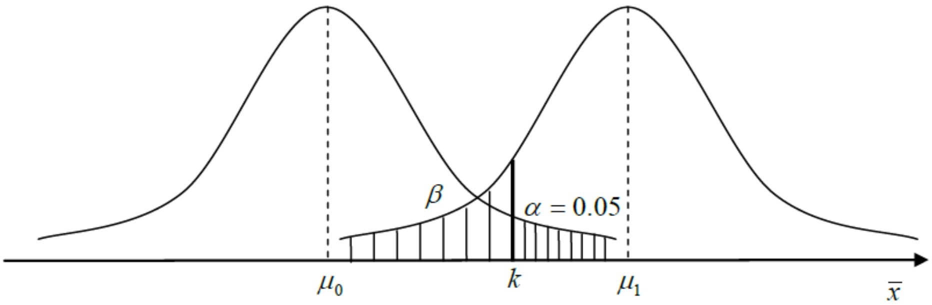
Probability and Statistics. IDEA. Answers to List 9.

1. Critical region for testing $\mu = \mu_0$ against $\mu = \mu_1$:



Under the null hypothesis,

$$P\{\bar{\boldsymbol{x}} \geq k; \mu_0\} = P\left\{\frac{\bar{\boldsymbol{x}} - \mu_0}{1/\sqrt{n}} \geq \frac{k-\mu_0}{1/\sqrt{n}}; \mu_0\right\} = P\left\{\tilde{z} \geq \frac{k-\mu_0}{1/\sqrt{n}}\right\} = 0.05,$$

where $\tilde{z} \sim \mathrm{N}(0,1)$. From the table we find that $z = 1.645$, corresponds to an entry of 0.4500. Thus,

$$1.645 = \frac{k-\mu_0}{1/\sqrt{n}}$$

and

$$k = \mu_0 + \frac{1.645}{\sqrt{n}},$$

so that the desired critical region of size $\alpha = 0.05$ is

$$\bar{x} \geq \mu_0 + \frac{1.645}{\sqrt{n}}.$$

When testing the null hypothesis $\mu = \mu_0$ against the alternative hypothesis $\mu = \mu_1$, when $\mu_1 > \mu_0$, and $\alpha = 0.05$, the probability β of the type II error is given by the area of the ruled region on the left of the figure above. For

$H_0 : \mu = 10$ and $H_1 : \mu = 11$, and $\alpha = 0.05$, we thus get under the alternative hypothesis that

$$\beta = P\left\{\bar{\boldsymbol{x}} < 10 + \frac{1.645}{\sqrt{n}}; 11\right\} = P\left\{\frac{\bar{\boldsymbol{x}} - 11}{1/\sqrt{n}} < \frac{\left(10 + \frac{1.645}{\sqrt{n}}\right) - 11}{1/\sqrt{n}}; 11\right\}$$

$$P\left\{\tilde{z} < \frac{\frac{1.645}{\sqrt{n}} - 1}{1/\sqrt{n}}\right\} = P\left\{\tilde{z} < 1.645 - \sqrt{n}\right\},$$

where $\tilde{z} \sim \mathrm{N}(0,1)$. Now, since

$$P\left\{\tilde{z} < -z_{0.05}\right\} = 0.05,$$

where $z_{0.05} = 1.645$, we set

$$1.645 - \sqrt{n} = -1.645,$$

from which we obtain $n = 10.824$. Therefore the minimum sample size needed to keep $\beta \leq 0.05$ is $n = 11$.

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2. The likelihoods under the null and the alternative hypotheses are

$$L_0 = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^n\left(\frac{x_i - \mu_0}{\sigma}\right)^2}$$

and

$$L_1 = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^n\left(\frac{x_i-\mu_1}{\sigma}\right)^2}.$$

After some simplifications their ratio becomes

$$\frac{L_0}{L_1} = \exp\left(\frac{n}{2\sigma^2}\left(\mu_1^2-\mu_0^2\right)+\left(\frac{\mu_0-\mu_1}{\sigma^2}\right)\sum_{i=1}^n x_i\right)$$

Thus, we must find a constant k and a region C of the sample space such that

$$\begin{aligned}\exp\left(\frac{n}{2\sigma^2}\left(\mu_1^2-\mu_0^2\right)+\left(\frac{\mu_0-\mu_1}{\sigma^2}\right)\sum_{i=1}^n x_i\right) &\leq k \quad \text{inside } C\\ \exp\left(\frac{n}{2\sigma^2}\left(\mu_1^2-\mu_0^2\right)+\left(\frac{\mu_0-\mu_1}{\sigma^2}\right)\sum_{i=1}^n x_i\right) &> k \quad \text{outside } C,\end{aligned}$$

and after taking logarithms, subtracting $\frac{n}{2\sigma^2}\left(\mu_1^2-\mu_0^2\right)$, and dividing by the negative quantity $\frac{n}{\sigma^2}\left(\mu_0-\mu_1\right)$, these two inequalities become

$$\begin{aligned}\frac{\sum_{i=1}^n x_i}{n} &= \bar{x} \geq K \quad \text{inside } C\\ \bar{x} &< K \quad \text{outside } C,\end{aligned}$$

where

$$K = \frac{\ln k + \frac{n}{2\sigma^2}\left(\mu_0^2-\mu_1^2\right)}{\frac{n}{\sigma^2}\left(\mu_0-\mu_1\right)} = \frac{\sigma^2\ln k}{n\left(\mu_0-\mu_1\right)}+\frac{\left(\mu_0+\mu_1\right)}{2}.$$

According to the Neyman-Pearson lemma this is the most powerful test among the tests with a level of significance no larger than that of this test.

To determine K we make use of the size α of the critical region and the distribution of the sample mean $\bar{\boldsymbol{x}}$. Since $\bar{\boldsymbol{x}} \sim \mathrm{N}\left(\mu_0, \sigma^2/n\right)$ under the null hypothesis, we obtain that the value K must satisfy

$$P\left\{\bar{\boldsymbol{x}} \geq K;\mu_0\right\} = P\left\{\frac{\bar{\boldsymbol{x}}-\mu_0}{\sigma/\sqrt{n}} \geq \frac{K-\mu_0}{\sigma/\sqrt{n}};\mu_0\right\} = P\left\{\tilde{z} \geq \frac{K-\mu_0}{\sigma/\sqrt{n}}\right\} = \alpha,$$

and $\tilde{z} = \frac{\bar{\boldsymbol{x}}-\mu_0}{\sigma/\sqrt{n}}$ is standard normal under the null hypothesis and $\frac{K-\mu_0}{\sigma/\sqrt{n}} = z_\alpha$. Therefore,

$$K = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

and, thus, the most powerful critical region of size α for testing the null hypothesis $\mu = \mu_0$ against the alternative $\mu = \mu_1$ (with $\mu_1 > \mu_0$) for the given normal population is

$$\bar{x} \geq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

and it should be noted that it does not depend on μ_1 (this is an important property).

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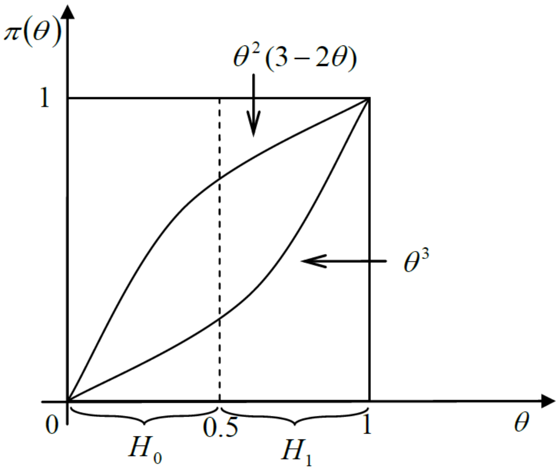
3. (a) The power function is the probability of rejecting the null hypothesis as a function of the parameter value.

$$\pi_1(\theta) = P(\{3 \text{ heads in } 3 \text{ flips}\};\theta) = \theta^3$$

(b)

$$\pi_2(\theta) = P(\{2 \text{ or } 3 \text{ heads in } 3 \text{ flips}\};\theta) = 3\theta^2(1-\theta) + \theta^3 = \theta^2(3-2\theta)$$



We see that

$$\pi_2(\theta) = 3\theta^2(1-\theta) + \theta^3 > \theta^3 = \pi_1(\theta) \quad \text{for } \theta \in (0,1)$$

and $\pi_1(0) = \pi_2(0) = 0$ and $\pi_1(1) = \pi_2(1) = 1$. Therefore, the power function of the second test is higher than that of the first test, not only on H_1 (which is desirable as it implies a lower probability of committing a type II error, i.e., of not rejecting H_0 when we should reject it) but also on H_0 (which is not desirable as it implies a higher probability of committing a type I error, i.e., of rejecting H_0 when we should not reject it). Thus, none of the two tests dominates unambiguously the other.

