ANSWER
Probability and Statistics. IDEA-UAB.
Final Exam 2022-23. Prof. J. Caballé.

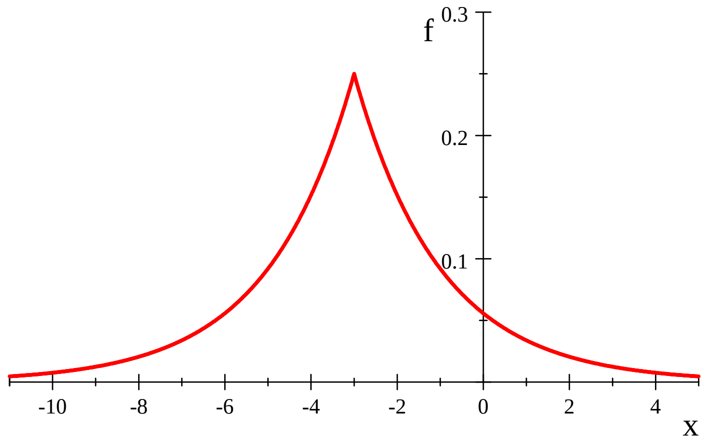
1. (a)

$$f(x;\mu,\theta)=\begin{cases}\dfrac{1}{2\theta}e^{(x-\mu)/\theta} & \text{for } x\leq\mu\\[2ex] \dfrac{1}{2\theta}e^{-(x-\mu)/\theta} & \text{for } x>\mu.\end{cases}$$

For $\mu=-3$ and $\theta=2$,

$$f(x;\mu,\theta)=\begin{cases}\dfrac{1}{4}e^{(x+3)/2} & \text{for } x\leq -3\\[2ex] \dfrac{1}{4}e^{-(x+3)/2} & \text{for } x>-3.\end{cases}$$

Note that this density achieves its maximum value $(1/4=0.25)$ when $x=-3$. This density is continuous everywhere but not differentiable at $x=-3$ since its left derivative is $1/8$ and the right derivative is $-1/8$ at $x=-3$. Moreover, the second derivative of the density is strictly positive for both $x<-3$ and $x>-3$.



(b) For $x\leq\mu$,

$$F(x)=\int_{-\infty}^{x}\frac{1}{2\theta}e^{(x-\mu)/\theta}dx=\frac{e^{-\mu/\theta}}{2\theta}\int_{-\infty}^{x}e^{x/\theta}dx=\frac{e^{-\mu/\theta}}{2\theta}\left[\theta e^{x/\theta}\right]_{-\infty}^{x}=\frac{e^{-\mu/\theta}}{2}e^{x/\theta}=\frac{1}{2}e^{(x-\mu)/\theta}$$

For $x>\mu$,

$$F(x)=F(\mu)+\int_{\mu}^{x}\frac{1}{2\theta}e^{-(x-\mu)/\theta}dx=\frac{1}{2}+\frac{e^{\mu/\theta}}{2\theta}\int_{\mu}^{x}e^{-x/\theta}dx=\frac{1}{2}+\frac{e^{\mu/\theta}}{2\theta}\left[-\theta e^{-x/\theta}\right]_{\mu}^{x}=$$

$$\frac{1}{2}+\frac{e^{\mu/\theta}}{2}\left[-e^{-x/\theta}\right]_{\mu}^{x}=\frac{1}{2}+\frac{e^{\mu/\theta}}{2}\left[-e^{-x/\theta}+e^{-\mu/\theta}\right]=\frac{1}{2}-\frac{1}{2}e^{-(x-\mu)/\theta}+\frac{1}{2}=1-\frac{1}{2}e^{-(x-\mu)/\theta}.$$

Thus,

$$F(x)=\begin{cases}\dfrac{1}{2}e^{(x-\mu)/\theta} & \text{for } x\leq\mu\\[2ex] 1-\dfrac{1}{2}e^{-(x-\mu)/\theta} & \text{for } x>\mu\end{cases}$$

The cdf is differentiable everywhere since the density is continuous. However, the cdf is not twice differentiable at $x = \mu$ since the density is not differentiable at $x = \mu$.

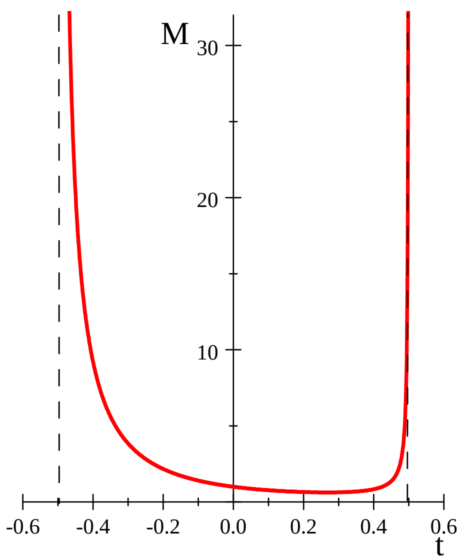
(c) For t sufficiently close to zero (i.e., when $t \in (-1/\theta, 1/\theta)$ so that $t + \dfrac{1}{\theta} > 0$ and $t - \dfrac{1}{\theta} < 0$), we have

