Probability and Statistics. IDEA-UAB.
Final Exam 2024-25. Prof. J. Caballé.

The exam lasts for 3 hours. Try to answer all the questions within each exercise. All the questions have the same weight for the final grade.

1. The density function of the random vector $(\tilde{x}, \tilde{y})$ is

$$f_{\tilde{x},\tilde{y}}(x,y) = \begin{cases} kx(y+1) & \text{for } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(**a**) Prove that the value of the constant k is $4/3$.

(**b**) Compute the product moment $\mathrm{E}(\tilde{x} \cdot \tilde{y})$ and the covariance $\mathrm{Cov}(\tilde{x}, \tilde{y})$ between $\tilde{x}$ and $\tilde{y}$.

(**c**) Use the marginal density of the random variable $\tilde{x}$ to find the moment-generating function $M_{\tilde{x}}(t)$ of $\tilde{x}$ and use it to find the expectation of $\tilde{x}$, $\mathrm{E}(\tilde{x})$.

(**d**) Find the (cumulative) distribution function $F_{\tilde{x},\tilde{y}}(x,y)$ of the random vector $(\tilde{x}, \tilde{y})$ and check that the cross partial derivative of $F_{\tilde{x},\tilde{y}}(x,y)$ is the density $f_{\tilde{x},\tilde{y}}(x,y)$ a.e. with respect to Lebesgue measure in $\mathbb{R}^2$.

(**e**) Use the marginal density of the random variable $\tilde{x}$ to compute the expectation of $\ln \tilde{x}$, $\mathrm{E}(\ln \tilde{x})$.

(**f**) Use the density function method to find the density $f_{\tilde{w},\tilde{z}}(w,z)$ of the random vector $(\tilde{w}, \tilde{z})$, where

$$\tilde{w} = \ln \tilde{x} \quad \text{and} \quad \tilde{z} = \ln \tilde{x} - \ln \tilde{y}.$$

(**g**) Use the density $f_{\tilde{w},\tilde{z}}(w,z)$ of the random vector $(\tilde{w}, \tilde{z})$ to find the density $f_{\tilde{w}}(w)$ of the random variable $\tilde{w}$ and the density $f_{\tilde{z}}(z)$ of the random variable $\tilde{z}$. Draw the densities $f_{\tilde{w}}(w)$ and $f_{\tilde{z}}(z)$.

THE EXAM CONTINUES ON THE REVERSE SIDE

2. The probability function of the random variable $\tilde{x}$ is

$$f(x;\theta) = \frac{1}{\theta}\left(\frac{\theta}{\theta+1}\right)^x, \quad \text{for } x = 1, 2, 3, \ldots$$

with $\theta > 0$. Note that the distribution of $\tilde{x}$ is discrete but it has infinite support.

(**a**) Find the moment-generating function $M_{\tilde{x}}(t)$ of the random variable $\tilde{x}$ and use it to find the mean $\mathrm{E}(\tilde{x})$, the second non-central moment $\mathrm{E}\left(\tilde{x}^2\right)$, and the variance $\mathrm{Var}(\tilde{x})$.

Assume that we observe a random sample $\{\tilde{x}_i\}_{i=1}^n$ of size n from the population $\tilde{x}$ and let $\bar{\boldsymbol{x}}$ be the sample mean of that random sample.

(**b**) Find the method of moments estimator $\hat{\boldsymbol{\theta}}_{\mathbf{MM}}$ for the parameter θ.

(**c**) Find the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_{\mathbf{ML}}$ for the parameter θ. Check that the second-order condition for a maximum holds.

(**d**) Find the maximum likelihood estimator $\hat{\boldsymbol{\sigma}}^{\mathbf{2}}_{\mathbf{ML}}$ for the population variance $\mathrm{Var}(\tilde{x})$.

(**e**) Is the statistic $\hat{\boldsymbol{\theta}} = \bar{\boldsymbol{x}} - 1$ an unbiased estimator for θ whose variance is equal to the Cramér-Rao lower bound? Remember that the Cramér-Rao lower bound for this case is

$$\left(-n\mathrm{E}\left[\frac{\partial^2 \ln f\left(\tilde{x};\theta\right)}{\partial\theta^2}\right]\right)^{-1}$$

or, equivalently,

$$\left(n\mathrm{E}\left[\left(\frac{\partial \ln f\left(\tilde{x};\theta\right)}{\partial\theta}\right)^2\right]\right)^{-1}.$$

(**f**) Is the statistic $\hat{\boldsymbol{\theta}} = \bar{\boldsymbol{x}} - 1$ a (weakly) consistent estimator for θ?

