Probability and Statistics. IDEA. Answers to List 7.

1.

$$\begin{aligned}
s^2 &= \frac{\sum_{i=1}^n (x_i - \overline{x})^2}{n-1} = \frac{1}{n-1}\left[\sum_{i=1}^n \left(x_i^2 - 2\overline{x}x_i + \overline{x}^2\right)\right] \\
&= \frac{\sum_{i=1}^n x_i^2}{n-1} - \frac{2\overline{x}\sum_{i=1}^n x_i}{n-1} + \frac{n\overline{x}^2}{n-1} = \frac{n\sum_{i=1}^n x_i^2}{n(n-1)} - \frac{\frac{2}{n}\left(\sum_{i=1}^n x_i\right)^2}{n-1} + \frac{n\frac{\left(\sum_{i=1}^n x_i\right)^2}{n^2}}{n-1} \\
&= \frac{n\sum_{i=1}^n x_i^2}{n(n-1)} - \frac{2\left(\sum_{i=1}^n x_i\right)^2}{n(n-1)} + \frac{\left(\sum_{i=1}^n x_i\right)^2}{n(n-1)} = \frac{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2}{n(n-1)}.
\end{aligned}$$

− − − − − − − − − − − − − − −

2. (a)

$$\begin{aligned}
\mathrm{E}(\bar{\boldsymbol{x}}_{\mathbf{n}}) &= \mathrm{E}(\tilde{x}_i) = \theta. \\
\mathrm{Var}(\bar{\boldsymbol{x}}_{\mathbf{n}}) &= \frac{\mathrm{Var}(\tilde{x}_i)}{n} = \frac{\theta(1-\theta)}{n}.
\end{aligned}$$

(b) $\tilde{y}_n = \sum\limits_{i=1}^n \tilde{x}_i$ and the $\tilde{x}_i$'s are independent. Then, since

$$M_{\tilde{x}_i}(t) = \mathrm{E}\left[e^{t\tilde{x}_i}\right] = \theta e^{1\cdot t} + (1-\theta)e^{0\cdot t} = 1 + \theta\left(e^t - 1\right),$$

we get

$$M_{\tilde{y}_n}(t) = M_{\sum\limits_{i=1}^n \tilde{x}_i}(t) = \left[M_{\tilde{x}_i}(t)\right]^n = \left[1 + \theta\left(e^t - 1\right)\right]^n$$

$\Longrightarrow \tilde{y}_n \sim \mathrm{B}(n, \theta)$ as $M_{\tilde{y}_n}(t)$ is the MGF of a binomial distribution with the parameters n and θ.

(c) Dividing numerator and denominator by n and, since $\mathrm{E}(\bar{\boldsymbol{x}}_{\mathbf{n}}) = \theta$ and

$\mathrm{Var}\left(\bar{\boldsymbol{x}}_{\mathbf{n}}\right) = \dfrac{\theta\left(1-\theta\right)}{n}$, from the Central Limit Theorem (CLT) we obtain

$$\frac{\tilde{y}_n - n\theta}{\sqrt{n\theta\left(1-\theta\right)}} = \frac{\frac{\tilde{y}_n}{n} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} = \frac{\bar{\boldsymbol{x}}_{\mathbf{n}} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \longrightarrow \mathrm{N}\left(0,1\right),$$

Note: This result (which is the the De Moivre-Laplace theorem) was already proved in a handout. Thus, we have shown in this exercise that the De Moivre-Laplace theorem is a particular case of the CLT.

