{O} & \text{if } h \in H_0 \text{ or } h \in H_t \text{ is such that } \forall 1 \le \tau \le t\ h_t^\tau = \hat{O}_n \\ & \text{for some } n \in N \\ \bar{F} & \text{if } h \in H_t \text{ is such that } \exists 1 \le \tau \le t \text{ such that } h_t^\tau = \hat{F}_n \\ & \text{for some } n \in N \\ \bar{A} & \text{if } h \in H_t \text{ is such that } \forall 1 \le \tau \le t\ h_t^\tau \in \{(\hat{O}_n)_{n\in N}, (\hat{A}_n)_{n\in N}\} \\ & \text{and } \exists \tau \text{ such that } h_t^\tau = \hat{A}_n \text{ for some } n \in N. \end{cases}$$

That is, if the monopolist has a reputation of being strong ($\bar{F}$) he never loses it, i.e., $\bar{F}$ is an absorbing state. (Notice the overall similarity and the symmetric role of A and F with respect to the information structure of Example 1). In this case there is no sequential equilibrium with reputation. The subgame-perfection requirement together with the fact that reputation cannot be lost implies that once the monopolist has a reputation he is not willing to fight any entrant and hence, he is unable to avoid entry. Since obtaining reputation is costly, and not valuable, he never fights.

To conclude this note, the difficulties of modeling reputation as an equilibrium behavior in a game-theoretical context seem rather deep: reputation is a subtle phenomenon very sensitive to different alternative modeling decisions. I have focused on what I see as a more realistic description of which information entrants have prior to their decisions in the Chain-Store Paradox. Finally, one may still see the result of the equilibrium analysis of the Chain Store with perfect information as paradoxical, and therefore one may be led to think that at least part of the difficulty lies with more fundamental aspects of the modeling, i.e., the concepts of strategy and equilibrium.

APPENDIX

This Appendix contains a detailed and complete proof of Result 1.1, and most of the important arguments on the proofs of Results 2.1, 3.1, 5.1, and 5.2. The proof of Result 3.2 is similar and therefore omitted. The proof of Result 4.1 is also omitted since it consists of a standard backward induction argument.

RESULT 1.1. *Suppose that* $-1 + a > 2c$ *and* p *is such that for every* $n \in N$,

$$\sum_{t=1}^{T-1} p(n,t)d + p(n,T)b < 0 \quad \textit{and} \quad \sum_{t=2}^{T-1} (t-1)p(n,t)d + (T-1)p(n,T)b < 0.$$

Then, there exists a set of beliefs B *such that* $((s,r), B)$ *is a sequential equilibrium.*

Proof. Consider the following set of beliefs for the entrants $\{B_n\}_{n\in N}$, where B_n is the triple $(\{p(n,t \mid \tilde{O}, (s,r))\}_{t=1}^{T}, \{p(n,t \mid \tilde{F}, (s,r))\}_{t=1}^{T}, \{p(n,t \mid \tilde{A}, (s,r))\}_{t=1}^{T})$ defined by: for every $n \in N$ and every $1 \leq t \leq T$,

$$p(n,t \mid \tilde{O}, (s,r)) = p(n,t)$$

and

$$p(n,t \mid \tilde{F}, (s,r)) = p(n,t \mid \tilde{A}, (s,r)) = \frac{(t-1)p(n,t)}{\sum_{\tau=2}^{T}(\tau-1)p(n,\tau)} .$$

Notice that, since potential entrants move only once and probabilities are computed before they move, $p(n,t \mid \cdot, (s,r))$ could also be read as $p(n,t \mid \cdot, (s,r_{-n}))$. This abuse of language will be used repeatedly in what follows.

To check consistency, first note that $p(\tilde{O} \mid (n,t), (s,r)) = 1$ and therefore, by Bayes Rule

$$p(n,t \mid \tilde{O}, (s,r)) = \frac{p(\tilde{O} \mid (n,t), (s,r))p(n,t)}{\sum_{\tau=1}^{T} p(\tilde{O} \mid (n,\tau), (s,r))p(n,\tau)} = p(n,t).$$

Given the strategies (s,r), the information sets $\tilde{F}$ and $\tilde{A}$ for the entrants have zero probability. Let $\{\varepsilon_m\}_{m\in\mathbb{N}}$ be any sequence converging to zero with the property that $0 < \varepsilon_m < 1$ for every $m \in \mathbb{N}$. Define the following sequence of completely mixed strategies: for every $m \geq 1$

$$r_n^m(x) = \begin{cases} \left\{\begin{array}{ll} O & \text{with probability } (1-\varepsilon_m) \\ I & \text{with probability } \varepsilon_m \end{array}\right\} & \text{if } x \in \{\tilde{O}, \tilde{F}\} \\ \left\{\begin{array}{ll} I & \text{with probability } (1-\varepsilon_m) \\ O & \text{with probability } \varepsilon_m \end{array}\right\} & \text{if } x = \tilde{A} \end{cases} \quad \text{for every } n \in N,$$

and

$$s^m(h,n) = \begin{cases} \left\{\begin{array}{ll} F & \text{with probability } (1-\varepsilon_m) \\ A & \text{with probability } \varepsilon_m \end{array}\right\} & \begin{array}{l}\text{if } h \in H_t, \text{ where } t < T-1 \\ \text{and } f(h) \in \{\tilde{O}, \tilde{F}\}\end{array} \\ \left\{\begin{array}{ll} A & \text{with probability } (1-\varepsilon_m) \\ F & \text{with probability } \varepsilon_m \end{array}\right\} & \begin{array}{l}\text{if either } h \in H_t, \text{ where} \\ t < T-1 \text{ and } f(h) = \tilde{A}, \\ \text{or } h \in H_{T-1}\end{array} \end{cases}$$

for every $n \in N(h)$. Notice that all players have the same probability of mistake through the sequence. It is easy to check that $\{r^m\}_{m=1}^{\infty} \to r$ and $\{s^m\}_{m=1}^{\infty} \to s$.

