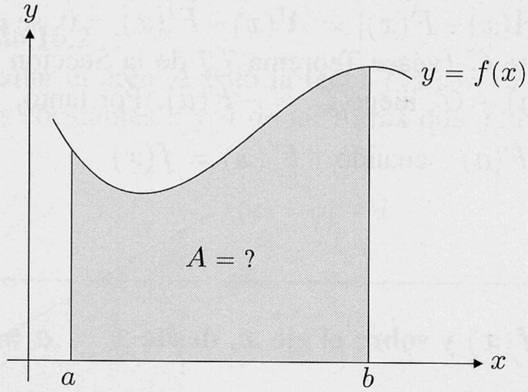
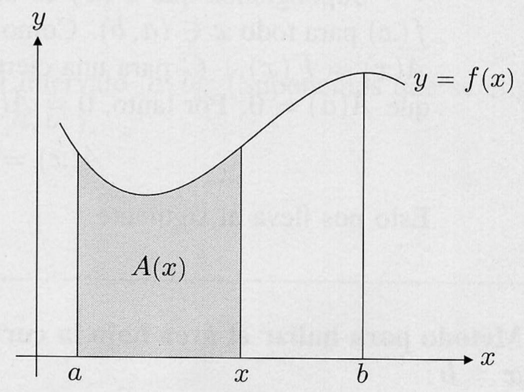
A Primer on Integration

1. The fundamental theorem of calculus

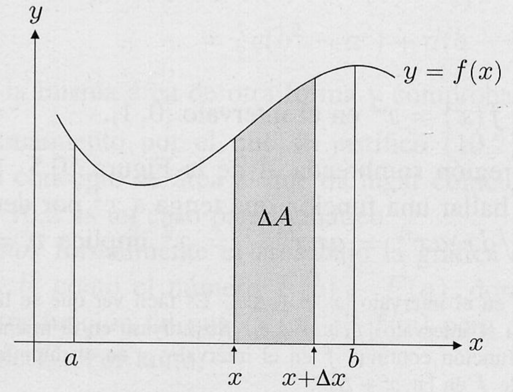
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and non-negative function ($f \geq 0$). We want to find the area A between the graph of the function and the horizontal axis on the interval $[a, b]$.



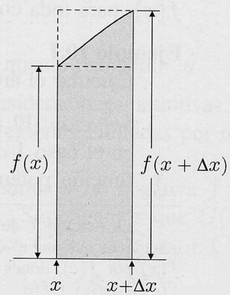
Let us fix the value a and make the area a function of the upper endpoint of the interval, $A(x)$,



Let us increase the value x by a small (infinitessimal) amount Δx. The increase in the area is $\Delta A = A(x + \Delta x) - A(x)$.



- Let us have a closer look to the increase of the area, ΔA,



- We see that

$$f(x)\cdot\Delta x \leq A(x+\Delta x)-A(x) \leq f(x+\Delta x)\cdot\Delta x, \quad \text{if } f \text{ is increasing,}$$

whereas

$$f(x)\cdot\Delta x \geq A(x+\Delta x)-A(x) \geq f(x+\Delta x)\cdot\Delta x, \quad \text{if } f \text{ is decreasing.}$$

- Note that for Δx sufficiently small, f is either increasing or decreasing on the interval $[x, x+\Delta x]$.

- Divide the three terms in the previous expression by Δx

$$f(x) \underset{\geq}{\leq} \frac{A(x+\Delta x)-A(x)}{\Delta x} \underset{\geq}{\leq} f(x+\Delta x)$$

and take the limit when $\Delta x \to 0$,

$$\lim_{\Delta x \to 0} f(x) \underset{\geq}{\leq} \underbrace{\lim_{\Delta x \to 0} \frac{A(x+\Delta x)-A(x)}{\Delta x}}_{A'(x)} \underset{\geq}{\leq} \lim_{\Delta x \to 0} f(x+\Delta x).$$

- Since f is continuous it holds that $\lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$. Therefore, we get

$$f(x) \underset{\geq}{\leq} A'(x) \underset{\geq}{\leq} f(x)$$

$$\Longrightarrow A'(x) = f(x). \qquad \textbf{(Fundamental theorem of calculus I)}$$

- Therefore, the derivative of the area $A(x)$ at the point x is equal to the function f evaluated at the point x, and this is true for all x.
- This means that the problem of finding the area on the interval $[a, x]$ is equivalent to the problem of finding a function F such that $F' = f$.
- **Definition.** F is a "primitive", "antiderivative" or "indefinite integral" of the function f if $F' = f$. We write the indefinite integral of f as

$$\int f(x)dx.$$

- Hence, the area $A(\cdot)$ (as a function of the upper endpoint x) is a "primitive", "antiderivative" or "indefinite integral" of the function f.

- **Lemma 1.** If F is a primitive of f then $G = F + C$, where C is a constant (or scalar), is also a primitive of f.
- **Lemma 2.** If F and G are primitives of f then $G(x) - F(x) = C$, for all x.

- Hence,

$$A(x) = F(x) + C, \quad \text{where } F' = f.$$

- Note that $A(a) = 0$ so that

$$0 = A(a) = F(a) + C \Longrightarrow C = -F(a).$$

  and, thus,

$$A(x) \quad = \quad F(x) - F(a), \quad \text{where } F' = f.$$

(**Fundamental theorem of calculus II**)

- In particular, the area A between the graph of the continuous and non-negative function f and the horizontal axis on the interval $[a, b]$ is

$$A = F(b) - F(a), \quad \text{where } F' = f.$$

- **Definition.** The definite integral of the continuous function f on the interval $[a, b]$ is

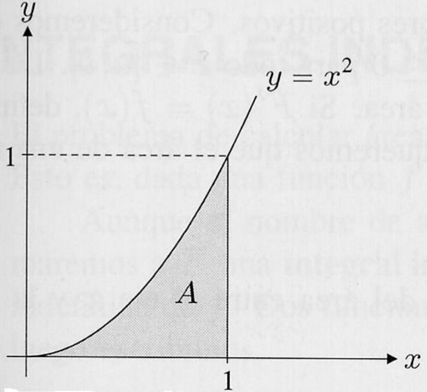
$$\int_a^b f(x)dx = F(b) - F(a) \equiv [F(x)]_a^b \equiv F(x)|_a^b, \quad \text{where } F' = f.$$

- Thus, when f is continuous and non-negative on the interval $[a, b]$, the area A between the graph of f and the horizontal axis on the interval $[a, b]$ is

$$A = \int_a^b f(x)dx.$$

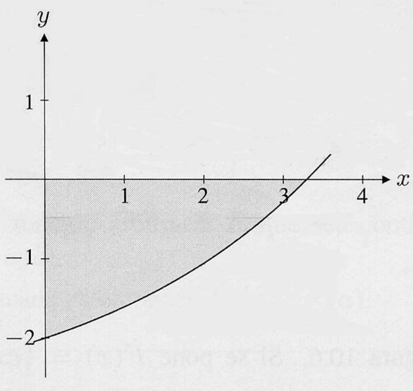
- This is the concept of integral due to Newton and Leibniz. Other more general concepts of integral are those of Riemann and of Lebesgue.