(**g**) Is the sample mean $\bar{\boldsymbol{x}}$ a sufficient statistic for θ? Is the statistic $\hat{\boldsymbol{\theta}} = \bar{\boldsymbol{x}} - 1$ a sufficient statistic for θ? Use the factorization theorem for this part.

ANSWER
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1. (a)

$$\int_0^1 \int_0^1 kx(y+1)dxdy = k\int_0^1 \int_0^1 x(y+1)dxdy = 1$$

$$\Longrightarrow k = \frac{1}{\displaystyle\int_0^1 \int_0^1 x(y+1)dxdy} = \frac{1}{\left[\displaystyle\int_0^1 xdx\right] \cdot \left[\displaystyle\int_0^1 (y+1)dy\right]} = \frac{1}{\frac{1}{2}\cdot\frac{3}{2}} = \frac{1}{3/4} = \frac{4}{3}.$$

(b) We can find the marginal densities:

$$f_{\tilde{x}}(x) = \begin{cases} \displaystyle\int_0^1 \frac{4}{3}x(y+1)dy = 2x \quad \text{for } 0 < x < 1 \\ 0 \quad \text{otherwise} \end{cases}$$

$$f_{\tilde{y}}(y) = \begin{cases} \displaystyle\int_0^1 \frac{4}{3}x(y+1)dx = \frac{2}{3}(y+1) \quad \text{for } 0 < y < 1 \\ 0 \quad \text{otherwise} \end{cases}$$

$$\mathrm{E}(\tilde{x}\cdot\tilde{y}) = \int_0^1 \int_0^1 xy\frac{4}{3}x(y+1)dxdy = \frac{4}{3}\left[\int_0^1 x^2dx\right]\cdot\left[\int_0^1 (y^2+y)dy\right] = \frac{10}{27},$$

$$\mathrm{E}(\tilde{x}) = \int_0^1 \int_0^1 x\frac{4}{3}x(y+1)dxdy = \int_0^1 x\cdot(2x)\,dx = \frac{2}{3},$$

$$\mathrm{E}(\tilde{y}) = \int_0^1 \int_0^1 y\frac{4}{3}x(y+1)dxdy = \int_0^1 y\cdot\frac{2}{3}(y+1)dy = \frac{5}{9}.$$

Thus,

$$\mathrm{Cov}(\tilde{x},\tilde{y}) = \mathrm{E}(\tilde{x}\cdot\tilde{y}) - \mathrm{E}(\tilde{x})\,\mathrm{E}(\tilde{y}) = \frac{10}{27} - \left(\frac{2}{3}\cdot\frac{5}{9}\right) = 0.$$

(c)

$$M_{\tilde{x}}(t) = \mathrm{E}\left(e^{t\tilde{x}}\right) = \int_0^1 e^{tx} f_{\tilde{x}}(x)dx = \int_0^1 e^{tx}\cdot(2x)\,dx.$$

The previous integral can be solved by parts by making $H(x) = 2x$ and $g(x) = G'(x) = e^{tx}$ so that $h(x) = H'(x) = 2$ and $G(x) = \frac{e^{tx}}{t}$. Then,

$$M_{\tilde{x}}(t) = \int_0^1 e^{tx}\cdot(2x)\,dx = \left[2x\frac{e^{tx}}{t}\right]_0^1 - \int_0^1 2\frac{e^{tx}}{t}dx = \left[2x\frac{e^{tx}}{t}\right]_0^1 - 2\left[\frac{e^{tx}}{t^2}\right]_0^1$$

$$= \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2} = \frac{2te^t - 2e^t + 2}{t^2}, \text{ for } t \neq 0,$$

and $M_{\tilde{x}}(0) = 1$. Note that

$$\lim_{t\to 0} M_{\tilde{x}}(t) = \frac{0}{0}$$

and, from l'Hôpital's rule, we get

$$\lim_{t\to 0} M_{\tilde{x}}(t) = \lim_{t\to 0}\frac{2te^t}{2t} = \lim_{t\to 0} e^t = 1.$$