We can now apply Bayes Rule along the sequence to the information sets $\tilde{F}$ and $\tilde{A}$. For $n \in N$ and $2 \leq t \leq T$,

$$p(n,t \mid \tilde{F},(s^m,r^m)) = \frac{p(\tilde{F} \mid (n,t),(s^m,r^m))p(n,t)}{\sum_{\tau=2}^{T} p(\tilde{F} \mid (n,\tau),(s^m,r^m))p(n,\tau)}$$
$$= \frac{[(t-1)\varepsilon_m(1-\varepsilon_m)^{t-1} + \circ(\varepsilon_m^2,t)]p(n,t)}{\sum_{\tau=2}^{T}[(\tau-1)\varepsilon_m(1-\varepsilon_m)^{\tau-1} + \circ(\varepsilon_m^2,\tau)]p(n,\tau)}, \quad (2)$$

which converges to $(t-1)p(n,t)/\sum_{\tau=2}^{T}(\tau-1)p(n,\tau)$ as ε_m tends to zero, where, in general, $\circ(\varepsilon_m^k,t)$ means a sum of terms multiplied by ε_m power a number greater than k if $t > 2$, and zero if $t = 2$. Those terms come from histories with more than two mistakes. Similarly,

$$p(n,t \mid \tilde{A},(s^m,r^m)) = \frac{p(\tilde{A} \mid (n,t),(s^m,r^m))p(n,t)}{\sum_{\tau=2}^{T} p(\tilde{A} \mid (n,\tau),(s^m,r^m))p(n,\tau)}$$
$$= \frac{\{\varepsilon_m^2[\sum_{\tau=0}^{t-2}(1-\varepsilon_m)^{t-2+\tau}] + \circ(\varepsilon_m^3,t)\}p(n,t)}{\sum_{\tau=2}^{T}\{\varepsilon_m^2[\sum_{\nu=0}^{\tau-2}(1-\varepsilon_m)^{\tau-2+\nu}] + \circ(\varepsilon_m^3,\tau)\}p(n,\tau)}, \quad (3)$$

which converges to $(t-1)p(n,t)/\sum_{\tau=2}^{T}(\tau-1)p(n,\tau)$ as ε_m tends to zero. Therefore, the set of beliefs B are consistent.

To check sequential rationality consider any entrant $n \in N$ and suppose first that the information set $\tilde{O}$ is reached. Then, the expected payoff of entering, given the strategies of the other players, is

$$E\pi(I \mid \tilde{O},(s,r_{-n})) = \sum_{t=1}^{T-1} p(n,t \mid \tilde{O},(s,r))d + p(n,T \mid \tilde{O},(s,r))b$$
$$= \sum_{t=1}^{T-1} p(n,t)d + p(n,T)b. \quad (4)$$

The expected payoff of staying out is $E\pi(O \mid \tilde{O},(s,r_{-n})) = 0$. Therefore, since expression (4) is, by assumption, strictly negative, it follows that $E\pi(O \mid \tilde{O},(s,r_{-n})) > E\pi(I \mid \tilde{O},(s,r_{-n}))$.

Suppose that the information set $\tilde{F}$ is reached. Then

$$E\pi(I \mid \tilde{F},(s,r_{-n})) = \sum_{t=2}^{T-1} p(n,t \mid \tilde{F},(s,r_{-n}))d + p(n,T)b$$
$$= \sum_{t=2}^{T-1} \frac{(t-1)p(n,t)}{\sum_{\tau=2}^{T}(\tau-1)p(n,\tau)}d + \frac{(T-1)p(n,T)}{\sum_{\tau=2}^{T}(\tau-1)p(n,\tau)}b \quad (5)$$

and $E\pi(O \mid \tilde{F}, (s, r_{-n})) = 0$. Therefore, since expression (5) is, by assumption, strictly negative, it follows that $E\pi(O \mid \tilde{F}, (s, r_{-n})) > E\pi(I \mid \tilde{F}, (s, r_{-n}))$.

Suppose now that the information set $\tilde{A}$ is reached. Then

$$E\pi(I \mid \tilde{A}, (s, r_{-n})) = \sum_{t=2}^{T} p(n, t \mid \tilde{A}, (s, r_{-n}))b = b. \tag{6}$$

Since $b > 0$ we have that (6) is strictly positive, and thus $E\pi(I \mid \tilde{A}, (s, r_{-n})) > E\pi(O \mid \tilde{A}, (s, r_{-n})) = 0$. Therefore, the entrants' strategies satisfy sequential rationality.