$$\begin{aligned}
M_{\tilde{x}}(t) &= \mathrm{E}\left(e^{t\tilde{x}}\right) = \frac{1}{2\theta}\left[\int_{-\infty}^{\mu} e^{tx} e^{(x-\mu)/\theta} dx + \int_{\mu}^{\infty} e^{tx} e^{-(x-\mu)/\theta} dx\right] \\
&= \frac{1}{2\theta}\left[e^{-\mu/\theta}\int_{-\infty}^{\mu} e^{x\left(t+\frac{1}{\theta}\right)} dx + e^{\mu/\theta}\int_{\mu}^{\infty} e^{x\left(t-\frac{1}{\theta}\right)} dx\right] \\
&= \frac{1}{2\theta}\left\{e^{-\mu/\theta}\left[\frac{e^{x\left(t+\frac{1}{\theta}\right)}}{t+\frac{1}{\theta}}\right]_{-\infty}^{\mu} + e^{\mu/\theta}\left[\frac{e^{x\left(t-\frac{1}{\theta}\right)}}{t-\frac{1}{\theta}}\right]_{\mu}^{\infty}\right\} \\
&= \frac{1}{2\theta}\left\{e^{-\mu/\theta}\left[\frac{e^{\mu\left(t+\frac{1}{\theta}\right)}}{t+\frac{1}{\theta}}\right] - e^{\mu/\theta}\left[\frac{e^{\mu\left(t-\frac{1}{\theta}\right)}}{t-\frac{1}{\theta}}\right]\right\} \\
&= \frac{e^{\mu t}}{2\theta}\left\{\frac{\theta e^{-\mu/\theta} e^{\mu/\theta}}{\theta t + 1} - \frac{\theta e^{\mu/\theta} e^{-\mu/\theta}}{\theta t - 1}\right\} = \frac{e^{\mu t}}{2}\left\{\frac{1}{1+\theta t} + \frac{1}{1-\theta t}\right\} \\
&= \frac{e^{\mu t}}{2}\left[\frac{1-\theta t + 1 + \theta t}{(1+\theta t)(1-\theta t)}\right] = \frac{e^{\mu t}}{1-\theta^2 t^2}.
\end{aligned}$$

$M_{\tilde{x}}(t)$ with $\mu = -3$ and $\theta = 2$ becomes

$$M_{\tilde{x}}(t) = \frac{e^{-3t}}{1-4t^2},$$

which has two asymptotes at $t = 1/\theta = 1/2$ and at $t = -1/\theta = -1/2$. It has negative slope at $t = 0$ since $M_{\tilde{x}}'(0) = \mathrm{E}(\tilde{x}) = \mu = -3 < 0$ (see next part (d)) and satisfies $M_{\tilde{x}}(0) = 1$.



(d)

$$\begin{aligned}
\mathrm{E}(\tilde{x}) &= M'_{\tilde{x}}(0) = \frac{\mu e^{\mu t}}{1-\theta^2 t^2} + e^{\mu t} 2\theta^2 t \left(1-\theta^2 t^2\right)^{-2} \Big|_{t=0} = \mu M_{\tilde{x}}(t) + M_{\tilde{x}}(t) \frac{2\theta^2 t}{1-\theta^2 t^2}\Big|_{t=0} \\
&= M_{\tilde{x}}(t)\left(\mu + \frac{2\theta^2 t}{1-\theta^2 t^2}\right)\Big|_{t=0} = 1\cdot(\mu - 0) = \mu.
\end{aligned}$$

In fact, we already know that $\mathrm{E}(\tilde{x}) = \mu$ from the symmetry of $f(x;\mu,\theta)$ with respect to μ.

$$\begin{aligned}
\mathrm{Var}(\tilde{x}) &= \sigma^2 = \mathrm{E}\left(\tilde{x}^2\right) - \mu^2 = M''_{\tilde{x}}(0) - \mu^2 \\
&= M'_{\tilde{x}}(t)\left(\mu + \frac{2\theta^2 t}{1-\theta^2 t^2}\right) + M_{\tilde{x}}(t)\left[\frac{2\theta^2\left(1-\theta^2 t^2\right) + (2\theta^2 t)^2}{\left(1-\theta^2 t^2\right)^2}\right]\Bigg|_{t=0} - \mu^2 \\
&= (\mu\cdot\mu) + \left(1\cdot 2\theta^2\right) - \mu^2 = 2\theta^2.
\end{aligned}$$

(e) If $\tilde{z} = |\tilde{x}-\mu| \geq 0$, then

$$\begin{aligned}
F_{\tilde{z}}(z) &= P\{\tilde{z} \leq z\} = P\{|\tilde{x}-\mu| \leq z\} = P\{-z \leq \tilde{x}-\mu \leq z\} = P\{-z+\mu \leq \tilde{x} \leq z+\mu\} \\
&= P_{\tilde{x}}[-z+\mu, z+\mu] = F_{\tilde{x}}(z+\mu) - \lim_{s\to -z^-} F_{\tilde{x}}(s+\mu), \text{ for } z \geq 0.
\end{aligned}$$

and $F_{\tilde{z}}(z) = 0$ for $z < 0$. Since $\tilde{x}$ is absolutely continuous, the cdf $F_{\tilde{x}}$ is continuous so that the distribution function of $\tilde{z} = |\tilde{x}-\mu|$ becomes

$$F_{\tilde{z}}(z) = \begin{cases} F_{\tilde{x}}(z+\mu) - F_{\tilde{x}}(-z+\mu) & \text{for } z \geq 0 \\ 0 & \text{for } z < 0, \end{cases}$$

and its density is

$$f_{\tilde{z}}(z) = \begin{cases} f(z+\mu;\mu,\theta) + f(-z+\mu;\mu,\theta) & \text{for } z > 0 \\ 0 & \text{for } z \leq 0. \end{cases}$$