$- - - - - - - - - - - - - - -$

3. $\tilde{x} \sim f\left(x\right),\ \mu = \mu_{\bar{\boldsymbol{x}}} = 75,\ \sigma^2 = 256,\ n = 100,\ \sigma_{\bar{\boldsymbol{x}}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{16}{10}.$

Chebyshev's inequality:

$$P\left\{\mu_{\bar{\boldsymbol{x}}} - k\sigma_{\bar{\boldsymbol{x}}} < \bar{\boldsymbol{x}} < \mu_{\bar{\boldsymbol{x}}} + k\sigma_{\bar{\boldsymbol{x}}}\right\} \geq 1 - \frac{1}{k^2}$$

$$\begin{array}{r|c}
\mu_{\bar{\boldsymbol{x}}} - k\sigma_{\bar{\boldsymbol{x}}} = 67 & \mu_{\bar{\boldsymbol{x}}} + k\sigma_{\bar{\boldsymbol{x}}} = 83 \\
75 - k\frac{16}{10} = 67 & 75 - k\frac{16}{10} = 83 \\
k = 5 & k = 5
\end{array}$$

Chebyshev's inequality tells us that:

$$P\left\{67 < \bar{\boldsymbol{x}} < 83\right\} \geq 1 - \frac{1}{5^2} = 0.96.$$

$- - - - - - - - - - - - - - -$

4. The Central Limit Theorem asserts that, if

$$\tilde{z} = \frac{\bar{\boldsymbol{x}} - \mu_{\bar{\boldsymbol{x}}}}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{\boldsymbol{x}} - 75}{\frac{16}{10}},$$

then

$$\tilde{z} \longrightarrow \mathrm{N}\left(0, 1\right).$$

Therefore, if

$$\begin{aligned} P\left\{67 < \bar{\boldsymbol{x}} < 83\right\} &= P\left\{\frac{67 - 75}{\frac{16}{10}} < \frac{\bar{\boldsymbol{x}} - 75}{\frac{16}{10}} < \frac{83 - 75}{\frac{16}{10}}\right\} \\ &\approx P\left\{-5 < \tilde{z} < 5\right\}, \end{aligned}$$

where $\tilde{z} \sim \mathrm{N}\left(0, 1\right)$. Then, from the table of the standard normal distribution we get,

$$P\left\{0 < \tilde{z} < 5\right\} = 0.4999$$

$\Longrightarrow$

$$P\left\{-5 < \tilde{z} < 5\right\} = 2 \cdot P\left\{0 < \tilde{z} < 5\right\} = 0.9998,$$

which is larger than 0.96.

– – – – – – – – – – – – – –

5. $\tilde{x} \sim \mathrm{N}\left(\mu, 25^2\right), n = 100, \mathrm{Var}(\bar{\boldsymbol{x}}) = \dfrac{25^2}{100}$

$\Longrightarrow \sigma_{\bar{\boldsymbol{x}}} = \sqrt{\mathrm{Var}\left(\bar{\boldsymbol{x}}\right)} = \sqrt{\dfrac{25^2}{100}} = \dfrac{25}{10} = 2.5 \Longrightarrow \bar{\boldsymbol{x}} \sim \mathrm{N}\left(\mu, 2.5\right).$

$$\begin{aligned}
\pi &\equiv P\left(\{(\bar{\boldsymbol{x}}-\mu)<-3\}\cup\{(\bar{\boldsymbol{x}}-\mu)>3\}\right)\\
&= 1-P\left\{-3<\bar{\boldsymbol{x}}-\mu<3\right\}=1-P\left\{-\frac{3}{\sigma_{\bar{\boldsymbol{x}}}}<\frac{\bar{\boldsymbol{x}}-\mu}{\sigma_{\bar{\boldsymbol{x}}}}<\frac{3}{\sigma_{\bar{\boldsymbol{x}}}}\right\}\\
&= 1-P\left\{-\frac{3}{2.5}<\tilde{z}<\frac{3}{2.5}\right\}=1-P\left\{-1.2<\tilde{z}<1.2\right\},
\end{aligned}$$

where $\tilde{z}\sim \mathrm{N}(0,1)$.

$$P\{0<\tilde{z}<1.2\}=0.3849 \text{ from the Table, so that}$$

$$\pi = 1-2\cdot(0.3849)=1-0.7698=0.2302.$$

$$- - - - - - - - - - - - - - - -$$

6. $\tilde{x}\sim\chi^2_{50}$, $\mathrm{E}(\tilde{x})=50$, $\mathrm{Var}(\tilde{x})=100$.