To check sequential rationality for the monopolist, suppose that $h \in H_t$ is such that $t < T - 1$, $f(h) \in \{\tilde{O}, \tilde{F}\}$, and $n \in N(h)$ just got in. Remember that, in this case, the monopolist will avoid future entrants if and only if he fights and also that, since he knows the period he is at, his information sets are singletons. Therefore, the expected payoff of following the strategy s, given r, is $E\pi(s \mid h, n, r) = -1 + a(T - t - 1)$, which is greater than or equal to $E\pi(s' \mid h, n, r)$, the expected payoff of following any other strategy s', since by assumption $-1 + a > 2c$. If $h \in H_t$ is such that $t < T - 1$, $f(h) \in \tilde{A}$ and $n \in N(h)$ just got in, it follows that $E\pi(s \mid h, n, r) = c(T - t - 1) \geq E\pi(s' \mid h, n, r)$ for any other strategy s', since every entrant will enter and it is dominant for the monopolist to accept every entrant. Finally, if $h \in H_{T-1}$ and $n \in N(h)$ just entered, it follows that $E\pi(s \mid h, n, r) = c \geq E\pi(s' \mid h, n, r)$, because entrant n is the last one and $s(h) = A$ is dominant. ∎

RESULT 2.1. *Suppose that $a > 6c+5$, and p is the uniform distribution. Then there exists a set of beliefs B such that $((\hat{s}, r), B)$ is a sequential equilibrium.*

Proof. As in the proof of Result 1.1 we can construct a sequence of completely mixed strategies, with ε_m as a uniform mistake, converging to $(\hat{s}, r)$.

Consider first a potential entrant $n \in \{1, 2, \ldots, 5\}$ at the information set $\tilde{O}$ (or $\hat{A}$). Since, in the limit, the monopolist is going to fight (or to acquiesce) with probability 1, n should stay out (or enter) independently of the limit beliefs (i.e., independently of the probability distribution on the set of nodes that constitute the information set $\tilde{O}$ (or $\tilde{A}$)). Suppose that n is at the information set $\tilde{F}$. Notice that the monopolist's strategy is anonymous. Therefore, from the point of view of n, his information set can be partitioned into four relevant subsets according to how many entrants the monopolist has already observed. Call them $\hat{F}^n$, $\hat{F}\hat{F}^n$, $\hat{F}\hat{F}\hat{F}^n$, and $\hat{F}\hat{F}\hat{F}\hat{F}^n$ (for instance, $\hat{F}^n$ is the union of the family of nodes of the form $\hat{F}_i$, for $i \neq n$). For every $m \in \mathbb{N}$ the probability $p(\tilde{F} \mid (\hat{s}^m, r^m))$ is strictly positive and, since $\tilde{F} = \hat{F}^n \cup \hat{F}\hat{F}^n \cup \hat{F}\hat{F}\hat{F}^n \cup \hat{F}\hat{F}\hat{F}\hat{F}^n$, the conditional distribution on $\tilde{F}$ has the property that $p(\hat{F}^n \mid \tilde{F}, (\hat{s}^m, r^m))$ converges to 1 as ε_m tends to zero. Since the monopolist will fight with probability 1, sequential rationality for n follows because $E\pi(I \mid \tilde{F}, (\hat{s}^m, r^m)) = d < 0 = E\pi(O \mid \tilde{F}, (\hat{s}^m, r^m))$.

Consider now the monopolist. His relevant uncertainty, in this case, is on the set of periods of time ({1, 2, 3, 4, 5}).

If z has at least one $\hat{A}_n$, to play A is a strictly dominant action, given r, independently of the beliefs about the period he is at.

If $z = \tilde{O}$, for every $1 \leq n \leq 5$ and $1 \leq t \leq 5$, $p(t \mid \tilde{O}, n, (\hat{s}, r)) = p(n, t) = 1/5$ since p is uniform. To check sequential rationality for this case, notice that the expected payoff of following the strategy $\hat{s}$, given r, is

$$E\pi(\hat{s} \mid \tilde{O}, n, r) = \sum_{t=1}^{5} p(t \mid \tilde{O}, n, (\hat{s}, r))[-1+(5-t)a] = \frac{1}{5}\sum_{t=1}^{5}[-1+(5-t)a]$$

and the maximum expected payoff of following any other strategy $\hat{s}'$, given r, is $\sum_{t=1}^{5} p(t \mid \tilde{O}, n, (\hat{s}, r))(6-t)c$ if $\hat{s}'(\tilde{O}, n) = A$ or $E\pi(\hat{s} \mid \tilde{O}, n, r)$ if $\hat{s}'(\tilde{O}, n) = F$. However, $(1/5)\sum_{t=1}^{5}[-1 + (5-t)a] > (1/5)\sum_{t=1}^{5}(6-t)c$, since $a > 6c + 5$ and $c > -1$. Therefore $E\pi(\hat{s} \mid \tilde{O}, n, r) \geq E\pi(\hat{s}' \mid \tilde{O}, n, r)$ for every $\hat{s}'$. Notice that the beliefs $\{p(t \mid \tilde{O}, n, (\hat{s}, r))\}_{t=1}^{5}$ are computed using the hypothesized strategies $(\hat{s}, r)$. Sequential rationality compares, at every information set, different strategies $(\hat{s}')$ for the remainder of the game, but using the beliefs produced by $(\hat{s}, r)$.

If z is a sequence of 4 $\hat{F}_i$'s $(i \neq n)$, then $p(t = 5 \mid \hat{F}_i\hat{F}_j\hat{F}_k\hat{F}_l, n, (\hat{s}, r)) = 1$. This implies that $\hat{s}$ is sequentially rational since $\hat{s}(z, n) = A$ is a strictly dominant action.