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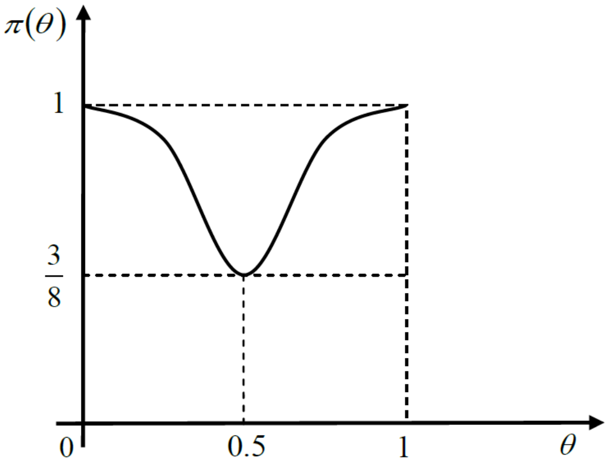
4.

$$\begin{aligned}\pi(\theta) &= P(\{\tilde{y} = 0, 1, 4 \text{ or } 5\}; \theta) = 1 - \binom{5}{2}\theta^2(1-\theta)^3 - \binom{5}{3}\theta^3(1-\theta)^2 \\ &= \frac{3}{8} + 5(\theta-0.5)^2 - 10(\theta-0.5)^4 = -10\theta^4 + 20\theta^3 - 10\theta^2 + 1.\end{aligned}$$



$$\alpha = \pi\,(0.5) = \frac{3}{8}.$$

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5. (a) Since the null hypothesis is simple, it follows that μ_0 maximizes the likelihood function under the null hypothesis H_0 and, since the parameter space is the set of all real numbers, it follows that the sample mean $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ maximizes the likelihood function L over the parameter space,

$$\bar{x} = \arg\max_{\mu\in\mathbb{R}} \underbrace{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}}_{L(\mu;x_1,\ldots,x_n)=\prod_{i=1}^{n} n(x_i;\mu,\sigma)} = \arg\min_{\mu\in\mathbb{R}} \sum_{i=1}^{n} (x_i-\mu)^2 .$$

Thus,

$$L_0 = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$$

and

$$\max_{\mu\in\mathbb{R}} L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2},$$

and the value of the likelihood ratio test statistic is

$$\lambda = \frac{L_0}{\max\limits_{\mu \in \mathbb{R}} L} = e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n \left[(x_i-\mu_0)^2-(x_i-\bar{x})^2\right]}.$$

Note that

$$\sum_{i=1}^n \left[(x_i-\mu_0)^2-(x_i-\bar{x})^2\right] = \sum_{i=1}^n \left(x_i^2+\mu_0^2-2x_i\mu_0-x_i^2-\bar{x}^2+2x_i\bar{x}\right)$$

$$= \sum_{i=1}^n x_i^2 + n\mu_0^2 - 2\mu_0 \underbrace{\sum_{i=1}^n x_i}_{n\bar{x}} - \sum_{i=1}^n x_i^2 - n\bar{x}^2 + 2\bar{x}\underbrace{\sum_{i=1}^n x_i}_{n\bar{x}}$$

$$= n\mu_0^2 - 2n\mu_0\bar{x} - n\bar{x}^2 + 2n\bar{x}^2 = n\mu_0^2 - 2n\mu_0\bar{x} + n\bar{x}^2$$

$$= n(\bar{x}^2 + \mu_0^2 - 2\mu_0\bar{x}) = n(\bar{x}-\mu_0)^2.$$

Thus,

$$\lambda = e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}.$$

Hence, the critical region is

$$e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2} \le k,$$

which after taking logarithms and dividing by $-1/2$ becomes

$$\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 \ge -2\ln k$$

or

$$\left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| \ge K$$

where K will have to be determined so that the size of the critical region is α.

Since the sample $\bar{\boldsymbol{x}}$ has a normal distribution with the mean μ_0 and the variance σ^2/n (or the standard deviation $\sigma/\sqrt{n}$) under the null hypothesis, $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ is $\mathrm{N}(0,1)$. Thus, we find that the critical region of the likelihood ratio test is

$$\left| \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \right| \geq z_{\frac{\alpha}{2}}$$

In other words, the null hypothesis must be rejected when $\tilde{z} = \dfrac{\bar{\boldsymbol{x}} - \mu_0}{\sigma/\sqrt{n}}$ takes on a value greater than or equal to $z_{\frac{\alpha}{2}}$, or a value less than or equal to $-z_{\frac{\alpha}{2}}$.

Note:

$$-2\ln\tilde{\lambda} = \frac{n}{\sigma^2}\left(\bar{\boldsymbol{x}} - \mu_0\right)^2 = \left(\frac{\bar{\boldsymbol{x}} - \mu_0}{\sigma/\sqrt{n}}\right)^2 \sim \chi_1^2.$$

In general, it is true that $-2\ln\tilde{\lambda} \overset{a}{\sim} \chi_M^2$, where $M = 1$ in this case. However, in this exercise we have that $-2\ln\tilde{\lambda} \sim \chi_1^2$.

(b) The Wald test statistic is

$$\widetilde{W} = \frac{\left(\bar{\boldsymbol{x}} - \mu_0\right)^2}{\mathrm{Var}\left(\bar{\boldsymbol{x}}\right)} = \left(\frac{\bar{\boldsymbol{x}} - \mu_0}{\sigma/\sqrt{n}}\right)^2 \sim \chi_1^2$$

or, equivalently,

$$\frac{\bar{\boldsymbol{x}} - \mu_0}{\sigma/\sqrt{n}} \sim \mathrm{N}\left(0,1\right).$$

so that the Wald test is exactly the same as the likelihood ratio test.

The score test statistic is

$$\widetilde{S} = \left. \frac{\left(\dfrac{\partial \ln L\left(\mu; \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n\right)}{\partial \mu}\right)^2}{-\mathrm{E}\left(\dfrac{\partial^2 \ln L\left(\mu; \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n\right)}{\partial \mu^2}\right)} \right|_{\mu=\mu_0}.$$

$$\ln L\left(\mu;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n\right)=n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\tilde{x}_i-\mu\right)^2,$$

Note that

$$\frac{\partial\ln L\left(\mu;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n\right)}{\partial\mu}=\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\tilde{x}_i-\mu\right),$$

$$\frac{\partial^2\ln L\left(\mu;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n\right)}{\partial\mu^2}=-\frac{n}{\sigma^2},$$

$$-\mathrm{E}\left(\frac{\partial^2\ln L\left(\mu;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n\right)}{\partial\mu^2}\right)=\frac{n}{\sigma^2}.$$

Therefore,

$$\widetilde{S}=\left.\frac{\left(\dfrac{\partial\ln L\left(\mu;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n\right)}{\partial\mu}\right)^2}{-\mathrm{E}\left(\dfrac{\partial^2\ln L\left(\mu;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n\right)}{\partial\mu^2}\right)}\right|_{\mu=\mu_0}=\frac{\left[\dfrac{1}{\sigma^2}\sum_{i=1}^{n}\left(x_i-\mu_0\right)\right]^2}{n/\sigma^2}=$$

$$\frac{\left[\dfrac{1}{\sigma^2}\left(\sum_{i=1}^{n}x_i-n\mu_0\right)\right]^2}{n/\sigma^2}=\frac{\left[\dfrac{1}{\sigma^2}\left(n\bar{\boldsymbol{x}}-n\mu_0\right)\right]^2}{n/\sigma^2}=\frac{n^2\left[\dfrac{1}{\sigma^2}\left(\bar{\boldsymbol{x}}-\mu_0\right)\right]^2}{n/\sigma^2}=$$

$$\left(\frac{\bar{\boldsymbol{x}}-\mu_0}{\sigma\,/\sqrt{n}}\right)^2\sim\chi_1^2$$

or

$$\frac{\bar{\boldsymbol{x}}-\mu_0}{\sigma\,/\sqrt{n}}\sim\mathrm{N}\left(0,1\right).$$

so that the likelihood ratio test, the Wald test, and the score test are identical in this exercise. We know that they are asymptotically identical but in this exercise they are identical even for finite random samples.

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6.

$H_0 : \mu = 8$

$H_1 : \mu \neq 8$

Critical region: $|z| \geq z_{0.005} = 2.575$, where $z = \dfrac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$.

Since $\bar{x} = 8.112$, $n = 25$, we get

$$z = \frac{8.112 - 8}{\frac{0.16}{\sqrt{25}}} = 3.5 > 2.575.$$

Therefore, we reject the null hypothesis.

The p-value is $2P\{\tilde{z} \geq 3.5\} = 0.000465 < \alpha = 0.01$, where $\tilde{z} \sim \mathrm{N}(0, 1)$. $(* * *)$

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7.

$H_0 : \mu = 22000$

$H_1 : \mu < 22000$

Critical region: $z \leq -z_{0.05} = -1.645$, where $z = \dfrac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$.

Using $\bar{x} = 21431$, $n = 100$, $\sigma \approx s = 1295$, we get

$$z = \frac{21431 - 22000}{\frac{1295}{\sqrt{100}}} = -4.3938.$$

Therefore, H_0 is rejected, i.e., the tires are not as good as claimed.

The p-value is $P\{\tilde{z} \leq -4.3938\} = 0.000005569 < \alpha = 0.05$, where $\tilde{z} \sim \mathrm{N}(0, 1)$. $(* * *)$

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8.

$H_0 : \mu = 185$

$H_1 : \mu < 185$

Critical region: $t \leq -t_{0.05,4} = -2.132$, where $t = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$.

Having $\bar{x} = 183.1$, $s = 8.2$, and $n = 5$, we get

$$t = \frac{183.1 - 185}{\frac{8.2}{\sqrt{5}}} = -0.52.$$

Since $t = -0.52$ is larger than $-t_{0.05,4} = -2.132$, the null hypothesis cannot be rejected.

The p-value is $P\left\{\tilde{t} \leq -0.52\right\} = 0.3153 > \alpha = 0.05$, where $\tilde{t} \sim \mathrm{t}_4$.

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9.

$H_0 : \mu_1 - \mu_2 = 0.2$

$H_1 : \mu_1 - \mu_2 \neq 0.2$

Critical region: $|z| \geq z_{0.025} = 1.96$, where

$$z = \frac{\bar{x}_1 - \bar{x}_2 - \delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Using $\bar{x}_1 = 2.61, s_1 = 0.12, n_1 = 50, \bar{x}_2 = 2.38, s_2 = 0.14$, and $n_2 = 40$,

$$z = \frac{2.61 - 2.38 - 0.2}{\sqrt{\frac{(0.12)^2}{50} + \frac{(0.14)^2}{40}}} = 1.08 < 1.96$$

Therefore, the H_0 cannot be rejected. Hence, we can either accept the null hypothesis, or merely say that the difference between $2.61 - 2.38 = 0.23$ and 0.2 is not large enough to reject the null hypothesis.

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10.