Therefore, the statement says that

$$P\{\tilde{x}>68\}=0.04596.$$

(a) Assuming that $\tilde{z}=\dfrac{\tilde{x}-50}{\sqrt{100}}$ is $\mathrm{N}(0,1)$, how close to 0.04596 does one get?

$$\begin{aligned}
\pi_1 &= P\{\tilde{x}>68\}=P\left\{\tilde{z}>\frac{68-50}{\sqrt{100}}\right\}=P\{\tilde{z}>1.8\}\\
&= 0.5-\overbrace{0.4641}^{\text{From Table}}=0.0359.
\end{aligned}$$

0.0359 is 78.1% of 0.04596.

(b) Assuming that $\tilde{w}\equiv\sqrt{2\tilde{x}}-\sqrt{2\cdot 50}$ is $\mathrm{N}(0,1)$, how close to 0.04596 does one get?

$$\begin{aligned}\pi_2 &= P\{\tilde{x} > 68\} = P\left\{\tilde{w} > \sqrt{2\cdot 68} - 10\right\} = P\{\tilde{w} > 1.66\} \\ &= 0.5 - \overbrace{0.4515}^{\text{From Table}} = 0.0485.\end{aligned}$$

0.0485 is 105.5% of 0.04596.

Therefore, in this case, $\left[\sqrt{2\tilde{x}} - \sqrt{2\nu}\right]$ seems better than $\left[\dfrac{\tilde{x}-\nu}{\sqrt{2\nu}}\right]$.

(c) Let $\tilde{x}_\nu \sim \chi^2_\nu$. We know that $\tilde{x}_v \overset{d}{=} \sum\limits_{i=1}^{\nu} \tilde{y}_i$, where $\tilde{y}_i \sim \chi^2_1$ and $\{\tilde{y}_i\}_{i=1}^{\nu}$ are independent. Thus,

$$\frac{\tilde{x}_v - \nu}{\sqrt{2\nu}} \overset{d}{=} \frac{\sum_{i=1}^{\nu} \tilde{y}_i - \nu}{\sqrt{2\nu}}.$$

Dividing by ν, we get

$$\frac{\sum_{i=1}^{\nu} \tilde{y}_i - \nu}{\sqrt{2\nu}} = \frac{\dfrac{\sum_{i=1}^{\nu} \tilde{y}_i}{\nu} - 1}{\sqrt{\dfrac{2}{\nu}}} \longrightarrow \mathrm{N}(0,1) \text{ as } \nu \to \infty \longleftarrow \text{Central Limit Theorem}$$

since $\dfrac{\sum_{i=1}^{\nu} \tilde{y}_i}{\nu}$ is the average of the i.i.d. random variables $\{\tilde{y}_i\}_{i=1}^{\nu}$ with

$$\mathrm{E}\left(\frac{\sum_{i=1}^{\nu} \tilde{y}_i}{\nu}\right) = \mathrm{E}\left(\frac{\tilde{x}_v}{\nu}\right) = \frac{1}{\nu}\cdot \nu = 1 = \mathrm{E}(\tilde{y}_i)$$

and

$$\mathrm{Var}\left(\frac{\sum_{i=1}^{\nu} \tilde{y}_i}{\nu}\right) = \frac{\mathrm{Var}(\tilde{x}_v)}{\nu^2} = \frac{2\nu}{\nu^2} = \frac{2}{\nu} = \frac{\mathrm{Var}(\tilde{y}_i)}{\nu}.$$

We have thus proved that

$$\frac{\tilde{x}_v - \nu}{\sqrt{2\nu}} \longrightarrow \mathrm{N}(0,1) \text{ as } \nu \to \infty,$$

which justifies the approximation given in (a).