Note that

$$f(z+\mu;\mu,\theta) = \frac{1}{2\theta}e^{-|z+\mu-\mu|/\theta} = \frac{1}{2\theta}e^{-|z|/\theta} = \frac{1}{2\theta}e^{-z/\theta} \text{ for } z > 0,$$

and

$$f(-z+\mu;\mu,\theta) = \frac{1}{2\theta}e^{-|-z+\mu-\mu|/\theta} = \frac{1}{2\theta}e^{-|-z|/\theta} = \frac{1}{2\theta}e^{-z/\theta} \text{ for } z > 0.$$

Therefore, $f(z+\mu;\mu,\theta) = f(-z+\mu;\mu,\theta)$, which also follows obviously from the symmetry of the density function $f(z;\mu,\theta)$ with respect to μ. Thus,

$$f_{\tilde{z}}(z) = \begin{cases} 2f(z+\mu;\mu,\theta) = \dfrac{1}{\theta}e^{-z/\theta} & \text{for } z > 0 \\ 0 & \text{for } z \leq 0. \end{cases}$$

$$\begin{aligned}
\mathrm{E}(\tilde{z}) &= \int_{-\infty}^{\infty} z\, f_{\tilde{z}}(z)dx = \int_{-\infty}^{0} z\cdot 0\cdot dx + \int_0^{\infty} \underbrace{z}_{H(z)} \underbrace{\frac{1}{\theta}e^{-z/\theta}}_{g(z)} dz \\
&= 0 + [H(z)G(z)]_0^{\infty} - \int_0^{\infty} h(z)G(z)dz
\end{aligned}$$

$$= \underbrace{\left[-ze^{-z/\theta}\right]_0^\infty}_{-\lim\limits_{z\to\infty}\left(\frac{z}{e^{z/\theta}}\right)+0=-\underbrace{\lim\limits_{z\to\infty}\left(\frac{1}{\frac{1}{\theta}e^{z/\theta}}\right)}_{\uparrow\ \text{L'Hôpital's rule}}=0} - \int_0^\infty \underbrace{1}_{h(z)} \cdot \underbrace{\left(-e^{-z/\theta}\right)}_{G(z)} dx = \left[\frac{e^{-z/\theta}}{-1/\theta}\right]_0^\infty = \theta,$$

where $H' = h$ and $G' = g$.

(f)

$$\begin{aligned} P\left(\{\tilde{x} \in (-4,1)\} | \{\tilde{x} \in (-5,-3)\}\right) &= \frac{P\left(\{\tilde{x} \in (-4,1)\} \cap \{\tilde{x} \in (-5,-3)\}\right)}{P\{\tilde{x} \in (-5,-3)\}} \\ &= \frac{P\{\tilde{x} \in (-4,-3)\}}{P\{\tilde{x} \in (-5,-3)\}}. \end{aligned}$$

Since the cdf of $\tilde{x}$ is $F(x) = \frac{1}{2}e^{(x+3)/2}$ for $x \leq -3$,

$$P\{\tilde{x} \in (-4,-3)\} = F(-3) - F(-4) = \frac{1}{2}e^{(-3+3)/2} - \frac{1}{2}e^{(-4+3)/2} = \frac{1}{2} - \frac{1}{2}e^{-1/2}.$$

$$P\{\tilde{x} \in (-5,-3)\} = F(-3) - F(-5) = \frac{1}{2}e^{(-3+3)/2} - \frac{1}{2}e^{(-5+3)/2} = \frac{1}{2} - \frac{1}{2}e^{-1}.$$

$$P\left(\{\tilde{x} \in (-4,1)\} | \{\tilde{x} \in (-5,-3)\}\right) = \frac{\frac{1}{2} - \frac{1}{2}e^{-1/2}}{\frac{1}{2} - \frac{1}{2}e^{-1}} = \frac{1 - e^{-1/2}}{1 - e^{-1}} = 0.6225.$$

(g) The method of moments consists of solving the following equations for the two parameters we want to estimate:

$$m'_k = \mu'_k, \quad k = 1,2$$

where

$$m'_k = \frac{\sum_{i=1}^n x_i^k}{n},$$

and $\mu'_k = \mu$ and $\mu'_2 = \mathrm{E}\left(\tilde{x}^2\right) = \mu^2 + \sigma^2 = \mu^2 + 2\theta^2$ as follows from part (d). Thus, we solve for μ and θ in

$$\frac{\sum_{i=1}^n x_i}{n} = \mu$$

and

$$\frac{\sum_{i=1}^n x_i^2}{n} = \mu^2 + 2\theta^2.$$

The solution is

$$\hat{\mu}_{MM} = \frac{\sum_{i=1}^n x_i}{n} \Longrightarrow \hat{\boldsymbol{\mu}}_{\mathbf{MM}} = \frac{\sum_{i=1}^n \tilde{x}_i}{n}$$

and

$$\begin{aligned} \hat{\theta}_{MM} &= \left[\frac{1}{2}\left(\frac{\sum_{i=1}^n x_i^2}{n} - \left[\frac{\sum_{i=1}^n x_i}{n}\right]^2\right)\right]^{1/2} \\ &\Longrightarrow \hat{\boldsymbol{\theta}}_{\mathbf{MM}} = \left[\frac{1}{2}\left(\frac{\sum_{i=1}^n \tilde{x}_i^2}{n} - \left[\frac{\sum_{i=1}^n \tilde{x}_i}{n}\right]^2\right)\right]^{1/2} \end{aligned}$$