$H_0 : \mu_1 - \mu_2 = 0$

$H_1 : \mu_1 - \mu_2 \neq 0$

Critical region: $|t| \geq t_{0.025,6} = 2.447$, where

$$t = \frac{\bar{x}_1 - \bar{x}_2 - \delta}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

Having $\bar{x}_1 = 512, s_1 = 31, n_1 = 4, \bar{x}_2 = 492, s_2 = 26$, and $n_2 = 4$, s_p becomes

$$s_p = \sqrt{\frac{3\,(31)^2 + 3\,(26)^2}{4 + 4 - 2}} = 28.609$$

and, since $\delta = 0$,

$$t = \frac{512 - 492}{28.609\sqrt{\frac{1}{4} + \frac{1}{4}}} = 0.98865 < t_{0.025,6} = 2.447.$$

Therefore, the null hypothesis cannot be rejected. Even though the difference between the two sample means seems to be large, the samples are so small that the results are not conclusive; that is, the difference may well be due to chance.

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11.

$H_0 : \sigma^2 = 0.36$

$H_1 : \sigma^2 > 0.36$

Critical region: $\chi^2 \geq \chi^2_{0.05,17} = 27.587$, where

$$\chi^2 = \frac{(n-1)\, s^2}{\sigma_0^2}$$

Since $s^2 = 0.68$ and $n = 18$, we get

$$\chi^2 = \frac{17 \cdot 0.68}{0.36} = 32.1 > \chi^2_{0.05,17} = 27.587$$

Therefore H_0 should be rejected.

The p-value is $P\left\{\widetilde{\chi}^2 \geq 32.1\right\} = 0.0146 < \alpha = 0.05$, where $\widetilde{\chi}^2 \sim \chi^2_{17}$. $(*)$

Remark: Note that if α had been 0.01, the null hypothesis could not have been rejected, since $\chi^2 = 32.1$ does not exceed $\chi^2_{0.01,17} = 33.409$. Note that the p-value is larger than 0.01. This serves to indicate that the choice of α is something which should always be specified in advance, so that we will avoid the temptation of choosing a level of significance which happens to suit our purpose.

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12.

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

Critical region: $\dfrac{s_1^2}{s_2^2} \geq F_{0.01,12,15} = 3.67$, since $s_1^2 > s_2^2$. Having $s_1^2 = 19.2$ and $s_2^2 = 3.5$, we get

$$\frac{s_1^2}{s_2^2} = \frac{19.2}{3.5} = 5.4857 > F_{0.01,12,15} = 3.67.$$

Therefore, the H_0 is rejected.

The p-value is $2P\left\{\widetilde{F} \geq 5.4857\right\} = 0.00272 < \alpha = 0.02$, where $\widetilde{F} \sim \mathrm{F}_{12,15}$. $(**)$

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13.

$H_0 : \theta \leq 0.20$

$H_1 : \theta > 0.20$

Critical region: $z \geq z_{0.01} = 2.33$, where

$$z = \frac{x - n\theta_0}{\sqrt{n\theta_0\,(1-\theta_0)}}.$$

Since $x = 58$, $n = 200$, $n\theta_0 = 200 \cdot 0.20 = 40$, so that

$$z = \frac{58 - 40}{\sqrt{200\,(0.20)\,(0.80)}} = 3.182 > z_{0.01} = 2.33.$$

Therefore, the H_0 is rejected and we can conclude that brand A seems to be bought by more than 20 percent of all automobile owners.

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14.

$H_0 : \theta_1 = \theta_2 = \theta_3$

H_1 : the three θ's are not all equal

Critical region: $\chi^2 \geq \chi^2_{0.05,2} = 5.991$, where

$$\chi^2 = \sum_{i=1}^{3}\sum_{j=1}^{2}\frac{(n_{ij} - e_{ij})^2}{e_{ij}}.$$

The pooled estimate of θ is given by

$$\hat{\theta} = \frac{232 + 260 + 197}{400 + 500 + 400} = \frac{689}{1300} = 0.53.$$

Thus, we estimate the expected cell frequencies as

$$\begin{aligned}
e_{11} &= 400 \cdot 0.53 = 212 \quad \text{and} \quad e_{12} = 400 \cdot 0.47 = 188 \\
e_{21} &= 500 \cdot 0.53 = 265 \quad \text{and} \quad e_{22} = 500 \cdot 0.47 = 235 \\
e_{31} &= 400 \cdot 0.53 = 212 \quad \text{and} \quad e_{32} = 400 \cdot 0.47 = 188
\end{aligned}$$

and substitution into the above formula for χ^2 yields

$$\begin{aligned}
\chi^2 &= \frac{(232 - 212)^2}{212} + \frac{(260 - 265)^2}{265} + \frac{(197 - 212)^2}{212} \\
&\quad + \frac{(168 - 188)^2}{188} + \frac{(240 - 235)^2}{235} + \frac{(203 - 188)^2}{188} \\
&= 6.4733.
\end{aligned}$$

Since $\chi^2 = 6.4733 > \chi^2_{0.05,2} = 5.991$, the H_0 is rejected, in other words, the true proportions of shoppers favoring detergent A over detergent B in the three cities does not seem to be the same.

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15.

H_0 : Ability in mathematics and interests in statistics are independent

H_1 : these two variables are not independent

Critical region: $\chi^2 \geq \chi^2_{0.01,4} = 13.277$, where

$$\chi^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{(n_{ij} - e_{ij})^2}{e_{ij}}.$$

The expected frequencies for the first row are $\frac{120 \cdot 135}{360} = 45$, 50, and 25; those for the second row are 56.25, 62.50, and 31.25; and those for the third row are 33.75, 37.5, and 18.75. Then, substitution in the formula for χ^2 yields

$$\begin{aligned}\chi^2 &= \frac{(63-45)^2}{45} + \frac{(42-50)^2}{50} + \frac{(15-25)^2}{25} \\ &\quad + \frac{(58-56.25)^2}{56} + \frac{(61-62.5)^2}{62} + \frac{(31-31.25)^2}{31} \\ &\quad + \frac{(14-33.75)^2}{34} + \frac{(47-37.5)^2}{38} + \frac{(29-18.75)^2}{19} \\ &= 31.95\end{aligned}$$

Since $\chi^2 = 31.95 > \chi^2_{0.01,4} = 13.277$, the H_0 is rejected and we conclude that a person's ability in mathematics and his or her interest in statistics are not independent.

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16. To determine a corresponding set of expected absolute frequencies for a random sample from a Poisson population, we first use the mean of the observed distribution to estimate the Poisson parameter λ, getting

$$\lambda = \frac{\sum_{x=0}^{9} x \cdot n(x)}{440} = \frac{1341}{440} = 3.0477.$$

Then, finding the Poisson probabilities for $\lambda = 3.0477$ and multiplying by 440, the total frequency, we get the expected absolute frequencies shown in the right-hand column of the table below:

Class i	Number x of errors	observed absolute frequencies n_i	Poisson probabilities with $\lambda = 3.0477$	expected absolute frequencies e_i generated by the Poisson: $440 \cdot p\,(x;\lambda)$
1	0	18	0.0475	20.9
2	1	53	0.1447	63.7
3	2	103	0.2205	97.0
4	3	107	0.2240	98.5
5	4	82	0.1706	75.1
6	5	46	0.1040	45.8
7	6	18	0.0528	23.2
8	7	10	0.0230	10.1
9	8	2	0.0088	3.9
	9	1	0.0030	1.3

H_0 : Population has a Poisson distribution with $\lambda = 3.0477$.

H_1 : Population does not have a Poisson distribution with $\lambda = 3.0477$.

Critical region: Combining the last two rows so that each expected absolute frequency is at least 5, we get the critical region

$$\chi^2 \geq \chi^2_{0.05,\underbrace{9-1-1}_{7}} = 14.067,$$

where

$$\chi^2 = \sum_{i=1}^{8} \frac{(n_i - e_i)^2}{e_i}$$

Computing χ^2:

$$\begin{aligned}\chi^2 \;=\;& \frac{(18-20.9)^2}{20.9} + \frac{(53-63.7)^2}{63.7} + \frac{(103-97.0)^2}{97.0} + \frac{(107-98.5)^2}{98.5} + \frac{(82-75.1)^2}{75.1} \\ & + \frac{(46-45.8)^2}{45.8} + \frac{(18-23.2)^2}{23.2} + \frac{(10-10.1)^2}{10.1} + \frac{(3-5.2)^2}{5.2} = 6.0365.\end{aligned}$$

Since $\chi^2 = 6.0365 < \chi^2_{0.05,7} = 14.067$, H_0 cannot be rejected. Indeed, the close agreement between the observed and expected absolute frequencies suggests that the Poisson distribution provides a "good fit".

