5. Functions of Random Variables

5.1. The distribution of a function of a random object

- Let $\tilde{x} : (\Omega, \mathcal{F}, P) \longrightarrow (\Omega', \mathcal{F}')$ be a random object with distribution $P_{\tilde{x}}$ and $g : (\Omega', \mathcal{F}') \longrightarrow (\Omega'', \mathcal{F}'')$ be a measurable function. Then, the distribution of the random object $\tilde{y} = g(\tilde{x}) : (\Omega, \mathcal{F}) \longrightarrow (\Omega'', \mathcal{F}'')$ is given by

$$P_{\tilde{y}}(B) = P_{g(\tilde{x})}(B) = P\{g(\tilde{x}) \in B\}$$

$$= P\left\{\tilde{x} \in g^{-1}(B)\right\} = P_{\tilde{x}}\left(g^{-1}(B)\right), \quad \text{for all } B \in \mathcal{F}''.$$

5.2. The distribution function of a vector-valued function of a random object

- Let $\tilde{x} : (\Omega, \mathcal{F}, P) \longrightarrow (\Omega', \mathcal{F}')$ be a random object with distribution $P_{\tilde{x}}$ and $g : (\Omega', \mathcal{F}') \longrightarrow (\mathbb{R}^n, \mathcal{B})$ be a vector-valued Borel measurable function. Then, the distribution function $F_{\tilde{y}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the random vector $\tilde{y} = g(\tilde{x}) : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^n, \mathcal{B})$ is given by

$$F_{\tilde{y}}(y) = P\{\tilde{y} \leqq y\} = P\{g(\tilde{x}) \leqq y\} = P\{g(\tilde{x}) \in (-\infty, y]\}$$
$$= P\left\{\tilde{x} \in g^{-1}(-\infty, y]\right\} = P_{\tilde{x}}\left(g^{-1}(-\infty, y]\right)$$
$$= P_{\tilde{x}}\left\{x \in \Omega' \,|g(x) \leqq y\right\}.$$

- Recall that $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ and

$$(-\infty, y] = \{z \in \mathbb{R}^n \,|z \leqq y\}$$
$$= \{z \in \mathbb{R}^n \,|z_1 \leq y_1, z_2 \leq y_2, ..., z_n \leq y_n\}.$$

- If $F_{\tilde{y}}(y)$ is absolutely continuous, then the density function will be

$$f_{\tilde{y}}(y_1, y_2, ..., y_n) = \frac{\partial^n F_{\tilde{y}}(y_1, y_2, ..., y_n)}{\partial y_1 \partial y_2 ... \partial y_n}.$$

when this nth crossed partial derivative of $F_{\tilde{y}}$ exists. Moreover, this derivative exists a.e. w.r.t. Lebesgue measure on $(\mathbb{R}^n, \mathcal{B})$.

- **Example:** Let $\tilde{z}$ be a random variable and $\tilde{x} = |\tilde{z}| \geq 0$. Then

$$F_{\tilde{x}}(x) = P\{\tilde{x} \leq x\} = P\{|\tilde{z}| \leq x\} = P\{-x \leq \tilde{z} \leq x\}$$

$$= P_{\tilde{z}}[-x, x] = F_{\tilde{z}}(x) - \lim_{s \to -x^-} F_{\tilde{z}}(s), \text{ for } x \geq 0.$$

and $F_{\tilde{x}}(x) = 0$ for $x < 0$.

- Assume now that $\tilde{z} \sim \mathrm{N}(0,1)$. Then, $F_{\tilde{z}}$ is absolutely continuous (and, hence, continuous), so that the distribution function of $\tilde{x} = |\tilde{z}|$ is

$$F_{\tilde{x}}(x) = \begin{cases} F_{\tilde{z}}(x) - F_{\tilde{z}}(-x) & \text{for } x \geq 0 \\ \\ 0 & \text{for } x < 0, \end{cases}$$

and its density is

$$f_{\tilde{x}}(x) = \begin{cases} f_{\tilde{z}}(x) + f_{\tilde{z}}(-x) = n(x;0,1) + n(-x;0,1) & \text{for } x > 0 \\ \\ 0 & \text{for } x \leq 0. \end{cases}$$

- Given the symmetry of the density function $n(\cdot; 0, 1)$ with respect to 0, we get

$$f_{\tilde{x}}(x) = \begin{cases} 2n(x; 0, 1) = \sqrt{\dfrac{2}{\pi}} e^{-\frac{1}{2}x^2} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

5.3. The probability function of a discrete vector-valued function of a random object

- Let $\tilde{x} : (\Omega, \mathcal{F}, P) \longrightarrow (\Omega', \mathcal{F}')$ be a random object with distribution $P_{\tilde{x}}$, $g : (\Omega', \mathcal{F}') \longrightarrow (\mathbb{R}^n, \mathcal{B})$ be a vector-valued Borel measurable function, and the random vector $\tilde{y} = g(\tilde{x}) : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^n, \mathcal{B})$ be discrete (i.e., $\tilde{y}(\Omega)$ is discrete or countable). Then, the probability function $f_{\tilde{y}} : \tilde{y}(\Omega) \longrightarrow [0, 1]$ of the random vector $\tilde{y}$ is given by

$$f_{\tilde{y}}(y) = P_{\tilde{y}}\{y\} = P_{\tilde{x}}\left(g^{-1}(y)\right), \quad \text{for all } y \in g(\tilde{x}(\Omega)) \equiv \tilde{y}(\Omega).$$

- If either $\tilde{x}$ is discrete (i.e., $\tilde{x}(\Omega)$ is discrete or countable) or g is discrete (i.e., $g(\Omega')$ is discrete or countable), then the random vector $\tilde{y} = g(\tilde{x})$ is discrete (i.e., $\tilde{y}(\Omega) \equiv g(\tilde{x}(\Omega))$ is discrete or countable).

- If $\tilde{x} : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}^m, \mathcal{B})$ is a discrete random vector with probability function $f_{\tilde{x}} : \tilde{x}(\Omega) \longrightarrow [0, 1]$ and $g : (\mathbb{R}^m, \mathcal{B}) \longrightarrow (\mathbb{R}^n, \mathcal{B})$, then the probability function $f_{\tilde{y}} : \tilde{y}(\Omega) \longrightarrow [0, 1]$ of the discrete random vector $\tilde{y} = g(\tilde{x}) : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^n, \mathcal{B})$ is given by

$$f_{\tilde{y}}(y) = P_{\tilde{y}}\{y\} = P_{\tilde{x}}\left(g^{-1}(y)\right)$$

$$= \sum_{x \in g^{-1}(y)} f_{\tilde{x}}(x), \quad \text{for all } y \in g(\tilde{x}(\Omega)) \equiv \tilde{y}(\Omega).$$

- If the restriction of the function g to the range of $\tilde{x}$, $g : \tilde{x}(\Omega) \longrightarrow g(\tilde{x}(\Omega))$, is a one-to-one correspondence, then

$$f_{\tilde{y}}(y) = P_{\tilde{y}}\{y\} = P_{\tilde{x}}\left(g^{-1}(y)\right)$$

$$= f_{\tilde{x}}\left(g^{-1}(y)\right), \quad \text{for all } y \in g(\tilde{x}(\Omega)) \equiv \tilde{y}(\Omega).$$

- **Example:** Let $\tilde{x}$ be a binomial random variable with probability function

$$b_{\tilde{x}}(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \quad x = 0, 1, ..., n$$

and let $\tilde{y} = g(\tilde{x}) = \dfrac{\tilde{x}}{n}$ be the percentage of successes in n independent trials. Note that $x = g^{-1}(y) = ny$. Then, $g(\tilde{x}(\Omega)) = \{0, 1/n, 2/n, ..., 1\}$ and

$$f_{\tilde{y}}(y; n, \theta) = b_{\tilde{x}}(g^{-1}(y); n, \theta)$$

$$= \binom{n}{ny} \theta^{ny} (1-\theta)^{\overbrace{n-ny}^{n(1-y)}}, \quad y = 0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1.$$

5.4. Convolutions

- **Definition.** The convolution of two independent random variables on the same probability space, $\tilde{x} : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$ and $\tilde{y} : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$, is the distribution of their sum, $\tilde{x} + \tilde{y}$, i.e., the distribution of the random variable $\tilde{z} = \tilde{x} + \tilde{y}$, with $\tilde{z} : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$.

- **Convolution distribution:** Consider two random variables $\tilde{x}$ and $\tilde{y}$ on the same probability space $(\Omega, \mathcal{F}, P)$, then the distribution of the random variable $\tilde{z} = \tilde{x} + \tilde{y}$ is

$$P_{\tilde{z}}(B) = P_{\tilde{x}+\tilde{y}}(B) = P\{\tilde{x} + \tilde{y} \in B\}$$

$$= P\{\omega \in \Omega \mid \tilde{x}(\omega) + \tilde{y}(\omega) \in B\}, \quad \text{for all } B \in \mathcal{B}.$$

- Independency is not required for the previous expression.

- **Convolution distribution function:** If the two random variables $\tilde{x}$ and $\tilde{y}$ on the same probability space $(\Omega, \mathcal{F}, P)$ are independent, then the distribution function of the random variable $\tilde{z} = \tilde{x} + \tilde{y}$ is

$$F_{\tilde{z}}(z) = F_{\tilde{x}+\tilde{y}}(z) = P\left\{\tilde{x} + \tilde{y} \leq z\right\}$$

$$= \int_{\mathbb{R}} P\left\{\tilde{x} + \tilde{y} \leq z \,|\tilde{x} = x\right\} dF_{\tilde{x}}(x) = \int_{\mathbb{R}} P\left\{x + \tilde{y} \leq z\right\} dF_{\tilde{x}}(x)$$

$$= \int_{\mathbb{R}} P\left\{\tilde{y} \leq z - x\right\} dF_{\tilde{x}}(x) = \int_{\mathbb{R}} F_{\tilde{y}}(z - x) dF_{\tilde{x}}(x),$$

where the third equality comes from the theorem of total probability and the fourth from the independence between $\tilde{x}$ and $\tilde{y}$. Note that, if these two variables where not independent, then we will have that $P\left\{\tilde{x} + \tilde{y} \leq z \,|\tilde{x} = x\right\} = P\left\{x + \tilde{y} \leq z \,|\tilde{x} = x\right\}$, which is not necessarily equal to $P\left\{x + \tilde{y} \leq z\right\}$.