(h)

$$CR = \left[-n\mathrm{E}\left(\frac{\partial^2 \ln f\left(\tilde{x};\theta\right)}{\partial\theta^2}\right)\right]^{-1}$$

$$f(\tilde{x};\mu,\theta)=\frac{1}{2\theta}e^{-|\tilde{x}-\mu|/\theta}$$

$$\ln f(\tilde{x};\mu,\theta)=-\ln 2-\ln\theta-\frac{|\tilde{x}-\mu|}{\theta}$$

$$\frac{\partial \ln f(\tilde{x};\theta)}{\partial\theta}=-\frac{1}{\theta}+\frac{|\tilde{x}-\mu|}{\theta^2}$$

$$\frac{\partial^2 \ln f(\tilde{x};\theta)}{\partial\theta^2}=\frac{1}{\theta^2}-\frac{2|\tilde{x}-\mu|}{\theta^3}$$

$$\mathrm{E}\left[\frac{\partial^2 \ln f(\tilde{x};\theta)}{\partial\theta^2}\right]=\frac{1}{\theta^2}-\frac{2\mathrm{E}(|\tilde{x}-\mu|)}{\theta^3}=\frac{1}{\theta^2}-\frac{2\theta}{\theta^3}=\frac{1}{\theta^2}-\frac{2}{\theta^2}=-\frac{1}{\theta^2}$$

$$-n\mathrm{E}\left[\frac{\partial^2 \ln f(\tilde{x};\theta)}{\partial\theta^2}\right]=\frac{n}{\theta^2}.$$

$$CR=\left[-n\mathrm{E}\left(\frac{\partial^2 \ln f(\tilde{x};\theta)}{\partial\theta^2}\right)\right]^{-1}=\frac{\theta^2}{n}.$$

Note that $\mathrm{E}(|\tilde{x}-\mu|)=\mathrm{E}(\tilde{z})=\theta$ as follows from part (e).

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2. The partition induced on Ω by $\tilde{x}$ is $F(\tilde{x})=\{\{1,2\},\{3,4,5,6\}\}$. The σ-algebra generated by the partition $F(\tilde{x})$ is the σ-algebra induced on Ω by $\tilde{x}$,

$$\mathcal{F}(\tilde{x})=\{\emptyset,\Omega,\{1,2\},\{3,4,5,6\}\}$$

$$\mathrm{E}(\tilde{x}\,|\mathcal{F}(\tilde{x}))=\tilde{x}=\begin{cases}2 & \text{if } \omega\in\{1,2\}\\ \\ 3 & \text{if } \omega\in\{3,4,5,6\}.\end{cases}$$

$$\mathcal{G}=\{\emptyset,\Omega,\{1,2,3\},\{4,5,6\}\}.$$

Since

$$P\{\tilde{x}=2\,|\{1,2,3\}\}=\frac{P(\{1,2\}\cap\{1,2,3\})}{P\{1,2,3\}}=\frac{P\{1,2\}}{P\{1,2,3\}}=\frac{1/3}{1/2}=\frac{2}{3},$$

$$P\{\tilde{x}=3\,|\{1,2,3\}\}=\frac{P(\{3,4,5,6\}\cap\{1,2,3\})}{P\{1,2,3\}}=\frac{P\{3\}}{P\{1,2,3\}}=\frac{1/6}{1/2}=\frac{1}{3},$$

$$P\{\tilde{x}=2\,|\{4,5,6\}\}=\frac{P(\{1,2\}\cap\{4,5,6\})}{P\{4,5,6\}}=\frac{P(\emptyset)}{P\{4,5,6\}}=0,$$

$$P\{\tilde{x}=3\,|\{4,5,6\}\}=\frac{P(\{3,4,5,6\}\cap\{4,5,6\})}{P\{4,5,6\}}=\frac{P\{4,5,6\}}{P\{4,5,6\}}=1,$$

$$\mathrm{E}(\tilde{x}\,|\mathcal{G})=\begin{cases}\left(\frac{2}{3}\cdot 2\right)+\left(\frac{1}{3}\cdot 3\right)=\frac{7}{3} & \text{if } \omega=\{1,2,3\}\\ \\ (2\cdot 0)+(3\cdot 1)=3 & \text{if } \omega\in\{4,5,6\}.\end{cases}$$

Since

$$P\{\tilde{x}=2\}=P\{1,2\}=\frac{1}{3},$$

$$P\{\tilde{x}=3\}=P\{3,4,5,6\}=\frac{2}{3},$$

$$\mathrm{E}(\tilde{x}\,|\mathcal{H}) = \mathrm{E}(\tilde{x}) = \left(\frac{1}{3}\cdot 2\right) + \left(\frac{2}{3}\cdot 3\right) = \frac{8}{3} \quad \text{for } \omega \in \{1,2,3,4,5,6\} = \Omega.$$

Note also that, since

$$P\{1,2,3\} = \frac{1}{2},$$

$$P\{3,4,5\} = \frac{1}{2},$$

we get

$$\mathrm{E}(\mathrm{E}(\tilde{x}\,|\mathcal{G})) = \mathrm{E}(\mathrm{E}(\tilde{x}\,|\mathcal{G})\,|\mathcal{H}) = \left(\frac{1}{2}\cdot\frac{7}{3}\right) + \left(\frac{1}{2}\cdot 3\right) = \frac{8}{3} = \mathrm{E}(\tilde{x}\,|\mathcal{H}) = \mathrm{E}(\tilde{x}).$$