The derivative of $M_{\tilde{x}}(t)$ is

$$M'_{\tilde{x}}(t) = \frac{2e^t}{t} - \frac{4e^t}{t^2} + \frac{4e^t}{t^3} - \frac{4}{t^3} = \frac{2t^2e^t - 4te^t + 4e^t - 4}{t^3} \text{ for } t \neq 0.$$

Therefore,

$$\lim_{t\to 0} M'_{\tilde{x}}(t) = \frac{0}{0}.$$

Let us apply l'Hôpital's rule,

$$\lim_{t\to 0} M'_{\tilde{x}}(t) = \lim_{t\to 0} \frac{2t^2e^t}{3t^2} = \lim_{t\to 0} \frac{2e^t}{3} = \frac{2}{3} = \mathrm{E}\left(\tilde{x}\right),$$

which agrees with what we have obtained in part (b).

(d) If either $x < 0$ or $y < 0$, it follows immediately that $F_{\tilde{x},\tilde{y}}(x, y) = 0$. For $0 < x < 1$ and $0 < y < 1$ (Region I of the figure) we get

$$F_{\tilde{x},\tilde{y}}(x, y) = \int_0^y \int_0^x \frac{4}{3}x(y + 1)dxdy = \frac{1}{3}x^2y\left(y + 2\right),$$

for $x > 1$ and $0 < y < 1$ (Region II of the figure) we get

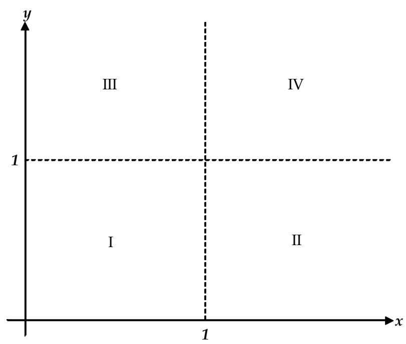
$$F_{\tilde{x},\tilde{y}}(x, y) = \int_0^y \int_0^1 \frac{4}{3}x(y + 1)dxdy = \frac{1}{3}y\left(y + 2\right),$$

for $0 < x < 1$ and $y > 1$ (Region III of the figure) we get

$$F_{\tilde{x},\tilde{y}}(x, y) = \int_0^1 \int_0^x \frac{4}{3}x(y + 1)dxdy = x^2,$$

and for $x > 1$ and $y > 1$ (Region IV of the figure) we get

$$F_{\tilde{x},\tilde{y}}(x, y) = \int_0^1 \int_0^1 \frac{4}{3}x(y + 1)dxdy = 1.$$



Since the joint distribution function is everywhere continuous, the boundaries between any two of these regions can be included in either one, and we can write

$$F_{\tilde{x},\tilde{y}}(x,y) = \begin{cases} 0 & \text{for } x \leq 0 \text{ or } y \leq 0 \\ \frac{1}{3}x^2 y\,(y+2) & \text{for } 0 < x < 1,\, 0 < y < 1 \\ \frac{1}{3}y\,(y+2) & \text{for } x \geq 1,\, 0 < y < 1 \\ x^2 & \text{for } 0 < x < 1,\, y \geq 1 \\ 1 & \text{for } x \geq 1,\, y \geq 1 \end{cases}$$

It is immediate to see that

$$\frac{\partial^2 F_{\tilde{x},\tilde{y}}(x,y)}{\partial x \partial y} = \frac{4}{3}x(y+1), \text{ for } x \in (0,1)\,,\ y \in (0,1)$$

and

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = 0, \text{ for } (x,y) \in C,$$

where C is the interior of the complement of Region I. Note that the boundary of Region I has zero Lebesgue measure. Therefore, $\frac{\partial^2 F_{\tilde{x},\tilde{y}}(x,y)}{\partial x \partial y} = f_{\tilde{x},\tilde{y}}(x,y)$ a.e. with respect to Lebesgue measure in $\mathbb{R}^2$. Note also that the discontinuities of the density function $f_{\tilde{x},\tilde{y}}(x,y)$ occur only at the boundary of Region I.