Alternative proof: Note that the chi-square distribution χ^2_ν is a particular case of the gamma distribution $\Gamma(\alpha, \beta)$ when $\alpha = \nu/2$ and $\beta = 2$. As we know from Exercise 17 of list 6, the standardized gamma random variable converges in distribution to a standard normal random variable when $\alpha \to \infty$ and β remains constant. Therefore if $\tilde{x}_\nu \sim \chi^2_\nu$ (i.e., $\tilde{x}_v \sim \Gamma(\nu/2, 2)$), then

$$\frac{\tilde{x}_\nu - \alpha\beta}{\sqrt{\alpha\beta^2}} = \frac{\tilde{x}_\nu - \nu}{\sqrt{2\nu}} \longrightarrow \mathrm{N}\,(0,1) \;\text{ as } \alpha \to \infty, \text{ i.e., as } \nu \to \infty.$$

To justify the approximation given in (b), observe that the inequality $\sqrt{2\tilde{x}_\nu} - \sqrt{2\nu} \leq k$ becomes $\sqrt{2\tilde{x}_\nu} \leq k + \sqrt{2\nu}$, which is equivalent to $2\tilde{x}_\nu \leq \left(k + \sqrt{2\nu}\right)^2 = k^2 + 2\nu + 2k\sqrt{2\nu}$. Rearranging, we get $2\tilde{x}_\nu - 2\nu \leq k^2 + 2k\sqrt{2\nu}$ and, dividing by $2\sqrt{2\nu}$, we finally obtain $\dfrac{\tilde{x}_\nu - \nu}{\sqrt{2\nu}} \leq k + \dfrac{k^2}{2\sqrt{2\nu}}$. Therefore,

$$\begin{aligned}
F_{\sqrt{2\tilde{x}_\nu}-\sqrt{2\nu}}(k) \quad &= \quad \underbrace{P\left\{\sqrt{2\tilde{x}_\nu} - \sqrt{2\nu} \leq k\right\}}_{\text{distribution function of } \sqrt{2\tilde{x}_\nu}-\sqrt{2\nu}} = P\left\{\frac{\tilde{x}_\nu - \nu}{\sqrt{2\nu}} \leq k + \frac{k^2}{2\sqrt{2\nu}}\right\} \\
&\longrightarrow \quad \underbrace{P\left\{\frac{\tilde{x}_\nu - \nu}{\sqrt{2\nu}} \leq k\right\}}_{\text{distribution function of } \frac{\tilde{x}_\nu-\nu}{\sqrt{2\nu}}} = F_{\frac{\tilde{x}_\nu-\nu}{\sqrt{2\nu}}}(k) \quad \text{as } \nu \longrightarrow \infty.
\end{aligned}$$

Thus,

$$\left(\sqrt{2\tilde{x}_\nu} - \sqrt{2\nu}\right) - \left(\frac{\tilde{x}_\nu - \nu}{\sqrt{2\nu}}\right) \xrightarrow{d} 0 \text{ as } \nu \longrightarrow \infty$$

so that

$$\sqrt{2\tilde{x}_\nu} - \sqrt{2\nu} \longrightarrow \mathrm{N}\,(0,1) \;\text{ as } \nu \to \infty.$$

— — — — — — — — — — — — — — —

7.

$$f(y) = \begin{cases} \dfrac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\dfrac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y^{\frac{\nu_1}{2}-1} \left(1+\dfrac{\nu_1}{\nu_2}y\right)^{-\frac{1}{2}(\nu_1+\nu_2)} & \text{for } y > 0 \\ \\ 0 & \text{elsewhere.} \end{cases}$$

For $\nu_1 = \nu_2 = 4$

$$f(y) = \begin{cases} \dfrac{\Gamma(4)}{\Gamma(2)\Gamma(2)} \cdot 1 \cdot y^1 (1 + 1 \cdot y)^{-\frac{1}{2}8} = 6y(1+y)^{-4} & \text{for } y > 0 \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Let

$$\tilde{w} = \frac{\frac{\boldsymbol{s_1^2}}{\sigma^2}}{\frac{\boldsymbol{s_2^2}}{\sigma^2}} = \frac{\boldsymbol{s_1^2}}{\boldsymbol{s_2^2}}, \text{ where } \boldsymbol{s_1} \text{ and } \boldsymbol{s_2} \text{ are independent}$$