- **Proposition.** Assume that the two random variables $\tilde{x}$ and $\tilde{y}$ on the same probability space $(\Omega, \mathcal{F}, P)$ are independent.

  (a) If $\tilde{x}$ and $\tilde{y}$ are discrete with the probability functions $f_{\tilde{x}} : \tilde{x}(\Omega) \longrightarrow [0,1]$ and $f_{\tilde{y}} : \tilde{y}(\Omega) \longrightarrow [0,1]$, respectively, then the probability function of their sum $\tilde{z} = \tilde{x} + \tilde{y}$ (i.e., the convolution pmf) is

$$f_{\tilde{x}+\tilde{y}}(z) = f_{\tilde{z}}(z) = \sum_{x \in \tilde{x}(\Omega)} f_{\tilde{y}}(z-x) f_{\tilde{x}}(x), \quad \text{for } z \in \tilde{z}(\Omega),$$

  with $z - x \in \tilde{y}(\Omega)$.

  (b) If $\tilde{x}$ and $\tilde{y}$ are absolutely continuous with the densities $f_{\tilde{x}} : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ and $f_{\tilde{y}} : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, respectively, then the density of their sum $\tilde{z} = \tilde{x} + \tilde{y}$ (i.e., the convolution pdf) is

$$f_{\tilde{x}+\tilde{y}}(z) = f_{\tilde{z}}(z) = \int_{\mathbb{R}} f_{\tilde{y}}(z-x) f_{\tilde{x}}(x) dx.$$

- **Proof. (a)** Define the discrete random variable $\tilde{z} = \tilde{x} + \tilde{y}$. Then, the probability function of $\tilde{z}$, $f_{\tilde{z}} : \tilde{z}(\Omega) \longrightarrow [0, 1]$, satisfies

$$f_{\tilde{z}}(z) = f_{\tilde{x}+\tilde{y}}(z) = P\left\{\tilde{x} + \tilde{y} = z\right\}$$

$$= \sum_{x \in \tilde{x}(\Omega)} P\left\{\tilde{x} + \tilde{y} = z \,|\tilde{x} = x\right\} \cdot P\left\{\tilde{x} = x\right\}$$

$$= \sum_{x \in \tilde{x}(\Omega)} P\left\{x + \tilde{y} = z\right\} \cdot P\left\{\tilde{x} = x\right\} = \sum_{x \in \tilde{x}(\Omega)} P\left\{\tilde{y} = z - x\right\} \cdot P\left\{\tilde{x} = x\right\}$$

$$= \sum_{x \in \tilde{x}(\Omega)} f_{\tilde{y}}(z - x) f_{\tilde{x}}(x), \quad \text{for } z \in \tilde{z}(\Omega), \text{ with } z - x \in \tilde{y}(\Omega),$$

where the third equality comes from the theorem of total probability and the fourth from the independence between $\tilde{x}$ and $\tilde{y}$. Note that, if these two variables where not independent, then we will have that $P\left\{\tilde{x} + \tilde{y} = z \,|\tilde{x} = x\right\} = P\left\{x + \tilde{y} = z \,|\tilde{x} = x\right\}$, which is not necessarily equal to $P\left\{x + \tilde{y} = z\right\}$.

(b) We know that the distribution function $F_{\tilde{z}}$ of the absolutely continuous random variable $\tilde{z} = \tilde{x} + \tilde{y}$ is

$$F_{\tilde{z}}(z) = \int_{\mathbb{R}} F_{\tilde{y}}(z - x) dF_{\tilde{x}}(x) = \int_{\mathbb{R}} F_{\tilde{y}}(z - x) f_{\tilde{x}}(x) dx.$$

Differentiating the last expression with respect to z, and exchanging the order of differentiation and integration, we obtain the density of the random variable $\tilde{z} = \tilde{x} + \tilde{y}$,

$$f_{\tilde{x}+\tilde{y}}(z) = f_{\tilde{z}}(z) = \frac{dF_{\tilde{z}}(z)}{dz} = \int_{\mathbb{R}} \frac{dF_{\tilde{y}}(z - x)}{dz} f_{\tilde{x}}(x) dx$$

$$= \int_{\mathbb{R}} f_{\tilde{y}}(z - x) f_{\tilde{x}}(x) dx, \quad \text{for } z \in \mathbb{R}. \qquad Q.E.D.$$

- See the handout for a non-trivial application of the previous proposition to the convolution of two independent absolutely continuous random variables.

5.5. The density of a vector-valued function of an absolutely continuous random vector

- Let $\tilde{x} : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ be a random vector with density $f_{\tilde{x}} : \mathbb{R}^m \longrightarrow \overline{\mathbb{R}}$ and $g : (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)) \longrightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be a vector-valued Borel measurable function. Then the distribution $P_{\tilde{y}}$ of the random vector $\tilde{y} = g(\tilde{x}) : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is given by

$$P_{\tilde{y}}(B) = P_{\tilde{x}}\left(g^{-1}(B)\right) = \int_{g^{-1}(B)} dP_{\tilde{x}} = \int_{g^{-1}(B)} f_{\tilde{x}}(x)dx,$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^m$, for all $B \in \mathcal{B}(\mathbb{R}^n)$.

- Let $m = n$ and assume now that (i) the function $g : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \longrightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ restricted to the (open) set A of values of its domain satisfying $f_{\tilde{x}}(x) \neq 0$, $g : A \longrightarrow g(A)$, is a one-to-one correspondence (or bijective function), and (ii) the inverse function $g^{-1} : g(A) \longrightarrow A$ is continuously differentiable (which implies that $g(A)$ is open as A is open). Then, the density function $f_{\tilde{y}}$ of the absolutely continuous random vector $\tilde{y} = g(\tilde{x}) : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is given by

$$f_{\tilde{y}}(y) = \begin{cases} f_{\tilde{x}}(g^{-1}(y)) \left| J_{g^{-1}}(y) \right| & \text{for } y \in g(A) \\ \\ 0 & \text{otherwise.} \end{cases}$$

- **Proof:** For every Borel set $B \subset g(A) \equiv \left\{ y \in \mathbb{R}^n \,\middle|\, f_{\tilde{x}}\left(g^{-1}(y)\right) \neq 0 \right\}$,

$$P_{\tilde{y}}(B) = P_{\tilde{x}}\left(g^{-1}(B)\right) = \int_{g^{-1}(B)} f_{\tilde{x}}(x)dx = \int_B \underbrace{f_{\tilde{x}}(g^{-1}(y)) \left| J_{g^{-1}}(y) \right|}_{f_{\tilde{y}}(y)} dy,$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. $Q.E.D.$

- **Example:** Let $\tilde{z} \sim \mathrm{N}(0,1)$ and $\tilde{y} = h\left(\tilde{z}\right) = \tilde{z}^2$. Find the density of $\tilde{y}$.

- Note that $h : \mathbb{R} \longrightarrow \mathbb{R}$ is not a one-to-one correspondence. However, we can make $\tilde{y} = g\left(\tilde{x}\right) = \tilde{x}^2$, where $\tilde{x} = \left|\tilde{z}\right|$ and, then, $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a one-to-one correspondence, whose inverse is $\tilde{x} = g^{-1}\left(\tilde{y}\right) = \tilde{y}^{1/2}$ and

$$\frac{dx}{dy} = \frac{dg^{-1}\left(y\right)}{dy} = \frac{1}{2} y^{-1/2} > 0.$$

- Recall that the density of $\tilde{x}$ is

$$f_{\tilde{x}}(x) = \begin{cases} 2n(x; 0, 1) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

- Then,

$$f_{\tilde{y}}(y) = \begin{cases} 2n(y^{1/2};0,1)\cdot\left(\frac{1}{2}y^{-1/2}\right) = n(y^{1/2};0,1)\cdot y^{-1/2} & \text{for } y > 0 \\ \\ 0 & \text{for } y \leq 0. \end{cases}$$

- Note that

$$n(y^{1/2};0,1)\cdot y^{-1/2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(y^{1/2}\right)^2}\cdot y^{-1/2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y}\cdot y^{-1/2}.$$

- Therefore,

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y}y^{-1/2} & \text{for } y > 0 \\ \\ 0 & \text{for } y \leq 0. \end{cases}$$

- Recall also that a random variable $\tilde{y}:(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ has the χ^2 (chi-square) distribution with ν degrees of freedom $\left(\tilde{y} \sim \chi^2_\nu\right)$ if its density is

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} y^{\frac{\nu-2}{2}} e^{-y/2} & \text{for } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Then, $\tilde{y} \sim \chi^2_1$ if

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{1}{2^{1/2}\Gamma\left(\frac{1}{2}\right)} y^{-1/2} e^{-y/2} = \dfrac{1}{\sqrt{2\pi}} e^{-y/2} y^{-1/2} & \text{for } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Therefore, the random variable $\tilde{y} = \tilde{z}^2$, with $\tilde{z} \sim \mathrm{N}(0,1)$, is chi-square with 1 degree of freedom $\left(\tilde{y} \sim \chi^2_1\right)$.

- Even if $m > n$ we can apply this method by introducing additional variables to enlarge the dimension of the image space and, thus, to make the function g a one-to-one correspondence for all the values $x \in \mathbb{R}^m$ of the domain of g for which $f_{\tilde{x}}(x) \neq 0$. Then, we will need to compute the marginal density to get rid of the additional variables we have introduced. See the handout.