(e) We apply the LOTUS,

$$\mathrm{E}\,(\ln \tilde{x}) = \int_0^1 \ln x \cdot (2x) dx \qquad \text{(integrating by parts)}$$

$$= \left[\ln x \cdot x^2\right]_0^1 - \int_0^1 \left(\frac{1}{x} \cdot x^2\right) dx = 0 - \lim_{x \to 0} \left(\ln x \cdot x^2\right) - \int_0^1 x dx = -\frac{1}{2}$$

since

$$\lim_{x \to 0} \left(\ln x \cdot x^2\right) = \lim_{x \to 0} \left(\frac{\ln x}{x^{-2}}\right) = \frac{-\infty}{\infty},$$

and, thus, we can apply L'Hôpital's rule,

$$\lim_{x \to 0} \left(\frac{\ln x}{x^{-2}}\right) = \lim_{x \to 0} \left(\frac{\frac{1}{x}}{-2x^{-3}}\right) = \lim_{x \to 0} \left(\frac{x^2}{-2}\right) = 0,$$

and

$$\int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

(f)

$$(w,z) = g(x,y) : \begin{cases} w = \ln x \in (-\infty, 0) \\ z = \ln x - \ln y \in (w, \infty) \end{cases}$$

$$(x,y) = g^{-1}(w,z) : \begin{cases} x = e^w \in (0,1) \\ y = e^{w-z} \in (0,1) \end{cases}$$

Note that $x = e^w \in (0,1) \Longrightarrow \ln x = w \in (\ln 0, \ln 1) \Longrightarrow w \in (-\infty, 0)\,.$ Moreover, $y = e^{w-z} \in (0,1) \Longrightarrow$

$$\ln y = w - z \in (-\infty, 0) \Longrightarrow -w + z \in (0, \infty) \Longrightarrow z \in (w, \infty).$$

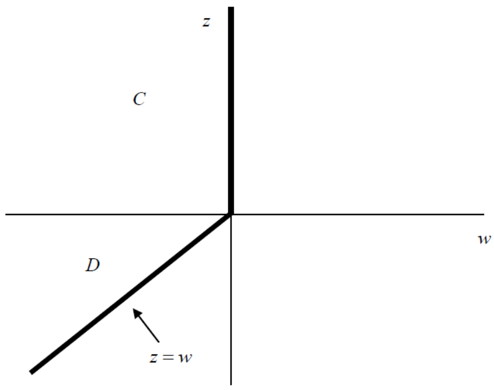
$$J_{g^{-1}}(w,z) = \begin{pmatrix} e^{w} & 0 \\ e^{w-z} & -e^{w-z} \end{pmatrix} \Longrightarrow \det J_{g^{-1}}(w,z) = -e^{2w-z} < 0$$

$$\Longrightarrow \left|\det J_{g^{-1}}\right| = e^{2w-z}$$

Then,

$$f_{\tilde{w},\tilde{z}}(w,z) = \begin{cases} \frac{4}{3}e^{w}(e^{w-z}+1)e^{2w-z} = \frac{4}{3}\left(e^{3w-z}+e^{4w-2z}\right), \text{ for } w \in (-\infty,0), \ z \in (w,\infty) \\ 0, \quad \text{otherwise.} \end{cases}$$

(g) The region of positive density for $f_{\tilde{w},\tilde{z}}(w,z)$ is $C \cup D$:



Marginal density $f_{\tilde{w}}(w)$:

$$f_{\tilde{w}}(w) = \begin{cases} \int_{w}^{\infty} \frac{4}{3}\left(e^{3w-z}+e^{4w-2z}\right) dz = 2e^{2w} \text{ for } w \in (-\infty,0) \\ 0, \quad \text{otherwise,} \end{cases}$$

Marginal density $f_{\tilde{z}}(z)$:
If $z \leq 0$, then

$$f_{\tilde{z}}(z) = \int_{-\infty}^{z} f_{\tilde{w},\tilde{z}}(w,z)dw$$
$$= \int_{-\infty}^{z} \frac{4}{3}\left(e^{3w-z}+e^{4w-2z}\right) dw = \frac{7}{9}e^{2z}.$$

