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On the invariance of the set of Core matchings with respect to preference profiles ☆

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ABSTRACT

We consider the general many-to-one matching model with ordinal preferences and give a procedure to partition the set of preference profiles into subsets with the property that all preference profiles in the same subset have the same Core. We also show how to identify a profile of (incomplete) binary relations containing the minimal information needed to generate as strict extensions all the (complete) preference profiles with the same Core. This is important for applications since it reduces the amount of information that agents have to reveal about their preference relations to centralized Core matching mechanisms; moreover, this reduction is maximal.

1. Introduction

The purpose of this paper is to study the invariance of the Core of ordinal many-to-one matching problems with respect to changes on firms' preference relations on subsets of workers. An ordinal many-to-one matching problem (a *matching problem* for short) consists of two non-empty and disjoint sets of agents: the set of *firms* (or institutions like schools, colleges, hospitals, etc.) and the set of *workers* (or individuals like children, students, medical interns, etc.). An allocation for a matching problem is a *matching* among firms and workers with the property that each worker can be matched to at most one firm and each firm is matched to a (possibly empty) subset of workers, keeping the bilateral nature of the relationships in the sense that if a worker is matched to a firm this firm is matched to a subset of workers that contains this worker. Each worker has a strict preference relation on the set of firms plus the prospect of remaining unmatched. Each firm has a strict preference relation on the set of all subsets of workers. A *preference profile* is a list of preference relations, one for each agent. The Core of a matching problem (at a given preference profile) is the set of matchings that are not blocked; namely, a matching belongs to the Core if there is no subset of agents (a coalition of firms and workers) such that, by rematching only among themselves, each agent gets a weakly better partner and at least one of them gets a strictly better one.

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The first result of the paper (Theorem 1) characterizes the family of equivalence classes of preference relations of each firm with the property that two preference relations are in the same class if and only if they have the same Core for all preference relations of the remaining agents. Our invariance result in Theorem 1 identifies those orderings between pairs of subsets of workers in a preference relation of a firm that, if inverted, the Core remains unchanged for *all* possible preference relations of the other agents. In other words, Theorem 1 identifies irrelevant changes on a preference relation of a firm that leave the Core invariant, irrespectively of the other agents' preference relations. The way of proceeding with this identification is as follows. Take a preference relation of a firm. First, construct the family of individually rational subsets of workers (a set of workers S belongs to the family if and only if the firm prefers the set S to all of its strict subsets). Second, define a binary relation on this family as follows: given two subsets of workers S and S' in the family declare that S is preferred to S' (according to the binary relation) if and only if S is the best subset (according to the original and complete preference relation of the firm) among all subsets of $S \cup S'$; otherwise, the two subsets of workers are left unordered by the binary relation. Observe that in general this binary relation is not only defined on a subfamily of subsets of workers but it is also incomplete. It turns out that this binary relation can be used as the representative of one equivalence class of preference relations of the firm because all preference relations that share the same binary relation constructed as we just described have the property that the Core is the same regardless of the other agents' preference relations.

Theorem 1 extends and generalizes our previous result in Martínez et al. (2008) where we construct this invariant partition only for the subclass of substitutable preference profiles.[1] If the preference profile is substitutable the Core and the set of stable matchings coincide, are non-empty, and the binary relations obtained from the preference relations of the firms, as we have described above, are partial orders.

In general, the binary relation used to represent the equivalence class formed by all preference relations of a firm that leave the Core invariant still relates too many pairs of subsets of workers. In centralized matching markets in which Core mechanisms (stable ones, whenever firms' preferences are substitutable) are used to suggest to the participants—after collecting and processing their preference relations—a matching in the Core, it would be very useful to use the smallest possible amount of information contained in the preference profile that still allows to compute a Core matching relative to this preference profile.[2] Thus, and in order to identify this minimal amount of information, we give a procedure to construct the minimal binary relation contained in the binary relation identified in Theorem 1, with the property that it still can generate all preference relations in the same equivalence class (that is, with the same Core) as their strict extensions. Furthermore, this binary relation is minimal in the sense that any strictly weaker (*i.e.*, strictly contained) binary relation has at least two strict extensions that belong to different equivalence classes and thus have different Cores for some preference relations of the other agents.

Observe that the question of finding the minimal binary relation that can generate all equivalent preference relations was not even asked in Martínez et al. (2008) for the subclass of substitutable preference relations. Thus, the marginal contribution of this paper in relation to our former one is two-fold. We first extend the result of Theorem 1 from substitutable preference relations to any preference relation. Second, we identify for each preference relation (substitutable or not) the minimal binary relation that can be used as the representative of each equivalence class of preference relations with an invariant Core. This binary relation contains the indispensable and, at the same time, minimal information to generate the full class.

Echenique (2008) answers a related question. Suppose we observe a set of matchings and we do not know agents' preference relations. Are there preference relations for the agents so that the observed sets of matchings are stable?[3] If yes, the set of matchings is said to be rationalizable. Echenique (2008) first shows that there are sets of matchings that are not rationalizable (and thus, the theory is testable) and second he identifies conditions that characterize the sets of matchings that are rationalizable: a necessary condition is that a certain graph has no odd cycles and a necessary and sufficient condition is in terms of no odd cycles and a certain system of polynomial inequalities. However, his results are different from ours in many respects. Echenique (2008)'s results apply only to the one-to-one matching model while ours apply to the more general many-to-one matching model. His results are in graph-theoretical terms and deal with the full preference profile by identifying how agents can rank potential partners given the set of matchings to be rationalizable. In contrast we identify, given a preference relation of a firm over subsets of workers (and independently of the other agents' preferences), those relations between pairs of subsets of workers that are critical from the point of view of the Core and those that are not.

Before finishing this Introduction we want to emphasize that, besides their intrinsic interest, our invariance and minimality results have a relevant informational implication. They show that the amount of information about firms' preferences required to compute the set of Core matchings may be significantly smaller than the amount needed to describe their complete preference relations. This may be specially relevant for running direct preference revelation Core mechanisms in

[1] A preference relation of a firm is *substitutable* if the desirability of a worker w in a particular set of workers does not come from the presence of another worker w' in that set because the firm still wants to hire worker w even when worker w' is not available anymore; *i.e.*, substitutable preference relations do not exhibit strong complementarities among workers. A preference profile is substitutable if the preference relations of all firms are substitutable. Kelso and Crawford (1982) were the first to define and use substitutability in a more general matching model with money.

[2] See Roth and Sotomayor (1990) for a general description and analysis of these centralized markets. Niederle et al. (2008) contains a recent overview on matching and market design in general.

[3] Chambers and Echenique (2008) ask a similar question for a cardinal matching model.

centralized entry-level professional labor markets. Moreover, our results may have computational and behavioral implications since they may simplify the task of computing the set of Core matchings as well as the analysis of the strategic behavior induced on firms by centralized Core matching mechanisms (in particular, to find either best-replies or unilateral deviations may be substantially easier). Finally, our results can be straightforwardly extended to the Core of ordinal many-to-many matching markets.

The paper is organized as follows. In Section 2 we present the notation, the basic definitions, and some preliminary results. In Section 3 we state and prove the invariance result for the set of Core matchings. In Section 4 we define the notions of minimal binary relation, strict extension and state and prove the minimality result. Finally, in Section 5 we conclude with final remarks, including a very preliminarily analysis of the computational aspect of our approach.

2. Preliminaries

2.1. Agents and preferences

Let W be the set of *workers* and let F be the set of *firms*. We assume that W and F are finite and disjoint. The set of *agents* is $W \cup F$. Each worker $w \in W$ has a preference relation P_w on the set of firms plus the prospect of remaining unemployed. We assume that P_w is strict. Specifically, P_w is a complete, irreflexive, and transitive binary relation on $F \cup \{\emptyset\}$, where $\emptyset$ means that w is not hired by any firm.[4] Given P_w, let R_w be the weak preference relation on $F \cup \{\emptyset\}$ induced by P_w as follows: for $f, f' \in F \cup \{\emptyset\}$, $f R_w f'$ if and only if either $f = f'$ or $f P_w f'$. Then, R_w is a complete, reflexive, antisymmetric, and transitive binary relation on $F \cup \{\emptyset\}$.[5] Each firm $f \in F$ has a preference relation P_f on the family of all subsets of workers. We assume that P_f is strict. Specifically, P_f is a complete, irreflexive, and transitive binary relation on 2^W, where the empty set is interpreted as the prospect of not hiring any worker. Given firm f's preference relation P_f and a subset of workers S, $Ch(S, P_f)$ denotes f's most-preferred subset of S according to P_f. Generically, we will refer to this set as the *choice set*. Given P_f, let R_f be the complete, reflexive, antisymmetric, and transitive binary relation induced similarly on 2^W by P_f. A *preference profile* $P = ((P_f)_{f \in F}, (P_w)_{w \in W})$ is an $|F| + |W|$-tuple of preference relations, one for each agent. Given a preference profile P and f's preference relation P'_f, we will denote by (P'_f, P_{-f}) the original preference profile P after replacing P_f by P'_f and refer to P_{-f} as a subprofile. Given a preference relation P_f of firm f, the subsets of workers preferred to the empty set by f are called *acceptable*. Similarly, given a preference relation P_w of worker w, the firms preferred to the empty set by w are called *acceptable*. By convention, we declare the empty set as being acceptable for all agents. Since the set of agents will be fixed throughout the paper, we identify a matching problem with a preference profile P.