$\tilde{w} \sim \mathrm{F}_{4,4}$ will have the density

$$f(w) = \begin{cases} 6w(1+w)^{-4} & \text{for } w > 0 \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Define

$$\pi \equiv P\left(\{\tilde{w} \leq 1/2\} \cup \{\tilde{w} \geq 2\}\right) = 1 - P\{1/2 < \tilde{w} < 2\} = 1 - \int_{\frac{1}{2}}^{2} 6w(1+w)^{-4}\, dw.$$

Evaluate

$$\int_{1/2}^{2} 6w(1+w)^{-4}\, dw$$

by integration by parts:

$$
\begin{array}{r|c}
F(w) = w & G'(w) = (1+w)^{-4} \\
F'(w) = 1 & G(w) = \frac{(1+w)^{-3}}{(-3)}
\end{array}
$$

$$
\begin{aligned}
6\int_{1/2}^{2} w\,(1+w)^{-4}\,dw &= 6\left\{\left[\frac{w}{-3\,(1+w)^3}\right]_{1/2}^{2} - \int_{1/2}^{2}\frac{(1+w)^{-3}}{(-3)}dw\right\} \\
&= (-2)\left[\frac{2}{27} - \frac{\frac{1}{2}}{\frac{27}{8}}\right] + 2\int_{1/2}^{2}(1+w)^{-3}\,dw \\
&= -2\left[\frac{2-4}{27}\right] + 2\left[\frac{(1+w)^{-2}}{(-2)}\right]_{1/2}^{2} = \frac{4}{27} - \left[\frac{1}{(1+w)^2}\right]_{1/2}^{2} \\
&= \frac{4}{27} - \left[\frac{1}{9} - \frac{1}{9/4}\right] = \frac{4}{27} - \left[\frac{1}{9} - \frac{4}{9}\right] = \frac{4}{27} + \frac{3}{9} = \frac{13}{27}.
\end{aligned}
$$

Hence,

$$
\pi \equiv P\left(\{\tilde{w} \le 1/2\} \cup \{\tilde{w} \ge 2\}\right) = 1 - \frac{13}{27} = \frac{14}{27}.
$$

$- - - - - - - - - - - - - - -$

8. $\tilde{x} \sim \mathrm{F}_{\nu_1,\nu_2}$

$$
f_{\tilde{x}}(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1}\left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{1}{2}(\nu_1+\nu_2)}, \text{ for } x > 0.
$$

and $f_{\tilde{x}}(x) = 0$, otherwise.

$\tilde{y} = \dfrac{1}{\tilde{x}} \to \tilde{x} = \dfrac{1}{\tilde{y}} \quad (\tilde{y} > 0)$. Then,

$\dfrac{dx}{dy} = -\dfrac{1}{y^2}, \qquad \left|\dfrac{dx}{dy}\right| = \dfrac{1}{y^2}.$