- Note that, in order to find the density of $\tilde{y} = g(\tilde{x})$ we could find first the distribution function of $\tilde{y}$ from the density of $\tilde{x}$ and, then, by performing the corresponding derivative, we obtain the density of $\tilde{y}$ (**distribution function method**). This method does not require that g be a bijection for all the values $x \in \mathbb{R}^n$ of the domain of g for which $f_{\tilde{x}}(x) \neq 0$ (see Section 5.2).

- In this Section 5.5 we find the density of $\tilde{y} = g(\tilde{x})$ directly from the density function of $\tilde{x}$ (**density function method**). This method requires that g be a bijection for all the values $x \in \mathbb{R}^n$ of the domain of g for which $f_{\tilde{x}}(x) \neq 0$.

5.6. Characteristic function, moment-generating function, and Laplace transform of a function of a random variable

- If $\tilde{y} = g(\tilde{x})$, then we can compute the characteristic function $\varphi_{\tilde{y}}$ of $\tilde{y}$ (or its moment-generating function $M_{\tilde{y}}$ or its Laplace transform $\Lambda_{\tilde{y}}$ if they are well-defined in a neighborhood of $t = 0$), using the distribution $P_{\tilde{x}}$ of $\tilde{x}$,

$$\varphi_{\tilde{y}}(t) = \mathrm{E}(e^{it\tilde{y}}) = \mathrm{E}(e^{itg(\tilde{x})}) = \int_{\mathbb{R}} e^{itg(x)} dP_{\tilde{x}}(x),$$

$$= (\mathrm{E}[\cos(tg(\tilde{x}))], \mathrm{E}[\sin(tg(\tilde{x}))])$$

$$= \left(\int_{\mathbb{R}} \cos(tg(x)) dP_{\tilde{x}}(x), \int_{\mathbb{R}} \sin(tg(x)) dP_{\tilde{x}}(x) \right),$$

$$M_{\tilde{y}}(t) = \mathrm{E}(e^{t\tilde{y}}) = \mathrm{E}(e^{tg(\tilde{x})}) = \int_{\mathbb{R}} e^{tg(x)} dP_{\tilde{x}}(x),$$

$$\Lambda_{\tilde{y}}(t) = \mathrm{E}(e^{-t\tilde{y}}) = \mathrm{E}(e^{-tg(\tilde{x})}) = \int_{\mathbb{R}} e^{-tg(x)} dP_{\tilde{x}}(x),$$

and try to identify the distribution $P_{\tilde{y}}$ associated with $\varphi_{\tilde{y}}$ (or with $M_{\tilde{y}}$ or with $\Lambda_{\tilde{y}}$).

- This is called **the characteristic function (or moment-generating function or Laplace transform) method.**

5.7. Mixture distributions

- Consider a discrete set of random variables $\{\tilde{x}_j\}_{j=1}^{N}$, with $\tilde{x}_j : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$ for $j = 1, ..., N$, where N can be either a finite number or infinity.
- **Definition.** The random variable $\tilde{x} : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$ is a (discrete) mixture of the random variables $\{\tilde{x}_j\}_{j=1}^{N}$, if $\tilde{x} = \tilde{x}_j$ with probability p_j.
- Obviously, $p_j \in [0, 1]$ and $\sum\limits_j p_j = 1$.
- The distribution of $\tilde{x}$ is called the **mixture distribution** and the random variables $\{\tilde{x}_j\}_{j=1}^{N}$ in the mixture are called the **components** of the mixture.
- **Notation for a mixture:**

$$\tilde{x} = (p_1 \odot \tilde{x}_1) \oplus (p_2 \odot \tilde{x}_2) \oplus (p_3 \odot \tilde{x}_3) \oplus ...$$

- **Proposition.** Assume that the component $\tilde{x}_j$ of the mixture $\tilde{x}$ has the distribution $P_j : \mathcal{B} \longrightarrow [0, 1]$ for $j = 1, ..., N$. Then, the mixture distribution $P_{\tilde{x}}$ is

$$P_{\tilde{x}}(B) = \sum_j p_j P_j(B), \quad \text{for all } B \in \mathcal{B}.$$

Proof.

$$P_{\tilde{x}}(B) = \sum_j p_j P\left\{\tilde{x} \in B \,|\tilde{x} = \tilde{x}_j\right\}$$

$$= \sum_j p_j P\left\{\tilde{x}_j \in B\right\} = \sum_j p_j P_j(B), \quad \text{for all } B \in \mathcal{B},$$

where the first equality comes from the theorem of total probability. $Q.E.D.$

- **Corollary.** Assume that the component $\tilde{x}_j$ of the mixture $\tilde{x}$ has the distribution function $F_j : \mathbb{R} \longrightarrow [0, 1]$ for $j = 1, ..., N$. Then, the distribution function $F_{\tilde{x}}$ of the mixture distribution is

$$F_{\tilde{x}}(x) = \sum_j p_j F_j(x), \quad \text{for } x \in \mathbb{R}.$$

- **Proof.** Using the previous proposition we get

$$F_{\tilde{x}}(x) = P\{\tilde{x} \leq x\} = P_{\tilde{x}}(-\infty, x] = \sum_j p_j P_j(-\infty, x]$$

$$= \sum_j p_j P\{\tilde{x}_j \leq x\} = \sum_j p_j F_j(x), \quad \text{for } x \in \mathbb{R}. \qquad Q.E.D.$$

- **Corollary.** Assume that the component $\tilde{x}_j$ of the mixture $\tilde{x}$ is discrete with the probability function $f_j : \tilde{x}_j(\Omega) \longrightarrow [0,1]$ for $j = 1, ..., N$. Then, the mixture distribution is discrete with the probability function

$$f_{\tilde{x}}(x) = \sum_j p_j f_j(x), \quad \text{for } x \in \tilde{x}(\Omega) \equiv \bigcup_j \tilde{x}_j(\Omega),$$

where we make $f_j(x) = 0$ if $x \notin \tilde{x}_j(\Omega)$.

- **Proof.** Using again the proposition we get

$$f_{\tilde{x}}(x) = P\{\tilde{x} = x\} = \sum_j p_j P\{\tilde{x}_j = x\} = \sum_j p_j f_j(x), \quad \text{for } x \in \tilde{x}(\Omega).$$

Q.E.D.

- **Proposition.** Assume that the component $\tilde{x}_j$ of the mixture $\tilde{x}$ is absolutely continuous with the density $f_j : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ for $j = 1, ..., N$. Then, the mixture distribution is absolutely continuous with the density

$$f_{\tilde{x}}(x) = \sum_j p_j f_j(x), \quad \text{for } x \in \mathbb{R}.$$

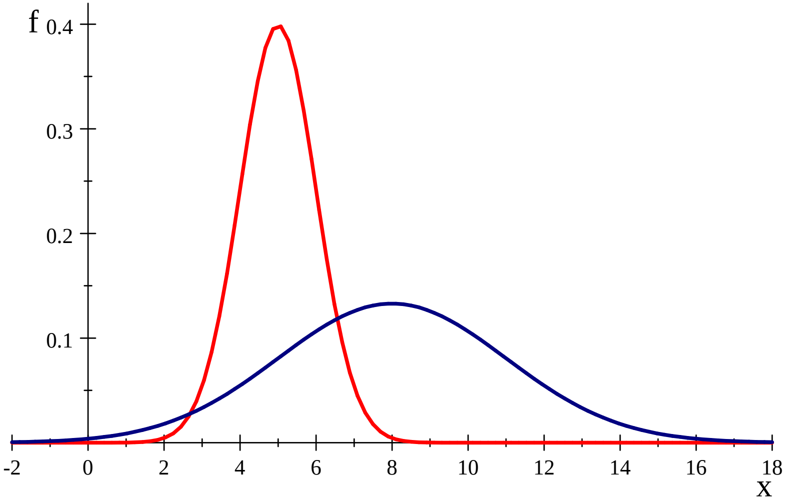
- **Proof.** Since the distribution function of the mixture satisfies $F_{\tilde{x}}(x) = \sum_j p_j F_j(x)$, we can take the derivative with respect to x to obtain the density of the mixture $\tilde{x}$,

$$f_{\tilde{x}}(x) = \frac{dF_{\tilde{x}}(x)}{dx} = \frac{d\left[\sum_j p_j F_j(x)\right]}{dx}$$

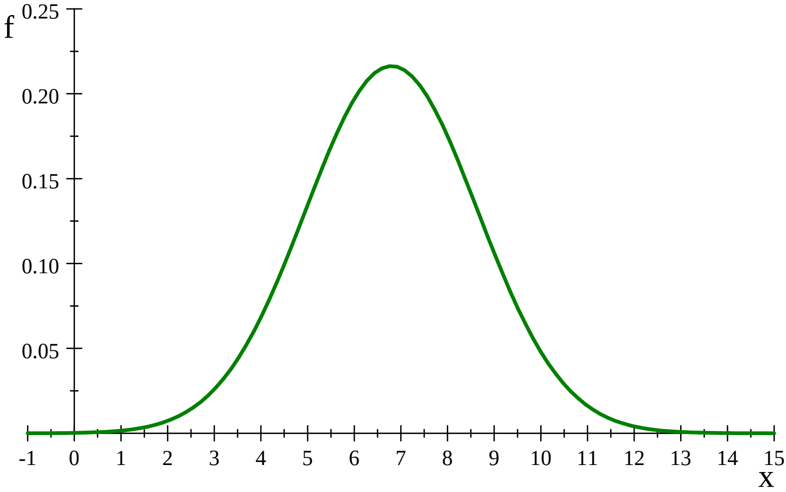
$$= \sum_j p_j \frac{dF_j(x)}{dx} = \sum_j p_j f_j(x), \quad \text{for } x \in \mathbb{R}. \qquad Q.E.D.$$

- Note that the distribution of a mixture of random variables is not the same as the distribution of their weighted sum.
- **Example:** Consider the two independent random variables $\tilde{x}_1 \sim \mathrm{N}\left(\mu_1, \sigma_1^2\right)$ and $\tilde{x}_2 \sim \mathrm{N}\left(\mu_2, \sigma_2^2\right)$. Then, $p_1\tilde{x}_1 + p_2\tilde{x}_2 \sim \mathrm{N}\left(p_1\mu_1 + p_2\mu_2, p_1^2\sigma_1^2 + p_2^2\sigma_2^2\right)$, while $\tilde{x} = (p_1 \odot \tilde{x}_1) \oplus (p_2 \odot \tilde{x}_2)$ is not normal.
- $\mu_1 = 5$, $\sigma_1 = 1$, $\mu_2 = 8$, $\sigma_2 = 3$, $p_1 = 0.4$, $p_2 = 0.6$.