2.2. Matchings and the Core

A matching assigns each firm to a subset of workers (possibly empty) and each worker to at most one firm, keeping the bilateral nature of the relationship; *i.e.*, worker w works for firm f if and only if firm f hires worker w.

Definition 1. A *matching* is a mapping $\mu : W \cup F \to 2^{F \cup W}$ with the properties:

(ma.1) $\mu(f) \in 2^W$ for all $f \in F$;
(ma.2) $\mu(w) \in 2^F$ and $|\mu(w)| \leqslant 1$ for all $w \in W$; and
(ma.3) $w \in \mu(f)$ if and only if $\mu(w) = f$.[6]

If matching is voluntary it should be immune to any secession of a coalition of agents that, by matching only amongst themselves, could obtain better partners by breaking the former partnerships and creating new ones (a block). The Core is the set of matchings that are not blocked by any coalition of agents.

Definition 2. Let P be a preference profile and let μ be a matching. Coalition $W' \cup F' \subseteq W \cup F$ *blocks* μ if there exists another matching μ' such that:

(bl.1) $\mu'(f) \subseteq W'$ for all $f \in F'$;
(bl.2) either $\mu'(w) = \emptyset$ or $\mu'(w) \in F'$ for all $w \in W'$; and
(bl.3) for all $f \in F'$,

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[4] A binary relation $\succ$ on X is (i) *complete* if for all $x, y \in X$ such that $x \neq y$, either $x \succ y$ or $y \succ x$, (ii) *irreflexive* if $x \not\succ x$ for all $x \in X$, and (iii) *transitive* if for all $x, y, z \in X$ such that $x \succ y \succ z$, $x \succ z$ holds.

[5] A binary relation $\succcurlyeq$ on X is (i) *reflexive* if $x \succcurlyeq x$ for all $x \in X$ and (ii) *antisymmetric* if, for all $x, y \in X$ such that $x \succcurlyeq y$ and $y \succcurlyeq x$, $x = y$ holds.

[6] With a slight abuse of notation we treat $\mu(w) \neq \emptyset$ as an element of F instead of one of its subsets; for instance, we write $\mu(w) = f$ instead of $\mu(w) = \{f\}$.

$$\mu'(f) R_f \mu(f), \tag{1}$$

for all $w \in W'$,

$$\mu'(w) R_w \mu(w), \tag{2}$$

and at least one of the weak preferences in (1) and (2) is strict.

Definition 3. Let P be a preference profile. A matching μ belongs to the *Core* (at P) if it is not blocked by any coalition.

A matching μ is *individually rational* (at P) if $\mu(w) R_w \emptyset$ for all $w \in W$ and $\mu(f) = Ch(\mu(f), P_f)$ for all $f \in F$. Denote by $IR(P)$ the set of individually rational matchings at P. A matching μ is *pair-wise stable* (at P) if there is no unmatched pair $(w, f) \in W \times F$ such that $f P_w \mu(w)$ and $w \in Ch(\mu(f) \cup \{w\}, P_f)$. The set of *stable* matchings (at P) is the set of individually rational matchings that are pair-wise stable. Let $S(P)$ denote the set of stable matchings (at P) and let $C(P)$ denote the set of matchings in the Core (at P). Obviously, $C(P) \subseteq S(P)$ for all P. It is well known that there are preference profiles for which the Core (and the set of stable matchings) is empty.

Kelso and Crawford (1982) proposed (in a more general many-to-one matching model) a condition on the preference relations of firms, called substitutability, with the property that if in a profile P all firms have substitutable preference relations then the Core is non-empty and coincides with the set of stable matchings. For this reason substitutability has played a central role in the analysis of many-to-one matching models.

Definition 4. A firm f's preference relation P_f satisfies *substitutability* if for any set S containing workers w and w' ($w \neq w'$), if $w \in Ch(S, P_f)$ then $w \in Ch(S \setminus \{w'\}, P_f)$.

Substitutability precludes strong complementarities among workers since it requires that the desirability of a worker w in a particular set S does not come exclusively from the presence of another worker w' in that set; *i.e.*, the firm still wants to hire worker w even though worker w' is not available anymore; thus, w is a good worker (in the context of the set S) not only because of the presence of w'. A preference profile P is *substitutable* if for each firm f, the preference relation P_f satisfies substitutability. Let $\mathcal{S}$ be the set of substitutable preference profiles. For any substitutable preference profile $P \in \mathcal{S}$, $C(P) = S(P) \neq \emptyset$. However, there are non-substitutable preference profiles P for which $C(P) \neq \emptyset$.

2.3. Extracting binary relations from firms' preferences

Consider a preference relation of a firm on the family of all subsets of workers. Our objective is to distinguish among all orderings between pairs of subsets of workers those that are irrelevant for the Core from those that are relevant in the following sense. Take S and S' and assume that $S P_f S'$. Consider a new preference relation P'_f with the property that it coincides with P_f except that $S' P'_f S$. Then either $C(P_f, P_{-f}) = C(P'_f, P_{-f})$ for all P_{-f} (in which case the ordering between S and S' is irrelevant for the Core) or else there exists at least one P'_{-f} such that $C(P_f, P'_{-f}) \neq C(P'_f, P'_{-f})$ (in which case the ordering is relevant). To attain this objective we proceed by first selecting from the family of all subsets of workers a subfamily on which we will then define a binary relation that keeps only the relevant orderings (from the point of view of the Core) between subsets of workers. But before, we need some additional notions and notation.

Let A be a non-empty subfamily of subsets of W containing the empty set; *i.e.*, $A \subseteq 2^W$ and $\emptyset \in A$. A *partial order* $\succcurlyeq$ on A is a reflexive, transitive, and antisymmetric binary relation on A. Observe that weak preference relations of firms are complete partial orders on 2^W. Given a binary relation $\succcurlyeq$ on A, let $\succ$ be the antireflexive and transitive binary relation on A induced by $\succcurlyeq$ on A as follows: for $S, S' \in A$, $S \succ S'$ if and only if $S \succcurlyeq S'$ and $S \neq S'$. A binary relation $\succcurlyeq$ on A is *acyclic* if for all $S_1, \ldots, S_k \in A$ such that $S_1 \neq S_k$, $S_1 \succcurlyeq \cdots \succcurlyeq S_k$ implies $S_k \not\succcurlyeq S_1$. A binary relation $\succcurlyeq$ on A has a *maximal element* on $B \subseteq A$ if there exists $S \in B$ such that for all $S' \in B$ with $S' \neq S$, $S \succ S'$ holds where $\succ$ is induced by $\succcurlyeq$. Then, given a preference relation P_f and a set $S \in 2^W$, P_f has a maximal element on the family of all subsets of S. We have denoted this set by $Ch(S, P_f)$ and called it the choice set of S according to P_f; namely, $Ch(S, P_f) P_f S'$ for all $S' \in 2^S \setminus Ch(S, P_f)$.

It will be useful to understand (and to denote) a binary relation $\succcurlyeq$ on A as a subset of $A \times A$; namely, for all $S, T \in A$, $(S, T) \in \succcurlyeq \subseteq A \times A$ if and only if $S \succcurlyeq T$. Hence, for two binary relations $\succcurlyeq$ and $\succcurlyeq'$ on A the notation $\succcurlyeq \subseteq \succcurlyeq'$ means that if $S, S' \in A$ and $S \succcurlyeq S'$ then, $S \succcurlyeq' S'$.