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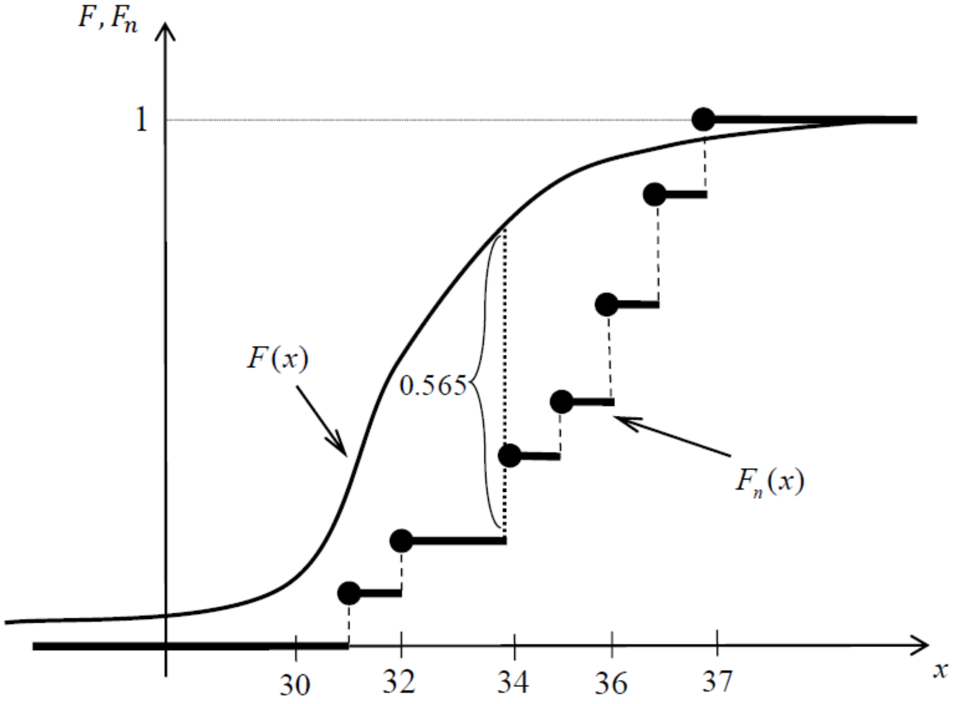
17. Standardizing the observations and using the table of the standard normal distribution we find that

$$D_n = \sup_x |F_n(x) - F(x)| = 0.565$$

which corresponds to $\lim\limits_{x \to 33.3^-} |F_n(x) - F(x)|$. According to the table the acceptance limit for D_{10} at 5% significance level is 0.410. Since $0.565 \geq 0.410$ the distribution being tested is rejected at 5% significance level.

In the next figure, the thin distribution function F corresponds to the population distribution and the thick distribution function F_n corresponds to the sample distribution. The length of the dotted line corresponds to the supremum of the distance between the two functions.



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18. We want to find the critical region C and a statistic such that

$$\frac{L_0}{L_1} \leq k \quad \text{inside } C$$

and

$$\frac{L_0}{L_1} > k \quad \text{outside } C$$

$$f(x; \mu, \sigma) = \frac{1}{\left(\sqrt{2\pi}\right)\sigma \cdot x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}$$

$H_0 : \mu = 80 \quad \sigma = 20$

$H_1 : \mu = 110 \quad \sigma = 20$

$$\begin{aligned} L_0 &= \prod_{i=1}^{n} f\left(x_i; 80, 20\right) = \left(\frac{1}{\sqrt{2\pi}\cdot 20}\right)^n \frac{1}{\prod_{i=1}^{n} x_i} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln x_i - 80}{20}\right)^2} \\ L_1 &= \prod_{i=1}^{n} f\left(x_i; 110, 20\right) = \left(\frac{1}{\sqrt{2\pi}\cdot 20}\right)^n \frac{1}{\prod_{i=1}^{n} x_i} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln x_i - 110}{20}\right)^2} \end{aligned}$$

$$\frac{L_0}{L_1} = \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln x_i - 80}{20}\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln x_i - 110}{20}\right)^2\right\}$$

$$= \exp\left\{\frac{1}{800}\left[\sum_{i=1}^{n}\left(\ln x_i - 110\right)^2 - \sum_{i=1}^{n}\left(\ln x_i - 80\right)^2\right]\right\} =$$

$$\exp\left\{\frac{1}{800}\left[5700n - 60\sum_{i=1}^{n}\ln x_i\right]\right\} = \exp\left\{\frac{57}{8}n - \frac{3}{40}\sum_{i=1}^{n}\ln x_i\right\}$$

$\Longrightarrow$

$$\exp\left\{\frac{57}{8}n - \frac{3}{40}\sum_{i=1}^{n}\ln x_i\right\} \leq k \quad \text{inside } C$$

Taking logs,

$$\frac{57}{8}n - \frac{3}{40}\sum_{i=1}^{n}\ln x_i \leq \ln k$$

$\Longrightarrow$

$$\sum_{i=1}^{n}\ln x_i \geq -\frac{40}{3}\ln k + \underbrace{\frac{40}{3}\frac{57}{8}}_{95}n$$

$$\hat{\theta} \equiv \frac{\sum_{i=1}^{n}\ln x_i}{n} \geq 95 - \frac{40\ln k}{3n} \equiv K \quad \text{inside } C$$

If $\tilde{x}_i$ is lognormal distributed, then $\ln\tilde{x}_i \sim \mathrm{N}\left(\mu, \sigma^2\right) \Longrightarrow$

$$\frac{\sum_{i=1}^{n}\ln\tilde{x}_i}{n} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right) \Longleftrightarrow \hat{\boldsymbol{\theta}} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

where $\dfrac{\sigma^2}{n}$ is the variance of the sample mean. Under the null hypothesis, we want

$$P\left\{\hat{\boldsymbol{\theta}} \geq K; \mu_0\right\} = P\left\{\tilde{z} \geq \frac{K-80}{20\,/\sqrt{n}}\right\} = 0.05, \quad \text{with } \mu_0 = 80$$

where $\tilde{z} = \dfrac{\hat{\boldsymbol{\theta}} - 80}{20\,/\sqrt{n}}$ is a random variable with standard normal distribution. Then,

$$\underset{z_{0.05}}{\underset{\parallel}{1.645}} = \frac{K-80}{20\,/\sqrt{n}} \Longrightarrow K = 80 + \frac{32.9}{\sqrt{n}}$$

Thus, we use the statistic $\hat{\boldsymbol{\theta}} = \dfrac{\sum_{i=1}^{n} \ln \tilde{x}_i}{n}$ and we reject the null hypothesis if

$$\hat{\theta} \geq 80 + \frac{32.9}{\sqrt{n}}$$

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19. (a) Let $\theta_0 = (1/12, 1/12, 1/12, 1/4, 1/6, 1/6, 1/6)$ and $\theta_1 = (a/3, b/3, c/3, 2/3, 0, 0, 0)$. The composite alternative hypothesis includes all the probability functions we get by assigning different values from 0 to 1 to a, b, c, subject to $a + b + c = 1$. Note that $\max\limits_{\theta \in B_0} L(\theta; x) = L(\theta_0; x)$. Note that the parameter space is

$$\Theta = \{(1/12, 1/12, 1/12, 1/4, 1/6, 1/6, 1/6) \cup (a/3, b/3, c/3, 2/3, 0, 0, 0)\}$$

for all $a \in [0,1]$, $b \in [0,1]$, $c \in [0,1]$ with $a + b + c = 1$.

To determine $\lambda = \dfrac{L(\theta_0; x)}{\max\limits_{\theta \in \Theta} L(\theta; x)}$ for each value of x, we first let $x = 1$. For this value we get $L(\theta_0; x) = 1/12$ and $\max\limits_{\theta \in \Theta} L(\theta; 1) = 1/3$, which corresponds to $a = 1$ under the alternative hypothesis, and, hence, $\lambda = 1/4$. Determining

λ for the other values of x in the same way, we get the following table:

x	1	2	3	4	5	6	7
λ	1/4	1/4	1/4	3/8	1	1	1

If the size of the critical region is $\alpha = 0.25$, we should find the value k for which $P\left\{\tilde{\lambda} \leq k; H_0\right\} = 0.25$. Clearly, we must have $k \in [1/4, 3/8)$. That is, the null hypothesis is rejected when $\lambda = 1/4$, namely, when $x = 1, x = 2$ or $x = 3$. Clearly, $f(1) + f(2) + f(3) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = 0.25$ as desired. The corresponding probability of a type II error is $g(4) + g(5) + g(6) + g(7) = 2/3$.

(b) The size of the critical region is also $\alpha = 0.25$ since $f(4) = 1/4$, but the corresponding probability of a type II error is

$$g(1) + g(2) + g(3) + g(5) + g(6) + g(7) = \frac{a}{3} + \frac{b}{3} + \frac{c}{3} + 0 + 0 + 0 = \frac{1}{3},$$

for all the distributions under the alternative hypothesis, which is less than 2/3.

(c) Obviously, we should prefer the test in (b) as the probability of a type II error is lower (i.e., the power is larger) for all the distributions under the alternative hypothesis. The Neyman-Pearson lemma does not apply here since the alternative hypothesis is "composite".

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20. $n = 535 \longleftarrow$ number of bombs thrown

$\theta = \dfrac{1}{576} \longleftarrow$ the probability that, if a bomb were thrown randomly, this bomb will hit a given square.

$$\lambda = n\theta = \frac{535}{576} = 0.9288 = \mathrm{E}\left(\tilde{x}\right) \longleftarrow \text{expected number of bombs per square}$$

Note that $p\left(x;\lambda\right) \approx b\left(x;n,\theta\right)$ gives us the probability that a square is hit by x bombs when $\lambda = 0.9288$ (or when $n = 535$ and $\theta = 1/576$), where $p\left(\cdot;\lambda\right)$ and $b\left(\cdot;n,\theta\right)$ are the Poisson and binomial probability functions, respectively. Multiplying by 576, the total number of squares, we get the expected absolute frequencies shown in the table below:

Class i	Number x of bombs per square	observed absolute frequencies n_i	(rounded) expected absolute frequencies e_i generated by the Poisson: $576 \cdot p\left(x;\lambda\right)$	(rounded) expected absolute frequencies e_i generated by the binomial: $576 \cdot b\left(x;n,\theta\right)$
1	0	229	228	227
2	1	211	211	212
3	2	93	98	98
4	3	35	30	30
5	4 or more	8	9	9

↖↗

They are very similar since the Poisson is a very good approximation of the binomial when n is large and θ is small (so we can use the Poisson directly).

$$\chi^2 = \sum_{i=1}^{5} \frac{\left(n_i - e_i\right)^2}{e_i} = \frac{1}{228} + 0 + \frac{25}{98} + \frac{25}{30} + \frac{1}{9} = 1.204,$$

$$\chi^2_{0.01,\underbrace{5-1-1}_{3}} = 11.345 > \chi^2 = 1.204$$

$\Longrightarrow$ Therefore, we cannot reject the hypothesis that the bombs were hitting randomly those squares.