Example:



$$A = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

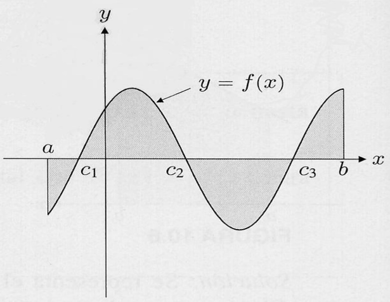
If the function f were negative,



then the area A of the shaded region will be

$$A = \int_a^b [-f(x)]\, dx = -\underbrace{\int_a^b f(x) dx}_{\text{negative}} \geq 0 \; .$$

If we want to find he area A between the graph of a continuous function f and the horizontal axis on the interval $[a, b]$, where the function takes both positive and negative values on that interval,



then the area of the shaded region will be

$$A = \underbrace{-\int_{a}^{c_1} f(x)dx}_{\text{negative}} + \underbrace{\int_{c_1}^{c_2} f(x)dx}_{\text{positive}} - \underbrace{\int_{c_2}^{c_3} f(x)dx}_{\text{negative}} + \underbrace{\int_{c_3}^{b} f(x)dx}_{\text{positive}} \geq 0.$$

2. The indefinite integral and its properties

- Recall that the indefinite integral of a continuous function f is

$$\int f(x)dx = F(x) + C, \quad \text{where } F' = f \text{ and } C \text{ is a constant.}$$

- **Properties of the indefinite integral of a continuous function:**
- **1.**

$$\int cf(x)dx = c\int f(x)dx, \quad \text{where } c \text{ is a constant.}$$

- **2.**

$$\int \left[f(x) + g(x)\right] dx = \int f(x)dx + \int g(x)dx.$$

Summing up:

$$\int \left(\sum_{n=1}^{N} \left[c_n f_n(x) \right] \right) dx = \sum_{n=1}^{N} \left[c_n \int f_n(x) dx \right], \text{ where } \{c_n\}_{n=1}^{N} \text{ are scalars.}$$

3. The definite integral and its properties

- Recall that the definite integral of the continuous function f is

$$\int_a^b f(x)dx = F(b) - F(a), \quad \text{where } F' = f.$$

- **Note:**

$$\int_a^b f(x)dx = \int_a^b f(z)dz = F(b) - F(a), \quad \text{where } F' = f.$$

- That is, the variable appearing as the argument of the function f is a "mute" variable.

Properties of the definite integral of a continuous function:

- **1.**
$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$
- **2.**
$$\int_a^a f(x)dx = 0.$$
- **3.**
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx, \quad \text{where } c \text{ is a constant.}$$
- **4.**
$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

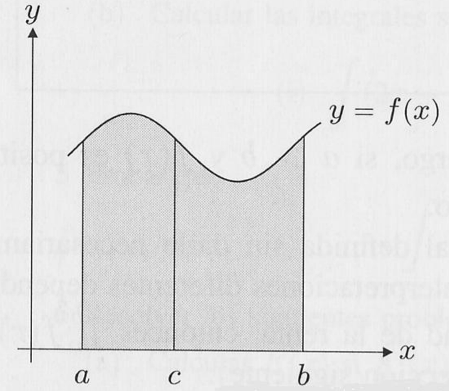
- **3 and 4 imply that**

$$\int_a^b \left(\sum_{n=1}^{N} \left[c_n f_n(x) \right] \right) dx = \sum_{n=1}^{N} \left[c_n \int_a^b f_n(x) dx \right],$$

where $\{c_n\}_{n=1}^{N}$ are scalars.

- **5.**

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$



- **6.**

$$\frac{d}{dx}\int_a^x f(z)dz = f(x)$$

and

$$\frac{d}{dx}\int_x^b f(z)dz = -f(x).$$

- **7.** Let $b > a$. If $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

- **8.**

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx \quad \text{for } b > a.$$

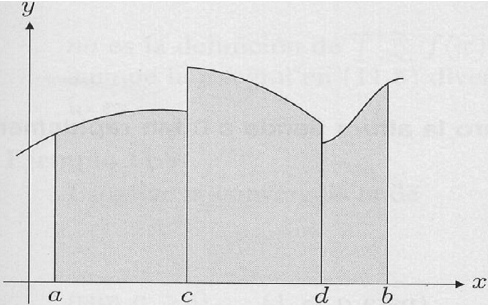
- **9. Cauchy-Schwarz Inequality:**

$$\left[\int_a^b |f(x)g(x)|\, dx\right]^2 \leq \left[\int_a^b [f(x)]^2\, dx\right] \cdot \left[\int_a^b [g(x)]^2\, dx\right]$$

or

$$\int_a^b |f(x)g(x)|\, dx \leq \left[\int_a^b [f(x)]^2\, dx\right]^{1/2} \cdot \left[\int_a^b [g(x)]^2\, dx\right]^{1/2} \text{ for } b > a.$$

- If the function f has a countable number of discontinuities on the interval $[a, b]$, then we perform the integral for each subinterval where the function is continuous and then we sum all the resulting integrals to obtain the integral on the whole interval $[a, b]$.



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx$$

- Note that it does not matter if we count a point twice since the integral on a single point is zero by property 2.