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3. (a) The probability $P(i)$ that there are exactly i white balls in the box is $\frac{1}{7}$ for $i = 0, 1, ..., 6$. Let $\{W_1 \cap W_2\}$ be the event where we pick two white balls in two extractions with replacement. Let $P(W_1 \cap W_2|i)$ be the probability of extracting two white balls with replacement given that there are i white balls in the box. Obviously, $P(W_1 \cap W_2|i) = \left(\frac{i}{6}\right)^2 = \frac{i^2}{36}$. We are asked to to find $P(5|W_1 \cap W_2)$. We apply Bayes' theorem,

$$P(5|W_1 \cap W_2) = \frac{P(5)P(W_1 \cap W_2|5)}{\sum_{i=0}^{6} P(i)P(W_1 \cap W_2|i)} = \frac{\frac{1}{7}\cdot\frac{25}{36}}{\sum_{i=0}^{6}\frac{1}{7}\cdot\frac{i^2}{36}} = \frac{25}{\sum_{i=0}^{6} i^2} = \frac{25}{91} = 0.2747.$$

(b)
$\{B1\}$ = the first extracted ball is black
$\{B2\}$ = the second extracted ball is black
$\{j\}$ = there are j black balls in the box, i.e, there are $6-j$ white balls in the box, $j = 0, 1, ..., 6$.

$$P(B1 \cap B2) = \sum_{j=0}^{6} P(j) \cdot P(B1 \cap B_2\,|j)$$

$$= \sum_{j=0}^{6} \frac{1}{7} \cdot \left(\frac{j}{6}\cdot\frac{j-1}{5}\right) = \frac{1}{210}\sum_{j=0}^{6} j\cdot(j-1) = \frac{1}{210}\cdot 70 = \frac{1}{3}.$$

(c) We use the multivariate hypergeometric distribution

$$h(2,2,0;4,6,3,2,1) = \frac{\binom{3}{2}\binom{2}{2}\binom{1}{0}}{\binom{6}{4}} = \frac{1}{5}.$$

(d) We use the multinomial distribution

$$m(2,2,0;4,1/2,1/3,1/6) = \frac{4!}{2!2!0!}\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right)^0 = \frac{1}{6}.$$

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1. By changing to polar coordinates,

$$\int_A k(x^2+y^2)^{1/2}d(x,y) = k\int_0^{\pi}\int_3^4 r\cdot r dr d\theta = k\left(\int_3^4 r^2 dr\right)\left(\int_0^{\pi} d\theta\right) = k\cdot\frac{37\pi}{3} = 1$$

Thus, $k = \dfrac{3}{37\pi}$.

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2. (a)

$$P\{\bar{\boldsymbol{x}} \geq k; 7\} = P\left\{\frac{\bar{\boldsymbol{x}}-7}{\sqrt{3}/\sqrt{12}} \geq \frac{k-7}{\sqrt{3}/\sqrt{12}}; 7\right\} = P\{\tilde{z} \geq 2k-14\} = 0.05,$$

where $\tilde{z} \sim \mathrm{N}(0,1)$. From the table we find that $P\{z \leq 1.645\} = 0.95$. Thus,

$$1.645 = 2k - 14$$

or

$$k = 7.8225,$$

so that the desired critical region of size $\alpha = 0.05$ is

$$\bar{x} = \frac{\sum_{i=1}^{12} x_i}{12} \geq 7.8225.$$

The power π of this test is the probability of rejecting H_0 when $\mu = 9$:

$$\pi = P\{\bar{\boldsymbol{x}} \geq 7.8225; 9\} = P\left\{\frac{\bar{\boldsymbol{x}}-9}{\sqrt{3}/\sqrt{12}} \geq \frac{7.8225-9}{\sqrt{3}/\sqrt{12}}; 9\right\}$$
$$P\{\tilde{z} \geq -2.355\} = 0.99074,$$

(b) Since the null hypothesis is simple, it follows that $\mu_0 = 7$ maximizes the likelihood function under the null hypothesis H_0. Moreover, the sample mean $\bar{x} = \frac{\sum_{i=1}^{12} x_i}{12}$ maximizes the likelihood function L over the parameter space,

$$\bar{x} = \arg\max_{\mu\in\mathbb{R}} \underbrace{\left(\frac{1}{\sqrt{3}\sqrt{2\pi}}\right)^{12} e^{-\frac{1}{2\cdot 3}\sum_{i=1}^{12}(x_i-\mu)^2}}_{L(\mu;x_1,\ldots,x_{12})=\prod_{i=1}^{12} n(x_i;\mu,\sqrt{3})} = \arg\min_{\mu\in\mathbb{R}} \sum_{i=1}^{12}(x_i-\mu)^2$$

since the FOC is

$$-2\sum_{i=1}^{12}(x_i-\mu) = 0 \implies \sum_{i=1}^{12} x_i - 12\mu = 0 \implies \hat{\mu}_{ML} = \frac{\sum_{i=1}^{12} x_i}{12} = \bar{x}.$$

and the SOC for a minimum is satisfied as the function $\sum_{i=1}^{12}(x_i-\mu)^2$ is convex in μ.

$$L_0 = \left(\frac{1}{\sqrt{3}\sqrt{2\pi}}\right)^{12} e^{-\frac{1}{6}\sum_{i=1}^{12}(x_i-7)^2}$$

and

$$\max_{\mu\in\mathbb{R}} L = \left(\frac{1}{\sqrt{3}\sqrt{2\pi}}\right)^{12} e^{-\frac{1}{6}\sum_{i=1}^{12}(x_i-\bar{x})^2},$$

and the value of the likelihood ratio test statistic is

$$\lambda = \frac{L_0}{\max\limits_{\mu\in\mathbb{R}} L} = e^{-\frac{1}{6}\sum_{i=1}^{12}\left[(x_i-7)^2-(x_i-\bar{x})^2\right]}.$$