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21. (a) We should use the Neyman-Pearson lemma. The probability function evaluated at the sample value under the null hypothesis is

$$h_0(x_1, x_2) = \frac{1}{9} \text{ for } (x_1, x_2) \in \{0,1,2\} \times \{0,1,2\},$$

whereas the probability function evaluated at the sample value under the alternative hypothesis is

$$h_1(x_1, x_2) = \begin{cases} 4/9 & \text{for } (x_1, x_2) = (1,1) \\ 2/9 & \text{for } (x_1, x_2) \in \{(1,2),(2,1)\} \\ 1/9 & \text{for } (x_1, x_2) = (2,2) \\ 0 & \text{otherwise (i.e., if } x_i = 0 \text{ for some } i = 1,2). \end{cases}$$

Therefore, we should consider the critical region of the type $C_k \equiv \left\{ \frac{h_0(x_1, x_2)}{h_1(x_1, x_2)} \le k \right\}$. Note that

$$\lambda \equiv \frac{h_0(x_1, x_2)}{h_1(x_1, x_2)} = \begin{cases} 1/4 & \text{for } (x_1, x_2) = (1,1) \\ 1/2 & \text{for } (x_1, x_2) \in \{(1,2),(2,1)\} \\ 1 & \text{for } (x_1, x_2) = (2,2) \\ \infty & \text{otherwise (i.e., if } x_i = 0 \text{ for some } i = 1,2). \end{cases}$$

The probability function of the extended real valued random variable $\tilde{\lambda} \equiv \dfrac{h_0(\tilde{x}_1, \tilde{x}_2)}{h_1(\tilde{x}_1, \tilde{x}_2)}$ under the null hypothesis is thus

$$f_{\tilde{\lambda}}(\lambda; H_0) = \begin{cases} 1/9 & \text{if } \lambda = 1/4, \text{ i.e., if } (x_1, x_2) = (1,1) \\ 2/9 & \text{if } \lambda = 1/2, \text{ i.e., if } (x_1, x_2) \in \{(1,2),(2,1)\} \\ 1/9 & \text{if } \lambda = 1, \text{ i.e., if } (x_1, x_2) = (2,2) \\ 5/9 & \text{if } \lambda = \infty, \text{ i.e., if } x_i = 0 \text{ for some } i = 1,2. \end{cases}$$

If $P\left\{\tilde{\lambda} \leq k; H_0\right\} = \alpha = 1/9$, then $k \in [1/4, 1/2)$. Thus, we reject the null hypothesis when $\lambda = 1/4$, that is, when $(x_1, x_2) = (1,1)$, and we do not reject it otherwise.

The power of this test is $P\left\{\tilde{\lambda} \leq k; H_1\right\}$ for $k \in [1/4, 1/2)$. Therefore, the power of the test is $P\{(\tilde{x}_1, \tilde{x}_2) = (1,1); H_1\} = 4/9$.

(b) The following table summarizes the probability function of the sample mean under the null and the alternative hypothesis:

$\overline{x}$	0	1/2	1	3/2	2
$f_{\overline{\boldsymbol{x}}}(\overline{x}; H_0)$	1/9	2/9	3/9	2/9	1/9
$f_{\overline{\boldsymbol{x}}}(\overline{x}; H_1)$	0	0	4/9	4/9	1/9

Therefore, if $P\{\overline{\boldsymbol{x}} \geq k; H_0\} = \alpha = 1/9$, then $k = (3/2, 2]$. Hence, we reject the null hypothesis when $\overline{x} = 2$, that is, when $(x_1, x_2) = (2,2)$, and we do not reject it otherwise.

The power of this test is $P\{\overline{\boldsymbol{x}} \geq 2; H_1\} = 1/9$. Note that this power is lower than the one of the test obtained in (a) as follows from the Neyman-Pearson lemma.

(c) The generic probability function of the population $\tilde{x}$ is

$$f_{\tilde{x}}(x; a, b) = \begin{cases} a & \text{if } x_1 = 0 \\ b & \text{if } x_1 = 1 \\ 1 - a - b \equiv c & \text{if } x_2 = 2, \end{cases}$$

with $a, b,$ and c non-negative. Note that $H_0 : a = 1/3, b = 1/3$. The generic probability function h of the random sample $\{\tilde{x}_1, \tilde{x}_2\}$ is thus summarized in the following table:

(x_1, x_2)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$h(x_1, x_2; a, b)$	a^2	ab	ac	ba	b^2	bc	ca	cb	c^2

Under the null hypothesis, the probability function of the random sample is

$$h(x_1, x_2; 1/3, 1/3) = f_{\tilde{x}}(x_1; 1/3, 1/3) \cdot f_{\tilde{x}}(x_2; 1/3, 1/3) = 1/9$$
$$\text{for } (x_1, x_2) \in \{0, 1, 2\} \times \{0, 1, 2\}.$$

Let us compute the maximum of the likelihood function $L(a, b; x_1, x_2) \equiv h(x_1, x_2; a, b)$. To this end note that

$$\max\ L(a, b; 0, 0) = \max\ a^2 \Longrightarrow a = 1, b = 0, c = 0, \text{ and } \max\ L(a, b; 0, 0) = 1;$$

$$\max\ L(a, b; 0, 1) = \max\ ab = \max\ a(1-a)$$
$$\Longrightarrow a = 1/2, b = 1/2, c = 0, \text{ and } \max\ L(a, b; 0, 1) = 1/4;$$

and so on. Therefore, we get

(x_1, x_2)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$\max\ L(a,b;x_1,x_2)$	1	1/4	1/4	1/4	1	1/4	1/4	1/4	1

Let us consider the likelihood ratio test

$$\tilde{\lambda} = \frac{L(1/3,1/3;\tilde{x}_1,\tilde{x}_2)}{\max\ L(a,b;\tilde{x}_1,\tilde{x}_2)} = \frac{h(\tilde{x}_1,\tilde{x}_2;1/3,1/3)}{\max\ L(a,b;\tilde{x}_1,\tilde{x}_2)} = \frac{1/9}{\max\ L(a,b;\tilde{x}_1,\tilde{x}_2)}.$$

Note that $\tilde{\lambda} = 1/9$ if $x_1 = x_2$, and $\tilde{\lambda} = 4/9$ if $x_1 \neq x_2$. Therefore, the probability function of the test statistic $\tilde{\lambda}$ under the null hypothesis is the following:

$$f_{\tilde{\lambda}}(\lambda; H_0) = \begin{cases} P\{\tilde{x}_1 = \tilde{x}_2; H_0\} = 1/3 & \text{if } \lambda = 1/9, \text{ i.e., if } x_1 = x_2 \\ P\{\tilde{x}_1 \neq \tilde{x}_2; H_0\} = 2/3 & \text{if } \lambda = 4/9, \text{ i.e., if } x_1 \neq x_2. \end{cases}$$

Therefore, if $P\left\{\tilde{\lambda} \leq k; H_0\right\} = \alpha = 1/3$, then $k \in [1/9, 4/9)$. Thus, we reject the null hypothesis when $\lambda = 1/9$, that is, when $x_1 = x_2$, and we do not reject it when $x_1 \neq x_2$.

Summing up, in part (a) we reject the null hypothesis when $(x_1, x_2) = (1,1)$, in part (b) when $(x_1, x_2) = (2,2)$, and in part (c) when $x_1 = x_2$.

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22. Note that in the table

	successes	failures
Sample 1	x_1	$n_1 - x_1$
Sample 2	x_2	$n_2 - x_2$
...	...	...
Sample k	x_k	$n_k - x_k$

the observed cell frequencies, n_{ij}, $i = 1, 2, \ldots, k$ and $j = 1, 2$, are $n_{i1} = x_i$ and $n_{i2} = n_i - x_i$. Let the null hypothesis be $\theta_1 = \theta_2 = \ldots = \theta_k = \theta_0$. If θ_0 is unknown, we substitute for it the value θ of the pooled estimator $\boldsymbol{\theta}$**,**

$$\boldsymbol{\theta} = \frac{\tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_k}{n_1 + n_2 + \ldots + n_k}.$$

and the estimated cell frequencies become

$$e_{i1} = n_i\theta \quad \text{and} \quad e_{i2} = n_i(1-\theta), \quad \text{for } i = 1, 2, \ldots, k,$$

Then, using the fact that

$$(x_i - n_i\theta)^2 = \left[(n_i - x_i) - n_i(1-\theta)\right]^2,$$

the value of the test statistic in (2) becomes

$$\sum_{i=1}^{k}\sum_{j=1}^{2}\frac{(n_{ij}-e_{ij})^2}{e_{ij}} = \sum_{i=1}^{k}\frac{(x_i - n_i\theta)^2}{n_i\theta} + \sum_{i=1}^{k}\frac{([(n_i - x_i) - n_i(1-\theta)])^2}{n_i(1-\theta)}$$

$$= \sum_{i=1}^{k}\frac{(x_i - n_i\theta)^2}{n_i\theta} + \sum_{i=1}^{k}\frac{(x_i - n_i\theta)^2}{n_i(1-\theta)} = \sum_{i=1}^{k}\frac{(x_i - n_i\theta)^2(1-\theta) + (x_i - n_i\theta)^2\theta}{n_i\theta(1-\theta)}$$

$$= \sum_{i=1}^{k} \frac{(x_i - n_i\theta)^2}{n_i\theta(1-\theta)},$$

which is the value of the statistic $\sum_{i=1}^{k} \frac{(\tilde{x}_i - n_i\boldsymbol{\theta})^2}{n_i\boldsymbol{\theta}(1-\boldsymbol{\theta})}$ given in (1).