4. Differentiation of integrals

- **On the differentiability and continuity of the integral**

$$H(z) \equiv \int_a^z f(t)dt, \quad \text{where } z \in [a, b].$$

- **1.** If f is continuous at x, then H is differentiable at x and $H'(x) = f(x)$.
- **2.** If f is discontinuous at x, then H is not differentiable at x.
- **3.** The function H is continuous on $[a, b]$,

$$\lim_{z \longrightarrow x} H(z) = \lim_{z \longrightarrow x} \int_a^z f(t)dt = \int_a^x f(t)dt = H(x), \quad \text{for all } x \in [a, b].$$

- Let $f(x,y)$ be a function such that the partial derivative $\dfrac{\partial f(x,y)}{\partial y}$ exists and is continuous. Then,

$$\frac{d}{dy}\int_a^b f(x,y)dx = \int_a^b \frac{\partial f(x,y)}{\partial y}dx.$$

  Therefore, one may interchange the integral and partial differential operators.

- **Proof.**

$$\frac{d}{dy}\int_a^b f(x,y)dx = \lim_{h\to 0}\frac{\int_a^b f(x,y+h)dx - \int_a^b f(x,y)dx}{h} =$$

$$\lim_{h\to 0}\frac{\int_a^b \left[f(x,y+h) - f(x,y)\right]dx}{h} =$$

$$\int_a^b \lim_{h\to 0}\left[\frac{f(x,y+h)-f(x,y)}{h}\right]dx = \int_a^b \frac{\partial f(x,y)}{\partial y}dx.$$

- **Leibniz rule.** Let $f(x,y)$ be a function such that the partial derivative $\dfrac{\partial f(x,y)}{\partial y}$ exists and is continuous, and $a(y)$ and $b(y)$ be differentiable functions. Then,

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)dx =$$

$$\int_{a(y)}^{b(y)} \frac{\partial f(x,y)}{\partial y} dx + f(b(y),y)\cdot b'(y) - f(a(y),y)\cdot a'(y).$$

- **Proof.** Since the variable y appears thrice in the integral $\int_{a(y)}^{b(y)} f(x,y)dx$, we apply the chain rule to obtain

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)dx = \int_{a(y)}^{b(y)} \frac{\partial f(x,y)}{\partial y} dx + \left[\frac{d}{db}\int_{a(y)}^{b(y)} f(x,y)dx\right]\cdot b'(y)$$

$$+\left[\frac{d}{da}\int_{a(y)}^{b(y)} f(x,y)dx\right]\cdot a'(y) =$$

$$\int_{a(y)}^{b(y)} \frac{\partial f(x,y)}{\partial y} dx + f(b(y),y)\cdot b'(y) - f(a(y),y)\cdot a'(y).$$

5. Integration rules: immediate integrals and integration by parts

- **Immediate integrals:**

$f(x)$	$\int f(x)dx = F(x) + C$
0	C (an arbitrary constant)
x^n, with $n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$1 \left(= x^0\right)$	$x + C$

$f(x)$	$\int f(x)dx = F(x) + C$
$\frac{1}{x}\left(= x^{-1}\right)$, with $x \neq 0$	$\ln\lvert x\rvert + C$
$f'(x) \cdot [f(x)]^n$, with $n \neq -1$	$\frac{[f(x)]^{n+1}}{n+1} + C$
$\frac{f'(x)}{f(x)}\left(= f'(x) \cdot [f(x)]^{-1}\right)$, with $f(x) \neq 0$	$\ln\lvert f(x)\rvert + C$

$f(x)$	$\int f(x)dx = F(x) + C$
e^x	$e^x + C$
$f'(x) \cdot e^{f(x)}$	$e^{f(x)} + C$
a^x with $a > 0$	$\frac{a^x}{\ln a} + C$

$f(x)$	$\int f(x)dx = F(x) + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\tan x$, with $\cos x \neq 0$	$-\ln \lvert\cos x\rvert + C$
$\ln x$, with $x > 0$	$x\left[(\ln x) - 1\right] + C$

- **Examples:**
- **1.**

$$\int x\cdot\underbrace{\left(x^2+4\right)}_{f(x)}^{1/2}dx=\int\frac{1}{2}\cdot\underbrace{2x}_{f'(x)}\underbrace{\left(x^2+4\right)^{1/2}}_{[f(x)]^{1/2}}dx$$

$$=\frac{1}{2}\int\underbrace{2x}_{f'(x)}\cdot\underbrace{\left(x^2+4\right)^{1/2}}_{[f(x)]^{1/2}}dx=\frac{1}{2}\left[\frac{\left(x^2+4\right)^{3/2}}{3/2}\right]+C$$

$$=\frac{1}{3}\left(x^2+4\right)^{3/2}+C.$$

- **2.**

$$\int(\tan x)\,dx=\int\frac{\sin x}{\underbrace{\cos x}_{f(x)}}dx=\int-\left(\frac{-\sin x}{\cos x}\right)dx=-\int\frac{\overbrace{-\sin x}^{f'(x)}}{\underbrace{\cos x}_{f(x)}}dx$$

$$=-\ln\left|\cos x\right|+C,\quad\text{if }\cos x\neq 0.$$

- **3.**

$$\int e^{\overbrace{3x+2}^{f(x)}}dx = \int \frac{1}{3}\cdot \underbrace{3}_{f'(x)} e^{\overbrace{3x+2}^{f(x)}}dx = \frac{1}{3}\int \underbrace{3}_{f'(x)} e^{\overbrace{3x+2}^{f(x)}}dx$$

$$= \frac{1}{3}e^{3x+2} + C.$$

- **Integration by parts:**
- Let F and G be the primitives of the continuous functions f and g, respectively. Then,

$$\frac{d\left[F(x)\cdot G(x)\right]}{dx} = \underbrace{f(x)}_{F'(x)}\cdot G(x) + F(x)\cdot \underbrace{g(x)}_{G'(x)}.$$

- Therefore, computing the indefinite integral of both sides we get

$$F(x)\cdot G(x) + C = \int f(x)\cdot G(x)dx + \int F(x)\cdot g(x)dx$$

or

$$\int F(x)\cdot g(x)dx = F(x)\cdot G(x) - \int f(x)\cdot G(x)dx + C.$$

- For definite integrals we have

$$\int_a^b F(x) \cdot g(x) dx = [F(x) \cdot G(x)]_a^b - \int_a^b f(x) \cdot G(x) dx.$$

- **Example:** Let $x > 0$,

$$\int (\ln x)\, dx = \int \left[\underbrace{(\ln x)}_{F(x)} \cdot \underbrace{1}_{g(x)} \right] dx = ...$$