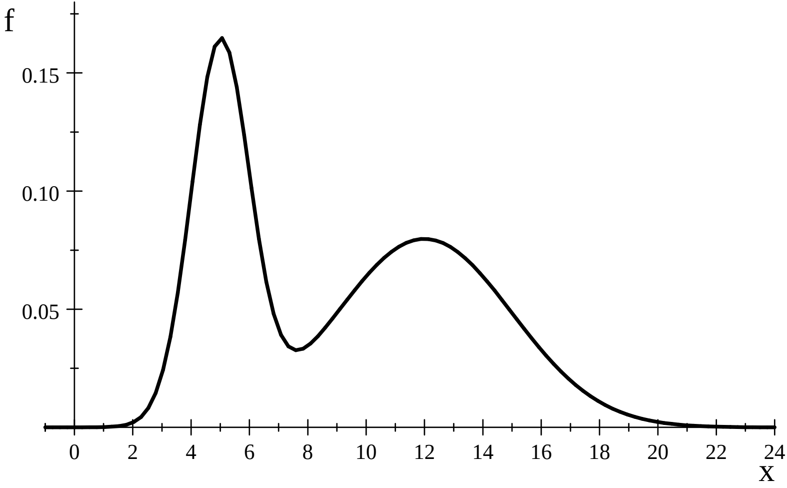
- Densities of $\tilde{x}_1$ and $\tilde{x}_2$, $n(x;5,1)$ and $n(x;8,3)$, respectively:



- Density of the weighted sum $p_1\tilde{x}_1 + p_2\tilde{x}_2$:
$n\left(x; p_1\mu_1 + p_2\mu_2, \sqrt{p_1^2\sigma_1^2 + p_2^2\sigma_2^2}\right) = n(x; 6.8, 1.8439)$:



- Density of the mixture $\tilde{x} = (p_1 \odot \tilde{x}_1) \oplus (p_2 \odot \tilde{x}_2)$: $0.6n(x; 5, 1) + 0.4n(x; 8, 3)$:



- **Proposition.** Assume that the component $\tilde{x}_j$ of the mixture $\tilde{x}$ has the finite mean μ_j (i.e., $\tilde{x}_j \in \mathrm{L}^1$) for $j = 1, ..., N$. Then, the mean μ of the mixture distribution is

$$\mu = \sum_j p_j \mu_j.$$

- **Proof.**

$$\mu = \mathrm{E}(\tilde{x}) = \sum_j p_j \mathrm{E}(\tilde{x} \,|\tilde{x} = \tilde{x}_j) = \sum_j p_j \mathrm{E}(\tilde{x}_j) = \sum_j p_j \mu_j,$$

where the second equality comes from the theorem of total expectation. *Q.E.D.*

- **Proposition.** Assume that the component $\tilde{x}_j$ of the mixture $\tilde{x}$ has the finite mean μ_j and the finite variance σ_j^2 (i.e., $\tilde{x}_j \in \mathrm{L}^2$) for $j = 1, ..., N$, and let $\mu = \sum_j p_j \mu_j$ be the mean of the mixture $\tilde{x}$. Then, the variance σ^2 of the mixture distribution is

$$\sigma^2 = \sum_j p_j \left(\sigma_j^2 + \mu_j^2 - \mu^2 \right) = \sum_j p_j \left(\sigma_j^2 + \mu_j^2 \right) - \mu^2. \qquad \text{(Exercise)}$$

- **Proposition.** Assume that the component $\tilde{x}_j$ of the mixture $\tilde{x}$ has a finite kth central moment (i.e., $\tilde{x}_j \in \mathrm{L}^k$) and the finite mean μ_j for $j = 1, ..., N$, and let $\mu = \sum_j p_j \mu_j$ be the mean of the mixture $\tilde{x}$. Then, the kth central moment of the mixture is given by

$$\mathrm{E}\left[(\tilde{x} - \mu)^k \right] =$$

$$\sum_j p_j \left(\sum_{n=0}^{k} \binom{k}{n} \left(\mu_j - \mu \right)^{k-n} \mathrm{E}\left[\left(\tilde{x}_j - \mu_j \right)^n \right] \right). \qquad \text{(Exercise)}$$

- **General mixtures (not necessarily discrete)**

- Consider a set of random variables. A generic element of this set is $\tilde{x}_\theta$, where $\theta \in \Theta \subset \mathbb{R}$. The index set Θ can be either discrete or continuous. The distribution of the random variable $\tilde{x}_\theta$ is $P_\theta : \mathcal{B} \longrightarrow [0,1]$, for $\theta \in \Theta$.

- Consider also a probability P on $(\mathbb{R}, \mathcal{B})$, where $\Theta \in \mathcal{B}$ with $P(\Theta) = 1$.

- In the mixture $\tilde{x}$, the random variables $\tilde{x}_\theta$, with $\theta \in \Theta$, are "mixed" according to the probability P.

- Thus, the **mixture distribution** $P_{\tilde{x}}$ is

$$P_{\tilde{x}}(B) = \int_\Theta P_\theta(B) dP(\theta) = \int_\mathbb{R} P_\theta(B) dP(\theta), \quad \text{for all } B \in \mathcal{B}.$$

- The previous two integrals are equal since the probability P is concentrated on Θ.

- If the random variables $\tilde{x}_\theta$, for $\theta \in \Theta$, are discrete (absolutely continuous) with probability function (density) f_θ, then the probability function (density) of the mixture distribution is

$$f_{\tilde{x}}(x) = \int_\Theta f_\theta(x) dP(\theta) = \int_{\mathbb{R}} f_\theta(x) dP(\theta).$$

- Moreover, if $\mu_\theta = \mathrm{E}\left(\tilde{x}_\theta\right)$ and $\sigma_\theta^2 = \mathrm{Var}\left(\tilde{x}_\theta\right)$, for $\theta \in \Theta$, then the mean of the mixture $\tilde{x}$ is

$$\mu = \mathrm{E}\left(\tilde{x}\right) = \int_\Theta \mu_\theta dP(\theta)$$

and the variance of the mixture $\tilde{x}$ is

$$\sigma^2 = \mathrm{Var}\left(\tilde{x}\right) = \int_\Theta \left(\sigma_\theta^2 + \mu_\theta^2 - \mu^2\right) dP(\theta) = \int_\Theta \left(\sigma_\theta^2 + \mu_\theta^2\right) dP(\theta) - \mu^2.$$

- **Particular case.** Assume now that the probability P on $(\mathbb{R}, \mathcal{B})$ is absolutely continuous with respect to Lebesgue measure with the density $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ so that

$$P(C) = \int_C f(\theta) d\theta, \quad \text{for all } C \in \mathcal{B}.$$

- Assume also that each component $\tilde{x}_\theta$ of the mixture $\tilde{x}$ is also absolutely continuous with the density $f_\theta : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, for $\theta \in \Theta \subset \mathbb{R}$.
- Then, the mixture distribution is absolutely continuous with the density

$$f_{\tilde{x}}(x) = \int_\Theta f_\theta(x) dP(\theta) = \int_{\mathbb{R}} f_\theta(x) f(\theta) d\theta, \quad \text{for } x \in \mathbb{R},$$

with the mean

$$\mu = \mathrm{E}(\tilde{x}) = \int_{\mathbb{R}} \mu_\theta f(\theta) d\theta$$

and the variance

$$\sigma^2 = \mathrm{Var}(\tilde{x}) = \int_{\mathbb{R}} \left(\sigma_\theta^2 + \mu_\theta^2 - \mu^2\right) f(\theta) d\theta = \int_{\mathbb{R}} \left(\sigma_\theta^2 + \mu_\theta^2\right) f(\theta) d\theta - \mu^2.$$

Example of Convolution

Consider two independent, absolutely continuous random variables $\tilde{x}_1$ and $\tilde{x}_2$ having both the uniform density on $(0, \theta)$,

$$\tilde{x}_i \sim f_i(x_i) = f(x_i) = \begin{cases} \dfrac{1}{\theta} & \text{for} \quad x_i \in (0, \theta) \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, 2$. Let us find the density $f_{\tilde{x}_1+\tilde{x}_2}$ of their sum, $\tilde{x}_1 + \tilde{x}_2$. This density will characterize the distribution of $\tilde{x}_1 + \tilde{x}_2$ (i.e., the convolution of $\tilde{x}_1$ and $\tilde{x}_2$).