Note that

$$\sum_{i=1}^{12}\left[(x_i-7)^2-(x_i-\bar{x})^2\right] = \sum_{i=1}^{12}\left(x_i^2+49-14x_i-x_i^2-\bar{x}^2+2x_i\bar{x}\right)$$

$$= \sum_{i=1}^{12}x_i^2 + (12\cdot 49) - 14\underbrace{\sum_{i=1}^{12}x_i}_{12\bar{x}} - \sum_{i=1}^{12}x_i^2 - 12\bar{x}^2 + \underbrace{2\bar{x}\underbrace{\sum_{i=1}^{12}x_i}_{12\bar{x}}}_{24\bar{x}^2}$$

$$= 588 - 168\bar{x} - 12\bar{x}^2 + 24\bar{x}^2 = 12\bar{x}^2 + 588 - 168\bar{x}.$$

and

$$12(\bar{x}-7)^2 = 12\cdot\left(\bar{x}^2+49-14\bar{x}\right) = 12\bar{x}^2+588-168\bar{x}.$$

Thus,

$$\sum_{i=1}^{12}\left[(x_i-7)^2-(x_i-\bar{x})^2\right] = 12(\bar{x}-7)^2.$$

Therefore, the value of the likelihood ratio becomes

$$\lambda = e^{-\frac{12}{6}(\bar{x}-7)^2} = e^{-2(\bar{x}-7)^2}.$$

Hence, the critical region is

$$e^{-2(\bar{x}-7)^2} \leq k,$$

which after taking logarithms and dividing by -2 becomes

$$\left(\frac{\bar{x}-7}{1/2}\right)^2 \geq -\frac{\ln k}{2}$$

or

$$\left|\frac{\bar{x}-7}{1/2}\right| \geq K$$

where K will have to be determined so that the size of the critical region is $\alpha = 0.05$.

Since the sample mean $\bar{\boldsymbol{x}}$ has a normal distribution with the mean 7 and the variance $\sigma^2/n = 3/12 = 1/4$ (or the standard deviation $1/2$) under the null hypothesis,

$\tilde{z} = \dfrac{\bar{x} - 7}{1/2} = 2\bar{x} - 14$ is $\mathrm{N}(0,1)$. Thus, we find that the critical region of the likelihood ratio test is

$$|\tilde{z}| = |2\bar{x} - 14| \geq z_{0.025} = 1.96,$$

where $P\{z \leq z_{0.025}\} = 0.975$. In other words, the null hypothesis must be rejected when $2\bar{x} - 14$ takes either on a value greater than or equal to 1.96 or on a value lower than or equal to -1.96. Equivalently, we reject the null hypothesis either when $\bar{x} \geq 7.98$ or when $\bar{x} \leq 6.02$.

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3. (a)

$$f_{\tilde{x}_1}(x_1) = \int_0^\infty 6e^{-3x_1-2x_2}dx_2 = 3e^{-3x_1} \text{ for } x_1 > 0 \text{ and } f_{\tilde{x}_1}(x_1) = 0 \text{ otherwise.}$$

$$f_{\tilde{x}_2}(x_2) = \int_0^\infty 6e^{-3x_1-2x_2}dx_1 = 2e^{-2x_2} \text{ for } x_2 > 0 \text{ and } f_{\tilde{x}_2}(x_2) = 0 \text{ otherwise.}$$

The random variables $\tilde{x}_1$ and $\tilde{x}_2$ are independent since

$$f_{\tilde{x}_1,\tilde{x}_2}(x_1,x_2) = 6e^{-3x_1-2x_2} = f_{\tilde{x}_1}(x_1) \cdot f_{\tilde{x}_2}(x_2) = 3e^{-3x_1} \cdot 2e^{-2x_2} \text{ for } x_1 > 0,\ x_2 > 0,$$

and, obviously, $f_{\tilde{x}_1,\tilde{x}_2}(x_1,x_2) = 0 = f_{\tilde{x}_1}(x_1) \cdot f_{\tilde{x}_2}(x_2)$, otherwise.

(b) If $f_{\tilde{x}}(x) = \theta e^{-\theta x}$ for $x > 0$ and $f_{\tilde{x}}(x) = 0$ otherwise, with $\theta > 0$, then

$$M_{\tilde{x}}(t) = \int_0^\infty \theta e^{-\theta x} e^{tx} dx = \theta \int_0^\infty e^{-(\theta - t)x} dx = -\theta \left[\frac{e^{-(\theta-t)x}}{(\theta - t)} \right]_0^\infty$$

$$= -\theta \left(0 - \frac{1}{(\theta - t)} \right) = \frac{\theta}{\theta - t} \quad \text{for } t < \theta.$$

$$\mathrm{E}(\tilde{x}) = M'_{\tilde{x}}(t)|_{t=0} = \left. \frac{\theta}{(\theta - t)^2} \right|_{t=0} = \frac{\theta}{\theta^2} = \frac{1}{\theta},$$

$$\mathrm{E}(\tilde{x}^2) = M''_{\tilde{x}}(t) = \left. \frac{2\theta}{(\theta - t)^3} \right|_{t=0} = \frac{2\theta}{\theta^3} = \frac{2}{\theta^2},$$

$$\mathrm{Var}(\tilde{x}) = \mathrm{E}\left(\tilde{x}^2\right) - [\mathrm{E}(\tilde{x})]^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2},$$