- Make $F(x) = \ln x$ and $g(x) = 1$ so that $f(x) = \dfrac{1}{x}$ and $G(x) = x$.
-

$$... = \underbrace{(\ln x)}_{F(x)} \cdot \underbrace{x}_{G(x)} - \int \underbrace{\frac{1}{x}}_{f(x)} \cdot \underbrace{x}_{G(x)} dx + C = (\ln x) \cdot x - \int 1 dx + C$$

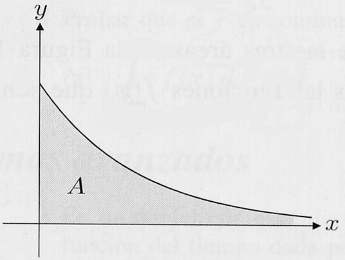
$$= (\ln x) \cdot x - x + C = x \left[(\ln x) - 1 \right] + C.$$

- Moreover, the corresponding definite integral is

$$\int_a^b (\ln x)\, dx = b \left[(\ln b) - 1 \right] - a \left[(\ln a) - 1 \right].$$

6. Improper integrals

- So far we have looked at integrals on a closed interval $[a, b]$, where a and b are real numbers.
- Let us consider integrals of continuous functions on non-closed intervals. These integrals are called **improper**.



- Integral on the interval $[a, \infty)$:

$$\int_a^\infty f(x)dx \equiv \lim_{b \to \infty} \int_a^b f(x)dx.$$

- Integral on the interval $(\infty, b]$:

$$\int_{-\infty}^{b} f(x)dx \equiv \lim_{a \longrightarrow -\infty} \int_{a}^{b} f(x)dx.$$

- Integral on the interval $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} f(x)dx \equiv \lim_{a \longrightarrow -\infty} \int_{a}^{0} f(x)dx + \lim_{b \longrightarrow \infty} \int_{0}^{b} f(x)dx.$$

- Integral on the right-semiclosed interval $(a, b]$:

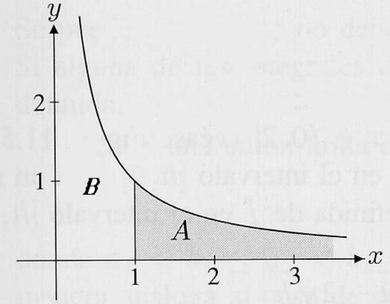
$$\int_{a^+}^{b} f(x)dx \equiv \lim_{z \longrightarrow a^+} \int_{z}^{b} f(x)dx, \quad \text{where } z > a.$$

- Integral on the left-semiclosed interval $[a, b)$:

$$\int_{a}^{b-} f(x)dx \equiv \lim_{z \longrightarrow b^-} \int_{a}^{z} f(x)dx, \quad \text{where } z < b.$$

- Integral on the open interval (a, b) :

$$\int_{a^+}^{b-} f(x)dx \equiv \underbrace{\lim_{z\to a^+}\int_z^c f(x)dx}_{B} + \underbrace{\lim_{z\to b^-}\int_c^z f(x)dx}_{A}, \quad \text{with } c \in (a, b)\,.$$



- All the previous limits might fail to exist (i.e., they could be equal to $\infty - \infty$) or be equal to $\pm\infty$. In the latter case, we say that the improper integral "diverges".
- **Examples:**
- **1.**

$$\int_{0^+}^{1} \frac{1}{x} dx = \lim_{z \to 0^+} \left[\ln|x|\right]_z^1 = \ln 1 - \lim_{z \to 0^+} \left(\ln|x|\right) = 0 - (-\infty) = \infty,$$

  so that the previous improper integral diverges.
- **2.**

$$\int_{1}^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \left[\ln|x|\right]_1^b = \lim_{b \to \infty} \left(\ln|x|\right) - \ln 1 = \infty - 0 = \infty,$$

  so that the previous improper integral diverges.
- **3.**

$$\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{b \to \infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{x}\right) - \left(-\frac{1}{1}\right) = 0 + 1 = 1.$$

7. Economic applications of integration

- **Investment and the stock of capital.**
- Under discrete time, $t = 0, 1, 2, \ldots$
- If K_t is the stock of capital at the beginning of period t, and I_t is the amount of investment during period t, we have

$$K_{t+1} = K_t + I_t \quad \text{or} \quad K_{t+1} - K_t = I_t .$$

- Therefore,

$$K_t = K_0 + \sum_{s=0}^{t-1} I_s .$$

- Under continuous time, $t \in [0, \infty)$.
- If $K(t)$ is the stock of capital at period t and $I(t)$ is the instantaneous amount of investment during the time interval $(t, t + \Delta t)$, where Δt is infinitessimal, we have

$$K(t + \Delta t) - K(t) = I(t) \cdot \Delta t$$

or

$$\frac{K(t + \Delta t) - K(t)}{\Delta t} = I(t),$$

so that

$$\lim_{\Delta t \to 0} \frac{K(t + \Delta t) - K(t)}{\Delta t} = K'(t) = I(t).$$

and, hence,

$$K(t) - K(0) = \int_0^t I(s) ds \quad \text{or} \quad K(t) = K(0) + \int_0^t I(s) ds.$$

- **Present value of income flows.**
- <u>Under discrete time, $t = 0, 1, 2, \ldots$</u>
- If $r \geq 0$ is the interest rate per period and $W_t > 0$ is the wealth at the beginning of period t, we have

$$W_{t+1} = (1+r)W_t \quad \text{or} \quad W_{t+1} - W_t = rW_t.$$

- Therefore,

$$W_t = (1+r)^t W_0 \quad \text{or} \quad W_0 = \frac{W_t}{(1+r)^t},$$

  where the latter equality means that the present value (at date $t = 0$) of having W_t euros at the future date t is $\dfrac{W_t}{(1+r)^t}$ euros.

- An implication of the previous formula is that the present value PV of a stream of income $\{y_t\}_{t=0}^{T}$ is

$$PV = \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}.$$

- If the time horizon of the stream is infinite, $T \to \infty$, then

$$PV = \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t},$$

where the previous sum could diverge.