Let us define the random variable $\tilde{y} = \tilde{x}_1 + \tilde{x}_2$. The density $f_{\tilde{y}}$ of $\tilde{y}$ is the following:

$$f_{\tilde{y}}(y) = f_{\tilde{x}_1+\tilde{x}_2}(y) = \int_{\mathbb{R}} f_{\tilde{x}_2}(y - x_1) f_{\tilde{x}_1}(x_1) dx_1 = \int_{\mathbb{R}} f(y - x_1) f(x_1) dx_1.$$

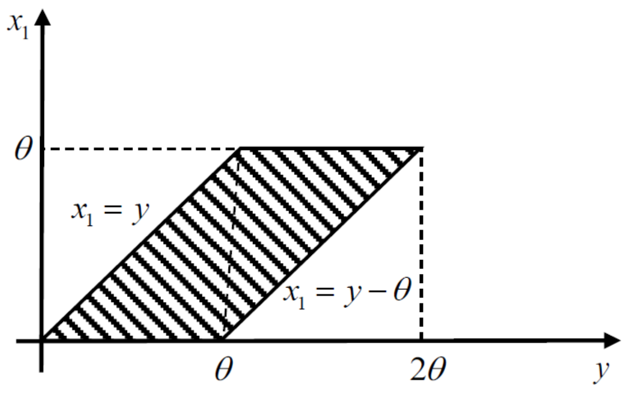
Note that $f(y - x_1)$ is equal to $1/\theta$ if $y - x_1 \in (0, \theta)$ and is equal to zero otherwise. Moreover, $f(x_1)$ is equal to $1/\theta$ if $x_1 \in (0, \theta)$ and is equal to zero otherwise. Therefore, $f(y - x_1) f(x_1)$ is equal to $(1/\theta)^2$ if $y - x_1 \in (0, \theta)$ and $x_1 \in (0, \theta)$ while it is equal to zero otherwise. Note that the set $C \subset \mathbb{R}^2$ where $f(y - x_1) f(x_1) = (1/\theta)^2 \neq 0$ is given by the points (y, x_1) satisfying the following two inequalities:

$$x_1 < y < x_1 + \theta \iff y - \theta < x_1 < y$$

and

$$0 < x_1 < \theta$$

This set C is given by the interior of the shaded region in the following figure:



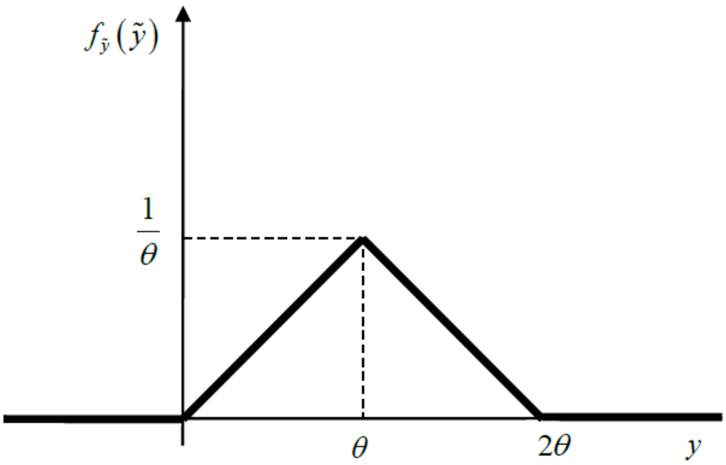
Therefore, the density of the convolution for $y \in (0, 2\theta)$ is

$$f_{\tilde{x}_1+\tilde{x}_2}(y) = f_{\tilde{y}}(y) = \int_{\mathbb{R}} \mathbb{I}_C(y, x_1) f(y - x_1) f(x_1) dx_1 = \int_{C(y)} f(y - x_1) f(x_1) dx_1,$$

where $C(y)$ is the section of y, $C(y) = \{x \in \mathbb{R} \,|\, (x,y) \in C\}$. Thus,

$$f_{\tilde{x}_1+\tilde{x}_2}(y) = f_{\tilde{y}}(y) = \begin{cases} \displaystyle\int_0^y \frac{1}{\theta^2} dx_1 = \frac{1}{\theta^2} y, & \text{for} \quad 0 < y \leq \theta \\ \displaystyle\int_{y-\theta}^{\theta} \frac{1}{\theta^2} dx_1 = -\frac{1}{\theta^2} y + \frac{2}{\theta}, & \text{for} \quad \theta < y < 2\theta, \\ 0 & \quad\quad \text{elsewhere,} \end{cases}$$

whose graph is



This type of densities (and their corresponding distributions) are called "triangular" for obvious reasons.

Examples of densities of functions of random vectors

Example 1.

Assume that the joint density function of the random variables $\tilde{x}_1$ and $\tilde{x}_2$ is

$$f_{\tilde{x}_1,\tilde{x}_2}(x_1,x_2) = \begin{cases} e^{-(x_1+x_2)} & \text{for } x_1 > 0,\ x_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the density function of the random variable $\tilde{y} = \dfrac{\tilde{x}_1}{\tilde{x}_1 + \tilde{x}_2}$.

Solution: For each given value of x_1, $\dfrac{x_1}{x_1+x_2}$ is strictly decreasing in x_2. Therefore, we can define the bijective function $g : \mathbb{R}_{++} \times \mathbb{R}_{++} \longrightarrow (0,1) \times \mathbb{R}_{++}$ as

$$(y,z) = g(x_1,x_2) : \begin{cases} y = \dfrac{x_1}{x_1+x_2} \in (0,1) \\ z = x_1 > 0, \end{cases}$$

and thus $g^{-1} : (0,1) \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++} \times \mathbb{R}_{++}$ is given by

$$(x_1,x_2) = g^{-1}(y,z) : \begin{cases} x_1 = z > 0 \\ x_2 = \dfrac{z(1-y)}{y} > 0. \end{cases}$$

The joint density of $\tilde{y}$ and $\tilde{z}$ is given by

$$f_{\tilde{y},\tilde{z}}(y,z) = \begin{cases} f_{\tilde{x}_1,\tilde{x}_2}\left(g^{-1}(y,z)\right)|J_{g^{-1}}(y,z)| & \text{for } y \in (0,1),\ z > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where,

$$|J_{g^{-1}}| = \left|\det\begin{pmatrix} \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_1}{\partial z} \\ \dfrac{\partial x_2}{\partial y} & \dfrac{\partial x_2}{\partial z} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 0 & 1 \\ -\dfrac{z}{y^2} & \dfrac{1-y}{y} \end{pmatrix}\right| = \left|\frac{z}{y^2}\right| = \frac{z}{y^2} > 0.$$

Therefore, since $x_1 + x_2 = z + \dfrac{z(1-y)}{y} = \dfrac{z}{y}$, we get

$$f_{\tilde{y},\tilde{z}}(y,z) = \begin{cases} e^{-z/y}\dfrac{z}{y^2} & \text{for } y \in (0,1) \text{ and } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

The set A where $f_{\tilde{x}_1,\tilde{x}_2}(x_1,x_2) \neq 0$ is defined by the points $(x_1,x_2) \in \mathbb{R}$ satisfying the following two inequalities:

$$x_1 > 0 \ \text{ and } \ x_2 > 0.$$

Thus, the set $g(A)$ where $f_{\tilde{y},\tilde{z}}(y,z) \neq 0$ is defined by the points $(y,z) \in \mathbb{R}$ satisfying the following two inequalities:

$$0 < y < 1 \ \text{ and } \ z > 0.$$

The marginal density of $\tilde{y}$ is

$$f_{\tilde{y}}(y) = \int_{-\infty}^{\infty} f_{\tilde{y},\tilde{z}}(y,z)dz = \int_0^{\infty} e^{-z/y}\,\frac{z}{y^2}\,dz, \quad \text{if } y \in (0,1)$$

and $f_{\tilde{y}}(y) = 0$, otherwise.

Let us make a change of variable: $u = \dfrac{z}{y} \Leftrightarrow z = yu \Rightarrow \dfrac{dz}{du} = y \in (0,1)$.

$$\Longrightarrow \ f_{\tilde{y}}(y) = \int_0^{\infty} e^{-u}\,\frac{u}{y} y\,du = \int_0^{\infty} e^{-u}\,u\,du = \Gamma(2) = (2-1)! = 1 \ \text{ if } y \in (0,1).$$

Alternatively, note that we can integrate by parts,

$$\begin{aligned}\int_0^{\infty} e^{-u}\,u\,du &= \left[-e^{-u}u\right]_0^{\infty} - \int_0^{\infty} -e^{-u}\,du = 0 + \int_0^{\infty} e^{-u}\,du \\ &= \left[-e^{-u}\right]_0^{\infty} = 0 - (-1) = 1.\end{aligned}$$

$$\Longrightarrow \ f_{\tilde{y}}(y) = \begin{cases} 1 & \text{for } y \in (0,1) \\ 0 & \text{otherwise.}\end{cases}$$

Therefore, $\tilde{y}$ is uniform on $(0,1)$.

Note that, in this example, we can name the new variable $\tilde{z}$ with the old name $\tilde{x}_1$ so that the function g would be

$$(y,x_1) = g(x_1,x_2) : \begin{cases} y = \dfrac{x_1}{x_1+x_2} \in (0,1) \\ x_1 = x_1 > 0.\end{cases}$$

Example 2.

Assume that the two independent random variables $\tilde{x}_1$ and $\tilde{x}_2$ have both the uniform density on $(0,\theta)$,

$$\tilde{x}_i \sim f_i(x_i) = f(x_i) = \begin{cases} \dfrac{1}{\theta} & \text{for } \ x_i \in (0,\theta) \\ 0 & \text{elsewhere,}\end{cases}$$

for $i = 1, 2$. Find the density function of the random variable $\tilde{y} = \tilde{x}_1 + \tilde{x}_2$. Note that this is the same example as in the previous handout on convolutions.