If z is a sequence of 3 $\hat{F}_i$'s, for example $\hat{F}_i\hat{F}_j\hat{F}_k$, then

$$\begin{aligned} p(\hat{F}_i\hat{F}_j\hat{F}_k \mid t &= 4, n, (\hat{s}^m, r^m)) = \frac{\varepsilon_m^4(1-\varepsilon_m)^3 p(ijknl)}{p(n,4)} = \varepsilon_m^4(1-\varepsilon_m)^3\frac{1/120}{1/5} \\ &= \varepsilon_m^4(1-\varepsilon_m)^3(1/24), \end{aligned}$$

and

$$\begin{aligned} &p(\hat{F}_i\hat{F}_j\hat{F}_k \mid t = 5, n, (\hat{s}^m, r^m)) \\ &\quad = \frac{\varepsilon_m^4(1-\varepsilon_m)^4[p(lijkn) + p(iljkn) + p(ijlkn) + p(ijkln)]}{p(n,5)} \\ &\quad = \varepsilon_m^4(1-\varepsilon_m)^4\frac{4/120}{1/5} = \varepsilon_m^4(1-\varepsilon_m)^4(1/6). \end{aligned}$$

Therefore, by Bayes Rule,

$$\begin{aligned} &p(t = 4 \mid \hat{F}_i\hat{F}_j\hat{F}_k, n, (\hat{s}^m, r^m)) \\ &\quad = \frac{(1/24)\varepsilon_m^4(1-\varepsilon_m)^3 p(n,4)}{(1/24)\varepsilon_m^4(1-\varepsilon_m)^3 p(n,4) + (1/6)\varepsilon_m^4(1-\varepsilon_m)^4 p(n,5)}, \end{aligned}$$

which converges to 1/5 as ε_m tends to zero, and

$$\begin{aligned} p(t = 5 \mid \hat{F}_i \hat{F}_j \hat{F}_k, n, (\hat{s}^m, r^m)) \\ = \frac{(1/6)\varepsilon_m^4 (1-\varepsilon_m)^4 p(n,5)}{(1/24)\varepsilon_m^4 (1-\varepsilon_m)^3 p(n,4) + (1/6)\varepsilon_m^4 (1-\varepsilon_m)^4 p(n,5)}, \end{aligned}$$

which converges to 4/5 as ε_m tends to zero. To check optimality in this information set, notice that

$$\begin{aligned} E\pi(\hat{s} \mid \hat{F}_i \hat{F}_j \hat{F}_k, n, r) &= p(t = 4 \mid \hat{F}_i \hat{F}_j \hat{F}_k, n, (\hat{s}, r))(-1 + a) \\ &\quad + p(t = 5 \mid \hat{F}_i \hat{F}_j \hat{F}_k, n, (\hat{s}, r))(-1). \end{aligned}$$

Substituting the probabilities by the limits 1/5 and 4/5, just obtained, it follows that

$$E\pi(\hat{s} \mid \hat{F}_i \hat{F}_j \hat{F}_k, n, r) = (1/5)(-1 + a) + (4/5)(-1) = (1/5)a - 1.$$

The maximum payoff following any other strategy $\hat{s}'$, given r is

$$(1/5)2c + (4/5)c = (6/5)c \text{ if } \hat{s}'(\hat{F}_i \hat{F}_j \hat{F}_k, n) = A,$$

and

$$(1/5)a - 1 \text{ if } \hat{s}'(\hat{F}_i \hat{F}_j \hat{F}_k, n) = F.$$

However, since by assumption $a > 6c + 5$ and $c > -1$, it follows that $(1/5)a - 1 > (6/5)c$, implying that $E\pi(\hat{s} \mid \hat{F}_i \hat{F}_j \hat{F}_k, n, r) \geq E\pi(\hat{s}' \mid \hat{F}_i \hat{F}_j \hat{F}_k, n, r)$ for every $\hat{s}'$.

For the other remaining cases, where z is of the form $\hat{F}_i \hat{F}_j$ or $\hat{F}_i$, the monopolist's beliefs are obtained in a similar way and his sequential rationality follows from arguments similar to the ones already made. ■

RESULT 3.1. *Suppose that* $3d + 2b < 0$, $2d + 3b > 0$, *and* $7a > 15c + 8$. *Then, there exists a set of beliefs* B *such that* $((s, r), B)$ *is a sequential equilibrium.*

Proof. First, I will find a set of beliefs for the entrants. With it I will check entrants' sequential rationality. Therefore, let (s, r) be given, and consider the information set $\tilde{O}$. For every $n \neq i$,

$$\begin{aligned} p(\tilde{O} \mid (n,1), (s,r)) &= 1, \\ p(\tilde{O} \mid (n,2), (s,r)) &= \frac{p(i \geq 3, (n,2))}{p(n,2)} \\ &= \frac{3!\,p(knijl) + 3!\,p(knjil) + 3!\,p(knjli)}{p(n,2)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{6(1/240)+6(1/240)+6(3/120)}{27/120} = 8/9,\\
p(\tilde{O} \mid (n,3),(s,r)) &= \frac{p(i \geq 4, (n,3))}{p(n,3)} = \frac{3!\,p(kjnil)+3!(kjnli)}{p(n,3)}\\
&= \frac{6(1/240)+6(3/120)}{27/120} = 7/9,\\
p(\tilde{O} \mid (n,4),(s,r)) &= \frac{p((i,5),(n,4))}{p(n,4)} = \frac{3!\,p(kjlni)}{p(n,4)} = \frac{6(3/120)}{27/120} = 6/9,
\end{aligned}
$$

and

$$p(\tilde{O} \mid (n,5),(s,r)) = 0.$$

Therefore, by Bayes Rule, $p((n,1) \mid \tilde{O},(s,r)) = 27/90$, $p((n,2) \mid \tilde{O},(s,r)) = 24/90$, $p((n,3) \mid \tilde{O},(s,r)) = 21/90$, $p((n,4) \mid \tilde{O},(s,r)) = 18/90$, and $p((n,5) \mid \tilde{O},(s,r)) = 0$. Furthermore,

$$
\begin{aligned}
E\pi(I \mid \tilde{O},(s,r_{-n})) &= (27/90)d + (24/90)d + (21/90)d + (18/90)b\\
&= (72d+18b)/90.
\end{aligned}
$$

Since the assumptions $3d+2b<0$ and $d<0$ imply $72d+18b<0=E\pi(O \mid \tilde{O},(s,r_{-n}))$, it follows that $r_n(\tilde{O}) = O$ is optimal for every $n \neq i$.