After these preliminaries we now turn to define the procedure to delete from the preference relation of a firm the orderings between those pairs of subsets of workers that are irrelevant with respect to the set of matchings in the Core. First, subsets that are not the choice set of themselves can be left unordered since no matching in the Core, regardless of the other agents' preference relations, matches this firm with any of these subsets. Formally, given the preference relation P_f on 2^W, define the family A_{P_f} of *individually rational subsets of workers relative to* P_f as the collection of sets that are choice sets of themselves; that is,

$$A_{P_f} = \{S \in 2^W \mid S = Ch(S, P_f)\}.$$

Second, some pairs of subsets of workers in A_{P_f} will be left unordered. Specifically, define the binary relation $\succcurlyeq_{P_f}$ on A_{P_f} *obtained from* P_f as follows: for all $S, S' \in A_{P_f}$,

$$S \succcurlyeq_{P_f} S' \quad \text{if and only if} \quad S = Ch\big(S \cup S', P_f\big).$$

Again, the binary relation $\succcurlyeq_{P_f}$ on A_{P_f} leaves as unordered (i) all sets in 2^W that are not the choice of themselves and (ii) those pairs of sets in A_{P_f} whose union contains a set that is preferred to each of the two sets.[7] Martínez et al. (2008) show that if P_f is substitutable then $\succcurlyeq_{P_f}$ is a partial order on A_{P_f} and $(A_{P_f}, \succcurlyeq_{P_f})$ is a semilattice; namely, for every $S, S' \in A_{P_f}$, $lub_{\succcurlyeq_{P_f}}\{S, S'\} \in A_{P_f}$ (where, given a family of subsets $\mathcal{T}$, $lub_{\succcurlyeq_{P_f}}\mathcal{T}$ is the least upper bound of $\mathcal{T}$). Example 1 below shows that if P_f is not substitutable then the binary relation $\succcurlyeq_{P_f}$ may not be transitive.

Example 1. Let $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the set of workers and let f be a firm. Consider the preference relation

$$P_f: \quad \{w_1, w_5\}, \{w_1, w_2\}, \{w_3, w_4\}, \{w_4, w_5\}, \{\emptyset\},$$

where we only list acceptable subsets of workers in decreasing order of preference. Observe that P_f is not substitutable since $w_5 \in Ch(W, P_f)$ and $w_5 \notin Ch(W \setminus \{w_1\}, P_f) = \{w_3, w_4\}$. Moreover, the family of individually rational subsets of workers relative to P_f is $A_{P_f} = \{\{w_1, w_5\}, \{w_1, w_2\}, \{w_3, w_4\}, \{w_4, w_5\}, \{\emptyset\}\}$ and

$$\begin{aligned}
&\{w_1, w_2\} = Ch\big(\{w_1, w_2\} \cup \{w_3, w_4\}, P_f\big),\\
&\{w_3, w_4\} = Ch\big(\{w_3, w_4\} \cup \{w_4, w_5\}, P_f\big), \quad \text{and}\\
&\{w_1, w_5\} = Ch\big(\{w_1, w_2\} \cup \{w_4, w_5\}, P_f\big).
\end{aligned}$$

Hence, $\{w_1, w_2\} \succcurlyeq_{P_f} \{w_3, w_4\} \succcurlyeq_{P_f} \{w_4, w_5\}$ but $\{w_1, w_2\} \not\succcurlyeq_{P_f} \{w_4, w_5\}$. Thus, the binary relation $\succcurlyeq_{P_f}$ is not transitive and $(A_{P_f}, \succcurlyeq_{P_f})$ is not a semilattice.

Example 1 shows that the binary relation $\succcurlyeq_{P_f}$ may be incomplete on A_{P_f} (both $\{w_1, w_2\} \not\succcurlyeq_{P_f} \{w_4, w_5\}$ and $\{w_4, w_5\} \not\succcurlyeq_{P_f} \{w_1, w_2\}$ hold) and that it may not inherit the transitivity of P_f. Nevertheless, Remarks 1 and 2 below establish that the binary relation $\succcurlyeq_{P_f}$ inherits some other properties from the preference relation P_f.

Remark 1. Let $S, S' \in A_{P_f}$ be such that $S \succcurlyeq_{P_f} S'$. Then, $S R_f S'$.

Remark 2. The binary relation $\succcurlyeq_{P_f}$ on A_{P_f} is reflexive, antisymmetric, acyclic, and has a maximal element on A_{P_f}.

3. The invariance result

Theorem 1 below gives a simple procedure to partition the set of firm f's preference relations into equivalence classes where each class contains exactly those preference relations for which the set of Core matchings is invariant regardless of the other agents' preference relations. Theorem 1 says that an equivalence class is composed of all firm f's preference relations for which the binary relations obtained from them coincide.

Theorem 1. *Let P_f and P'_f be two preference relations on 2^W. Then,*

$$\succcurlyeq_{P_f} = \succcurlyeq_{P'_f} \quad \textit{if and only if} \quad C(P_f, P_{-f}) = C\big(P'_f, P_{-f}\big) \quad \textit{for all } P_{-f}.$$

Proof. ($\Rightarrow$) Let P_f and P'_f be two preference relations such that $\succcurlyeq_{P_f} = \succcurlyeq_{P'_f}$. Thus, $A_{P_f} = A_{P'_f}$. Assume there exist P_{-f} and μ such that $\mu \in C(P_f, P_{-f}) \setminus C(P'_f, P_{-f})$. Since $A_{P_f} = A_{P'_f}$ and $\mu \in C(P_f, P_{-f})$, $\mu(f) \in A_{P_f} = A_{P'_f}$; *i.e.*, $\mu(f) = Ch(\mu(f), P'_f)$. Thus,

$$\mu \in IR(P_f, P_{-f}) \cap IR\big(P'_f, P_{-f}\big). \tag{3}$$

Define $P' = (P'_f, P_{-f})$ and let (F', W', μ') be a block of μ at P'. By (3), there exist $f' \in F'$ and $S' \subseteq W'$ such that $\mu'(f') = S'$ and for all $v \in \{f'\} \cup S'$,

$$\mu'(v) R'_v \mu(v), \tag{4}$$

and there exists $\bar{v} \in \{f'\} \cup S'$ such that

[7] Blair (1988) was the first to use this binary relation when showing that the set of stable matchings with multiple partners has a lattice structure.

$$\mu'(\bar{v})P'_{\bar{v}}\mu(\bar{v}). \tag{5}$$

If $f' \neq f$, by (4) and (5), $\mu'(v)R_v\mu(v)$ and $\mu'(\bar{v})P_{\bar{v}}\mu(\bar{v})$. Hence $(\{f'\}, S', \mu')$ blocks μ at (P_f, P_{-f}). This contradicts the hypothesis that $\mu \in C(P_f, P_{-f})$. Thus, $f' = f$. Hence $(\{f\}, \mu'(f), \mu')$ blocks μ at P'. If $\mu'(f) \neq Ch(\mu'(f), P'_f)$ then $(\{f\}, Ch(\mu'(f), P'_f), \mu'')$ also blocks μ at P, where μ'' is such that $\mu''(f) = Ch(\mu'(f), P'_f)$ and $\mu''(\hat{f}) = \emptyset$ for all $\hat{f} \neq f$. Hence, and since $A_{P_f} = A_{P'_f}$, we can assume without loss of generality that $\mu'(f) = Ch(\mu'(f), P'_f) = Ch(\mu'(f), P_f)$. Since $\mu \in C(P_f, P_{-f})$,

$$\mu(f)P_f\mu'(f) \quad \text{and} \quad \mu'(f)P'_f\mu(f).$$

Thus,

$$\mu'(f) \not\succcurlyeq_{P_f} \mu(f) \quad \text{and} \quad \mu(f) \not\succcurlyeq_{P'_f} \mu'(f). \tag{6}$$

By the hypothesis that $\succcurlyeq_{P_f} = \succcurlyeq_{P'_f}$,

$$\mu'(f) \not\succcurlyeq_{P'_f} \mu(f) \quad \text{and} \quad \mu(f) \not\succcurlyeq_{P_f} \mu'(f). \tag{7}$$

Consider any matching $\hat{\mu}$ with the property that $\hat{\mu}(f) = Ch(\mu(f) \cup \mu'(f), P_f)$. We now show that, $(\{f\}, Ch(\mu(f) \cup \mu'(f), P_f), \hat{\mu})$ blocks μ at (P_f, P_{-f}). Since (7) we have that $\hat{\mu}(f)P_f\mu(f)$. Let $w \in \hat{\mu}(f)$. Either $w \in \mu(f)$, in which case $w \in S'$ and $\hat{\mu}(w)R_w\mu(w)$ since $\hat{\mu}(w) = \mu(w)$ or else, $w \in \mu'(f)$, in which case $\hat{\mu}(w) = f = \mu'(w)R_w\mu(w)$ by (4) and $R_w = R'_w$. This contradicts the hypothesis that $\mu \in C(P_f, P_{-f})$.