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23. Likelihood function under the null hypothesis:

$$L\left(\frac{1}{2};x\right) = \binom{3}{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{3-x} = \binom{3}{x}\left(\frac{1}{2}\right)^{3}.$$

$$\arg\sup_{\theta\in[0,1]} L\left(\theta;x\right) = \arg\max_{\theta\in[0,1]} \binom{3}{x}\theta^{x}\left(1-\theta\right)^{3-x} = \hat{\theta}_{ML} = \frac{x}{3}$$

$$\sup_{\theta\in[0,1]} L\left(\theta;x\right) = \binom{3}{x}\left(\frac{x}{3}\right)^{x}\left(\frac{3-x}{3}\right)^{3-x}$$

Thus, the test statistic is

$$\tilde{\lambda} = \frac{L\left(\frac{1}{2};\tilde{x}\right)}{\sup_{\theta\in[0,1]} L\left(\theta;\tilde{x}\right)} = \frac{\binom{3}{\tilde{x}}\left(\frac{1}{2}\right)^{3}}{\binom{3}{\tilde{x}}\left(\frac{\tilde{x}}{3}\right)^{\tilde{x}}\left(\frac{3-\tilde{x}}{3}\right)^{3-\tilde{x}}} = \frac{\left(\frac{1}{2}\right)^{3}}{\left(\frac{\tilde{x}}{3}\right)^{\tilde{x}}\left(\frac{3-\tilde{x}}{3}\right)^{3-\tilde{x}}}$$

$$= \frac{\frac{1}{8}}{\frac{\tilde{x}^{\tilde{x}}\left(3-\tilde{x}\right)^{3-\tilde{x}}}{3^3}} = \frac{\frac{27}{8}}{\tilde{x}^{\tilde{x}}\left(3-\tilde{x}\right)^{3-\tilde{x}}}.$$

The critical region is defined by a constant $k > 0$ such that we reject the null hypothesis if

$$\tilde{\lambda} = \frac{\frac{27}{8}}{\tilde{x}^{\tilde{x}}\left(3-\tilde{x}\right)^{3-\tilde{x}}} \leq k$$

If $\tilde{x} = 0$, then $\tilde{\lambda} = \lim\limits_{x\to 0} \dfrac{27/8}{\tilde{x}^{\tilde{x}} (3-\tilde{x})^{3-\tilde{x}}} = \dfrac{27/8}{1 \cdot 3^3} = 1/8$. Under the null hypothesis this occurs with probability $\binom{3}{0}\left(\frac{1}{2}\right)^3 = 1/8$.

If $\tilde{x} = 1$, then $\tilde{\lambda} = \dfrac{27/8}{\tilde{x}^{\tilde{x}} (3-\tilde{x})^{3-\tilde{x}}} = 27/32$. Under the null hypothesis this occurs with probability $\binom{3}{1}\left(\frac{1}{2}\right)^3 = 3/8$.

If $\tilde{x} = 2$, then $\tilde{\lambda} = \dfrac{27/8}{\tilde{x}^{\tilde{x}} (3-\tilde{x})^{3-\tilde{x}}} = 27/32$. Under the null hypothesis this occurs with probability $\binom{3}{1}\left(\frac{1}{2}\right)^3 = 3/8$.

If $\tilde{x} = 3$, then $\tilde{\lambda} = \lim\limits_{x\to 3} \dfrac{27/8}{\tilde{x}^{\tilde{x}} (3-\tilde{x})^{3-\tilde{x}}} = \dfrac{27/8}{3^3 \cdot 1} = 1/8$. Under the null hypothesis this occurs with probability $\binom{3}{0}\left(\frac{1}{2}\right)^3 = 1/8$.

Note that

$$\lim_{x\to 0}(x \cdot \ln x) = \lim_{x\to 0}\left(\frac{\ln x}{\frac{1}{x}}\right) = \frac{\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)}{\lim\limits_{x\to 0}\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0}\left(-\frac{x^2}{x}\right) = \lim_{x\to 0}(-x) = 0.$$

Thus,

$$\lim_{x\to 0} x^x = \lim_{x\to 0}\exp(x \cdot \ln x) = \exp\left[\lim_{x\to 0}(x \cdot \ln x)\right] = e^0 = 1.$$

Therefore, the probability function of $\tilde{\lambda}$ under the null hypothesis $\theta = \dfrac{1}{2}$ is $f_{\tilde{\lambda}}\left(\dfrac{27}{32}\right) = \dfrac{3}{8} + \dfrac{3}{8} = \dfrac{3}{4}$ and $f_{\tilde{\lambda}}\left(\dfrac{1}{8}\right) = \dfrac{1}{8} + \dfrac{1}{8} = \dfrac{1}{4}$.

If we want that the level of significance be $1/4$, then

$$P\left\{\tilde{\lambda} \leq k; \frac{1}{2}\right\} = \frac{1}{4}.$$

Therefore, we must choose for the threshold value k any real number lying in the left semiclosed interval $\left[\dfrac{1}{8}, \dfrac{27}{32}\right)$. Thus, we reject the null hypothesis when $\lambda = 1/8$, that is, when we get 0 or 3 successes in 3 trials.

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24. We know that

$$\frac{\sum_{i=1}^{n}\left(\tilde{x}_i - \bar{\boldsymbol{x}}\right)^2}{\sigma^2} = \frac{(n-1)\boldsymbol{s^2}}{\sigma^2} = \frac{n\boldsymbol{\hat{s}^2}}{\sigma^2} \sim \chi^2_{n-1}.$$

Therefore,

$$P\left\{\chi^2_{19,0.975} < \frac{n\boldsymbol{\hat{s}^2}}{\sigma^2} < \chi^2_{19,0.25}; H_0\right\} = P\left\{8.907 < \frac{20\boldsymbol{\hat{s}^2}}{5} < 32.852\right\}$$
$$= P\left\{2.227 < \boldsymbol{\hat{s}^2} < 8.213\right\} = 0.95$$

The acceptance interval is

$$a = 2.227 < \hat{s}^2 < 8.213 = b$$

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25. We now that, in this case, the sample mean $\bar{\boldsymbol{x}}$ is a sufficient estimator for the population mean μ (see Exercise 33 of List 8). Since $\bar{\boldsymbol{x}}$ ∼N$(\mu, \sigma^2/n)$, the pdf of $\bar{\boldsymbol{x}}$ is

$$g(\overline{x}, \mu) = \frac{1}{\frac{\sigma}{\sqrt{n}}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\overline{x}-\mu}{\sigma/\sqrt{n}}\right)^2}.$$

If $\mu_2 > \mu_1$, then

$$\frac{g(\overline{x}, \mu_2)}{g(\overline{x}, \mu_1)} = e^{-\frac{1}{2}\left[\left(\frac{\overline{x}-\mu_2}{\sigma/\sqrt{n}}\right)^2 - \left(\frac{\overline{x}-\mu_1}{\sigma/\sqrt{n}}\right)^2\right]} = \exp\left\{-\frac{n}{2\sigma^2}\left[(\overline{x}-\mu_2)^2 - (\overline{x}-\mu_1)^2\right]\right\}$$

$$= \exp\left\{\frac{n}{2\sigma^2}\left[2\overline{x}(\mu_2 - \mu_1) - \mu_2^2 + \mu_1^2\right]\right\},$$

which is non-decreasing (in fact, strictly increasing) in $\overline{x}$. Therefore, $g(\overline{x}, \mu)$ satisfies the MLR property.

According to the Karlin-Rubin theorem, the test

$$\widehat{\phi}\left(\bar{\boldsymbol{x}}\right)=\begin{cases}1 & \text{if } \bar{\boldsymbol{x}}<\overline{x}_0\\ \\ 0 & \text{if } \bar{\boldsymbol{x}}\geq\overline{x}_0,\end{cases}$$

is a UMP test with size

$$\alpha=\sup_{\mu\geq\mu_0}P\left\{\bar{\boldsymbol{x}}<\overline{x}_0;\mu\right\}=\sup_{\mu\geq\mu_0}\left[1-P\left\{\bar{\boldsymbol{x}}\geq\overline{x}_0;\mu\right\}\right]=1-\inf_{\mu\geq\mu_0}P\left\{\bar{\boldsymbol{x}}\geq\overline{x}_0;\mu\right\}$$

$$=1-P\left\{\bar{\boldsymbol{x}}\geq\overline{x}_0;\mu_0\right\}=P\left\{\bar{\boldsymbol{x}}<\overline{x}_0;\mu_0\right\}$$

where the fourth inequality comes from the MLR property.

The value $\overline{x}_0$ is such that

$$P\left\{\bar{\boldsymbol{x}}<\overline{x}_0;\mu_0\right\}=P\left\{\frac{\bar{\boldsymbol{x}}-\mu_0}{\sigma/\sqrt{n}}<\frac{\overline{x}_0-\mu_0}{\sigma/\sqrt{n}};\mu_0\right\}=P\left\{\tilde{z}<-z_\alpha\right\}=\alpha.$$

with $\tilde{z}\sim$N$(0,1)$. Thus,

$$\frac{\overline{x}_0-\mu_0}{\sigma/\sqrt{n}}=-z_\alpha\Longrightarrow\overline{x}_0=\mu_0-z_\alpha\frac{\sigma}{\sqrt{n}}.$$

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26.

$H_0:\tilde{x}\sim\text{N}\left(700,200^2\right)$

$H_1:\tilde{x}\sim\text{N}\left(800,200^2\right)$

$\alpha=P\left\{\text{type I error}\right\}$

$\beta=P\left\{\text{type II error}\right\}$

(a) Under the null hypothesis, $\mu = \mu_0 = 700$,

$$\bar{\boldsymbol{x}} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right) = \mathrm{N}\left(\mu, \frac{200^2}{25}\right) = \mathrm{N}\left(\mu, 40^2\right)$$

$$\alpha_1 = P\left\{\bar{\boldsymbol{x}} \geq k; \mu_0\right\} = P\left\{\tilde{z} \geq \underbrace{\frac{k-700}{40}}_{z_\alpha}\right\} = P\left\{\tilde{z} \geq z_\alpha\right\}$$

$\Longrightarrow$

$$k = 700 + 40 \cdot z_\alpha$$

Under the alternative hypothesis, $\mu = \mu_1 = 800$,

$$\begin{aligned} \beta_1 &= P\left\{\bar{\boldsymbol{x}} < k; \mu_1\right\} = P\left\{\tilde{z} < \frac{k-800}{40}\right\} \\ &= P\left\{\tilde{z} < \frac{700 + 40 \cdot z_\alpha - 800}{40}\right\} = P\left\{\tilde{z} < \frac{-100 + 40 \cdot z_\alpha}{40}\right\} \\ &= P\left\{\tilde{z} < -2.5 + z_\alpha\right\} = P\left\{\tilde{z} \geq 2.5 - z_\alpha\right\} \end{aligned}$$

$$\begin{aligned} \alpha_1 &= \beta_1 \Longrightarrow z_\alpha = 2.5 - z_\alpha \Longrightarrow z_\alpha = 1.25 \\ \alpha_1 &= \beta_1 = P\left\{\tilde{z} \geq 1.25\right\} = P\left\{\tilde{z} < -1.25\right\} = 0.10565 \end{aligned}$$

Note that $k = 700 + 40 \cdot z_\alpha = 700 + 40 \cdot 1.25 = 750$. Thus, the null hypothesis is rejected when $\bar{\boldsymbol{x}} \geq 750$.