For $n=i$, $p((i,t) \mid \tilde{O},(s,r)) = p(i,t)$ is satisfied for every $1 \leq t \leq 5$. Therefore,

$$
\begin{aligned}
E\pi(I \mid \tilde{O},(s,r_{-i})) &= (1/10)d + (1/10)d + (1/10)d + (1/10)d + (6/10)b\\
&= (4d+6b)/10.
\end{aligned}
$$

Since the assumption $2d+3b>0$ implies $4d+6b>0=E\pi(O \mid \tilde{O},(s,r_{-i}))$, it follows that $r_i(\tilde{O}) = I$ is optimal for entrant i.

Consider the information set $\tilde{F}$. For every $n \neq i$

$$
\begin{aligned}
p(\tilde{F} \mid (n,2),(s,r)) &= \frac{3!\,p(inkjl)}{p(n,2)} = \frac{6(1/240)}{27/120} = 1/9,\\
p(\tilde{F} \mid (n,3),(s,r)) &= \frac{3!\,p(iknjl)+3!\,p(kinjl)}{p(n,3)} = 2/9,\\
p(\tilde{F} \mid (n,4),(s,r)) &= \frac{3!\,p(ikjnl)+3!(kijnl)+3!\,p(kjinl)}{p(n,4)} = 3/9,
\end{aligned}
$$

and

$$
\begin{aligned}
p(\tilde{F} \mid (n,5),(s,r)) &= \frac{3!\,p(ikjln)+3!\,p(kijln)+3!\,p(kjiln)+3!\,p(kjlin)}{p(n,5)}\\
&= 1.
\end{aligned}
$$

Therefore, by Bayes Rule, $p((n,2) \mid \tilde{F}, (s,r)) = 1/10$, $p((n,3) \mid \tilde{F}, (s,r)) = 2/10$, $p((n,4) \mid \tilde{F}, (s,r)) = 3/10$, and $p((n,5) \mid \tilde{F}, (s,r)) = 4/10$. Thus,

$$E\pi(I \mid \tilde{F}, (s, r_{-n})) = (1/10)d + (2,10)d + (3/10)d + (4/10)b = (6d+4b)/10.$$

Since the assumption $3d+2b<0$ implies $6d+4b<0=E\pi(O \mid \tilde{F}, (s, r_{-n}))$, it follows that $r_n(\tilde{F}) = O$ is optimal for every $n \neq i$.

In the case of $n = i$, the information set $\tilde{F}$ has zero probability. However, using completely mixed strategies as in the previous proofs, the conditional probabilities are

$$\begin{aligned} p(\tilde{F} \mid (i,2), (s^m, r^m)) &= \varepsilon_m(1-\varepsilon_m), \\ p(\tilde{F} \mid (i,3), (s^m, r^m)) &= 2\varepsilon_m(1-\varepsilon_m)^2, \\ p(\tilde{F} \mid (i,4), (s^m, r^m)) &= 3\varepsilon_m(1-\varepsilon_m)^3, \end{aligned}$$

and

$$p(\tilde{F} \mid (i,5), (s^m, r^m)) = 3\varepsilon_m(1-\varepsilon_m)^4 + \varepsilon_m^2(1-\varepsilon_m)^3.$$

Therefore, using Bayes Rule,

$$\begin{aligned} & p((i,2) \mid \tilde{F}, (s^m, r^m)) \\ &\quad = \frac{p(\tilde{F} \mid (i,2), (s^m, r^m))\, p(i,2)}{\sum_{t=2}^{5} p(\tilde{F} \mid (i,t), (s^m, r^m))\, p(i,t)} \\ &\quad = \frac{\varepsilon_m(1-\varepsilon_m)(1/10)}{\begin{array}{c}\varepsilon_m(1-\varepsilon_m)(1/10) + 2\varepsilon_m(1-\varepsilon_m^2)(1/10) + 3\varepsilon_m(1-\varepsilon_m)^3(1/10) \\ +[3\varepsilon_m(1-\varepsilon_m)^4 + \varepsilon_m^2(1-\varepsilon_m)^3](6/10)\end{array}}, \end{aligned}$$

which converges to $1/24$ as ε_m tends to zero. Similarly $p((i,3) \mid \tilde{F}, (s^m, r^m))$, $p((i,4) \mid \tilde{F}, (s^m, r^m))$, and $p((i,5) \mid \tilde{F}, (s^m, r^m))$ converge to $2/24$, $3/24$, and $18/24$, respectively. Therefore,

$$E\pi(I \mid \tilde{F}, (s, r_{-i})) = (1/24)d + (2/24)d + (3/24)d + (18/24)b = (6d+18b)/24.$$

Since the assumptions $2d+3b>0$ and $b>0$ imply $6d+18b>0=E\pi(O \mid \tilde{F}, (s, r_{-i}))$, it follows that $r_i(\tilde{F}) = I$ is optimal for entrant i.