($\Leftarrow$) Let P_f and P'_f be such that $C(P_f, P_{-f}) = C(P'_f, P_{-f})$ for all P_{-f}. We first show that $A_{P_f} = A_{P'_f}$. Assume $S \in A_{P_f}$. We want to show that $S \in A_{P'_f}$ (by symmetry, this will suffice). Consider the following subprofile P_{-f}: for all $w \in S$, all $w' \notin S$, and all $\hat{f} \neq f$,

$$\begin{array}{ccc} \underline{P_w} & \underline{P_{w'}} & \underline{P_{\hat{f}}} \\ f & \emptyset & \emptyset \\ \emptyset. & & \end{array}$$

The unique Core matching at (P_f, P_{-f}) is μ, where $\mu(f) = S$ and $\mu(\hat{f}) = \emptyset$ for all $\hat{f} \neq f$ (obviously, $\mu(w') = \emptyset$ for all $w' \notin S$). By hypothesis, $C(P_f, P_{-f}) = C(P'_f, P_{-f})$. Hence, μ is individually rational at (P'_f, P_{-f}). Thus, $S \in A_{P'_f}$. To show that $\succcurlyeq_{P_f} = \succcurlyeq_{P'_f}$ assume $S_1, S_2 \in A_{P_f} = A_{P'_f}$ are such that $S_1 \succcurlyeq_{P_f} S_2$, but $S_1 \not\succcurlyeq_{P'_f} S_2$. Then,

$$S_1 \neq Ch\big(S_1 \cup S_2, P'_f\big).$$

Consider the following preference profile P_{-f}: for all $w \in S_1 \cup S_2$, all $w' \notin S_1 \cup S_2$, and all $\hat{f} \neq f$,

$$\begin{array}{ccc} \underline{P_w} & \underline{P_{w'}} & \underline{P_{\hat{f}}} \\ f & \emptyset & \emptyset \\ \emptyset. & & \end{array}$$

Let μ be the matching where $\mu(f) = S_1$, $\mu(\hat{f}) = \emptyset$ for all $\hat{f} \neq f$, and $\mu(w') = \emptyset$ for all $w' \notin S_1$. Since $S_1 \succcurlyeq_{P_f} S_2$, $S_1 = Ch(S_1 \cup S_2, P_f)$. It is easy to check that $\mu \in C(P_f, P_{-f})$. Since $S_1 \neq Ch(S_1 \cup S_2, P'_f)$, $Ch(S_1 \cup S_2, P'_f)P'_f S_1$. Thus, $(\{f\}, Ch(S_1 \cup S_2, P'_f), \mu')$, where μ' is any matching such that $\mu'(f) = Ch(S_1 \cup S_2, P'_f)$, blocks μ at (P'_f, P_{-f}) since

$$\mu'(f)P'_f\mu(f) = S_1,$$

and for all $w \in Ch(S_1 \cup S_2, P'_f)$,

$$f = \mu'(w)R_w\mu(w).$$

Hence, $\mu \notin C(P'_f, P_{-f})$. This contradicts the hypothesis that $C(P_f, P_{-f}) = C(P'_f, P_{-f})$. □

An alternative way of describing Theorem 1 in terms of the Core correspondence is as follows. For a firm f and its preference relation P_f denote by T_{f,P_f} the Core mapping that takes as arguments all subprofiles of preferences P_{-f} of workers and remaining firms and such that $T_{f,P_f}(P_{-f}) = C(P_f, P_{-f})$. Theorem 1 partitions the set of preference relations of firm f into equivalence classes such that all preference relations in a class have the same Core mapping.

4. The minimality result

An implication of Theorem 1 is that an incomplete binary relation can be used as the representative of each equivalence class of all preference relations of a firm that leave the Core invariant. In general, the amount of information contained in the incomplete binary relation is substantially smaller than the one contained in any of its associated preference relations. However, this binary relation still contains redundant information (some pairs of subsets of workers are unnecessarily ordered) since the same equivalence class could be recovered by extending appropriately a strictly weaker binary relation. Example 2 below illustrates this fact and how we will proceed.

Example 2. Let $W = \{w_1, w_2, w_3\}$ be the set of workers and let f be a firm. Consider the preference relation P_f on 2^W

$$\{w_1, w_2, w_3\}P_f\{w_1, w_2\}P_f\{w_1, w_3\}P_f\{w_2, w_3\}P_f\{w_1\}P_f\{w_2\}P_f\{w_3\}P_f\{\emptyset\}.$$

Observe that $A_{P_f} = 2^W$. Obviously, $S \succcurlyeq_{P_f} S$ for all $S \in A_{P_f}$. In addition, $\succcurlyeq_{P_f}$ consists of the following orderings:

$$\begin{array}{lll}
\{w_1, w_2, w_3\} \succcurlyeq_{P_f} \{w_1, w_2\}, & \{w_1, w_2\} \succcurlyeq_{P_f} \{w_1\}, & \{w_1\} \succcurlyeq_{P_f} \{\emptyset\},\\
\{w_1, w_2, w_3\} \succcurlyeq_{P_f} \{w_1, w_3\}, & \{w_1, w_2\} \succcurlyeq_{P_f} \{w_2\}, & \{w_2\} \succcurlyeq_{P_f} \{\emptyset\},\\
\{w_1, w_2, w_3\} \succcurlyeq_{P_f} \{w_2, w_3\}, & \{w_1, w_2\} \succcurlyeq_{P_f} \{\emptyset\}, & \{w_3\} \succcurlyeq_{P_f} \{\emptyset\}.\\
\{w_1, w_2, w_3\} \succcurlyeq_{P_f} \{w_1\}, & \{w_1, w_3\} \succcurlyeq_{P_f} \{w_1\}, & \\
\{w_1, w_2, w_3\} \succcurlyeq_{P_f} \{w_2\}, & \{w_1, w_3\} \succcurlyeq_{P_f} \{w_3\}, & \\
\{w_1, w_2, w_3\} \succcurlyeq_{P_f} \{w_3\}, & \{w_1, w_3\} \succcurlyeq_{P_f} \{\emptyset\}, & \\
\{w_1, w_2, w_3\} \succcurlyeq_{P_f} \{\emptyset\}, & \{w_2, w_3\} \succcurlyeq_{P_f} \{w_2\}, & \\
 & \{w_2, w_3\} \succcurlyeq_{P_f} \{w_3\}, & \\
 & \{w_2, w_3\} \succcurlyeq_{P_f} \{\emptyset\}, &
\end{array}$$

Note that $\{w_1, w_2\}P_f\{w_1, w_3\}$, $\{w_1, w_2\}P_f\{w_2, w_3\}$, and $\{w_1, w_3\}P_f\{w_2, w_3\}$ but $\{w_1, w_2\} \not\succcurlyeq_{P_f} \{w_1, w_3\}$, $\{w_1, w_2\} \not\succcurlyeq_{P_f} \{w_2, w_3\}$, and $\{w_1, w_3\} \not\succcurlyeq_{P_f} \{w_2, w_3\}$ because $\{w_1, w_2, w_3\} = Ch(\{w_1, w_2\} \cup \{w_1, w_3\}, P_f) = Ch(\{w_1, w_2\} \cup \{w_2, w_3\}, P_f) = Ch(\{w_1, w_3\} \cup \{w_2, w_3\}, P_f)$. From the point of view of the Core, the only relevant information contained in P_f (together with the fact that $A_{P_f} = 2^W$) is that the best subset of workers is W itself (this is true as long as we extend the binary relation by making sure that if one set of workers is strictly contained in another set then, the larger set is strictly preferred to the smaller one in the extension). The relative orderings among subsets of cardinality two and the relative orderings among subsets of cardinality one are irrelevant for the set of Core matchings. For instance, in this case the three preference relations P'_f, P''_f, and P'''_f defined by

$$\{w_1, w_2, w_3\}P'_f\{w_1, w_2\}P'_f\{w_2, w_3\}P'_f\{w_1, w_3\}P'_f\{w_1\}P'_f\{w_2\}P'_f\{w_3\}P'_f\{\emptyset\},$$

$$\{w_1, w_2, w_3\}P''_f\{w_1, w_2\}P''_f\{w_1, w_3\}P''_f\{w_2, w_3\}P''_f\{w_2\}P''_f\{w_1\}P''_f\{w_3\}P''_f\{\emptyset\},$$

$$\{w_1, w_2, w_3\}P'''_f\{w_1, w_2\}P'''_f\{w_2, w_3\}P'''_f\{w_1, w_3\}P'''_f\{w_2\}P'''_f\{w_1\}P'''_f\{w_3\}P'''_f\{\emptyset\}$$

have the property that $C(P_f, P_{-f}) = C(P'_f, P_{-f}) = C(P''_f, P_{-f}) = C(P'''_f, P_{-f})$ for any subprofile P_{-f}. Indeed, we will show that from the information conveyed by the fact that $A_{P_f} = 2^W$ and the much weaker (and minimal) binary relation $\succcurlyeq^m_{P_f} = \{\emptyset\}$, where no pair of subsets of workers are related, we will be able to extract the class of (complete) preference relations on 2^W that leave the Core invariant. Observe that the number of pairs related by $\succcurlyeq_{P_f}$ (nineteen or twenty seven if we include those that follow from reflexivity) is much larger than the number of pairs related by $\succcurlyeq^m_{P_f}$ (none). But this is an extreme case.