Solution: From independence, we obtain the joint density of $\tilde{x}_1$ and $\tilde{x}_2$,

$$f_{\tilde{x}_1,\tilde{x}_2}(x_1, x_2) = f(x_1) \cdot f(x_2) = \begin{cases} \dfrac{1}{\theta^2} & \text{for} \quad x_1 \in (0,\theta) \ \text{ and } x_2 \in (0,\theta) \\ 0 & \text{elsewhere,} \end{cases}$$

We have

$$g : \begin{cases} y = x_1 + x_2 \in (x_1, x_1 + \theta) \\ z = x_1 \in (0, \theta) \end{cases} \Longrightarrow g^{-1} : \begin{cases} x_1 = z \in (0, \theta) \\ x_2 = y - z \in (0, \theta) \end{cases}$$

Note that $y - z \in (0, \theta) \Longleftrightarrow y \in (z, z + \theta) \Longleftrightarrow z \in (y - \theta, y)$.

$$|J_{g^{-1}}(y, z)| = \left| \det \begin{pmatrix} \dfrac{\partial x_1}{\partial z} & \dfrac{\partial x_1}{\partial y} \\ \dfrac{\partial x_2}{\partial z} & \dfrac{\partial x_2}{\partial y} \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right| = 1.$$

Then,

$$f_{\tilde{y},\tilde{z}}(y, z) = \frac{1}{\theta^2} \cdot 1 = \frac{1}{\theta^2}, \text{ for } 0 < z < \theta \text{ and } y - \theta < z < y.$$

Therefore,

$$f_{\tilde{y},\tilde{z}}(y, z) = \begin{cases} \dfrac{1}{\theta^2} & \text{for} \quad 0 < z < \theta \text{ and } y - \theta < z < y \\ 0 & \text{elsewhere} \end{cases}$$

The set A where $f_{\tilde{x}_1,\tilde{x}_2} \neq 0$ is defined by the points $(x_1, x_2) \in \mathbb{R}$ satisfying the following two inequalities:

$$0 < x_1 < \theta \quad \text{and} \quad 0 < x_2 < \theta.$$

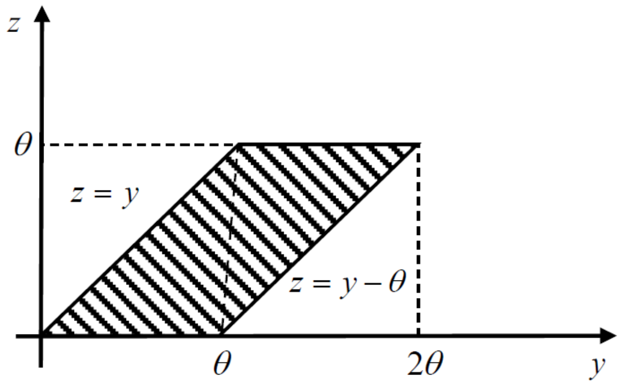
Thus, the set $g(A)$ where $f_{\tilde{y},\tilde{z}} \neq 0$ is defined by the points $(y, z) \in \mathbb{R}$ satisfying the following two inequalities:

$$0 < z < \theta \quad \text{and} \quad z < y < z + \theta$$

or, equivalently, the following two inequalities:

$$0 < z < \theta \quad \text{and} \quad y - \theta < z < y.$$

Therefore, $g(A) = C$, where the set C is given by the interior of the shaded region in the following figure:



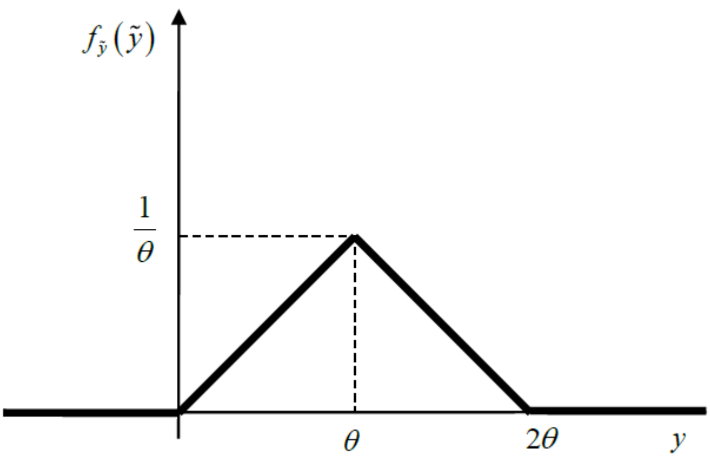
Then, the density of y for $y \in (0, 2\theta)$ is

$$f_{\tilde{y}}(y) = \int_{-\infty}^{\infty} f_{\tilde{y},\tilde{z}}(y,z)\,dz = \int_{\mathbb{R}} \mathbb{I}_C(y,z)\underbrace{f_{\tilde{y},\tilde{z}}(y,z)}_{1/\theta^2}dz = \int_{C(y)} f_{\tilde{y},\tilde{z}}(y,z)\,dz,$$

where $C(y)$ is the section of y, $C(y) = \{z \in \mathbb{R}\,|\,(y,z) \in C\}$. Therefore

$$f_{\tilde{x}_1+\tilde{x}_2}(y)\,(y) = \begin{cases} \displaystyle\int_0^y \frac{1}{\theta^2}dz = \frac{1}{\theta^2}y, & \text{for} \quad 0 < y \leq \theta \\ \displaystyle\int_{y-\theta}^{\theta} \frac{1}{\theta^2}dz = -\frac{1}{\theta^2}y + \frac{2}{\theta}, & \text{for} \quad \theta < y < 2\theta, \\ 0 & \text{elsewhere,} \end{cases}$$

so that the density is "triangular",



Again, in this example, we can name the new variable $\tilde{z}$ with the old name $\tilde{x}_1$ so that the function g would be

$$(y, x_1) = g(x_1, x_2) : \begin{cases} y = x_1 + x_2 \in (x_1, x_1 + \theta) \\ x_1 = x_1 \in (0, \theta). \end{cases}$$

Exercises. Probability and Statistics. IDEA.
5. Functions of Random Variables

1. Let $\tilde{x} = \ln \tilde{y}$ have a normal distribution with the mean μ and the variance σ^2.

   (a) find the density of the strictly positive random variable $\tilde{y}$, which is said to have the **log-normal distribution.**

   (b) find the mean m, the variance s^2, and the coefficient of variation $\mathrm{CV}_{\tilde{y}}$ of $\tilde{y}$ as functions of μ and σ^2.

   Let $\tilde{y}$ be log-normally distributed random variable with the mean m and the variance s^2.

   (c) find the mean μ and the variance σ^2 of the normal random variable $\tilde{x} = \ln \tilde{y}$ as functions of m and s^2.

2. If $\tilde{x}$ has a uniform density on the interval $(0, 1)$, show that the random variable $\tilde{y} = -2 \cdot \ln \tilde{x}$ has a gamma distribution. What are its parameters?

3. If the joint probability function of $\tilde{x}_1$ and $\tilde{x}_2$ is given by $f(x_1, x_2) = \dfrac{x_1 x_2}{36}$ for $x_1 = 1, 2, 3$, and $x_2 = 1, 2, 3$, find

   (a) the probability function of $\tilde{y} = \tilde{x}_1 \tilde{x}_2$;

   (b) the probability function of $\tilde{z} = \dfrac{\tilde{x}_1}{\tilde{x}_2}$.

4. If n independent random variables have the same gamma distribution with the parameters α and β, find the moment-generating function of their sum, and, if possible, identify its distribution. What is the distribution of the sum of n independent random variables having the same exponential distribution with the parameter θ?

5. If $\tilde{\theta}$ is the angle that a "random spinner" makes with the positive side of the horizontal axis, what is the density function of $\tan \tilde{\theta}$? In other words, what is the density of the "random tangent"?

   Hint: If $\psi(x) = \arctan(x)$, where the function $\arctan(\cdot) : \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$ is the inverse function of $\tan(\cdot) : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$, then the derivative of ψ is $\psi'(x) = \dfrac{1}{1+x^2}$. Recall also that $\tan\left(\dfrac{\pi}{4}\right) = 1$, $\tan\left(-\dfrac{\pi}{4}\right) = -1$, $\lim\limits_{\theta \to \pi/2} \tan(\theta) = \infty$, $\lim\limits_{\theta \to -\pi/2} \tan(\theta) = -\infty$ and $\tan(0) = 0$.

6. If the joint density of the random variables $\tilde{x}_1$ and $\tilde{x}_2$ is

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

   find the joint density of the random variables $\tilde{u}_1$ and $\tilde{u}_2$, where

$$\tilde{u}_1 = (-2 \ln \tilde{x}_1)^{\frac{1}{2}} \sin(2\pi \cdot \tilde{x}_2)$$

$$\tilde{u}_2 = (-2\ln\tilde{x}_1)^{\frac{1}{2}}\cos(2\pi\cdot\tilde{x}_2).$$

Note: You should use the hint of the previous exercise.

7. If the joint density of $\tilde{x}_1$ and $\tilde{x}_2$ is given by

$$f(x_1,x_2)=\begin{cases}1 & \text{for } 0<x_1<1 \text{ and } 0<x_2<1\\ \\ 0 & \text{elsewhere}\end{cases}$$

find the density $h(y)$ of $\tilde{y}=\tilde{x}_1+\tilde{x}_2$. Sketch the density $h(y)$.

8. Consider the two-dimensional vector $(\tilde{x},\tilde{y})$, which has coordinates that are independent and bivariate normally distributed with zero mean and variance σ^2.

(a) Prove that the variable "modulus" $\tilde{r}=\left(\tilde{x}^2+\tilde{y}^2\right)^{1/2}$ has the **Rayleigh** distribution with the scale parameter σ, whose density is given by

$$f_{\tilde{r}}(r)=\begin{cases}\dfrac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}} & \text{for } r\in(0,\infty)\\ \\ 0 & \text{otherwise,}\end{cases}$$

and find the density function of the random variable "angle" $\tilde{\theta}=\arctan(\tilde{y}/\tilde{x})$ with $0\le\tilde{\theta}<2\pi$.

(b) Are $\tilde{r}$ and $\tilde{\theta}$ independent?

(c) Prove that the square of the modulus $\tilde{r}^2=\tilde{x}^2+\tilde{y}^2$ has an exponential distribution with parameter $2\sigma^2$ and, if $\sigma^2=1$, then $\tilde{r}^2$ has the chi-square distribution with 2 degrees of freedom χ^2_2.

(d) Find the expected value and the variance of the modulus $\tilde{r}$.