At the information set $\tilde{A}$, since the monopolist will accept any entrant, sequential rationality states (independently of the beliefs) that all entrants have to enter. That is, for every n and for any belief,

$$E\pi(I \mid \tilde{A}, (s, r_{-n})) = b > 0 = E\pi(O \mid \tilde{A}, (s, r_{-n})).$$

Finally, I will obtain monopolist's beliefs and check his sequential rationality. Notice that for the monopolist what is crucial now is when player i will make his decision, because $r_i(x) = I$ for every x.

Suppose that $h \in H_0$ ($f(h) = \tilde{O}$) and i is the entrant. Then, $E\pi(s \mid (i,1), r) = -1 + 4a$ and the maximum expected payoff of following any other strategy s', given r, is $5c$ if $s'(h,i) = A$, and $-1+4a$ if $s'(h,i) = F$. Since the assumptions $7a > 15c + 8$ and $c > -1$ imply $-1 + 4a > 5c$, it follows that $E\pi(s \mid (i,1), r) \geq E\pi(s' \mid (i,1), r)$ for every s'. Therefore $s(h,i) = F$ satisfies optimality. If i is not the entrant ($i \neq 1$), the relevant conditional probabilities are $p((i,2) \mid (i \neq 1)) = \frac{1}{10}/\frac{9}{10} = 1/9$, $p((i,3) \mid (i \neq 1)) = \frac{1}{10}/\frac{9}{10} = 1/9$, $p((i,4) \mid (i \neq 1)) = \frac{1}{10}/\frac{9}{10} = 1/9$, and $p((i,5) \mid (i \neq 1)) = \frac{6}{10}/\frac{9}{10} = 6/9$. Therefore,

$$\begin{aligned} E\pi(s \mid (i \neq 1), r) &= (1/9)(-1 - 1 + 3a) + (1/9)(-1 + a - 1 + 2a) \\ &\quad + (1/9)(-1 + 2a - 1 + a) + (6/9)(-1 + 3a + c) \\ &= (1/3)(9a + 2c - 4), \end{aligned}$$

and the maximum expected payoff of following any other strategy s', given r, is $5c$ if $s'(h,n) = A$, and $(1/3)(9a + 2c - 4)$ if $s'(h,n) = F$. The assumptions $7a > 15c + 8$ and $c > -1$ imply $(1/3)(9a + 2c - 4) > 5c$. Therefore, $E\pi(s \mid (i \neq 1), r) \geq E\pi(s' \mid (i \neq 1), r)$ for every s', which is the monopolist's sequential rationality at those information sets.

Suppose that $h \in H_1$ and $f(h) \in \{\tilde{O}, \tilde{F}\}$. If i was the entrant at $t = 1$ (respectively, is the entrant now), $E\pi(s \mid (i = 1), n, r) = -1 + 3a$ (respectively, $E\pi(s \mid (i = 2), r) = -1 + 3a$), which by assumption is strictly larger than $4c$. Therefore, by a similar argument used in the previous case, $s(h,n) = F$ (respectively, $s(h,i) = F$) is optimal. If i has not decided yet ($i > 2$), the relevant conditional probabilities are

$$p((i,3) \mid (i > 2)) = \frac{12/120}{96/120} = 1/8,\ p((i,4) \mid (i > 2)) = \frac{12/120}{96/120} = 1/8,$$

and

$$p((i,5) \mid (i > 2)) = \frac{72/120}{96/120} = 6/8.$$

Therefore,

$$\begin{aligned} E\pi(s \mid (i > 2), r) &= (1/8)(-1 - 1 + 2a) + (1/8)(-1 + a - 1 + a) \\ &\quad + (6/8)(-1 + 2a + c) \\ &= (1/4)(8a + 3c - 5), \end{aligned}$$

and the maximum expected payoff of following any other strategy s', given r, is $4c$ if $s'(h,n) = A$, and $(1/4)(8a + 3c - 5)$ if $s'(h,n) = F$. The assumptions $7a > 15c + 8$ and $c > -1$ imply $(1/4)(8a + 3c - 5) > 4c$. Therefore, $E\pi(s \mid (i > 2), r) \geq E\pi(s' \mid (i > 2), r)$ for every s'.

Suppose that $h \in H_2$ and $f(h) \in \{\tilde{O}, \tilde{F}\}$. If i was the entrant at $t = 1$ or at $t = 2$ or is the entrant now $(1 \leq i \leq 3)$, $E\pi(s \mid (1 \leq i \leq 3), r) = -1 + 2a$, which by assumption is strictly larger than $3c$. Hence, $s(h, n) = F(s(h, i) = F)$ is optimal. If i has not decided yet $(i > 3)$, the relevant conditional probabilities are $p((i, 4) \mid (i > 3)) = \frac{12}{120}/\frac{84}{120} = 1/7$, and $p((i, 5) \mid (i > 3)) = \frac{72}{120}/\frac{84}{120} = 6/7$. Therefore,

$$E\pi(s \mid (i > 3), r) = (1/7)(-1-1+a)+(6/7)(-1+a+c) = (1/7)(7a+6c-8),$$

and the maximum expected payoff of following any other strategy s', given r, is $3c$ if $s'(h, n) = A$ and $(1/7)(7a + 6c - 8)$ if $s'(h, n) = F$. The assumptions $7a > 15c + 8$ and $c > -1$ imply $(1/7)(7a + 6c - 8) > 3c$. Therefore, $E\pi(s \mid (i > 3), r) \geq E\pi(s' \mid (i > 3), r)$ for every s'.