Consider now the preference relation $\hat{P}_f$ on 2^W

$$\{w_1, w_2\}\hat{P}_f\{w_1\}\hat{P}_f\{w_2\}\hat{P}_f\{w_3\}\hat{P}_f\{\emptyset\},$$

where we only list acceptable partners. Then, $A_{\hat{P}_f} = \{\{w_1, w_2\}, \{w_1\}, \{w_2\}, \{w_3\}, \{\emptyset\}\}$ and $\{w_1, w_2\} \succcurlyeq_{\hat{P}_f} \{w_1\}$, $\{w_1, w_2\} \succcurlyeq_{\hat{P}_f} \{w_2\}$, $\{w_1, w_2\} \succcurlyeq_{\hat{P}_f} \{w_3\}$, $\{w_1\} \succcurlyeq_{\hat{P}_f} \{w_3\}$, $\{w_2\} \succcurlyeq_{\hat{P}_f} \{w_3\}$, and for any $S \in A_{\hat{P}_f}$, $S \succcurlyeq_{\hat{P}_f} S$ and $S \succcurlyeq_{\hat{P}_f} \{\emptyset\}$. Observe first that $\{w_1\} \not\succcurlyeq_{\hat{P}_f} \{w_2\}$ and $\{w_2\} \not\succcurlyeq_{\hat{P}_f} \{w_1\}$, and that the orderings $\{w_1, w_2\} \succcurlyeq_{\hat{P}_f} \{w_3\}$ and $\{w_1, w_2\} \succcurlyeq_{\hat{P}_f} \{\emptyset\}$ could be recovered by transitivity (using either $\{w_1\}$ or $\{w_2\}$ as intermediate subset). In addition, the orderings $\{w_1, w_2\} \succcurlyeq_{\hat{P}_f} \{w_1\}$, $\{w_1, w_2\} \succcurlyeq_{\hat{P}_f} \{w_2\}$ and for any $S \in A_{\hat{P}_f}$, $S \succcurlyeq_{\hat{P}_f} S$ and $S \succcurlyeq_{\hat{P}_f} \{\emptyset\}$ could also be recovered because they relate a set with one of its subsets. Thus, we could define a much weaker binary relation $\succcurlyeq^m_{\hat{P}_f}$ on $A_{\hat{P}_f}$ with only two elements: $\{w_1\} \succcurlyeq^m_{\hat{P}_f} \{w_3\}$ and $\{w_2\} \succcurlyeq^m_{\hat{P}_f} \{w_3\}$. Moreover, given $A_{\hat{P}_f}$ (this conveys a very important information), the two preference relations $\hat{P}_f$ (the one that we started with) and $\overline{P}_f$ on 2^W (again, we only list acceptable partners)

$$\{w_1, w_2\}\hat{P}_f\{w_1\}\hat{P}_f\{w_2\}\hat{P}_f\{w_3\}\hat{P}_f\{\emptyset\},$$
$$\{w_1, w_2\}\overline{P}_f\{w_2\}\overline{P}_f\{w_1\}\overline{P}_f\{w_3\}\overline{P}_f\{\emptyset\}$$

can be obtained from $\succcurlyeq^m_{\hat{P}_f}$ as what we will call strict extensions. Our results will say that $C(\hat{P}_f, P_{-f}) = C(\overline{P}_f, P_{-f})$ for all subprofiles P_{-f}. On the other hand, if we had left the two subsets $\{w_1\}$ and $\{w_3\}$ unordered we could have found two strict extensions $\check{P}_f$ and $\tilde{P}_f$ with the property that $C(\check{P}_f, P_{-f}) \neq C(\tilde{P}_f, P_{-f})$ for some subprofile P_{-f}; in this sense the binary relation $\succcurlyeq^m_{\hat{P}_f}$ on $A_{\hat{P}_f}$ will be called minimal. □

In the sequel we define a minimal binary relation that will declare as unordered (i) any two subsets of workers with the property that one is a strict subset of the other and (ii) any two subsets of workers whose relative ordering (for instance, $S \succcurlyeq_{P_f} S''$) could be obtained by transitivity (*i.e.*, S and S'' will be left unordered whenever there exists $S' \in A_{P_f}$ such that $S \succcurlyeq_{P_f} S' \succcurlyeq_{P_f} S''$).

In this section we identify the minimal binary relation (weaker than the one used as the representative of the class) with the following two properties: (i) all preference relations in the class can be obtained from this minimal binary relation by what we call a strict extension, and (ii) any strictly weaker binary relation has at least two strict extensions that belong to two different equivalence classes.

4.1. Transitive closure

To make the proof of Theorem 2 below simpler, we will now first enlarge $\succcurlyeq_{P_f}$ with its transitive closure $\succcurlyeq^T_{P_f}$ and then reduce it by identifying a minimal binary relation $\succcurlyeq^m_{P_f}$ so that $\succcurlyeq^m_{P_f} \subseteq \succcurlyeq_{P_f} \subseteq \succcurlyeq^T_{P_f}$.

Definition 5. Let $\succcurlyeq$ be an acyclic binary relation on A. The binary relation $\succcurlyeq^T$ on A is the *transitive closure* of $\succcurlyeq$ if it is the smallest transitive binary relation on A that contains $\succcurlyeq$.

Notice that the all-relation on $A \times A$ is transitive and contains all binary relations on A. The intersection of transitive binary relations on A is again transitive; that is, given $\succcurlyeq$, $\succcurlyeq^T = \bigcap\{\succcurlyeq' \subseteq A \times A \mid \succcurlyeq' \supseteq \succcurlyeq$ and $\succcurlyeq'$ is transitive$\}$ is transitive. Finally, let $\succcurlyeq$ be an acyclic binary relation on A. Then, $\succcurlyeq^T = \bigcup_{n\in\mathbb{N}} \succcurlyeq^n$, where $S \succcurlyeq^n S'$ if there exist $S_1, \ldots, S_n \in A$ such that $S = S_1 \succcurlyeq \cdots \succcurlyeq S_n = S'$.[8]

Before proceeding we state and prove a lemma that will be useful in the sequel.

Lemma 1. *Let P_f be a preference relation on 2^W and assume that $S_1, S_2 \in A_{P_f}$, $S_1 \succcurlyeq^T_{P_f} S_2$, and $S_1 \subseteq S_2$. Then, $S_1 = S_2$.*

Proof. Since $S_1 \subseteq S_2$ and $S_2 \in A_{P_f}$, $Ch(S_1 \cup S_2, P_f) = Ch(S_2, P_f) = S_2$. Hence, $S_2 \succcurlyeq_{P_f} S_1$ and $S_2 \succcurlyeq^T_{P_f} S_1$. By hypothesis, $S_1 \succcurlyeq^T_{P_f} S_2$. By Proposition 1, $\succcurlyeq_{P_f}$ is antisymmetric. By its definition, $\succcurlyeq^T_{P_f}$ is antisymmetric as well. Thus, $S_1 = S_2$. □

4.2. Minimal binary relation

To identify the minimal binary relation associated to the preference relation P_f of firm f we proceed as follows. First, obtain A_{P_f}. Second, compute $\succcurlyeq_{P_f}$ and its transitive closure $\succcurlyeq^T_{P_f}$. Then, delete from $\succcurlyeq^T_{P_f}$ all ordered pairs of subsets of workers that (i) are related by inclusion and (ii) are related as the consequence of the transitivity of $\succcurlyeq^T_{P_f}$. Formally,

Definition 6. Let P_f be a preference relation on 2^W. The binary relation $\succcurlyeq \subseteq \succcurlyeq^T_{P_f}$ on A_{P_f} is *minimal* if for all $S, S' \in A_{P_f}$ such that $S \succcurlyeq^T_{P_f} S'$ the following condition holds:

(mi) $S \succcurlyeq S'$ if and only if $S \cap S' \notin \{S, S'\}$ and there does not exist $S'' \in A_{P_f} \setminus \{S, S'\}$ such that, $S \succcurlyeq^T_{P_f} S'' \succcurlyeq^T_{P_f} S'$.

Given P_f, there is a unique minimal binary relation on A_{P_f}. Denote it by $\succcurlyeq^m_{P_f}$. Next lemma states that $\succcurlyeq^m_{P_f}$ is not only weaker than $\succcurlyeq^T_{P_f}$ but it is also weaker than the original $\succcurlyeq_{P_f}$.

Lemma 2. *Let P_f be a preference relation on 2^W. Then, $\succcurlyeq^m_{P_f} \subseteq \succcurlyeq_{P_f}$.*

[8] See Harzheim (2005).