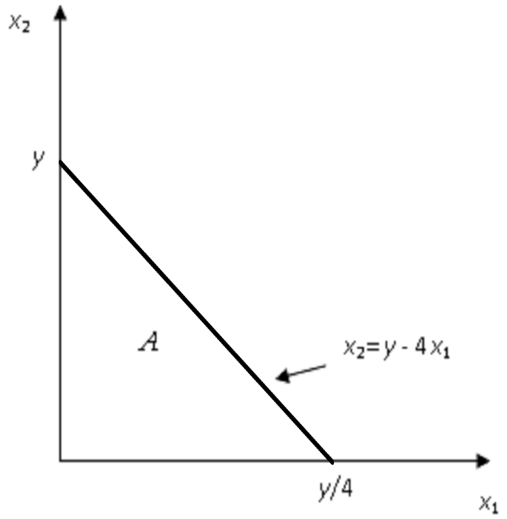
$$\mathrm{CV}_{\tilde{x}} = \frac{\sqrt{\mathrm{Var}(\tilde{x})}}{|\mathrm{E}(\tilde{x})|} = \frac{1/\theta}{1/\theta} = 1.$$

Thus,

$$M_{\tilde{x}_1}(t) = \frac{3}{3-t} \text{ for } t < 3, \quad \mathrm{E}(\tilde{x}_1) = \frac{1}{3}, \quad \mathrm{Var}(\tilde{x}_1) = \frac{1}{9}, \quad \mathrm{CV}_{\tilde{x}_1} = 1,$$

$$M_{\tilde{x}_2}(t) = \frac{2}{2-t}, \text{ for } t < 2, \quad \mathrm{E}(\tilde{x}_2) = \frac{1}{2}, \quad \mathrm{Var}(\tilde{x}_2) = \frac{1}{4}, \quad \mathrm{CV}_{\tilde{x}_2} = 1.$$

(c)



For $y > 0$,

$$F_{\tilde{y}}(y) = P\{\tilde{y} \le y\} = P\{4\tilde{x}_1 + \tilde{x}_2 \le y\} = P\{(\tilde{x}_1, \tilde{x}_2) \in A\}$$
$$= \int_0^{y/4} \int_0^{y-4x_1} 6e^{-3x_1-2x_2} dx_2 dx_1 = 1 - \frac{8}{5}e^{-\frac{3}{4}y} + \frac{3}{5}e^{-2y}.$$

Therefore,

$$F_{\tilde{y}}(y) = \begin{cases} 1 - \frac{8}{5}e^{-\frac{3}{4}y} + \frac{3}{5}e^{-2y} & \text{if } y > 0 \\ 0 & \text{if } y \le 0 \end{cases}$$

$$f_{\tilde{y}}(y) = F'_{\tilde{y}}(y) = \frac{d\left[1 - \frac{8}{5}e^{-\frac{3}{4}y} + \frac{3}{5}e^{-2y}\right]}{dy} = \frac{6}{5}\left(e^{-\frac{3}{4}y} - e^{-2y}\right) \text{ if } y > 0.$$

Thus,

$$f_{\tilde{y}}(y) = \begin{cases} \frac{6}{5}\left(e^{-\frac{3}{4}y} - e^{-2y}\right) & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(d) Consider the following change of variable for $x_1 > 0$ and $x_2 > 0$:

$$(y, x_2) = g(x_1, x_2) : \begin{cases} y = 4x_1 + x_2 \in (x_2, \infty) \\ x_2 = x_2 \in (0, \infty) \end{cases}$$

so that

$$(x_1, x_2) = g^{-1}(y, x_2) : \begin{cases} x_1 = \frac{1}{4}(y - x_2) \in (0, \infty) \\ x_2 = x_2 \in (0, \infty) \end{cases}$$

$$J_{g^{-1}}(y, x_2) = \begin{pmatrix} 1/4 & -1/4 \\ 0 & 1 \end{pmatrix} \Longrightarrow |J_{g^{-1}}(y, x_2)| = \frac{1}{4}.$$

Then,

$$f_{\tilde{y},\tilde{x}_2}(y,x_2)=\begin{cases}6e^{-3\overbrace{\left[\frac{1}{4}(y-x_2)\right]}^{x_1}-2x_2}\cdot\frac{1}{4}=\frac{3}{2}e^{-\frac{3}{4}y-\frac{5}{4}x_2} & \text{if } y>x_2 \text{ and } x_2>0 \\ & \iff x_2\in(0,y) \\ 0 & \text{otherwise}\end{cases}$$

(e)

$$f_{\tilde{y}\,|\,\tilde{x}_2}\left(y\,|\,x_2\right)=\frac{f_{\tilde{y},\tilde{x}_2}\left(x_1,x_2\right)}{f_{\tilde{x}_2}\left(x_2\right)}=\frac{\frac{3}{2}e^{-\frac{3}{4}y-\frac{5}{4}x_2}}{2e^{-2x_2}}=\frac{3}{4}e^{-\frac{3}{4}y+\frac{3}{4}x_2}$$

for $y>x_2$ and $f_{\tilde{y}\,|\,\tilde{x}_2}\left(y\,|\,x_2\right)=0$ otherwise, with $x_2>0$.