Suppose that $h \in H_3$, $f(h) \in \{\tilde{O}, \tilde{F}\}$ and player i has already made his decision $(1 \leq i \leq 4)$. Then, $E\pi(s \mid (1 \leq i \leq 4), r) = -1 + a$, which by assumption is strictly larger than $2c$. Hence, $s(h, n) = F(s(h, i) = F)$ is optimal.

For all remaining $h \in H$, $s(h, n) = A$ is optimal, since it is a dominant action and the monopolist can not avoid any entrance by fighting. ■

RESULT 5.1. *Suppose that* $T = 4$, $a > 3c + 2$, $19d + 9b < 0$ *and* p *is the uniform distribution. Then, there exists a set of beliefs* B *such that* $((s, r), B)$ *is a sequential equilibrium.*

Proof. Consider a potential entrant n. The argument made in Result 1.1 can be applied here, with the new interpretation of the probabilities $p(n, t \mid \cdot, (s, r))$, since strategies are the same. Therefore, the beliefs found there will also satisfy consistency here. Since p is uniform, sequential rationality would follow, and thus, at the information set $\tilde{O}$, it is satisfied that

$$\begin{aligned}\sum_{t=1}^{3} p(n, t)d + p(n, 4)b &= (16/64)d + (14/64)d + (12/64)d + (6/64)b \\ &= (42/64)d + (6/64)b < 0,\end{aligned}$$

because $19d + 9b < 0$ implies $42d + 6b < 0$. Moreover, at the information set $\tilde{F}$,

$$\begin{aligned}\sum_{t=2}^{3}(t-1)p(n, t)d + 3p(n, 4)b &= (14/64)d + 2(12/64)d + 3(6/64)b \\ &= (38/64)d + (18/64)b < 0,\end{aligned}$$

since $19d + 9b < 0$. Finally, at the information set $\tilde{A}$, player n should enter since, in the limit, the monopolist will accept the entrant.

Consider now the monopolist. His relevant uncertainty, given (h, n), is which period will be the last one. For $1 \leq t \leq 4$ and (h, n) let $q_t(h, n)$ be the probability that period t will be the last one.

Suppose that $h \in H_0(f(h) = \tilde{O})$ and n got in. Then, $q_1(h, n) = p(n)/[p(n) + p(n\cdot) + p(n\cdot\cdot) + p(n\cdot\cdot\cdot)] = 1/16$, and similarly, $q_2(h, n) = 3/16$, $q_3(h, n) = 6/16$, and $q_4(h, n) = 6/16$. Therefore, since $f(h) = \tilde{O}$, the expected payoff of following strategy s is

$$\begin{aligned} E\pi(s \mid h, n, r) &= (1/16)(-1) + (3/16)(-1 + a) + (6/16)(-1 + 2a) \\ &\quad + (6/16)(-1 + 3a) \\ &= (33/16)a - 1, \end{aligned}$$

and the maximum expected payoff of following any other strategy s', given r is $(1/16)c + (3/16)2c + (6/16)3c + (6/16)4c = (49/16)c$ if $s'(h, n) = A$ and $(33/16)a - 1$ if $s'(h, n) = F$. However, since by assumption $a > 3c + 2$ and $c > -1$, $(33/16)a - 1 > (49/16)c$ has to be satisfied, implying that $E\pi(s \mid h, n, r) \geq E\pi(s' \mid h, n, r)$ for every s'.

Suppose that $h \in H_1$, $f(h) \in \{\tilde{O}, \tilde{F}\}$, and $n \in N(h)$ got in. Let j be the player that has already made his decision at period 1 (i.e., if $f(h) = \tilde{O}$ then $h = \hat{O}_j$, and if $f(h) = \tilde{F}$ then $h = \hat{F}_j$). It follows that

$$\begin{aligned} q_2(h, n) &= \frac{p(jn)}{p(jn) + p(jn\cdot) + p(jn\cdot\cdot)} \\ &= 1/5,\ q_3(h, n) = 2/5,\ \text{and } q_4(h, n) = 2/5. \end{aligned}$$

Therefore, since $f(h) \in \{\tilde{O}, \tilde{F}\}$, the expected payoff of following the strategy s is

$$E\pi(s \mid h, n, r) = (1/5)(-1) + (2/5)(-1 + a) + (2/5)(-1 + 2a) = -1 + (6/5)a,$$

and the maximum expected payoff of following any other strategy s', given r, is $(1/5)c + (2/5)2c + (2/5)3c = (11/5)c$ if $s'(h, n) = A$, and $(6/5)a - 1$ if $s'(h, n) = F$. Nevertheless, since by assumption $a > 3c + 2$ and $c > -1$, $(6/5)a - 1 > (11/5)c$ has to be satisfied, implying that $E\pi(s \mid h, n, r) \geq E\pi(s' \mid h, n, r)$ for every s'.