(b)

$$\frac{\hat{\boldsymbol{\theta}} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \overset{a}{\sim} \mathrm{N}(0,1)$$

Under the null hypothesis, the expected proportion of observations with a value smaller than 800 is

$$\theta_0 = P\left\{\tilde{x} < 800; \mu_0\right\} = P\left\{\tilde{z} < \frac{800-700}{200}\right\} = P\left\{\tilde{z} < 0.5\right\} = 0.6915$$

Under the alternative hypothesis, the expected proportion of observations with a value smaller than 800 is

$$\theta_1 = P\left\{\tilde{x} < 800; \mu_1\right\} = P\left\{\tilde{z} < \frac{800-800}{200}\right\} = P\left\{\tilde{z} < 0\right\} = 0.5$$

Therefore,

$$\begin{aligned} H_0 &: \theta = 0.6915 \quad \Longleftrightarrow \quad \tilde{x} \sim \mathrm{N}\left(700, 200^2\right) \\ H_1 &: \theta = 0.5 \qquad\;\; \Longleftrightarrow \quad \tilde{x} \sim \mathrm{N}\left(800, 200^2\right). \end{aligned}$$

Note that

$$\begin{aligned} &\text{If } \ \theta = \theta_0 = 0.6915 \quad \rightarrow \quad \sigma_0 = \sqrt{\frac{\theta_0\left(1-\theta_0\right)}{n}} = 0.0660 \quad (n = 49) \\ &\text{If } \ \theta = \theta_1 = 0.5 \qquad\;\; \rightarrow \quad \sigma_1 = \sqrt{\frac{\theta_1\left(1-\theta_1\right)}{n}} = 0.0714 \quad (n = 49) \end{aligned}$$

Therefore, under the null hypothesis,

$$\alpha_2 = P\left\{\hat{\boldsymbol{\theta}} \leq k; \theta_0\right\} \approx P\left\{\tilde{z} \leq \underbrace{\frac{k-0.6915}{0.0660}}_{z_\alpha}\right\} = P\left\{\tilde{z} \leq z_\alpha\right\} = P\left\{\tilde{z} > -z_\alpha\right\}$$

$\Longrightarrow$

$$k = 0.6915 + 0.0660 \cdot z_\alpha$$

Under the alternative hypothesis,

$$\begin{aligned}\beta_2 &= P\left\{\hat{\boldsymbol{\theta}} > k; \theta_1\right\} \approx P\left\{\tilde{z} > \frac{k-0.5}{0.0714}\right\} \\ &= P\left\{\tilde{z} > \frac{0.6915 + 0.0660 \cdot z_\alpha - 0.5}{0.0714}\right\} = P\left\{\tilde{z} > 2.682 + 0.924 \cdot z_\alpha\right\}\end{aligned}$$

$$\begin{aligned}\alpha_2 &= \beta_2 \Longrightarrow -z_\alpha = 2.682 + 0.924 \cdot z_\alpha \Longrightarrow z_\alpha = -1.394 \\ \alpha_2 &= \beta_2 = P\left\{\tilde{z} > 1.394\right\} = P\left\{\tilde{z} \leq -1.394\right\} = 0.0817\end{aligned}$$

Note that $k = 0.6915 + 0.0660 \cdot z_\alpha = 0.6915 + 0.0660 \cdot (-1.394) = 0.5995$. Thus, the null hypothesis is rejected when $\hat{\boldsymbol{\theta}} \leq 0.5995$.

We should prefer the statistic given in (b) since

$$\alpha_2 = \beta_2 = 0.0817 < \alpha_1 = \beta_1 = 0.10565$$

Note that, according the Karlin-Rubin theorem, the test (a) is a UMP test since the sample mean $\bar{\boldsymbol{x}}$ is a sufficient estimator for μ and the density of $\bar{\boldsymbol{x}}$ satisfies the MLR property (see Exercise 25 of this list). The reason why the statistic in (b) is preferred here is that the sample size in (b) is larger than in (a). You can check that if, for instance, the sample size in (a) were $n = 49$, then we would indeed prefer the test (a).

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27. (a) Note that we can write the likelihood function as

$$L(\lambda;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n)=\prod_{i=1}^{n}p(x_i;\lambda)=\frac{\lambda^{x_1}e^{-\lambda}}{x_1!}\cdot\frac{\lambda^{x_2}e^{-\lambda}}{x_2!}\cdot...\cdot\frac{\lambda^{x_n}e^{-\lambda}}{x_n!}$$

$$=\frac{\lambda^{\left(\sum_{i=1}^{n}\tilde{x}_i\right)}e^{-n\lambda}}{\Pi_{i=1}^{n}x_i!}=\frac{\lambda^{n\bar{\boldsymbol{x}}}e^{-n\lambda}}{\Pi_{i=1}^{n}x_i!}.$$

We construct the test statistic $\tilde{l}$,

$$\tilde{l}=\frac{L(5;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_{70})}{L(6;\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_{70})}=\frac{\frac{5^{70\bar{\boldsymbol{x}}}e^{-70\cdot 5}}{\Pi_{i=1}^{n}\tilde{x}_i!}}{\frac{6^{70\bar{\boldsymbol{x}}}e^{-70\cdot 6}}{\Pi_{i=1}^{n}\tilde{x}_i!}}=\frac{5^{70\bar{\boldsymbol{x}}}e^{-350}}{6^{70\bar{\boldsymbol{x}}}e^{-420}}=\left(\frac{5}{6}\right)^{70\bar{\boldsymbol{x}}}e^{70}.$$

We reject H_0 when $\tilde{l}\le k$ with $P\left(\tilde{l}\le k;5\right)=0.05$.

Note that

$$\tilde{l}\le k\iff\ln\tilde{l}\le\ln k\iff 70\bar{\boldsymbol{x}}(\ln 5-\ln 6)+70\le\ln k\iff\bar{\boldsymbol{x}}\ge\frac{70-\ln k}{70\cdot(\ln 6-\ln 5)}\equiv K.$$

Thus, since $\mathrm{E}(\bar{\boldsymbol{x}})=\mathrm{E}(\tilde{x})=\lambda$ and $\mathrm{Var}(\bar{\boldsymbol{x}})=\dfrac{\mathrm{Var}(\tilde{x})}{n}=\dfrac{\lambda}{n}=\dfrac{\lambda}{70}$,

$$P\left\{\tilde{l}\le k;5\right\}=P\{\bar{\boldsymbol{x}}\ge K;5\}=P\left\{\frac{\bar{\boldsymbol{x}}-\mathrm{E}(\bar{\boldsymbol{x}})}{\sqrt{\mathrm{Var}(\bar{\boldsymbol{x}})}}\ge\frac{K-\mathrm{E}(\bar{\boldsymbol{x}})}{\sqrt{\mathrm{Var}(\bar{\boldsymbol{x}})}};5\right\}$$

$$=P\left\{\frac{\bar{\boldsymbol{x}}-5}{\sqrt{5/70}}\ge\frac{K-5}{\sqrt{5/70}};5\right\}\approx P\left\{\tilde{z}\ge\frac{K-5}{\sqrt{5/70}}\right\}=1-\mathrm{N}\left(\frac{K-5}{\sqrt{5/70}}\right)=0.05,$$

where $\tilde{z}\sim\mathrm{N}(0,1)$.

$$\Longrightarrow\mathrm{N}\left(\frac{K-5}{\sqrt{5/70}}\right)=0.95\Longrightarrow\frac{K-5}{\sqrt{5/70}}=1.645.$$

Thus, we reject when $\dfrac{\bar{\boldsymbol{x}}-5}{\sqrt{5/70}} \geq 1.645$, that is, when $\bar{\boldsymbol{x}} \geq 5+1.645\cdot\sqrt{5/70} = 5.44 \equiv K$.

