Probability and Statistics. IDEA-UAB.
Final Exam 2022-23. Prof. J. Caballé.

The exam lasts for 3 hours. Try to answer all the questions within each exercise.

1. Assume that the distribution of the random variable $\tilde{x}$ is characterized by the following density:

$$f(x;\mu,\theta)=\frac{1}{2\theta}e^{-|x-\mu|/\theta}, \quad -\infty<x<\infty.$$

Note that in the exponent of the previous density we have the negative of the absolute value $|x-\mu|$ so that $-|x-\mu| = -(x-\mu)$ if $x \geq \mu$, whereas $-|x-\mu| = x-\mu$ if $x \leq \mu$.

(a) **(0.5 points)** Draw the density of $\tilde{x}$ for the particular case where $\mu=-3$ and $\theta=2$.

(b) **(0.75 points)** Find the distribution function (cdf) of $\tilde{x}$. Is the cdf continuous? Is it differentiable everywhere? Is it twice differentiable everywhere?

(c) **(0.75 points)** Find the moment generating function of $\tilde{x}$? Draw it for he particular case where $\mu=-3$ and $\theta=2$.

(d) **(0.5 points)** Find the mean and the variance σ^2 of $\tilde{x}$.

(e) **(1 point)** Find the density of the random variable $\tilde{z}=|\tilde{x}-\mu|$. Find the mean of $\tilde{z}$.

(f) **(0.75 points)** Let $\mu=-3$ and $\theta=2$. Find the probability that $\tilde{x}\in(-4,1)$ given that we know that the event $\tilde{x}\in(-5,-3)$ has occurred. *Note:* You can use for the computations of this part the distribution function you have found in part (b).

Assume for the rest of the exercise that you have a random sample $\{\tilde{x}_i\}_{i=1}^n$ of size n from the population $\tilde{x}$.

(g) **(0.75 points)** Find the method of moments estimator for the parameter vector (μ,θ).

Assume now that the value of the parameter μ is known.

(h) **(0.75 points)** Find the value of the Cramér-Rao lower bound for an unbiased estimator for the parameter θ of the distribution of $\tilde{x}$.

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THE EXAM CONTINUES ON THE REVERSE SIDE

2. **(1.25 points)** Consider the random experiment consisting of rolling a balanced dice. Let $\tilde{x} : \left(\Omega, 2^{\Omega}\right) \longrightarrow (\mathbb{R}, \mathcal{B})$ be the random variable representing the prize in euros you get as a function of the number of dots $\omega \in \Omega$ obtained when rolling the dice. The random variable $\tilde{x}$ is defined as follows:

$$\tilde{x}(\omega) = \begin{cases} 2 & \text{if } \omega \in \{1, 2\} \\ \\ 3 & \text{if } \omega \in \{3, 4, 5, 6\} \end{cases}$$

and $\mathcal{F}(\tilde{x})$ is the σ-algebra induced on Ω by the random variable $\tilde{x}$.

Consider the following partition of the sample space Ω :

$$G = \{\{1, 2, 3\}, \{4, 5, 6\}\}$$

and $\mathcal{G}$ is the σ-algebra generated by the partition G.

Compute the following three conditional expectations of $\tilde{x}$ given a σ-algebra: $\mathrm{E}(\tilde{x} \,|\mathcal{F}(\tilde{x}))$, $\mathrm{E}(\tilde{x} \,|\mathcal{G})$, and $\mathrm{E}(\tilde{x} \,|\mathcal{H})$, where $\mathcal{H} = \{\Omega, \emptyset\}$ is the coarsest σ-algebra on Ω.

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3. Assume that there are 6 balls in a box. Each ball is either black or white and we assume that the 7 different possible compositions of the box are equally likely.

   (a) **(0.75 points)** We randomly pick two balls **with replacement** (i.e., we put back in the box the picked ball after each extraction) and it turns out that both balls are white. What is the probability that 5 out of the 6 balls were white in the initial composition of the box?

   (b) **(0.75 points)** Assume now that we extract two balls **with NO replacement** from the box. What is the probability that both balls will be black?

   Assume for the rest of this exercise that there are 6 balls in the box and 3 of them are white, 2 are black, and 1 is green.

   (c) **(0.75 points)** Assume now that we extract 4 balls **with NO replacement** from the box. What is the probability of extracting 2 white balls and 2 black balls.

   (d) **(0.75 points)** Assume now that we extract 4 balls **with replacement** from the box. What is the probability of extracting 2 white balls and 2 black balls.

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The exam lasts for 3 hours. Try to answer all the questions within each exercise.