- Under continuous time, $t \in [0, \infty)$.
- If $r \geq 0$ is the instantaneous interest rate, $W(t)$ is the wealth at date t, and Δt is an infinitessimal time length, then we have

$$W(t+\Delta t) - W(t) = rW(t)\Delta t$$

or

$$\frac{W(t+\Delta t) - W(t)}{\Delta t} = rW(t).$$

- Thus,

$$\lim_{\Delta t \to 0} \frac{W(t+\Delta t) - W(t)}{\Delta t} = W'(t) = rW(t),$$

which can be written as

$$\frac{W'(t)}{W(t)} = r.$$

- Finding the definite integral in both sides, we have

$$\int_0^t \frac{W'(s)}{W(s)} ds = \int_0^t r ds$$

so that

$$\left[\ln W(s)\right]_0^t = \left[rs\right]_0^t .$$

Thus,

$$\ln W(t) - \ln W(0) = \ln\left(\frac{W(t)}{W(0)}\right) = rt,$$

which implies that

$$\frac{W(t)}{W(0)} = e^{rt} \quad \text{or} \quad W(t) = W(0)e^{rt} \quad \text{or} \quad W(0) = \frac{W(t)}{e^{rt}} = W(t)e^{-rt},$$

where the last equality says that the present value (at date $t = 0$) of having $W(t)$ euros at the future date t is $W(t)e^{-rt}$ euros.

- An implication of the previous formula is that the present value PV of a stream of income $y(t)$ for $t \in [0, T]$ is

$$PV = \int_0^T y(t)e^{-rt}dt.$$

- If the time horizon of the stream is infinite, $T \to \infty$, then

$$PV = \int_0^\infty y(t)e^{-rt}dt,$$

  where the previous improper integral may diverge.

- Note that, if r_c is the instantaneous interest rate in continuous time and r_d is the interest rate per period in discrete time, then

$$1 + r_d = e^{r_c}$$

  so that

$$r_c = \ln(1 + r_d) \quad \text{or} \quad r_d = e^{r_c} - 1.$$

8. Integration with respect to several variables

- Assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function.
- The definite integral over a rectangle $A = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_{n-1}, b_{n-1}] \times [a_n, b_n]$ is

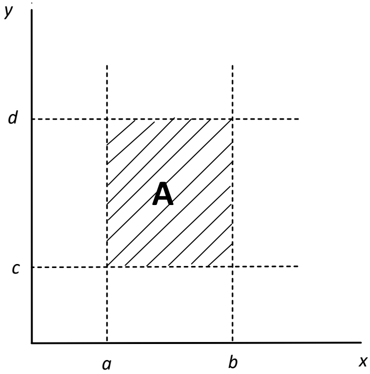
$$\int\limits_A f(x_1, x_2, ..., x_{n-1}, x_n)\, d(x_1, x_2, ..., x_{n-1}, x_n) =$$

$$\int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} .... \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, ..., x_{n-1}, x_n)\, dx_1 dx_2 ... dx_{n-1} dx_n =$$

$$\int_{a_n}^{b_n} \left[\int_{a_{n-1}}^{b_{n-1}} .... \left[\int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x_1, x_2, ..., x_{n-1}, x_n)\, dx_1 \right] dx_2 \right] ... dx_{n-1} \right] dx_n,$$

with $b_i > a_i,\ i = 1, 2, ...n$.

Example:



$$\int_A f\left(x,y\right) d\left(x,y\right), \quad \text{where } f(x,y) = x \cdot y \ \text{ and } \ A = [a,b] \times [c,d].$$

$$\int_A f(x,y)\,d(x,y) = \int_c^d \int_a^b xydxdy = \int_c^d \left[\int_a^b xydx\right] dy$$

$$= \int_c^d y \left[\int_a^b xdx\right] dy = \int_c^d y \left[\frac{x^2}{2}\right]_a^b dy$$

$$= \int_c^d y \left(\frac{b^2}{2} - \frac{a^2}{2}\right) dy = \left(\frac{b^2}{2} - \frac{a^2}{2}\right) \int_c^d ydy = \left(\frac{b^2}{2} - \frac{a^2}{2}\right) \cdot \left[\frac{y^2}{2}\right]_c^d$$

$$= \left(\frac{b^2}{2} - \frac{a^2}{2}\right) \cdot \left(\frac{d^2}{2} - \frac{c^2}{2}\right) = \frac{\left(b^2 - a^2\right) \cdot \left(d^2 - c^2\right)}{4}.$$

Properties of the multiple integral (or integral with respect to several variables) over a rectangle:

- **1.**

$$\int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} \underbrace{f_1(x_1) \cdot f_2(x_2) \cdot \ldots \cdot f_n(x_n)}_{f(x_1, x_2, \ldots, x_n)} dx_1 dx_2 \ldots dx_n =$$

$$\left[\int_{a_1}^{b_1} f_1(x_1)\, dx_1\right] \cdot \left[\int_{a_2}^{b_2} f_2(x_2)\, dx_2\right] \cdot \ldots \cdot \left[\int_{a_n}^{b_n} f_n(x_n)\, dx_n\right].$$

- **2.** Fubini's theorem:

$$\int_c^d \int_a^b f(x,y) dxdy = \int_a^b \int_c^d f(x,y) dydx.$$

- **3.**

$$\frac{\partial^n}{\partial x_1 \partial x_2 ... \partial x_n} \int_{a_n}^{x_n} ... \int_{a_2}^{x_2} \int_{a_1}^{x_1} f\left(t_1, t_2, ..., t_{n-1}, t_n\right) dt_1 dt_2 ... dt_n$$

$$= f\left(x_1, x_2, ..., x_{n-1}, x_n\right).$$

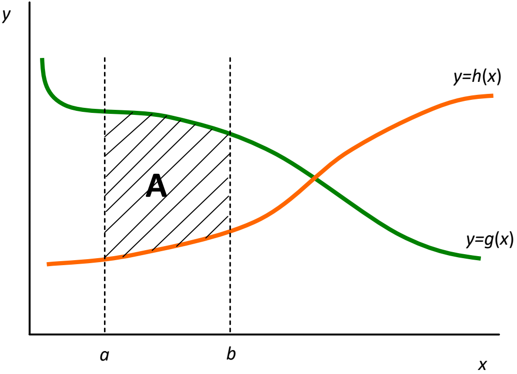
- **Integral over a non-rectangular region:**

$$\int\limits_A f\left(x_1, x_2, ..., x_{n-1}, x_n\right) d\left(x_1, x_2, ..., x_{n-1}, x_n\right),$$

where

$$A \neq \left[a_1, b_1\right] \times \left[a_2, b_2\right] \times ... \times \left[a_{n-1}, b_{n-1}\right] \times \left[a_n, b_n\right].$$

- **Example:**
- Consider the following non-rectangular region A :



$$\int_A f(x,y)\, d(x,y) = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\, dy dx$$

$$= \int_a^b \left[\int_{h(x)}^{g(x)} f(x,y)\, dy \right] dx.$$

Exercises. Probability and Statistics. IDEA.
A Primer on Integration

1. Let f be a continuous function on $[a, b]$. Prove the following integral version of the mean value theorem:

$$\text{There exists an } \varepsilon \in [a, b] \text{ such that } \int_a^b f(x)dx = f(\varepsilon)(b - a).$$

2. Let us assume that f is a continuous function on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Prove that

$$\int_a^b f(x)dx = 0$$

if and only if $f(x) = 0$ for all $x \in [a, b]$.