Suppose that $h \in H_2$, $f(h) \in \{\tilde{O}, \tilde{F}\}$ and $n \in N(h)$ got in. Let j and k be the players that have already made their decisions at period 1 and 2, respectively, (i.e., if $f(h) = \tilde{O}$ then $h = \hat{O}_j\hat{O}_k$ and if $f(h) = \tilde{F}$ then either $h = \hat{F}_j\hat{O}_k$, $\hat{O}_j\hat{F}_k$ or $\hat{F}_j\hat{F}_k$). Then, for $i \neq j, k, n$, $q_3(h, n) = p(jkn)/[p(jkn) + p(jkni)] = 1/2$, and $q_4(h, n) = 1/2$. Therefore, since $f(h) \in \{\tilde{O}, \tilde{F}\}$, the expected payoff of following strategy s is

$$E\pi(s \mid h, n, r) = (1/2)(-1) + (1/2)(-1 + a) = -1 + (1/2)a,$$

and the maximum expected payoff of following any other strategy s', given r, is $(1/2)c + (1/2)2c = (3/2)c$ if $s'(h, n) = A$ and $(1/2)a - 1$ if $s'(h, n) = F$. However, since by hypothesis $a > 3c + 2$, $E\pi(s \mid h, n, r) \geq E\pi(s' \mid h, n, r)$ for every s'.

For the remaining h the monopolist cannot avoid future entries, and therefore, to play A is sequentially rational. ■

RESULT 5.2. *Suppose that $T = 4$, $a > 6c + 5$, and p is the uniform distribution. Then, there exists a set of beliefs B such that $((\hat{s}, r), B)$ is a sequential equilibrium.*

Proof. Arguments similar to the ones already used in Result 2.1 permit to obtain a set of consistent beliefs for the potential entrants as well as to show that r satisfies sequential rationality.

Consider the monopolist. His relevant uncertainty, given (z, n), is about how many periods are left (including the current one). For $1 \leq t \leq 4$ and (z, n) let $q_t(z, n)$ be the probability that there are t periods left.

If $z = \tilde{O}$, for every $1 \leq n \leq 4$ and every $1 \leq t \leq 4$, the conditional probabilities are, $q_t(\tilde{O}, n) = q_t$. Therefore, the expected payoff of following strategy $\hat{s}$ is

$$\begin{aligned} E\pi(\hat{s} \mid z, n, r) &= (1/16)(-1) + (3/16)(-1 + a) + (6/16)(-1 + 2a) \\ &\quad + (6/16)(-1 + 3a) \\ &= (33/16)a - 1, \end{aligned}$$

and the maximum expected payoff of following any other strategy $\hat{s}'$, given r, is $(1/16)c + (3/16)2c + (6/16)3c + (6/16)4c = (49/16)c$ if $\hat{s}'(z, n) = A$, and $(33/16)a - 1$ if $\hat{s}'(z, n) = F$. Nevertheless, since by assumption $a > 6c + 5$, $(33/16)a - 1 > (49/16)c$ has to be satisfied, implying that $E\pi(\hat{s} \mid z, n, r) \geq E\pi(\hat{s}' \mid z, n, r)$ for every $\hat{s}'$.

If z is a sequence of 2 $\hat{F}_i$'s for example $\hat{F}_i \hat{F}_j$, then

$$q_1(z, n \mid (\hat{s}^m, r^m)) = \frac{\varepsilon_m^3(1 - \varepsilon_m)^2 p(ijn) + \varepsilon_m^3(1 - \varepsilon_m)^3[p(kijn) + p(ikjn) + p(ijkn)]}{\varepsilon_m^3(1 - \varepsilon_m)^2 p(ijn) + \varepsilon_m^3(1 - \varepsilon_m)^3[p(kijn) + p(ikjn) + p(ijkn)] + \varepsilon_m^3(1 - \varepsilon_m)^2 p(ijnk)},$$

which converges to 4/5 as ε_m tends to zero. Therefore, the expected payoff of following strategy $\hat{s}$ is

$$E\pi(\hat{s} \mid z, n, r) = (4/5)(-1) + (1/5)(-1 + a) = (1/5)a - 1,$$

and the maximum expected payoff of following any other strategy $\hat{s}'$, given r, is $(4/5)c + (1/5)2c = (6/5)c$ if $\hat{s}'(z, n) = A$, and $(1/5)a - 1$ if $\hat{s}'(z, n) = F$.

But, since by assumption $a > 6c + 5$, $(1/5)a - 1 > (6/5)c$ has to be satisfied, implying that $E\pi(\hat{s} \mid z, n, r) \geq E\pi(\hat{s}' \mid z, n, r)$ for every $\hat{s}'$.

If $z = \hat{F}_i$, then, with ε_m as common probability of mistake, $q_1(z, n \mid (\hat{s}^m, r^m))$, $q_2(z, n \mid (\hat{s}^m, r^m))$ and $q_3(z, n \mid (\hat{s}^m, r^m))$ converge to 9/14, 4/14, and 1/14, respectively. Therefore, the expected payoff of following strategy $\hat{s}$ is

$$E\pi(\hat{s} \mid z, n, r) = (9/14)(-1) + (4/14)(-1+a) + (1/14)(-1+2a) = (6/14)a - 1,$$

and the maximum expected payoff of following any other strategy $\hat{s}'$, given r, is $(9/14)c + (4/14)2c + (1/14)3c = (20/14)c$ if $\hat{s}'(z, n) = A$, and $(6/14)a - 1$ if $\hat{s}'(z, n) = F$. Still, since by assumption $a > 6c + 5$, $(6/14)a - 1 > (20/14)c$ has to be satisfied, implying that $E\pi(\hat{s} \mid z, n, r) \geq E\pi(\hat{s}' \mid z, n, r)$ for every $\hat{s}'$.

If z is a sequence of 3 $\hat{F}_i$'s, then $q_4(z, n) = 1$. This implies that $\hat{s}$ is sequentially rational since $\hat{s}(z, n) = A$ is strictly dominant action.

Finally, if z has at least one $\hat{A}_i$, to play A is also a strictly dominant action. Therefore, $\hat{s}$ is sequentially rational. ∎

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