(b) The power of the test is

$$P\left\{\bar{\boldsymbol{x}} \geq K; 6\right\} = P\left\{\bar{\boldsymbol{x}} \geq 5.44; 6\right\} = P\left\{\frac{\bar{\boldsymbol{x}}-6}{\sqrt{6/70}} \geq \frac{5.44-6}{\sqrt{6/70}}; 6\right\}$$

$$\approx P\left\{\tilde{z} \geq -1.913\right\} = P\left\{\tilde{z} < 1.913\right\} = \mathrm{N}\left(1.913\right) = 0.972.$$

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28. Note that

$$u\left(0\right) = 1,$$

$$u\left(1/2\right) = -\left(1/2\right)^2 + 2\cdot\left(1/2\right) + 1 = \frac{7}{4},$$

and

$$u(1) = -(1)^2 + (2\cdot 1) + 1 = 2.$$

Let us first use the standard formula

$$\mathrm{E}\left[u(\tilde{x})\right] = \int_{[a,b]} u\left(x\right) dF_{\tilde{x}}(x) = \int_{[0,1]} u\left(x\right) dF_{\tilde{x}}(x)$$

$$= \frac{1}{2}u(0) + \frac{1}{4}u(1/2) + \frac{1}{4}u(1) = \left(\frac{1}{2}\cdot 1\right) + \left(\frac{1}{4}\cdot\frac{7}{4}\right) + \left(\frac{1}{4}\cdot 2\right) = \frac{23}{16}.$$

Let us now use the formula

$$\mathrm{E}\left[u(\tilde{x})\right] = u(b) - \int_{[a,b]} F_{\tilde{x}}(x) du(x) = u(1) - \int_{[0,1]} F_{\tilde{x}}(x) du(x)$$

$$= u(1) - \int_{[0,1]} F_{\tilde{x}}(x)\left(-2x+2\right) dx$$

$$= u(1) - \left[\int_{[0,1/2)} F_{\tilde{x}}(x)\,(-2x+2)\,dx + \int_{[1/2,1]} F_{\tilde{x}}(x)\,(-2x+2)\,dx\right]$$

$$= 2 - \left[\int_{[0,1/2)} \frac{1}{2}\,(-2x+2)\,dx + \int_{[1/2,1)} \frac{3}{4}\,(-2x+2)\,dx\right]$$

$$= 2 - \left[\int_{[0,1/2]} \frac{1}{2}\,(-2x+2)\,dx + \int_{[1/2,1]} \frac{3}{4}\,(-2x+2)\,dx\right]$$

$$= 2 - \frac{1}{2}\left[-x^2+2x\right]_0^{1/2} - \frac{3}{4}\left[-x^2+2x\right]_{1/2}^{1}$$

$$= 2 - \frac{1}{2}\left(-\frac{1}{4}+1-0\right) - \frac{3}{4}\left(-1+2+\frac{1}{4}-1\right)$$

$$= 2 - \left(\frac{1}{2}\cdot\frac{3}{4}\right) - \left(\frac{3}{4}\cdot\frac{1}{4}\right) = \frac{23}{16},$$

where the third equality follows since

$$\int_{[0,1]} F_{\tilde{x}}(x)du(x) = \int_{[0,1]} F_{\tilde{x}}(x)u'(x)dx = \int_{[0,1]} F_{\tilde{x}}(x)\,(-2x+2)\,dx$$

and the fifth equality follows since

$$F_{\tilde{x}}(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1/2 & \text{for } x \in [0,1/2) \\ 3/4 & \text{for } x \in [1/2,1) \\ 1 & \text{for } x \geq 1. \end{cases}$$

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29. (a)

$$P\left\{\bar{\boldsymbol{x}} \geq k; 7\right\} = P\left\{\frac{\bar{\boldsymbol{x}}-7}{\sqrt{3}\,/\sqrt{12}} \geq \frac{k-7}{\sqrt{3}\,/\sqrt{12}}; 7\right\} = P\left\{\tilde{z} \geq 2k-14\right\} = 0.05,$$

where $\tilde{z} \sim \mathrm{N}(0,1)$. From the table we find that $P\{z \leq 1.645\} = 0.95$. Thus,

$$1.645 = 2k - 14$$

or

$$k = 7.8225,$$

so that the desired critical region of size $\alpha = 0.05$ is

$$\bar{x} = \frac{\sum_{i=1}^{12} x_i}{12} \geq 7.8225.$$

The power π of this test is the probability of rejecting H_0 when $\mu = 9$:

$$\pi = P\left\{\bar{\boldsymbol{x}} \geq 7.8225; 9\right\} = P\left\{\frac{\bar{\boldsymbol{x}} - 9}{\sqrt{3}\,/\sqrt{12}} \geq \frac{7.8225 - 9}{\sqrt{3}\,/\sqrt{12}}; 9\right\}$$

$$P\{\tilde{z} \geq -2.355\} = 0.99074,$$

(b) Since the null hypothesis is simple, it follows that $\mu_0 = 7$ maximizes the likelihood function under the null hypothesis H_0. Moreover, the sample mean $\bar{x} = \frac{\sum_{i=1}^{12} x_i}{12}$ maximizes the likelihood function L over the parameter space,

$$\bar{x} = \arg\max_{\mu \in \mathbb{R}} \underbrace{\left(\frac{1}{\sqrt{3}\sqrt{2\pi}}\right)^{12} e^{-\frac{1}{2\cdot 3}\sum_{i=1}^{12}(x_i - \mu)^2}}_{L(\mu; x_1, \ldots, x_{12}) = \prod_{i=1}^{12} n(x_i; \mu, \sqrt{3})} = \arg\min_{\mu \in \mathbb{R}} \sum_{i=1}^{12} (x_i - \mu)^2$$

since the FOC is

$$-2\sum_{i=1}^{12}(x_i - \mu) = 0 \implies \sum_{i=1}^{12} x_i - 12\mu = 0 \implies \hat{\mu}_{ML} = \frac{\sum_{i=1}^{12} x_i}{12} = \bar{x}.$$

and the SOC for a minimum is satisfied as the function $\sum_{i=1}^{12}(x_i-\mu)^2$ is convex in μ.

$$L_0=\left(\frac{1}{\sqrt{3}\sqrt{2\pi}}\right)^{12}e^{-\frac{1}{6}\sum_{i=1}^{12}(x_i-7)^2}$$

and

$$\max_{\mu\in\mathbb{R}}L=\left(\frac{1}{\sqrt{3}\sqrt{2\pi}}\right)^{12}e^{-\frac{1}{6}\sum_{i=1}^{12}(x_i-\bar{x})^2},$$

and the value of the likelihood ratio test statistic is

$$\lambda=\frac{L_0}{\max\limits_{\mu\in\mathbb{R}}L}=e^{-\frac{1}{6}\sum_{i=1}^{12}\left[(x_i-7)^2-(x_i-\bar{x})^2\right]}.$$

Note that

$$\sum_{i=1}^{12}\left[(x_i-7)^2-(x_i-\bar{x})^2\right]=\sum_{i=1}^{12}\left(x_i^2+49-14x_i-x_i^2-\bar{x}^2+2x_i\bar{x}\right)$$

$$=\sum_{i=1}^{12}x_i^2+(12\cdot 49)-14\underbrace{\sum_{i=1}^{12}x_i}_{12\bar{x}}-\sum_{i=1}^{12}x_i^2-12\bar{x}^2+\underbrace{2\bar{x}\underbrace{\sum_{i=1}^{12}x_i}_{12\bar{x}}}_{24\bar{x}^2}$$

$$=588-168\bar{x}-12\bar{x}^2+24\bar{x}^2=12\bar{x}^2+588-168\bar{x}.$$

and

$$12(\bar{x}-7)^2=12\cdot\left(\bar{x}^2+49-14\bar{x}\right)=12\bar{x}^2+588-168\bar{x}.$$

Thus,

$$\sum_{i=1}^{12}\left[(x_i-7)^2-(x_i-\bar{x})^2\right]=12(\bar{x}-7)^2.$$

Therefore, the value of the likelihood ratio becomes

$$\lambda=e^{-\frac{12}{6}(\bar{x}-7)^2}=e^{-2(\bar{x}-7)^2}.$$

Hence, the critical region is

$$e^{-2(\bar{x}-7)^2} \leq k,$$

which after taking logarithms and dividing by -2 becomes

$$\left(\frac{\bar{x}-7}{1/2}\right)^2 \geq -\frac{\ln k}{2}$$

or

$$\left|\frac{\bar{x}-7}{1/2}\right| \geq K$$

where K will have to be determined so that the size of the critical region is $\alpha = 0.05$.

Since the sample mean $\bar{\boldsymbol{x}}$ has a normal distribution with the mean 7 and the variance $\sigma^2/n = 3/12 = 1/4$ (or the standard deviation $1/2$) under the null hypothesis, $\tilde{z} = \dfrac{\bar{x}-7}{1/2} = 2\bar{x} - 14$ is $\mathrm{N}(0,1)$. Thus, we find that the critical region of the likelihood ratio test is

$$|\tilde{z}| = |2\bar{x} - 14| \geq z_{0.025} = 1.96,$$

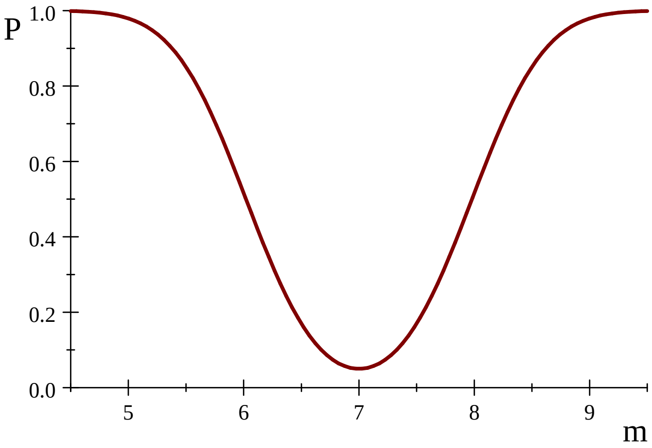
where $P\{z \leq z_{0.025}\} = 0.975$. In other words, the null hypothesis must be rejected when $2\bar{x} - 14$ takes either on a value greater than or equal to 1.96 or on a value lower than or equal to -1.96. Equivalently, we reject the null hypothesis either when $\bar{x} \geq 7.98$ or when $\bar{x} \leq 6.02$.

(c) The power function of the test is

$$\pi(\mu) = P\{\bar{\boldsymbol{x}} \geq 7.98; \mu\} + P\{\bar{\boldsymbol{x}} \leq 6.02; \mu\} = 1 - P(\{6.02 < \bar{\boldsymbol{x}} < 7.98; \mu\})$$

$$= 1 - \int_{6.02}^{7.98} \left(\frac{1}{\sqrt{3/12}\sqrt{2\pi}}\right) \exp\left(-\frac{1}{2}\left(\frac{\bar{x}-\mu}{\sqrt{3/12}}\right)^2\right) d\bar{x}.$$

Note that $\pi(7) = 0.05$.



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