3. Prove the Cauchy-Schwarz inequality: Let f and g continuous functions on $[a, b]$, then it holds that

$$\left(\int_a^b |f(x)g(x)|\, dx\right)^2 \leq \left(\int_a^b [f(x)]^2\, dx\right) \cdot \left(\int_a^b [g(x)]^2\, dx\right).$$

4. Taking into account that the term being integrated is a square and, thus, it is positive, the following argument must be erroneous. Explain why.

$$\int_0^2 \frac{1}{(x-1)^2}dx = \left[-\frac{1}{x-1}\right]_0^2 = -1 - 1 = -2.$$

5. Let f and g be two continuous and differentiable functions on the real line. Find the derivative of $h(x)$ and the second derivative of $k(x)$, where

$$\text{(a) } h(x) = x^2 \int_0^x f(t)g(t)dt \qquad \text{(b) } k(x) = e^{\int_0^x f(t)g(t)dt}.$$

6. Find the derivative of $G(x)$ in the following cases:

(a) $G(x) = \int_1^e \ln(xt)\, dt$

(b) $G(x) = \int_0^{2x} t^3 dt$

(c) $G(x) = \int_{x^2}^{x} \frac{\sin(xt)}{t}\, dt \qquad (x > 0)$.

7. . Compute the following definite integrals:

$$\text{(a) } \int_0^{\ln 2} 8e^{2x+3}dx \qquad \text{(b) } \int_0^{\pi/2} \frac{\sin x}{1 + \cos x}dx \qquad \text{(c) } \int_1^2 x^2 \ln x\, dx.$$

8. Find the following primitives (or indefinite integrals or antiderivatives):

(a) $\displaystyle\int x \ln x \, dx$ (b) $\displaystyle\int \frac{1}{\tan x} dx$ (c) $\displaystyle\int \sin^3 x \, dx$

(d) $\displaystyle\int \frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}} dx$ (e) $\displaystyle\int x^2 e^{2x} dx$ (f) $\displaystyle\int e^{3x} \cos 2x \, dx$

9. Compute the following integrals (some of them could be divergent or fail to exist):

(a) $\int_{-\infty}^{\infty} \dfrac{x}{1+x^2} \, dx$ (b) $\int_{0^+}^{1} x^{-2/3} dx$

(c) $\int_{-\infty}^{0} e^{rx} dx$ (analyze the result for all r) (d) $\int_{1}^{5} \dfrac{1}{x-2} \, dx.$

10. We have seen that the present value of an infinite income stream (or flow) $y(t)$ discounted at a constant interest rate $r > 0$ is given by $\int_0^\infty y(t)e^{-rt}dt$. Let us assume that the income stream $y(t)$ has the functional form $y(t) = A\left(2^t\right)$, that is, the income in $t = 0$ is A and it doubles in each time unit. For which values of the interest rate the present value of this income flow is finite? Find the present value in this case.

11. Compute the following double integrals on the specified domains:

(a) $\int\limits_D (2x+3y+4) \; d(x,y)$, where $D = [0,1] \times [0,2]$, that is, $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

(b) $\int\limits_C \dfrac{1}{y} \; d(x,y)$, where C is the region of $\mathbb{R}^2$ such that $0 \leq x \leq y, \; 0 \leq y \leq 1, \; x+y \geq 1/2$.

(c) $\int\limits_B xy^2 \; d(x,y)$, where B is the region of $\mathbb{R}^2$ such that $0 \leq x \leq 1, \; x \leq y \leq x^2+1$.

12. It is known that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

and

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}.$$

Find $\int_0^\infty e^{-x^2} dx$ and $\int_0^\infty e^{-\frac{1}{2}x^2} dx$.

Probability and Statistics. IDEA. Answers.

A Primer on Integration

1. By the Weierstrass theorem, since f is continuous, f has a maximum M and a minimum m on $[a,b]$. Thus,

$$\int_a^b m dx = m(b-a) \leq \int_a^b f(x)dx \leq \int_a^b M dx = M(b-a),$$

so that

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M.$$

Let $K = \frac{1}{b-a}\int_a^b f(x)dx$. There exists an $\varepsilon \in [a,b]$ such that $f(\varepsilon) = K$. Then, multiplying by $b-a$ one gets the result.

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2. If $f(x) = 0$ for all x, then the integral is obviously 0. In the other direction we make the proof by contradiction, that is, we suppose that there exists an $y \in [a,b]$, such that $f(y) > 0$. Then, since f is continuous, there exists an interval $[c,d] \subset [a,b]$ around y such that f is strictly positive on that interval. Then,

$$\int_a^b f(x)dx \geq \int_c^d f(x)dx = f(\varepsilon)(d-c) > 0,$$

by Exercise 1, where $\varepsilon \in [c,d]$.

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3. For every constant A and any pair of continuous functions f and g on $[a,b]$, we have

$$\int_a^b \left[A\left|f(x)\right| + \left|g(x)\right|\right]^2 dx \geq 0$$

and, thus,

$$\int_a^b A^2 \left|f(x)\right|^2 dx + \int_a^b 2A \left|f(x)\right| \left|g(x)\right| dx + \int_a^b \left|g(x)\right|^2 dx \geq 0$$

or

$$A^2 \underbrace{\int_a^b \left[f(x)\right]^2 dx}_{c} + A \cdot \underbrace{2\int_a^b \left|f(x)g(x)\right| dx}_{d} + \underbrace{\int_a^b \left[g(x)\right]^2 dx}_{h} \geq 0.$$

This means that the above second order polynomial of A can have 1 or 0 roots. That is, the discriminant $D = d^2 - 4ch$ has to be less than or equal to 0. Therefore,

$$\left[2\int_a^b \left|f(x)g(x)\right| dx\right]^2 - 4\left(\int_a^b \left[f(x)\right]^2 dx\right)\left(\int_a^b \left[g(x)\right]^2 dx\right) \leq 0.$$

Rearranging the above inequality and dividing by 4, one obtains the Cauchy-Schwarz inequality.