Proof. To obtain a contradiction assume $\succcurlyeq^m_{P_f} \nsubseteq \succcurlyeq_{P_f}$; namely, there exist $S, S' \in A_{P_f}$ such that $S \succcurlyeq^m_{P_f} S'$ and $S \not\succcurlyeq_{P_f} S'$. Since, by Remark 2, $\succcurlyeq_{P_f}$ is reflexive, $S \neq S'$. Observe that $\succcurlyeq^m_{P_f} \subseteq \succcurlyeq^T_{P_f}$ and $S \succcurlyeq^m_{P_f} S'$ imply $S \succcurlyeq^T_{P_f} S'$. Since $S \not\succcurlyeq_{P_f} S'$ and $S \succcurlyeq^T_{P_f} S'$, by definition of $\succcurlyeq^T_{P_f}$ there exists $S_1, \ldots, S_n \in A_{P_f}$ such that $S \neq S_1 \neq \cdots \neq S_n \neq S'$ and $S \succcurlyeq_{P_f} S_1 \succcurlyeq_{P_f} \cdots \succcurlyeq_{P_f} S_n \succcurlyeq_{P_f} S'$. Since $\succcurlyeq^T_{P_f}$ is transitive by definition, this implies that $S \succcurlyeq^T_{P_f} S_1 \succcurlyeq^T_{P_f} S'$. By (mi) in Definition 6, $S \not\succcurlyeq^m_{P_f} S'$, a contradiction. □

Alternatively, we could directly define $\succcurlyeq^m_{P_f} \subseteq \succcurlyeq_{P_f}$ replacing condition (mi) in Definition 6 above by

(mi′) $S \succcurlyeq S'$ if and only if $S \cap S' \notin \{S, S'\}$ and there does not exist $S'' \in A_{P_f} \setminus \{S, S'\}$ such that, $S \succcurlyeq_{P_f} S'' \succcurlyeq_{P_f} S'$.

However, the arguments would become more involved since instead of using the transitivity of $\succcurlyeq^T_{P_f}$ we should use the acyclicity of $\succcurlyeq_{P_f}$ by identifying (and working with) sequences $S_1 \succcurlyeq_{P_f} \cdots \succcurlyeq_{P_f} S_k$ with the property that $S_k \not\succcurlyeq_{P_f} S_1$.

4.3. Strict extension

We next give a procedure to obtain from the minimal binary relation all preference relations that would generate it. The procedure consists of completing the acyclic minimal binary relation by declaring a set in the family to be (strictly) preferred to all its subsets and if a set is not in the family of individually rational subsets of workers then it must have a strict subset that belongs to the family and is strictly preferred to it. Formally,

Definition 7. Let $\succcurlyeq$ be an acyclic binary relation on $A \subset 2^W$ with $\emptyset \in A$. The (strict) preference relation $\overline{P}_f$ on 2^W is a *strict extension of* $\succcurlyeq$ if for all $S', S'' \in A$ such that $S' \neq S''$:

(se.1) if $S' \succ S''$ then, $S' \overline{P}_f S''$;
(se.2) if $S'' \subsetneq S'$ then, $S' \overline{P}_f S''$; and
(se.3) if $S \notin A$ then, there exists $\hat{S} \in A$ such that $\hat{S} \subsetneq S$ and $\hat{S} \overline{P}_f S$.

Definition 7 can be seen as a set of instructions on how to extend an acyclic binary relation on A to a preference relation on 2^W. First, it preserves all the ordered pairs (this corresponds to the standard notion of an extension used by Szpilrajn, 1930). Second, a set is preferred to all its subsets. Third, if a set is not in A then, we have freedom on how to order it but the set has to be worse than one of its subsets (perhaps the empty set). Finally, all the remaining pairs that are not ordered by the acyclic binary relation can be freely ordered by the preference relation (this is one of the reasons of why in general there are many strict extensions of an acyclic binary relation). Before proceeding, we state and prove two results: Lemma 3 will be useful in the proof of Theorem 2 below and Lemma 4 states that indeed P_f is obtained as a strict extension of $\succcurlyeq^m_{P_f}$.

Lemma 3. *Let P_f be a preference relation on 2^W. Suppose $\overline{P}_f$ is a strict extension of $\succcurlyeq^m_{P_f}$. Furthermore, assume $S, S' \in A_{P_f}$, $S \neq S'$ and $S \succcurlyeq^T_{P_f} S'$. Then, $S \overline{P}_f S'$.*

Proof. If $S \succcurlyeq^m_{P_f} S'$ then, by (se.1) in Definition 7, $S \overline{P}_f S'$.

Assume $S \not\succcurlyeq^m_{P_f} S'$. By Lemma 1, and since $S \succcurlyeq^T_{P_f} S'$ and $S \neq S'$, $S \nsubseteq S'$. Thus, either $S \supseteq S'$ or there exists $S_1 \in A_{P_f}$ such that $S \succcurlyeq^T_{P_f} S_1 \succcurlyeq^T_{P_f} S'$. If $S \supseteq S'$ then, by (se.2) in Definition 7, $S \overline{P}_f S'$.

Assume there exists $S_1 \in A_{P_f}$ such that $S_1 \neq S, S'$ and $S \succcurlyeq^T_{P_f} S_1 \succcurlyeq^T_{P_f} S'$. We assume without lost of generality that S_1 is maximal with respect to $\succcurlyeq^T_{P_f}$; *i.e.*, there does not exist $\hat{S} \in A$, $\hat{S} \neq S_1, S, S'$ such that $S \succcurlyeq^T_{P_f} \hat{S} \succcurlyeq^T_{P_f} S_1 \succcurlyeq^T_{P_f} S'$. Observe that by Remark 2, $\succcurlyeq_{P_f}$ is acyclic and hence, by Lemma 2, $\succcurlyeq^m_{P_f}$ is also acyclic. Thus, this maximal set S_1 does exist.

Assume $S \not\succcurlyeq^m_{P_f} S_1$. Then, if there would exist $\hat{S} \in A_{P_f}$ such that $S \succcurlyeq^T_{P_f} \hat{S} \succcurlyeq^T_{P_f} S_1$ then, $S \succcurlyeq^T_{P_f} \hat{S} \succcurlyeq^T_{P_f} S_1 \succcurlyeq^T_{P_f} S'$, contradicting the maximality of S_1. Because $S \succcurlyeq^T_{P_f} S_1$, by (mi) in Definition 6, either $S \supseteq S_1$ or $S_1 \supseteq S$. Note that if $S_1 \supseteq S$, and since $S \succcurlyeq^T_{P_f} S_1$, Lemma 1 implies $S = S'$. This contradicts $S \not\succcurlyeq^m_{P_f} S_1$ because, by Remark 2, $\succcurlyeq_{P_f}$ is reflexive. Hence, and since $\succcurlyeq^m_{P_f} \subseteq \succcurlyeq^T_{P_f}$, $\succcurlyeq^m_{P_f}$ is reflexive as well. Thus, $S \supseteq S_1$. By (se.2) in Definition 7, any strict extension $\overline{P}_f$ of $\succcurlyeq^m_{P_f}$ satisfies $S \overline{P}_f S_1$.

Assume $S \succcurlyeq^m_{P_f} S_1$. By (se.1) in Definition 7, any strict extension $\overline{P}_f$ of $\succcurlyeq^m_{P_f}$ satisfies $S \overline{P}_f S_1$.

We have already shown that $S \overline{P}_f S_1$ and $S_1 \succcurlyeq^T_{P_f} S'$. If $S_1 \succcurlyeq^m_{P_f} S'$, by (se.1) in Definition 7, $S_1 \overline{P}_f S'$, in which case, $S \overline{P}_f S'$. If $S_1 \not\succcurlyeq^m_{P_f} S'$, we repeat the argument above replacing the former role of S by S_1. Since A_{P_f} is finite, there exists a finite sequence $\{S_1, \ldots, S_k\}$ such that $S \overline{P}_f S_1 \overline{P}_f \cdots \overline{P}_f S^k \overline{P}_f S'$. By transitivity of $\overline{P}_f$, $S \overline{P}_f S'$. □

Lemma 4. *Let P_f be a preference relation on 2^W. Then, P_f is a strict extension of the minimal binary relation $\succcurlyeq^m_{P_f}$.*

Proof. We consider separately the three cases in Definition 7. For the first two, let $S_1, S_2 \in A_{P_f}$ be such that $S_1 \neq S_2$.

(se.1) Assume $S_1 \succsim^m_{P_f} S_2$. By Lemma 2, $S_1 \succsim_{P_f} S_2$; *i.e.*, $S_1 = Ch(S_1 \cup S_2, P_f)$. Since $S_1 \neq S_2$, $S_1 P_f S_2$.

(se.2) Assume $S_2 \subsetneq S_1$. To obtain a contradiction, assume $S_2 P_f S_1$. Then, $S_1 \neq Ch(S_1 \cup S_2, P_f) = Ch(S_1, P_f)$, where the equality follows because $S_2 \subsetneq S_1$. But this is a contradiction with $S_1 \in A_f$.