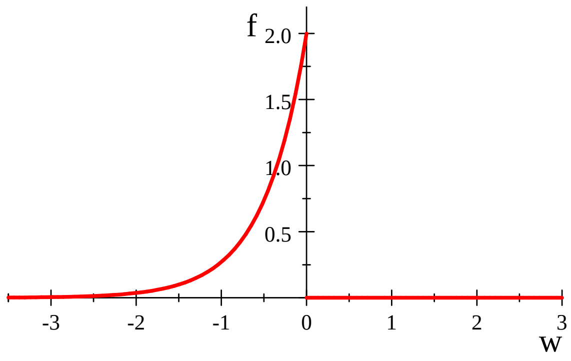
If $z > 0$, then

$$f_{\tilde{z}}(z) = \int_{-\infty}^{0} f_{\tilde{w},\tilde{z}}(w,z)dw$$
$$= \int_{-\infty}^{0} \frac{4}{3}\left(e^{3w-z}+e^{4w-2z}\right) dw = \frac{1}{9}\left(3e^{-2z}+4e^{-z}\right).$$

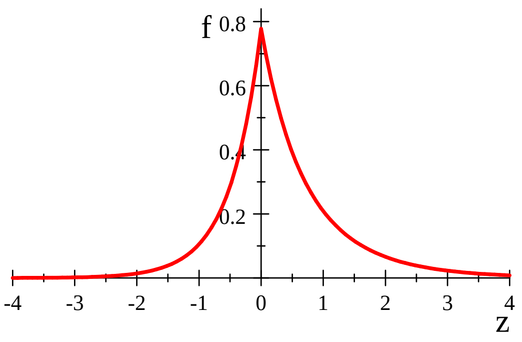
Note that $f_{\tilde{z}}(0) = \dfrac{7}{9}$ and that the density is continuous (but not differentiable) at $z = 0$. Thus,

$$f_{\tilde{z}}(z) = \begin{cases} \dfrac{7}{9}e^{2z} & \text{for } z \leq 0 \\ \dfrac{1}{9}\left(3e^{-2z} + 4e^{-z}\right) & \text{for } z > 0. \end{cases}$$

Plot of $f_{\tilde{w}}(w)$:



Plot of $f_{\tilde{z}}(z)$:



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2. (a)

$$M_{\tilde{x}}(t) = \sum_{x=1}^{\infty} e^{tx} f(x;\theta) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{\theta}\left(\frac{\theta}{\theta+1}\right)^x = \frac{1}{\theta}\sum_{x=1}^{\infty}\left(\frac{e^t\theta}{\theta+1}\right)^x$$

$$= \frac{1}{\theta}\left[\frac{\left(\frac{e^t\theta}{\theta+1}\right)}{1-\left(\frac{e^t\theta}{\theta+1}\right)}\right] = \frac{e^t}{1+\theta(1-e^t)}, \quad \text{for } \frac{e^t\theta}{\theta+1} < 1 \Longleftrightarrow t < \ln\left(\frac{\theta+1}{\theta}\right)$$

Thus, $M_x(t)$ is finite in a neighborhood of $t = 0$ since $\ln\left(\frac{\theta+1}{\theta}\right) > \ln 1 > 0$ as $\theta > 0$.

$$M'_{\tilde{x}}(t) = \frac{e^t\left[1+\theta(1-e^t)\right] + \theta e^{2t}}{[1+\theta(1-e^t)]^2} = \frac{(\theta+1)e^t}{[1+\theta(1-e^t)]^2}.$$

$$\begin{aligned} M''_{\tilde{x}}(t) &= \frac{(\theta+1)e^t\left[1+\theta(1-e^t)\right]^2 + 2\left[1+\theta(1-e^t)\right]\theta e^t(\theta+1)e^t}{[1+\theta(1-e^t)]^4} \\ &= \frac{(\theta+1)e^t\left[1+\theta(e^t+1)\right]}{[1-\theta(e^t-1)]^3}. \end{aligned}$$

$$\mathrm{E}(\tilde{x}) = M'_{\tilde{x}}(0) = \theta + 1,$$

$$\mathrm{E}(\tilde{x}^2) = M''_{\tilde{x}}(0) = (\theta+1) + 2\theta(\theta+1) = (\theta+1)(1+2\theta),$$

$$\sigma^2 = \mathrm{Var}(\tilde{x}) = \mathrm{E}(\tilde{x}^2) - [\mathrm{E}(\tilde{x})]^2 = (\theta+1)(1+2\theta) - (\theta+1)^2 = \theta\,(\theta+1)\,.$$