(e) Find the expected value and the variance of the angle $\tilde{\theta}$.

9. The joint density function of the two random variables $\tilde{x}$ and $\tilde{y}$ is

$$f(x,y)=\begin{cases}\dfrac{1}{12}(x+2y) & \text{if } x\in(0,2),\ y\in(0,2)\\ \\ 0 & \text{otherwise.}\end{cases}$$

(a) Find the joint distribution function of $\tilde{x}$ and $\tilde{y}$, $F(x,y)$.

(b) Find the marginal density functions of $\tilde{x}$ and $\tilde{y}$, $f_{\tilde{x}}(x)$ and $f_{\tilde{y}}(y)$.

(c) Find the conditional density function of $\tilde{y}$ given $\tilde{x}=1/2$, $f_{\tilde{y}|\tilde{x}}(y|\,1/2)$.

(d) Find the expectation of $\tilde{y}$, $\mathrm{E}(\tilde{y})$, and the conditional expectation of $\tilde{y}$ given $\tilde{x}=1/2$, $\mathrm{E}(\tilde{y}|\,\tilde{x}=1/2)$.

(e) Find $P(\{0<\tilde{x}<1\}\cap\{1<\tilde{y}<3/2\})$.

(f) Could you find a value x for which the conditional density of $\tilde{y}$ given $\tilde{x} = x$ is equal to the marginal density of $\tilde{y}$? Are $\tilde{x}$ and $\tilde{y}$ independent?

(g) Find the density function of the random variable $\tilde{w} = 2 - \tilde{y}^{1/3}$ and the expectation of $\tilde{w}$, $\mathrm{E}(\tilde{w})$.

(h) Find the density function of the random variable $\tilde{z} = \tilde{x} \cdot \tilde{y}$.

10. The density function of an absolutely continuous random variable $\tilde{x}$ is

$$f_{\tilde{x}}(x) = \frac{1}{2} e^{-|x|} , \qquad -\infty < x < \infty$$

Find the distribution function of $\tilde{x}$ and the density function of

$$\tilde{y} = (\tilde{x})^3 \qquad \text{and} \qquad \tilde{z} = \ln (|\tilde{x}| + 1) .$$

11. Let $\tilde{x}$ be an absolutely continuous random variable with the density function

$$f(x) = \begin{cases} \dfrac{1}{3} & \text{if } x \in (0, 3) \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation $\tilde{y} = 9 - (\tilde{x})^2$. Find and draw the distribution and the density functions of $\tilde{y}$.

12. Let $F_{\tilde{x}}(x)$ be the distribution function of the univariate random variable $\tilde{x}$.

(a) Assume that $\tilde{x}$ is a mixed random variable (that is, $\tilde{x}$ is neither discrete nor absolutely continuous). Find the distribution function of the absolute value of $\tilde{x}$, that is, find the distribution function $F_{\tilde{y}}(y)$ of the random variable $\tilde{y} = |\tilde{x}|$.

(b) Answer the same question of part (a) but assuming now that the distribution function $F_{\tilde{x}}(x)$ is continuous for all $x \in (-\infty, \infty)$.

(c) Assume now that the random variable $\tilde{x}$ is absolutely continuous with the density function $f_{\tilde{x}}(x)$. Moreover, assume that the distribution function $F_{\tilde{x}}(x)$ of the variable $\tilde{x}$ is differentiable for all $x \in (-\infty, \infty)$. Using the density function $f_{\tilde{x}}(x)$ of the variable $\tilde{x}$, find the density function $f_{\tilde{y}}(y)$ of the random variable $\tilde{y} = |\tilde{x}|$.

(d) Answer the same question of part (c) but assuming now that the density function $f_{\tilde{x}}(x)$ is also symmetric with respect to $x = 0$, that is, $f_{\tilde{x}}(x) = f_{\tilde{x}}(-x)$ for all $x \in (-\infty, \infty)$.

13. The density function of the random variable $\tilde{x}$ is

$$f_{\tilde{x}}(x) = \begin{cases} \dfrac{x}{8} & \text{if } x \in (0, 4) \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the density function $f_{\tilde{y}}(y)$ and the distribution function $F_{\tilde{y}}(y)$ of the random variable $\tilde{y} = \tilde{x}^2 - 8$.

(b) Compute $P\{1 < \tilde{y} < 3\}$ and $P\{\tilde{y} \geq -2\}$.

14. The absolutely continuous random variable $\tilde{x}$ has the following density function:

$$f_{\tilde{x}}(x) = \begin{cases} 6x(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean and the variance of $\tilde{y} = (\tilde{x})^3$.

(b) Find the density function and the distribution function of $\tilde{y}$.

15. The density function of the random vector $(\tilde{x}, \tilde{y})$ is

$$f(x,y) = \begin{cases} k\,(x+y) & \text{for } x > 0,\ x^2 + y^2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant k.

(b) Find the density function of the random variable $\tilde{z} = (\tilde{x}^2 + \tilde{y}^2)^{1/2}$.

16. The density function of the random vector $(\tilde{x}_1, \tilde{x}_2)$ is

$$f_{\tilde{x}_1,\tilde{x}_2}(x_1, x_2) = \begin{cases} 6e^{-3x_1 - 2x_2} & \text{for } x_1 > 0,\ x_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the conditional expectation of $\tilde{x}_2$ given $\tilde{x}_1 = 5$, $\mathrm{E}(\tilde{x}_2 \mid \tilde{x}_1 = 5)$. Are the random variables $\tilde{x}_1$ and $\tilde{x}_2$ independent?

(b) Find the moment-generating function for the random variable $\tilde{x}_1$ and use it to find the mean, the variance, and the coefficient of variation of $\tilde{x}_1$. Do the same for the random variable $\tilde{x}_2$.

For the rest of this exercise, consider the random variable $\tilde{y} = 4\tilde{x}_1 + \tilde{x}_2$.

(c) Without finding first the density of the random variable $\tilde{y}$, find directly the distribution function $F_{\tilde{y}}$ of the random variable $\tilde{y}$.

(d) Find the joint density of the random variables $\tilde{y}$ and $\tilde{x}_2$, $f_{\tilde{y},\tilde{x}_2}(y, x_2)$.

(e) Use the joint density obtained in (d) to find the density $f_{\tilde{y}}$ of the random variable $\tilde{y}$.

(f) Check that the results you have obtained in parts (c) and (e) are mutually consistent, i.e., check that $F_{\tilde{y}}$ is the distribution function associated with the density $f_{\tilde{y}}$, and that $f_{\tilde{y}}$ is the density associated with the distribution function $F_{\tilde{y}}$.

(g) Find the conditional density of the random variable $\tilde{y}$ given $\tilde{x}_2 = 3$, $f_{\tilde{y}\,|\,\tilde{x}_2}(y \,|\, 3)$.

(h) Find both the (unconditional) expectation of $\tilde{y}$, $\mathrm{E}(\tilde{y})$, and the conditional expectation of $\tilde{y}$ given $\tilde{x}_2 = 3$, $\mathrm{E}(\tilde{y} \mid \tilde{x}_2 = 3)$.

(i) Find the covariance between $\tilde{y}$ and $\tilde{x}_2$, $\mathrm{Cov}(\tilde{y}, \tilde{x}_2)$.

17. The density function of the random variable $\tilde{x}$ is

$$f_{\tilde{x}}(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the random variable $\tilde{y} = \tilde{x}^{1/2}$, i.e., $\tilde{y}$ is the positive square root of $\tilde{x}$.

(a) Without finding first the density of the random variable $\tilde{y}$, find directly the distribution function $F_{\tilde{y}}$ of the random variable $\tilde{y}$.

(b) Without making use of the distribution function you have found in part (a), find directly the density $f_{\tilde{y}}$ of the random variable $\tilde{y}$.

(c) Check that the results you have obtained in parts (a) and (b) are mutually consistent, i.e., check that $F_{\tilde{y}}$ is the distribution function associated with the density $f_{\tilde{y}}$, and that $f_{\tilde{y}}$ is the density associated with the distribution function $F_{\tilde{y}}$.

(d) Is the distribution function $F_{\tilde{x}}(x)$ of the random variable $\tilde{x}$ differentiable for all $x \in \mathbb{R}$? Is the distribution function $F_{\tilde{y}}(y)$ of the random variable $\tilde{y}$ differentiable for all $y \in \mathbb{R}$? Draw the functions $f_{\tilde{x}}(x)$, $F_{\tilde{x}}(x)$, $f_{\tilde{y}}(y)$, and $F_{\tilde{y}}(y)$.

18. The density function of the random vector $(\tilde{x}, \tilde{y})$ is

$$f(x, y) = \begin{cases} e^{-(x+y)} & \text{if } \; x > 0, \; y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) The distribution function of the random vector $(\tilde{x}, \tilde{y})$.

(b) The marginal density functions of $\tilde{x}$ and $\tilde{y}$. Are the random variables $\tilde{x}$ and $\tilde{y}$ independent?

(c) The conditional density function of $\tilde{y}$ given $\tilde{x} = 3$.

(d) $P(\{0 < \tilde{x} < 1\} \cap \{\tilde{x} < \tilde{y} < 1\})$.

Consider the random variable $\tilde{z} = \tilde{x} - 3\tilde{y}$.

(e) Without finding first the density of the random variable $\tilde{z}$, find the distribution function $F_{\tilde{z}}$ of the random variable $\tilde{z}$. That is, use the distribution function method.

(f) Without making use of the distribution function you have found in part (a), find the density $f_{\tilde{z}}$ of the random variable $\tilde{z}$. That is, use the density function method.

(g) Check that the results you have obtained in parts (e) and (f) are mutually consistent, i.e., check that $F_{\tilde{z}}$ is the distribution function associated with the density $f_{\tilde{z}}$, and that $f_{\tilde{z}}$ is the density associated with the distribution function $F_{\tilde{z}}$.