(se.3) Let $S \notin A_{P_f}$. We want to show that there exists $\hat{S} \in A_{P_f}$ such that $\hat{S} \subsetneq S$ and $\hat{S} P_f S$. Note that $Ch(S, P_f) \in A_{P_f}$ and $Ch(S, P_f) \subseteq S$. Since $S \notin A_{P_f}$, $Ch(S, P_f) P_f S$. Thus, set the desired $\hat{S}$ be equal to $Ch(S, P_f)$. Then, $\hat{S} P_f S$. □

4.4. Results

We are now ready to state and prove the two results of this section.

Theorem 2. *Let P_f be a preference relation on 2^W and assume $\overline{P}_f$ is a strict extension of the minimal binary relation $\succsim^m_{P_f}$. Then, $A_{\overline{P}_f} = A_{P_f}$ and $\succsim_{\overline{P}_f} = \succsim_{P_f}$.*

Before proving Theorem 2 two remarks are in order. First, the statement of Theorem 2 implicitly contains the following procedure that we want to make explicit before we proceed to its proof. Given a preference relation P_f on 2^W, construct the family A_{P_f} of individually rational subsets of workers relative to P_f. From A_{P_f}, obtain sequentially the binary relation $\succsim_{P_f}$ on A_{P_f}, its transitive closure $\succsim^T_{P_f}$, and the minimal binary relation $\succsim^m_{P_f}$. Then, take an arbitrary strict extension $\overline{P}_f$ of $\succsim^m_{P_f}$, construct the family $A_{\overline{P}_f}$ and its associated binary relation $\succsim_{\overline{P}_f}$. Theorem 2 says that $A_{P_f} = A_{\overline{P}_f}$ and $\succsim_{P_f} = \succsim_{\overline{P}_f}$ hold. Second, Theorem 2 implies (together with Theorem 1) that the minimal binary relation can be used as the representative of all preference relations that leave the set of Core matchings invariant; namely, $\succsim^m_{P_f}$ still contains all information needed to obtain the full equivalence class of preference relations as strict extensions of $\succsim^m_{P_f}$.

We now turn to prove Theorem 2.

Proof of Theorem 2. We first prove that $A_{\overline{P}_f} = A_{P_f}$. To show that $A_{\overline{P}_f} \subseteq A_{P_f}$, we will show that $S \notin A_{P_f}$ implies $S \notin A_{\overline{P}_f}$. Assume $S \notin A_{P_f}$. Hence, and since $\overline{P}_f$ is a strict extension of $\succsim^m_{P_f}$, by (se.3), there exists $\hat{S} \in A_{P_f}$ such that $\hat{S} \subsetneq S$ and $\hat{S} \overline{P}_f S$. Thus, $S \neq Ch(S, \overline{P}_f)$ and $S \notin A_{\overline{P}_f}$.

To show that $A_{P_f} \subseteq A_{\overline{P}_f}$, assume $S \in A_{P_f} \setminus A_{\overline{P}_f}$. Hence,

$$S = Ch(S, P_f) \tag{8}$$

and $S \neq Ch(S, \overline{P}_f)$. Let $S' \subsetneq S$ be such that $S' = Ch(S, \overline{P}_f)$. Obviously, $S' = Ch(S', \overline{P}_f)$ and

$$S' \overline{P}_f S. \tag{9}$$

Thus, $S' \in A_{\overline{P}_f}$. Hence, and since we have already proved that $A_{\overline{P}_f} \subseteq A_{P_f}$, $S' \in A_{P_f}$. Thus, $S' = Ch(S', P_f)$. By $S' \subsetneq S$ and (8), $S = Ch(S \cup S', P_f)$. Hence, $S \succ_{P_f} S'$. Thus, $S \succ^T_{P_f} S'$. By Lemma 3, $S \overline{P}_f S'$, a contradiction with (9).

Second, to prove $\succsim_{\overline{P}_f} = \succsim_{P_f}$ we will show that for all $S_1, S_2 \in A_{\overline{P}_f} = A_{P_f}$,

$$S_1 = Ch(S_1 \cup S_2, \overline{P}_f) \quad \text{if and only if} \quad S_1 = Ch(S_1 \cup S_2, P_f).$$

($\Rightarrow$) Assume $S_1 \neq Ch(S_1 \cup S_2, P_f)$. Hence, $Ch(S_1 \cup S_2, P_f) \succ_{P_f} S_1$. By definition of $\succsim^T_{P_f}$, $Ch(S_1 \cup S_2, P_f) \succsim^T_{P_f} S_1$. By Lemma 3, $Ch(S_1 \cup S_2, P_f) \overline{P}_f S_1$. Since $Ch(S_1 \cup S_2, P_f) \subseteq S_1 \cup S_2$, $S_1 \neq Ch(S_1 \cup S_2, \overline{P}_f)$.

($\Leftarrow$) To obtain a contradiction, assume $S_1 = Ch(S_1 \cup S_2, P_f)$ and $S_1 \neq Ch(S_1 \cup S_2, \overline{P}_f)$. Then, $Ch(S_1 \cup S_2, \overline{P}_f) \succ_{\overline{P}_f} S_1$. Since $Ch(S_1 \cup S_2, \overline{P}_f) \subseteq S_1 \cup S_2$ and $S_1 = Ch(S_1 \cup S_2, P_f)$, $S_1 \succsim_{P_f} Ch(S_1 \cup S_2, \overline{P}_f)$. By definition of $\succsim^T_{P_f}$, $S_1 \succsim^T_{P_f} Ch(S_1 \cup S_2, \overline{P}_f)$. By Lemma 3, $S_1 \overline{P}_f Ch(S_1 \cup S_2, \overline{P}_f)$, a contradiction with $Ch(S_1 \cup S_2, \overline{P}_f) \succ_{\overline{P}_f} S_1$. □

Theorem 3 below states that $\succsim^m_{P_f}$ is indeed minimal in the sense that any strictly weaker binary relation generates, as one of its strict extensions, a preference relation of firm f that belongs to a different equivalence class of the one to which P_f belongs to and thus, with a different Core for some subprofile P_{-f}.

Theorem 3. *Let P_f be a preference relation and assume $\widehat{\succsim} \subsetneq \succsim^m_{P_f}$. Then, there exists a strict extension $\hat{P}_f$ of $\widehat{\succsim}$ such that $\succsim_{\hat{P}_f} \neq \succsim_{P_f}$.*

Proof. Since $\widehat{\succsim} \subsetneq \succsim^m_{P_f}$, there exist $S_1, S_2 \in A_{P_f}$ such that $S_1 \succsim^m_{P_f} S_2$, but $S_1 \,\widehat{\not\succsim}\, S_2$. By (mi) in Definition 6, $S_1 \succsim^m_{P_f} S_2$, implies that neither $S_1 \supsetneq S_2$ nor $S_2 \supsetneq S_1$. Let $\hat{P}_f$ be a strict extension of $\widehat{\succsim}$ with the property that $S_2 \hat{P}_f S_1$. Observe that none of the hypotheses of conditions (se.1), (se.2), and (se.3) in Definition 7 hold; thus, there exists such extension $\hat{P}_f$. Then, $S_1 \neq Ch(S_1 \cup S_2, \hat{P}_f)$. Hence, $\succsim^m_{P_f} \subseteq \succsim_{P_f}$ and $S_1 \succsim^m_{P_f} S_2$ imply that, by Lemma 2, $S_1 \succsim_{P_f} S_2$; namely, $S_1 = Ch(S_1 \cup S_2, P_f)$. Thus, $\succsim_{P_f} \neq \succsim_{\hat{P}_f}$. $\square$

5. Final remarks

We finish the paper with four remarks.

First, our approach has focused only on preference relations of firms. Hence, one may ask whether a symmetric analysis could be performed from the point of view of the workers. The answer is yes, although the analysis is trivial. Given a preference relation of a worker, we could similarly construct its corresponding binary relation on the set of acceptable firms. However, this binary relation on the set of acceptable firms coincides with the initial complete preference relation (on the set of acceptable firms) since the best firm of the union of two different firms is always equal to the best of the two firms. Thus, from the point of view of the workers' preference relations all orderings (between pairs of acceptable firms) are relevant for the set of Core matchings. This is the reason why preference relations of workers have remained fixed while we identified equivalence classes of preference relations of firms.

Second, an analogous literature in multi-unit auctions has evolved during the last years wondering about the complexity for bidders of revealing their valuations of all subsets of objects (see for instance Milgrom, 2009). However, this literature applies to settings where bidders have *cardinal* preference relations on objects and/or subsets of objects. Then, bids are related to valuations on those. In contrast here, as in a large literature on two-sided matching models, we consider agents with *ordinal* preference relations.