(b) The first non-central moment of the population (or mean) $\tilde{x}$ is $\mu'_1 = \mathrm{E}\,(\tilde{x}) = \theta + 1$. The first sample moment is $m'_1 = \frac{\sum_{i=1}^n x_i}{n} = \overline{x}$, where $\overline{x}$ is the value of the sample mean evaluated at the sample values. Thus, $\mu'_1 = m'_1$, i.e., $\theta + 1 = \overline{x}$, implies that $\hat{\theta}_{MM} = \overline{x} - 1$ or $\hat{\boldsymbol{\theta}}_{\mathbf{MM}} = \bar{\boldsymbol{x}} - 1$.

(c) The likelihood function is

$$L\,(\theta; x_1, x_2, ...x_n) = \prod_{i=1}^n f\,(x_i; \theta) = \prod_{i=1}^n \frac{1}{\theta}\left(\frac{\theta}{1+\theta}\right)^{x_i} = \left(\frac{1}{\theta}\right)^n \cdot \left(\frac{\theta}{\theta+1}\right)^{\overbrace{\sum_{i=1}^n x_i}^{n\overline{x}}}$$

$$\hat{\theta}_{ML} = \underset{\theta\in\mathbb{R}_{++}}{\arg\max}\left(\frac{1}{\theta}\right)^n \cdot \left(\frac{\theta}{\theta+1}\right)^{n\overline{x}} = \underset{\theta\in\mathbb{R}_{++}}{\arg\max}\left\{\ln\left[\left(\frac{1}{\theta}\right)^n \cdot \left(\frac{\theta}{\theta+1}\right)^{n\overline{x}}\right]\right\}$$

$$= \underset{\theta\in\mathbb{R}_{++}}{\arg\max}\left\{-n\ln\theta + n\overline{x}\cdot\ln\left(\frac{\theta}{\theta+1}\right)\right\} = \underset{\theta\in\mathbb{R}_{++}}{\arg\max}\left\{-n\ln\theta + n\overline{x}\ln\theta - n\overline{x}\ln\left(\theta+1\right)\right\}.$$

FOC:

$$\frac{d\left[-n\ln\theta + n\overline{x}\ln\theta - n\overline{x}\ln\left(\theta+1\right)\right]}{d\theta} = -\frac{n}{\theta} + \frac{n\overline{x}}{\theta} - \frac{n\overline{x}}{\theta+1} = -\frac{n}{\theta} + \frac{n\overline{x}}{\theta(\theta+1)} = 0$$

$$\Longleftrightarrow -1 + \frac{\overline{x}}{\theta+1} = 0 \Longrightarrow \hat{\theta}_{ML} = \overline{x} - 1$$

or $\hat{\boldsymbol{\theta}}_{\mathbf{ML}} = \bar{\boldsymbol{x}} - 1$.

Note that the SOC is satisfied as

$$\left.\frac{d^2\left[-n\ln\theta + n\overline{x}\ln\theta - n\overline{x}\ln\left(\theta+1\right)\right]}{d\theta^2}\right|_{\theta=\hat{\theta}_{ML}} = \left.\frac{d\left[-\frac{n}{\theta} + \frac{n\overline{x}}{\theta(\theta+1)}\right]}{d\theta}\right|_{\theta=\hat{\theta}_{ML}} = \left.\frac{n}{\theta^2} - \frac{(2\theta+1)\,n\overline{x}}{\theta^2(\theta+1)^2}\right|_{\theta=\hat{\theta}_{ML}}$$

$$= \frac{n}{(\overline{x}-1)^2} - \frac{(2(\overline{x}-1)+1)\,n\overline{x}}{(\overline{x}-1)^2\overline{x}^2} = -\frac{n}{(\overline{x}-1)\,\overline{x}} < 0 \quad \text{as } \overline{x} > 1.$$

(d) Since the variance of $\tilde{x}$ is $\sigma^2 = \theta\,(\theta+1)$ (see part (a)), we can use the invariance principle of maximum likelihood estimation so that

$$\hat{\boldsymbol{\sigma}}^{\mathbf{2}}_{\mathbf{ML}} = \hat{\boldsymbol{\theta}}_{\mathbf{ML}}\left(\hat{\boldsymbol{\theta}}_{\mathbf{ML}} + 1\right) = (\bar{\boldsymbol{x}} - 1)\,\bar{\boldsymbol{x}}.$$