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4. The function that is being integrated is discontinuous at $x = 1$ (in fact, it is not finite at $x = 1$). Then,

$$\int_0^2 \frac{1}{(x-1)^2} dx = \int_0^{1-} \frac{1}{(x-1)^2} dx + \int_{1^+}^2 \frac{1}{(x-1)^2} dx =$$

$$\lim_{b \to 1^-} \left[-(x-1)^{-1}\right]_0^b + \lim_{a \to 1^+} \left[-(x-1)^{-1}\right]_a^2 = \lim_{b \to 1^-} \left[-(b-1)^{-1}\right] - 1 - 1 + \lim_{a \to 1^+} \left[(a-1)^{-1}\right].$$

Note that

$$\lim_{b \to 1^-} \left[-(b-1)^{-1}\right] = \lim_{b \to 1^-} \left[(1-b)^{-1}\right] = \infty \text{ and } \lim_{a \to 1^+} \left[(a-1)^{-1}\right] = \infty.$$

Therefore, the integral diverges (it tends to ∞).

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5. (a)

$$h'(x) = 2x\int_0^x f(t)g(t)dt + x^2 f(x)g(x).$$

(b) Let $L(x) = \int_0^x f(t)g(t)dt$

$$k'(x) = L'(x)e^{L(x)},$$

and $L'(x) = f(x)g(x)$. Then,

$$k''(x) = e^{\int_0^x f(t)g(t)dt}[f(x)g(x)]^2 + e^{\int_0^x f(t)g(t)dt}[f'(x)g(x) + f(x)g'(x)].$$

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6. A straightforward application of the Leibniz's rule gives the results:

$$\text{(a) } G'(x) = \int_1^e \frac{1}{x}dt = \frac{e-1}{x}, \qquad \text{(b) } G'(x) = (2x)^3 \cdot 2 = 16x^3.$$

(c) Denote $f(x,t) = \dfrac{\sin xt}{t}$. Then,

$$G'(x) = f(x,x)\cdot 1 - f(x,x^2)\cdot 2x + \int_{x^2}^x f_x(x,t)dt,$$

where $f_x(x,t) = \cos(xt)$ is the partial derivative of f with respect to t. Hence, we get

$$G'(x) = \frac{\sin x^2}{x} - \frac{2\sin x^3}{x} + \frac{\sin x^2}{x} - \frac{\sin x^3}{x} = \frac{2\sin x^2}{x} - \frac{3\sin x^3}{x}.$$

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7. (a) $\int_0^{\ln 2} 8e^{2x+3}dx = 8e^3\int_0^{\ln 2} e^{2x}dx = 8e^3\int_0^{\ln 2}\frac{1}{2}2e^{2x}dx = \dfrac{8}{2}e^3\int_0^{\ln 2} 2e^{2x}dx$

$= 4e^3\left[e^{2x}\right]_0^{\ln 2} = 4e^3\left(e^{2\ln 2} - e^0\right) = 4e^3\left(e^{\ln 2^2} - e^0\right) = 4e^3\left(e^{\ln 4} - e^0\right)$

$= 4e^3(4-1) = 12e^3.$

We have used here the fact that a primitive of $f'(x)e^{f(x)}$ is $e^{f(x)}$.

(b) $\int_0^{\pi/2} \dfrac{\sin x}{1+\cos x}dx = [-\ln|1+\cos x|]_0^{\pi/2} = [-\ln(1+\cos x)]_0^{\pi/2} = \ln 2.$ Observe that $1+\cos x > 0$ for $x \in [0, \pi/2]$.

(c) Integral by parts, with $f(x) = x^2$ and $G(x) = \ln x$:

$$\left[\frac{x^3}{3}\ln x\right]_1^2 - \int_1^2 \frac{x^3}{3}\frac{1}{x}dx = \frac{8}{3}\ln 2 - \frac{7}{9}.$$

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8. (a) Integrating by parts, $f(x) = x$ and $G(x) = \ln x$, to obtain the primitive,

$$\frac{x^2}{2}\ln x - \int \frac{x}{2}dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C.$$

(b) Given that $\tan x = \dfrac{\sin x}{\cos x}$ so that $\dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$, the integrand has the form $\dfrac{f'(x)}{f(x)}$ and the primitive is thus

$$\ln|\sin x| + C,$$

which is well defined if $\sin x \neq 0$.

We have used here the fact that the primitive of $\dfrac{f'(x)}{f(x)}$ is $\ln|f(x)| + C$.

(c) We see that

$$\sin^3 x = \sin x \cdot \sin^2 x = \sin x \cdot \left(1 - \cos^2 x\right) = \sin x - \sin x \cos^2 x.$$

Therefore, the primitive is simply

$$-\cos x + \frac{\cos^3 x}{3} + C.$$

We have used here the fact that the primitive of $f'(x)\,[f(x)]^n$ is $\dfrac{[f(x)]^{n+1}}{n+1} + C$.

(d) Obvious, after dividing both the numerator and the denominator by $\sqrt{x^2+4}$. The result is

$$x + \sqrt{x^2+4} + C,$$

given that $\dfrac{x}{\sqrt{x^2+4}} = x\left(x^2+4\right)^{-1/2} = \dfrac{1}{2}2x\left(x^2+4\right)^{-1/2}$ is the derivative of $\sqrt{x^2+4} = \left(x^2+4\right)^{1/2} = \dfrac{1}{2}\dfrac{\left(x^2+4\right)^{1/2}}{1/2}$.

We have also used here the fact that the primitive of $f'(x)\left[f(x)\right]^n$ is $\dfrac{\left[f(x)\right]^{n+1}}{n+1} + C$.