Thus,

$$f_{\tilde{y}\,|\,\tilde{x}_2}\left(y\,|\,3\right)=\begin{cases}\frac{3}{4}e^{-\frac{3}{4}y+\frac{9}{4}} & \text{if } y>3 \\ 0 & \text{otherwise}\end{cases}$$

(f)

$$\begin{aligned}\mathrm{E}\left(\tilde{y}\right) &= \mathrm{E}\left(4\tilde{x}_1+\tilde{x}_2\right)=\mathrm{E}\left(4\tilde{x}_1\right)+\mathrm{E}\left(\tilde{x}_2\right)=4\mathrm{E}\left(\tilde{x}_1\right)+\mathrm{E}\left(\tilde{x}_2\right) \\ &= \left(4\cdot\frac{1}{3}\right)+\frac{1}{2}=\frac{11}{6}.\end{aligned}$$

Alternatively, we can compute

$$\mathrm{E}\left(\tilde{y}\right)=\int_0^\infty y\frac{6}{5}\left(e^{-\frac{3}{4}y}-e^{-2y}\right)dy=\frac{11}{6}.$$

$$\begin{aligned}\mathrm{E}\left(\tilde{y}\mid\tilde{x}_2=3\right) &= \mathrm{E}\left(4\tilde{x}_1+\tilde{x}_2\mid\tilde{x}_2=3\right)=\mathrm{E}\left(4\tilde{x}_1\mid\tilde{x}_2=3\right)+\mathrm{E}\left(\tilde{x}_2\mid\tilde{x}_2=3\right) \\ &= \mathrm{E}\left(4\tilde{x}_1\right)+3=4\mathrm{E}\left(\tilde{x}_1\right)+3=\frac{4}{3}+3=\frac{13}{3}\end{aligned}$$

since $\tilde{x}_1$ and $\tilde{x}_2$ are independent.

Alternatively, we can compute

$$\mathrm{E}\left(\tilde{y}\mid\tilde{x}_2=3\right)=\int_3^\infty y\frac{3}{4}e^{-\frac{3}{4}y+\frac{9}{4}}dy=\frac{13}{3}.$$

(g)

$$\begin{aligned}&\mathrm{Cov}\left(\tilde{y},\tilde{x}_2\right)=\mathrm{Cov}\left(4\tilde{x}_1+\tilde{x}_2,\tilde{x}_2\right)=\mathrm{Cov}\left(4\tilde{x}_1,\tilde{x}_2\right)+\mathrm{Cov}\left(\tilde{x}_2,\tilde{x}_2\right) \\ &=4\mathrm{Cov}\left(\tilde{x}_1,\tilde{x}_2\right)+\mathrm{Var}\left(\tilde{x}_2\right)=0+\mathrm{Var}\left(\tilde{x}_2\right)=\frac{1}{4}.\end{aligned}$$

as $\mathrm{Cov}\left(\tilde{x}_1,\tilde{x}_2\right)=0$ due to the independence between $\tilde{x}_1$ and $\tilde{x}_2$.

Alternatively, we can compute

$$\mathrm{E}\left(\tilde{y}\cdot\tilde{x}_2\right)=\int_0^\infty\int_0^y yx_2\frac{3}{2}e^{-\frac{3}{4}y-\frac{5}{4}x_2}dx_2dy=\frac{7}{6}$$

or

$$\mathrm{E}\left(\tilde{y}\cdot\tilde{x}_2\right)=\int_0^{\infty}\int_{x_2}^{\infty} yx_2\frac{3}{2}e^{-\frac{3}{4}y-\frac{5}{4}x_2}dydx_2=\frac{7}{6}.$$

Then,

$$\mathrm{Cov}\left(\tilde{y},\tilde{x}_2\right)=\mathrm{E}\left(\tilde{y}\cdot\tilde{x}_2\right)-\mathrm{E}\left(\tilde{y}\right)\mathrm{E}\left(\tilde{x}_2\right)=\frac{7}{6}-\left(\frac{11}{6}\cdot\frac{1}{2}\right)=\frac{1}{4}.$$

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4. In this exercise, we are basically applying Bayes' theorem to find the conditional pmf of $\tilde{y}$ given $\tilde{x}_1=x_1,...,\tilde{x}_n=x_n$.

The conditional probability function of $\tilde{y}$, given $\tilde{x}_1=x_1,...,\tilde{x}_n=x_n$, is

$$f_{\tilde{y}|\tilde{x}_1,...,\tilde{x}_n}(y|x_1,...,x_n)=\frac{f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y,x_1,...,x_n)}{f_{\tilde{x}_1,...,\tilde{x}_n}(x_1,...,x_n)}.$$

Since $\{\tilde{x}_1,...,\tilde{x}_n\}$ is a random sample and, thus, it is composed of identically and independently distributed random variables,

$$\begin{aligned} f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y,x_1,...,x_n) &= g(y)\cdot f_{\tilde{x}_1,...,\tilde{x}_n|\tilde{y}}(x_1,...,x_n|y) \\ &= g(y)\cdot\prod_{i=1}^{n}h(x_i;y). \end{aligned}$$

Therefore,

$$f_{\tilde{x}_1,...,\tilde{x}_n}(x_1,...,x_n)=\sum_{y\in\tilde{y}(\Omega)}f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y,x_1,...,x_n)=\sum_{y\in\tilde{y}(\Omega)}\left[g(y)\cdot\prod_{i=1}^{n}h(x_i;y)\right]>0.$$

Then,

$$\mathrm{E}\left(\tilde{y}\,|\tilde{x}_1=x_1,...,\tilde{x}_n=x_n\right)=\sum_{y\in\tilde{y}(\Omega)}yf_{\tilde{y}|\tilde{x}_1,...,\tilde{x}_n}(y|x_1,...,x_n)=$$

$$\sum_{y\in\tilde{y}(\Omega)}y\cdot\left(\frac{g(y)\cdot\prod_{i=1}^{n}h(x_i;y)}{\sum_{y\in\tilde{y}(\Omega)}\left[g(y)\cdot\prod_{i=1}^{n}h(x_i;y)\right]}\right)=\frac{\sum_{y\in\tilde{y}(\Omega)}y\cdot g(y)\cdot\prod_{i=1}^{n}h(x_i;y)}{\sum_{y\in\tilde{y}(\Omega)}\left[g(y)\cdot\prod_{i=1}^{n}h(x_i;y)\right]}.$$

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