(h) Draw the functions $F_{\tilde{z}}(z)$ and $f_{\tilde{z}}(z)$.

19. The density of the random vector $(\tilde{x}_1, \tilde{x}_2)$ is given by

$$f(x_1, x_2) = \begin{cases} \dfrac{1}{2} & \text{for } 0 < x_1 < 2 \text{ and } 0 < x_2 < 1 \\ \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the density $f_{\tilde{y}}(y)$ of the random variable $\tilde{y} = \tilde{x}_1 + \tilde{x}_2$. To do so, find first the density of the vector $(\tilde{y}, \tilde{x}_2)$. Draw the density $f_{\tilde{y}}(y)$ and check that the area below this density is equal to one.

(b) Find the distribution function of the random variable $\tilde{y}$, $F_{\tilde{y}}(y)$.

(c) Find the conditional density of $\tilde{y}$ given $\tilde{x}_2 = x_2$, $f_{\tilde{y}|\tilde{x}_2}(y\,|x_2)$, and the conditional density of $\tilde{x}_2$ given $\tilde{y} = y$, $f_{\tilde{x}_2|\tilde{y}}(x_2\,|y)$. Compute the following conditional expectations: $\mathrm{E}(\tilde{y}\,|\tilde{x}_2 = 1/4)$, $\mathrm{E}(\tilde{x}_2\,|\tilde{y} = 5/2)$ and $\mathrm{E}(\tilde{x}_2\,|\tilde{y} = 3/2)$.

20. The density function of the random vector $(\tilde{x}, \tilde{y})$ is

$$f(x, y) = \begin{cases} \dfrac{4}{3}(x + xy) & \text{for } 0 < x < 1,\ 0 < y < 1 \\ \\ 0 & \text{elsewhere} \end{cases}$$

(a) Are $\tilde{x}$ and $\tilde{y}$ independent?

(b) Compute the conditional expectation of $\tilde{y}$ given $\tilde{x} = 1/3$, $\mathrm{E}(\tilde{y}\,|\tilde{x} = 1/3)$.

(c) Using just the density of the random variable $\tilde{x}$, compute the expectation of $\ln \tilde{x}$, $\mathrm{E}(\ln \tilde{x})$.

(d) Find the (cumulative) distribution function $F(x, y)$ of the random vector $(\tilde{x}, \tilde{y})$ and check that the cross partial derivative of $F(x, y)$ is the density $f(x, y)$ a.e. with respect to Lebesgue measure in $\mathbb{R}^2$.

(e) Find the density of the random variable $\tilde{z} = \ln \tilde{x}$. Draw this density.

(f) Find the moment-generating function of the random variable $\tilde{z}$ and use this moment-generating function to compute both the expectation and the variance of $\tilde{z}$, $\mathrm{E}(\tilde{z})$ and $\mathrm{Var}(\tilde{z})$.

21. Assume that the density function of the random variable $\tilde{x}$ is

$$f_{\tilde{x}}(x) = \begin{cases} \dfrac{2 - x}{4} & \text{for } x \in (0, 2) \\ \\ \dfrac{x + 2}{4} & \text{for } x \in (-2, 0] \\ \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the distribution function $F_{\tilde{x}}(x)$ of the random variable $\tilde{x}$. Draw both $f_{\tilde{x}}(x)$ and $F_{\tilde{x}}(x)$.

(b) Find and draw the distribution function $F_{\tilde{y}}(y)$ and the density function $f_{\tilde{y}}(y)$ of the random variable $\tilde{y} = \tilde{x}^2$.

(c) Consider the random vector $(\tilde{x}, \tilde{y})$, where $\tilde{x}$ has the density defined above and $\tilde{y}$ is defined in part (b). Is the random vector $(\tilde{x}, \tilde{y})$ absolutely continuous, i.e., does its distribution have a density function?

(d) Compute the covariance between $\tilde{x}$ and $\tilde{y}$, $\mathrm{Cov}\,(\tilde{x}, \tilde{y})$. Are $\tilde{x}$ and $\tilde{y}$ independent? Answer the latter question using the definition of independence of two random variables.

22. Assume that the three random variables $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ on the same probability space $(\Omega, \mathcal{F}, P)$ are independent with the probability/density functions f_1, f_2 and f_3, respectively. Find the probability function/density of their convolution (i.e., of the distribution of their sum).

23. Consider a mixture $\tilde{x}$ of a discrete set $\{\tilde{x}_j\}_{j=1}^{N}$ of random variables, where N can be either a finite number or infinity and $\tilde{x} = \tilde{x}_j$ with probability p_j, for $j = 1, ..., N$. Let μ_j and σ_j^2 be the finite mean and the finite variance, respectively, of each component $\tilde{x}_j$ of the mixture, for $j = 1, ..., N$, and $\mu = \sum_j p_j \mu_j$ is thus the mean of the mixture $\tilde{x}$.

(a) Prove that the kth central moment of the mixture $\tilde{x}$ is

$$\mathrm{E}\left[(\tilde{x} - \mu)^k\right] = \sum_j p_j \left(\sum_{n=0}^{k} \binom{k}{n} (\mu_j - \mu)^{k-n} \, \mathrm{E}\left[(\tilde{x}_j - \mu_j)^n\right] \right).$$

(b) Prove that the variance σ^2 of the mixture $\tilde{x}$ is

$$\sigma^2 = \sum_j p_j \left(\sigma_j^2 + \mu_j^2 - \mu^2\right) = \sum_j p_j \left(\sigma_j^2 + \mu_j^2\right) - \mu^2.$$

24. The density function of the random vector $(\tilde{x}, \tilde{y})$ is

$$f_{\tilde{x},\tilde{y}}(x, y) = \begin{cases} \dfrac{4}{3}(x + xy) & \text{for } 0 < x < 1, \ 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Compute the covariance between $\tilde{x}$ and $\tilde{y}$, $\mathrm{Cov}(\tilde{x}, \tilde{y})$.

(b) Find the moment generation function $M_{\tilde{x}}(t)$ of the random variable $\tilde{x}$ and use it to find the expectation of $\tilde{x}$, $\mathrm{E}(\tilde{x})$.

(c) Use the density function method to find the density $f_{\tilde{w},\tilde{z}}(w, z)$ of the random vector $(\tilde{w}, \tilde{z})$, where

$$\tilde{w} = \ln \tilde{x} \quad \text{and} \quad \tilde{z} = \ln\left(\frac{\tilde{x}}{\tilde{y}}\right).$$

(d) Find the density $f_{\tilde{w}}(w)$ of the random variable $\tilde{w}$ and the density $f_{\tilde{z}}(z)$ of the random variable $\tilde{z}$. Draw these two densities.

25. The density function of the random vector $(\tilde{x}, \tilde{y})$ is

$$f(x,y) = \begin{cases} \dfrac{2x+y}{12} & \text{if } x \in (0,2) \text{ and } y \in (0,2) \\ \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the covariance between $\tilde{x}$ and $\tilde{y}$. Are $\tilde{x}$ and $\tilde{y}$ independent?

(b) Compute the conditional expectation of $\tilde{x}$ given $\tilde{y} = 1/2$, $\mathrm{E}(\tilde{x}\,|\tilde{y} = 1/2)$.

(c) Use the density function method to find the density $f_{\tilde{z}}(z)$ of the random variable

$$\tilde{z} = 3\tilde{y} - \frac{\tilde{x}}{2}$$

and draw the density $f_{\tilde{z}}(z)$ of $\tilde{z}$.

– – – – – – – – – – – – – –

26. Prove that, if $\tilde{x}$ has the logistic or the generalized extreme value type I, type II or type III distribution with location parameter m and scale parameter $s > 0$, then $\tilde{z} = c\tilde{x}+b$, with $c > 0$, has the the logistic or the generalized extreme value type I, type II or type III distribution, respectively, with location parameter $cm + b$ and scale parameter $cs > 0$. In particular, if $\tilde{x}$ is standard logistic or standard extreme value type I, type II or type III then $\tilde{z}$ is logistic or generalized extreme value type I, type II or type III, respectively, with location parameter b and scale parameter $c > 0$. Conversely, if $c = 1/s$ and $b = -m/s$, (i.e., $\tilde{z} = (\tilde{x} - m)/s$) then $\tilde{z}$ is standard logistic or standard extreme value type I, type II or type III, respectively. Moreover, if $\tilde{x}$ has the generalized extreme value type II or type III distribution, then the respective shape parameter α remains the same after applying that affine transformation.

– – – – – – – – – – – – – –

27. (a) Prove that $\tilde{w}$ has the exponential distribution with parameter equal to 1 (i.e., it has unitary mean) if and only if the random variable $\tilde{x} = m - s\ln\tilde{w}$ has the generalized extreme value type I (or Gumbel) distribution with location parameter m and scale parameter $s > 0$. Note that $\tilde{w} = e^{-(\tilde{x}-m)/s}$.

(b) Assume that $\tilde{w}_1$ and $\tilde{w}_2$ are independent random variables having the exponential distribution with parameter equal to 1. Find the distribution function and the density function of the random variable $\tilde{z} = \tilde{w}_1/\tilde{w}_2$.

(c) Use the results of parts (a) and (b) to prove that, if $\tilde{x}_1$ and $\tilde{x}_2$ are independent and have generalized extreme value type I (or Gumbel) distributions with location parameters m_1 and m_2, respectively, with the same scale parameter $s > 0$, then the random variable $\tilde{y} = \tilde{x}_1 - \tilde{x}_2$ has the logistic distribution with location parameter (or mean) $m_1 - m_2$ and scale parameter s, $\tilde{y} \sim l(y; m_1 - m_2, s)$. In particular, if $\tilde{x}_1$ and $\tilde{x}_2$ are independent and have standard extreme value type I distributions, then the random variable $\tilde{y} = \tilde{x}_1 - \tilde{x}_2$ has the standard logistic distribution, $\tilde{y} \sim l(y; 0, 1)$.