(e) Applying twice the method of integral by parts, that is, first $G(x) = x^2$ and $f(x) = e^{2x}$, and second $G(x) = x$ and $f(x) = e^{2x}$, one gets the primitive

$$\frac{e^{2x}}{2}\left(x^2 - x + \frac{1}{2}\right) + C.$$

(f) Integrate by parts with $G(x) = \cos(2x)$ and $f(x) = e^{3x}$. Then,

$$\int e^{3x}\cos(2x)\,dx = \frac{e^{3x}}{3}\cos(2x) + \frac{2}{3}\int e^{3x}\sin(2x)\,dx.$$

Next, integrate by parts $\int e^{3x}\sin(2x)\,dx$ similarly. Then,

$$\int e^{3x}\cos(2x)\,dx = \frac{e^{3x}}{3}\cos(2x) + \frac{2}{3}\left(\frac{e^{3x}}{3}\sin(2x) - \frac{2}{3}\int e^{3x}\cos(2x)\,dx\right).$$

Solving the equation for $\int e^{3x}\cos(2x)\,dx$ one gets

$$\int e^{3x}\cos(2x)\,dx = \frac{3}{13}e^{3x}\cos(2x) + \frac{2}{13}e^{3x}\sin(2x) + C.$$

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9. (a)

$$\lim_{b\to\infty}\left[\frac{1}{2}\ln\left|1+x^2\right|\right]_0^b - \lim_{a\to-\infty}\left[\frac{1}{2}\ln\left|1+x^2\right|\right]_a^0 = \infty,$$

so that the integral diverges.

(b)

$$\lim_{a\to0^+}\left[3x^{\frac{1}{3}}\right]_a^1 = 3$$

(c)

$$\frac{1}{r} - \lim_{a\to-\infty}\frac{e^{ra}}{r} = \frac{1}{r}, \quad \text{if } r > 0,$$

and the integral diverges otherwise.

(d) The function is discontinuous at $x = 2$. In fact, it is not defined at $x = 2$. Splitting the interval in two sub-intervals, we have:

$$\int_1^{5^-} \frac{1}{x-2} dx = \int_{2^+}^{5} \frac{1}{x-2} dx + \int_1^{2^-} \frac{1}{x-2} dx =$$

$$\ln 3 - \lim_{a \to 2^+} (\ln |a-2|) + \lim_{b \to 2^-} (\ln |b-2|) - \underbrace{\ln 1}_{0} = \infty - \infty,$$

which is not well defined.

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10.

$$\int_0^{\infty} A(2e^{-r})^t dt = \frac{A}{\ln 2 - r} \lim_{b \to \infty} \left[\left(2e^{-r}\right)^t \right]_0^b$$

If $r > \ln 2$ (or $2e^{-r} < 1$), then the integral is equal to

$$\frac{A}{r - \ln 2},$$

and it diverges otherwise.

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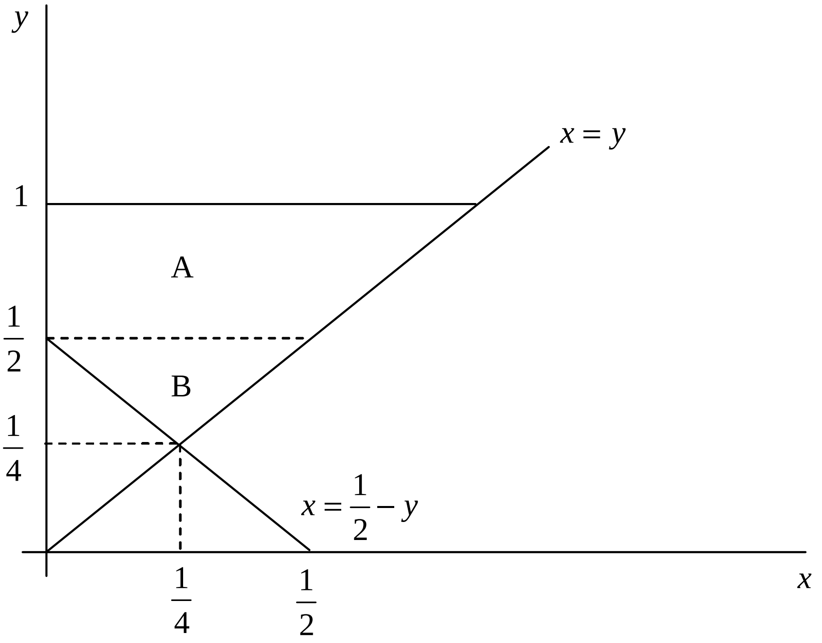
11. (a) Calculating the double integral, we get

$$\int_0^2 \left[\int_0^1 (2x + 3y + 4)\, dx \right] dy = \int_0^2 (3y + 5)\, dy = 16.$$

(b) The integral over region C is the sum of the integrals over regions A and B:



$$\int_A f(x,y)d(x,y)+\int_B f(x,y)d(x,y) = \int_{1/2}^{1}\int_{0}^{y}\frac{1}{y}dxdy+\int_{1/4}^{1/2}\int_{\frac{1}{2}-y}^{y}\frac{1}{y}dxdy = 1-\frac{\ln 2}{2} = 0.65343.$$

Note that

$$\int_{\frac{1}{2}}^{1}\int_{0}^{y}\frac{1}{y}dxdy = \int_{\frac{1}{2}}^{1}\left[\frac{1}{y}x\right]_0^y dy = \int_{1/2}^{1}1dy = [y]_{1/2}^{1} = 1-\frac{1}{2} = \frac{1}{2}$$

and

$$\begin{aligned}\int_{\frac{1}{4}}^{\frac{1}{2}}\int_{\frac{1}{2}-y}^{y}\frac{1}{y}dxdy &= \int_{\frac{1}{4}}^{\frac{1}{2}}\left[\frac{1}{y}x\right]_{\frac{1}{2}-y}^{y}dy = \int_{\frac{1}{4}}^{\frac{1}{2}}\left[\frac{y}{y}-\frac{\left(\frac{1}{2}-y\right)}{y}\right]dy \\ &= \int_{\frac{1}{4}}^{\frac{1}{2}}\left[1-\frac{1}{2y}+1\right]dy = \left[2y-\frac{1}{2}\ln y\right]_{1/4}^{1/2} = \frac{1}{2}-\frac{1}{2}\ln 2.\end{aligned}$$

(c) The integral can be simply written as

$$\int_0^1\int_x^{x^2+1}xy^2dydx = \frac{67}{120}.$$

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12. (a) Since e^{-x^2} is a symmetric function with respect to 0, we get

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

Similarly, since $e^{-\frac{1}{2}x^2}$ is also a symmetric function with respect to 0, we get

$$\int_0^\infty e^{-\frac{1}{2}x^2}dx = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}.$$

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