For $y>0$ we have

$$
\begin{aligned}
f_{\tilde{y}}(y) &= \underbrace{\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}}_{\equiv K}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y^{1-\frac{\nu_1}{2}}\left(1+\frac{\nu_1}{\nu_2}\frac{1}{y}\right)^{-\frac{1}{2}(\nu_1+\nu_2)}\frac{1}{y^2} \\
&= K\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y^{-1}y^{-\frac{\nu_1}{2}}\left(1+\frac{\nu_1}{\nu_2}\frac{1}{y}\right)^{-\frac{1}{2}(\nu_1+\nu_2)} \\
&= K\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}\overbrace{\left(\frac{\nu_2}{\nu_1}\right)^{-\frac{\nu_2}{2}}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}}}^{=1}\left(y^{-1}\right)\left(y^{-\frac{\nu_1}{2}}\right)\overbrace{\left(y^{-\frac{\nu_2}{2}}\right)\left(y^{\frac{\nu_2}{2}}\right)}^{=1}\left[1+\frac{\nu_1}{\nu_2}\frac{1}{y}\right]^{-\frac{1}{2}(\nu_1+\nu_2)} \\
&= K\left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}}\left[\left(\frac{\nu_2}{\nu_1}\right)^{-\frac{1}{2}(\nu_1+\nu_2)}\left(y^{-\frac{1}{2}(\nu_1+\nu_2)}\right)\right]\left(y^{\frac{\nu_2}{2}-1}\right)\left[1+\frac{\nu_1}{\nu_2}\frac{1}{y}\right]^{-\frac{1}{2}(\nu_1+\nu_2)} \\
&= K\left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}} y^{\frac{\nu_2}{2}-1}\left\{\left(\frac{\nu_2}{\nu_1}\right)y\left[1+\frac{\nu_1}{\nu_2}\frac{1}{y}\right]\right\}^{-\frac{1}{2}(\nu_1+\nu_2)} \\
&= K\left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}} y^{\frac{\nu_2}{2}-1}\left\{1+\frac{\nu_2}{\nu_1}y\right\}^{-\frac{1}{2}(\nu_1+\nu_2)}.
\end{aligned}
$$

Hence,

$$
f_{\tilde{y}}(y)=\begin{cases}\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}\left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}} y^{\frac{\nu_2}{2}-1}\left[1+\frac{\nu_2}{\nu_1}y\right]^{-\frac{1}{2}(\nu_1+\nu_2)}, & \text{for } y>0 \\ \\ 0 \quad \text{otherwise} & \end{cases}
$$

$\Longrightarrow \tilde{y}\sim \mathrm{F}_{\nu_2,\nu_1}$.

Alternatively, we know that if $\tilde{x}\sim \mathrm{F}_{\nu_1,\nu_2}$ then $\tilde{x}\overset{d}{=}\dfrac{\tilde{u}_1/\nu_1}{\tilde{u}_2/\nu_2}$ where $\tilde{u}_1\sim\chi^2_{\nu_1},\quad \tilde{u}_2\sim\chi^2_{\nu_2}$, and $\tilde{u}_1$ and $\tilde{u}_2$ are independent. Therefore,

$$
\tilde{y}=\frac{1}{\tilde{x}}\overset{d}{=}\frac{\tilde{u}_2/\nu_2}{\tilde{u}_1/\nu_1}\sim \mathrm{F}_{\nu_2,\nu_1}.
$$

$-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-$

9. If $\tilde{x} \sim \mathrm{F}_{\nu_1,\nu_2}$, we should show that

$$\nu_1 \tilde{x} \to \chi^2_{\nu_1} \text{ as } \nu_2 \to \infty.$$

We know that

$$\tilde{x} \stackrel{d}{=} \frac{\tilde{u}_1/\nu_1}{\tilde{u}_2/\nu_2}, \text{ where } \tilde{u}_1 \sim \chi^2_{\nu_1}, \quad \tilde{u}_2 \sim \chi^2_{\nu_2}, \quad \text{and } \tilde{u}_1 \text{ and } \tilde{u}_2 \text{ are independent.}$$

Note that $\mathrm{E}(\tilde{u}_2) = \nu_2$ and $\mathrm{Var}(\tilde{u}_2) = 2\nu_2$.