(e) The statistic $\hat{\boldsymbol{\theta}} = \bar{\boldsymbol{x}} - 1$ is an unbiased estimator for θ since

$$\mathrm{E}\left(\hat{\boldsymbol{\theta}}\right) = \mathrm{E}\,(\bar{\boldsymbol{x}} - 1) = \mathrm{E}\,(\bar{\boldsymbol{x}}) - 1 = \mathrm{E}\,(\tilde{x}) - 1 = \theta + 1 - 1 = \theta.$$

The Cramér-Rao lower bound for an unbiased estimator of θ is

$$\left(-n\mathrm{E}\left[\frac{\partial^2\ln f\,(\tilde{x};\theta)}{\partial\theta^2}\right]\right)^{-1}$$

or, equivalently,

$$\left(n\mathrm{E}\left[\left(\frac{\partial\ln f\,(\tilde{x};\theta)}{\partial\theta}\right)^2\right]\right)^{-1}.$$

Since

$$\ln f(\tilde{x};\theta) = -\ln\theta + \tilde{x}\ln\left(\frac{\theta}{\theta+1}\right) = -\ln\theta + \tilde{x}\ln\theta - \tilde{x}\ln(\theta+1),$$

and

$$\frac{\partial \ln f(\tilde{x};\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{\tilde{x}}{\theta} - \frac{\tilde{x}}{\theta+1},$$

$$\left(\frac{\partial \ln f(\tilde{x};\theta)}{\partial\theta}\right)^2 = \frac{(1+\theta-\tilde{x})^2}{\theta^2(\theta+1)^2} = \frac{(\theta+1)^2+\tilde{x}^2-2(\theta+1)\tilde{x}}{\theta^2(\theta+1)^2},$$

$$\mathrm{E}\left[\left(\frac{\partial \ln\ln f(\tilde{x};\theta)}{\partial\theta}\right)^2\right] = \frac{(\theta+1)^2+\mathrm{E}\left(\tilde{x}^2\right)-2(\theta+1)\mathrm{E}(\tilde{x})}{\theta^2(\theta+1)^2}$$

$$= \frac{(\theta+1)^2+(\theta+1)(1+2\theta)-2(\theta+1)(\theta+1)}{\theta^2(\theta+1)^2} = \frac{1}{\theta(\theta+1)},$$

$$n\mathrm{E}\left[\left(\frac{\partial \ln\ln f(\tilde{x};\theta)}{\partial\theta}\right)^2\right] = \frac{n}{\theta(\theta+1)},$$

$$CR = \left(n\mathrm{E}\left[\left(\frac{\partial \ln\ln f(\tilde{x};\theta)}{\partial\theta}\right)^2\right]\right)^{-1} = \frac{\theta(\theta+1)}{n}.$$

Alternatively,

$$\frac{\partial^2 \ln\ln f(\tilde{x};\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{\tilde{x}}{\theta^2} + \frac{\tilde{x}}{(\theta+1)^2},$$

$$\mathrm{E}\left[\frac{\partial^2 \ln\ln f(\tilde{x};\theta)}{\partial\theta^2}\right] = \frac{1}{\theta^2} - \frac{\mathrm{E}(\tilde{x})}{\theta^2} + \frac{\mathrm{E}(\tilde{x})}{(\theta+1)^2}$$

$$= \frac{1}{\theta^2} - \frac{\theta+1}{\theta^2} + \frac{\theta+1}{(\theta+1)^2} = -\frac{1}{\theta(\theta+1)},$$

$$-n\mathrm{E}\left[\frac{\partial^2 \ln f(\tilde{x};\theta)}{\partial\theta^2}\right] = \frac{n}{\theta(\theta+1)},$$

$$CR = \left(-n\mathrm{E}\left[\frac{\partial^2 \ln f(\tilde{x};\theta)}{\partial\theta^2}\right]\right)^{-1} = \frac{\theta(\theta+1)}{n},$$

which is the same CR lower bound that we have found above.

$$\mathrm{Var}\left(\hat{\boldsymbol{\theta}}\right) = \mathrm{Var}(\bar{\boldsymbol{x}}-1) = \mathrm{Var}(\bar{\boldsymbol{x}}) = \frac{\mathrm{Var}(\tilde{x})}{n} = \frac{\theta(\theta+1)}{n} = CR.$$

Therefore $\hat{\boldsymbol{\theta}} = \bar{\boldsymbol{x}} - 1$ is an efficient estimator for the parameter θ.