Third, our results extend to the same partial orders in many-to-many matching markets since the proofs of Theorems 1, 2 and 3 can be translated straightforwardly to the setting where preferences of workers are defined on 2^F instead of $F \cup \{\emptyset\}$ and matchings are many-to-many instead of many-to-one. Then our analysis can also be used to identify equivalence classes of preference relations of workers (on all subsets of firms) leaving invariant the set of Core matchings.

Fourth, we preliminarily address the computational aspect of our approach.[9] Given an arbitrary preference relation P_f, to obtain its family of individually rational subset of workers A_{P_f} and its minimal binary relation $\succsim^m_{P_f}$ may be a complex task, difficult to describe by a simple and systematic procedure. However, whenever the preference relation P_f is substitutable the first goal becomes easier.[10]

Assume P_f is substitutable. We ask the following question: is there any simple and systematic procedure to compute the family of individually rational subsets of workers A_{P_f}? We answer the question affirmatively by defining an algorithm that computes A_{P_f}.

Algorithm
Input: A substitutable preference relation P_f on 2^W.
Initialization: Set $T_0 = 2^W$ and $A_0 = \emptyset$.

Step 1. Given $T_0 \neq \emptyset$ and A_0 obtain $S = Ch(W, P_f)$. Define the families of subsets of workers

$$T_1 = T_0 \setminus \{T \in T_0 \mid \text{either } S \subsetneq T \subseteq W \text{ or } T \subseteq S\}$$

and

$$A_1 = A_0 \cup \{T \in 2^W \mid T \subseteq S\}.$$

If $T_1 = \emptyset$ stop and let A_1 be the outcome of the algorithm; otherwise go to Step 2.

Step *k*. Given $T_{k-1} \neq \emptyset$ and A_{k-1}, take $S' \in T_{k-1}$ with the property that $\#S' \geqslant \#S''$ for all $S'' \in T_{k-1}$ and obtain $S = Ch(S', P_f)$. Define the families of subsets of workers

$$T_k = T_{k-1} \setminus \{T \in T_{k-1} \mid \text{either } S \subsetneq T \subseteq S' \text{ or } T \subseteq S\}$$

and

$$A_k = A_{k-1} \cup \{T \in 2^W \mid T \subseteq S\}.$$

If $T_k = \emptyset$ stop and let A_k be the outcome of the algorithm; otherwise go to Step $k+1$.

[9] What follows has to be seen as a first step towards a more general and systematic analysis that is left for future research.

[10] Although restrictive, we think that our analysis still has interest because substitutability is a plausible restriction in settings where workers exhibit low complementarities in shaping firms' preference relations on 2^W; this is the reason why a very large proportion of the literature on ordinal many-to-one matching models assumes that firms have substitutable preference relations or even stronger conditions like responsiveness.

Observe that the algorithm can be understood as a set of simple and precise instructions given to the firm to compute A_{P_f}. Although it takes as input the preference relation P_f on 2^W, it tries to minimize the number of times that the firm has to be asked about how it orders pairs of subsets of workers and it only requires that the firm has three abilities. Given any subset of workers, the firm is able to calculate its cardinality, to compute its subsets, and to identify its most-preferred subset.

The algorithm has several executions depending on the particular subsets $S' \in T_{k-1}$ with largest cardinality chosen at each step $k > 1$, if any. The table below illustrates a particular execution of the algorithm for the following substitutable preference relation (in the table we omit brackets and commas when writing subsets of workers; for instance, $w_2 w_3$ should be read as $\{w_2, w_3\}$):

$$P_f: \quad \{w_1, w_2\}, \{w_1\}, \{w_2\}, \{w_1, w_3\}, \{w_3\}, \{\emptyset\}.$$

Step	S'	S	T_k	A_k
1	$w_1 w_2 w_3$	$w_1 w_2$	$\{w_1 w_3, w_2 w_3, w_3\}$	$\{w_1 w_2, w_1, w_2, \emptyset\}$
2	$w_1 w_3$	w_1	$\{w_2 w_3, w_3\}$	$\{w_1 w_2, w_1, w_2, \emptyset\}$
3	$w_2 w_3$	w_2	$\{w_3\}$	$\{w_1 w_2, w_1, w_2, \emptyset\}$
4	w_3	w_3	$\emptyset$	$\{w_1 w_2, w_1, w_2, w_3, \emptyset\}$

Observe that the outcome of this execution is $A_4 = A_{P_f}$. The next proposition states that, as long as P_f is substitutable, this is always the case.

Proposition 1. *Let P_f be a substitutable preference relation. Then, any execution of the algorithm terminates in a finite number of steps and its outcome is A_{P_f}.*

Proof. Let P_f be a substitutable preference relation. It is immediate to check that, at any step $k > 0$ of the algorithm, $T_k \subsetneq T_{k-1}$. Thus, any execution of the algorithm terminates after a finite number of steps, denoted by K. Moreover, $A_{k-1} \subseteq A_k$. We first prove that $A_K \subseteq A_{P_f}$. Assume $T \in A_K$. Then, there exist $0 < k \leqslant K$ and $S' \in T_{k-1}$ such that $T \subseteq S = Ch(S', P_f)$. Hence, by substitutability of P_f, $S = Ch(S, P_f)$ and, by substitutability again, $T = Ch(T, P_f)$. Thus, $T \in A_{P_f}$. To prove that $A_{P_f} \subseteq A_K$, assume $T \notin A_K$. Note that $T \in T_0$. Let $0 < k \leqslant K$ be the step where $T \in T_{k-1}$ but $T \notin T_k$. Hence, there exists $S' \in T_{k-1}$ such that $S = Ch(S', P_f)$ and either $S \subsetneq T \subseteq S'$ or $T \subseteq S$. If the later holds then, $T \in A_k$ which would contradict the hypothesis that $T \notin A_K$ since $A_k \subseteq A_K$. Hence, $S \subsetneq T \subseteq S'$ holds. Since $S = Ch(S', P_f)$, by substitutability, $S = Ch(T, P_f) \neq T$. Thus, $T \notin A_{P_f}$. $\square$

If P_f is not substitutable the outcome of the algorithm A_K is not necessarily equal to A_{P_f}. To see that consider the non-substitutable preference relation on $2^{\{w_1, w_2\}}$

$$P_f: \quad \{w_1, w_2\}, \{\emptyset\}$$

and take $S' = \{w_1, w_2\}$ in Step 1. Then, $S = \{w_1, w_2\}$, $T_1 = \{\emptyset\}$, $K = 1$, and $A_1 = \{\{w_1, w_2\}, \{w_1\}, \{w_2\}, \{\emptyset\}\}$. But, $A_{P_f} = \{\{w_1, w_2\}, \{\emptyset\}\}$. However, if the algorithm is modified by setting, at each step $k \geqslant 1$,

$$T_k = T_{k-1} \setminus \{T \in T_{k-1} \mid S \subseteq T \subseteq S'\}$$

and

$$A_k = A_{k-1} \cup S$$

then, the output of the algorithm A_K coincides with A_{P_f} for any preference relation P_f.[11] Notice that in this case, the execution of the algorithm may require to ask to the firm all binary comparisons and hence, the algorithm may not represent a computational improvement with respect to directly ask P_f.

Finally, we leave for further research the computational aspect of obtaining the minimal binary relation $\succ^m_{P_f}$ on A_{P_f}. Preliminary results suggest that, even for the case of substitutable preference relations, this may not be an easy task.

References

Blair, C., 1988. The lattice structure of the set of stable matchings with multiple partners. Math. Operations Res. 13, 619–628.
Chambers, C.P., Echenique, F., 2008. The core matchings of markets with transfers. CalTech Social Science Working Paper 1298.
Echenique, F., 2008. What matchings can be stable? The testable implications of matching theory. Math. Operations Res. 33, 757–768.
Harzheim, E., 2005. Ordered Sets. Advances Math.. Springer, New York.

[11] The proof of Proposition 1 can be easily adapted to deal with this alternative algorithm by using some properties of the choice set instead of substitutability.

Kelso, A., Crawford, V., 1982. Job matchings, coalition formation, and gross substitutes. Econometrica 50, 1483–1504.

Milgrom, P., 2009. Assignment messages and exchanges. Amer. Econ. Rev. 1, 95–113.

Martínez, R., Massó, J., Neme, A., Oviedo, J., 2008. On the invariance of the set of stable matchings with respect to substitutable preference profiles. Int. J. Game Theory 36, 497–518.

Niederle, M., Roth, A., Sönmez, T., 2008. Matching and market design. In: Durlauf, Steven N., Blume, Lawrence E. (Eds.), The New Palgrave Dictionary of Economics. 2nd ed. Palgrave Macmillan.

Roth, A., Sotomayor, M., 1990. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Cambridge University Press, Cambridge.

Szpilrajn, E., 1930. Sur l'extension de l'ordre partiel. Fund. Math. 16, 386–389.