Let us define the random variable $\tilde{w}$ as

$$\tilde{w} = \frac{\tilde{u}_2}{\nu_2}.$$

Then

$$\begin{aligned} \mathrm{E}(\tilde{w}) &= \frac{\nu_2}{\nu_2} = 1 \\ \mathrm{Var}(\tilde{w}) &= \frac{1}{\nu_2^2} 2\nu_2 = \frac{2}{\nu_2} \end{aligned}$$

so that

$$\lim_{\nu_2 \to \infty} \mathrm{Var}(\tilde{w}) = 0.$$

Therefore,

$$\tilde{w} = \frac{\tilde{u}_2}{\nu_2} \xrightarrow{m} 1 \Longrightarrow \frac{\tilde{u}_2}{\nu_2} \xrightarrow{p} 1 \Longleftrightarrow \frac{\tilde{u}_2}{\nu_2} \xrightarrow{d} 1 \text{ when } \nu_2 \to \infty.$$

Then, due to Slutsky's theorem,

$$\tilde{x} \xrightarrow{d} \frac{\tilde{u}_1 / \nu_1}{1} = \frac{\tilde{u}_1}{\nu_1} \quad \text{when } \nu_2 \to \infty.$$

or

$$\nu_1 \tilde{x} \xrightarrow{d} \tilde{u}_1 \text{ when } \nu_2 \to \infty.$$

Therefore, as $\nu_2 \to \infty$

$$\nu_1 \tilde{x} \to \chi^2_{\nu_1}.$$

$$- - - - - - - - - - - - - - -$$

10. $\tilde{x} \sim \mathrm{t}_\nu$. Let $\tilde{w} = \tilde{x}^2$. We want to show that $\tilde{w} \sim \mathrm{F}_{1,\nu}$.

$$\tilde{x} \overset{d}{=} \frac{\tilde{z}}{\sqrt{\tilde{y} / \nu}} \text{ with } \tilde{z} \sim \mathrm{N}\left(0, 1\right), \; \tilde{y} \sim \chi^2_\nu, \text{ with } \tilde{z} \text{ and } \tilde{y} \text{ independent.}$$

$$\tilde{w} = \tilde{x}^2 \overset{d}{=} \frac{\tilde{z}^2}{\tilde{y} / \nu} \text{ where } \tilde{z}^2 \sim \chi^2_1, \quad \tilde{y} \sim \chi^2_\nu, \quad \text{with } \tilde{z}^2 \text{ and } \tilde{y} \text{ independent.}$$

$$\Longrightarrow \tilde{w} \sim \mathrm{F}_{1,\nu}.$$

$$- - - - - - - - - - - - - - -$$

11. The claim $\sigma^2 = 25$ is rejected if either $s^2_{16} < 12.102$ or $s^2_{16} > 54.668$.

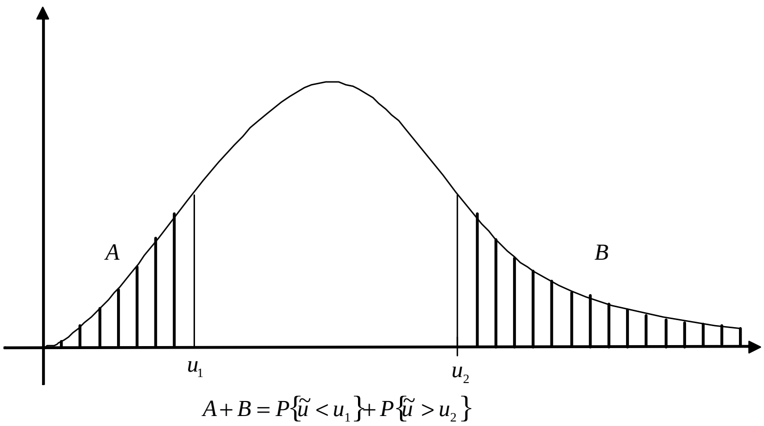
Define

$$\pi \equiv P\left(\left\{\boldsymbol{s}^{\mathbf{2}}_{16} < 12.102\right\} \cup \left\{\boldsymbol{s}^{\mathbf{2}}_{16} > 54.668\right\}\right).$$

$$\tilde{u} \equiv \frac{15 \boldsymbol{s}^{\mathbf{2}}_{16}}{\sigma^2} = \frac{15}{25} \boldsymbol{s}^{\mathbf{2}}_{16} = \frac{3}{5} \boldsymbol{s}^{\mathbf{2}}_{16} \to \boldsymbol{s}^{\mathbf{2}}_{16} = \frac{5}{3} \tilde{u}.$$