1. **(1 point)** The distribution of a random vector $(\tilde{x}, \tilde{y})$ is characterized by the following density function:

$$f(x,y) = \begin{cases} k\left(x^2+y^2\right)^{1/2} & \text{when } (x,y) \in A \\ 0 & \text{when } (x,y) \notin A, \end{cases}$$

where $A = \left\{(x,y) \in \mathbb{R}^2 \mid 9 < x^2+y^2 < 16,\ y > 0\right\}.$

Find the value of the constant k.

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2. Consider a normal population with the known variance $\sigma^2 = 3$ and a random sample $\{\tilde{x}_1, ..., \tilde{x}_n\}$ with size $n = 12$.

(a) **(1 point)** We want to test the null hypothesis that the mean μ of this population is equal to 7 ($H_0 : \mu = 7$) against the alternative simple hypothesis that it is 9 ($H_1 : \mu = 9$). Let $\bar{x}$ be the value of the mean of the random sample. Find the value of k such that $\bar{x} > k$ provides a critical region of size $\alpha = 0.05$. Compute the power of this test.

(b) **(1.5 points)** Construct a likelihood ratio test with a level of significance $\alpha = 0.05$ for testing the null hypothesis $H_0 : \mu = 7$ against the composite alternative $H_1 : \mu \neq 7$. You should provide the appropriate test statistic and the precise critical region. *Hint:* for this part, you should prove that

$$\sum_{i=1}^{12}\left[(x_i - 7)^2 - (x_i - \bar{x})^2\right] = 12(\bar{x}-7)^2.$$

Reminder: the density of a normal distribution with the mean μ and the variance $\sigma^2 > 0$ (or the standard deviation $\sigma > 0$) is

$$n(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \text{for } -\infty < x < \infty.$$

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THE EXAM CONTINUES ON THE REVERSE SIDE

3. The density function of the random vector $(\tilde{x}_1, \tilde{x}_2)$ is

$$f_{\tilde{x}_1,\tilde{x}_2}(x_1, x_2) = \begin{cases} 6e^{-3x_1-2x_2} & \text{for } x_1 > 0,\ x_2 > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

(a) **(0.75 points)** Are the random variables $\tilde{x}_1$ and $\tilde{x}_2$ independent?

(b) **(0.75 points)** Find the moment-generating function for the random variable $\tilde{x}_1$ and use it to find the mean, the variance, and the coefficient of variation of $\tilde{x}_1$. Do the same for the random variable $\tilde{x}_2$.

For the rest of this exercise, consider the random variable $\tilde{y} = 4\tilde{x}_1 + \tilde{x}_2$.

(c) **(1 point)** Without finding first the corresponding density function, find directly the distribution function $F_{\tilde{y}}$ of the random variable $\tilde{y}$. Use this distribution function to find the density function $f_{\tilde{y}}$ of the random variable $\tilde{y}$. *Note:* in this question, you are asked to use the "distribution function method".

(d) **(0.75 points)** Find the joint density of the random variables $\tilde{y}$ and $\tilde{x}_2$, $f_{\tilde{y},\tilde{x}_2}(y, x_2)$.

(e) **(0.75 points)** Find the conditional density of the random variable $\tilde{y}$ given $\tilde{x}_2 = 3$, $f_{\tilde{y}\,|\,\tilde{x}_2}(y\,|\,3)$.

(f) **(0.75 points)** Find both the (unconditional) expectation of $\tilde{y}$, $\mathrm{E}(\tilde{y})$, and the conditional expectation of $\tilde{y}$ given $\tilde{x}_2 = 3$, $\mathrm{E}(\tilde{y}\mid\tilde{x}_2 = 3)$.

(g) **(0.75 points)** Find the covariance between $\tilde{y}$ and $\tilde{x}_2$, $\mathrm{Cov}(\tilde{y}, \tilde{x}_2)$.

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4. **(1 point)** Let $\tilde{y}$ be a discrete random variable with probability function (pmf) $g(y)$, $g : \tilde{y}(\Omega) \longrightarrow [0, 1]$. If $\tilde{y} = y$, you observe a random sample $\{\tilde{x}_1, ..., \tilde{x}_n\}$ from a discrete population $\tilde{x}$ having the probability function $h(x; y)$, $h(\cdot; y) : \tilde{x}(\Omega) \longrightarrow [0, 1]$. Explain how you would compute the conditional expectation of $\tilde{y}$, given $\tilde{x}_1 = x_1, ..., \tilde{x}_n = x_n$ by just using the functions $g(\cdot)$ and $h(\cdot;\cdot)$ and the sample values $\{x_1, ..., x_n\}$.

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