(f) We have seen in part (e) that the statistic $\hat{\boldsymbol{\theta}} = \bar{\boldsymbol{x}} - 1$ is an unbiased estimator for θ, $\mathrm{E}\left(\hat{\boldsymbol{\theta}}\right) = \theta$, and that $\mathrm{Var}\left(\hat{\boldsymbol{\theta}}\right) = \frac{\theta(\theta+1)}{n}$, so that $\lim_{n\to\infty}\mathrm{Var}\left(\hat{\boldsymbol{\theta}}\right) = \lim_{n\to\infty}\frac{\theta(\theta+1)}{n} = 0$.

Thus,

$$\mathrm{E}\left[(\hat{\boldsymbol{\theta}}-\theta)^2\right] = \mathrm{Var}\left(\hat{\boldsymbol{\theta}}-\theta\right) + \left(\mathrm{E}\left[\hat{\boldsymbol{\theta}}-\theta\right]\right)^2$$

$$= \mathrm{Var}\left(\hat{\boldsymbol{\theta}}\right) + \left[\mathrm{E}(\hat{\boldsymbol{\theta}})-\theta\right]^2 = \frac{\theta(\theta+1)}{n} + 0 \longrightarrow 0 \text{ as } n\to\infty.$$

This means that $\hat{\boldsymbol{\theta}} \xrightarrow{m} \theta$, which implies in turn that $\hat{\boldsymbol{\theta}} \xrightarrow{p} \theta$.

(g) The density of the sample $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)$ is

$$f_{\tilde{X}}(x_1, x_2, ..., x_n; \theta) = f(x_1;\theta)\cdot f(x_2;\theta)\cdot ...\cdot f(x_n;\theta)$$

$$= \begin{cases} \prod_{i=1}^{n} \frac{1}{\theta}\left(\frac{\theta}{1+\theta}\right)^{x_i} = \left(\frac{1}{\theta}\right)^{n}\left(\frac{\theta}{\theta+1}\right)^{\overbrace{\sum_{i=1}^{n} x_i}^{n\overline{x}}}, & \text{for } x_i = 1, 2, \ldots, \\ & i = 1, ..., n \\ 0 & \text{elsewhere.} \end{cases}$$

We see that

$$f_{\tilde{X}}\left(x_1, x_2, ..., x_n; \theta\right) = h(x_1, x_2, ..., x_n) g\left(\overline{x}; \theta\right),$$

with $h(x_1, x_2, ..., x_n) = 1$ and

$$g\left(\overline{x}; \theta\right) = \left(\frac{1}{\theta}\right)^{n}\left(\frac{\theta}{\theta+1}\right)^{n\overline{x}},$$

with $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, for $x_i = 1, 2, \ldots,$ and $i = 1, ..., n$. Clearly, $h(x_1, x_2, ..., x_n)$ does not depend on θ, which means that $\overline{\boldsymbol{x}}$ is a sufficient statistic for θ.

We also see that

$$f_{\tilde{X}}\left(x_1, x_2, ..., x_n; \theta\right) = h(x_1, x_2, ..., x_n) g\left(\hat{\theta}; \theta\right),$$

with $h(x_1, x_2, ..., x_n) = 1$ and

$$g\left(\hat{\theta}; \theta\right) = \left(\frac{1}{\theta}\right)^{n}\left(\frac{\theta}{\theta+1}\right)^{n(\overline{x}-1+1)} = \left(\frac{1}{\theta}\right)^{n}\left(\frac{\theta}{\theta+1}\right)^{n(\hat{\theta}+1)},$$

with $\hat{\theta} = \overline{x} - 1 = \frac{\sum_{i=1}^{n} x_i}{n} - 1$, for $x_i = 1, 2, \ldots,$ and $i = 1, ..., n$. Clearly, $h(x_1, x_2, ..., x_n)$ does not depend on θ, which means that $\hat{\boldsymbol{\theta}}$ is also a sufficient statistic for θ.

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