$$\tilde{u} \sim \chi^2_{15}.$$

$$
\begin{aligned}
\pi &= P\left(\left\{\frac{5}{3}\tilde{u} < 12.102\right\} \cup \left\{\frac{5}{3}\tilde{u} > 54.668\right\}\right) \\
&= P\left(\{\tilde{u} < 7.2612\} \cup \{\tilde{u} > 32.80\}\right) \\
&= P\{\tilde{u} < 7.2612\} + P\{\tilde{u} > 32.80\} \\
&= \left(1 - \overset{\text{Table}}{0.95}\right) + \overset{\text{Table}}{(0.005)} \\
&= 0.055.
\end{aligned}
$$



— — — — — — — — — — — — — — — —

12.

$$\tilde{x}_n = \frac{(n-1)\,\boldsymbol{s}_n^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\sqrt{2\tilde{x}_n} - \sqrt{2(n-1)} \overset{a}{\sim} \mathrm{N}(0,1)$$

$$\sqrt{\frac{2(n-1)\,\boldsymbol{s}_n^2}{\sigma^2}} - \sqrt{2(n-1)} \overset{a}{\sim} \mathrm{N}(0,1)$$

$$\left(\sqrt{\frac{2(n-1)}{\sigma^2}}\right)\boldsymbol{s}_n - \sqrt{2(n-1)} \overset{a}{\sim} \mathrm{N}(0,1)$$

$$\frac{2(n-1)}{\sigma^2}\mathrm{Var}(\boldsymbol{s}_n) \approx 1 \text{ for } n \text{ large}$$

$\Longrightarrow$

$$\mathrm{Var}\left(\boldsymbol{s}_n\right) \approx \frac{\sigma^2}{2\left(n-1\right)} \text{ for } n \text{ large.}$$

$- - - - - - - - - - - - - - - -$

13 (a)

$$f\left(x_1, x_2, ..., x_n\right) = \frac{1}{N\left(N-1\right)...\left(N-n+1\right)} \quad \text{for } x_i = c_1, c_2, ..., c_N$$
$$i = 1, 2, ..., n$$
$$x_1 \neq x_2 \neq x_3 \neq ... \neq x_n.$$

(b)

$$\frac{n!}{N\left(N-1\right)...\left(N-n+1\right)} = \frac{1}{\binom{N}{n}}$$

(c)

$$f\left(x_i\right) = \frac{1}{N}, \text{ for } x_i = c_1, c_2, ..., c_N.$$

(d)

$$\begin{aligned} \mu &= \sum_{i=1}^{N} c_i \frac{1}{N}. \\ \sigma^2 &= \sum_{i=1}^{N} \left(c_i - \mu\right)^2 \frac{1}{N} = \frac{1}{N}\sum_{i=1}^{N} c_i^2 - \mu^2. \end{aligned}$$

(e)

$$g\left(x_i, x_j\right) = \frac{1}{N\left(N-1\right)} \quad \text{for } x_k = c_1, c_2, ..., c_N; k = i, j; x_i \neq x_j.$$

(f)

$$
\begin{aligned}
\operatorname{Cov}\left(\tilde{x}_i, \tilde{x}_j\right) &= \sum_{i \neq j} \sum_{j=1}^{N} \frac{1}{N\left(N-1\right)}\left(c_i-\mu\right)\left(c_j-\mu\right) \\
&= \frac{1}{N\left(N-1\right)}\left[\sum_{i=1}^{N} c_i \sum_{j \neq i} c_j - \underbrace{\sum_{j \neq i} \sum_{i=1}^{N} c_i \mu}_{N(N-1)\mu^2} - \underbrace{\sum_{i \neq j} \sum_{j=1}^{N} c_j \mu}_{N(N-1)\mu^2} + \underbrace{\sum_{i \neq j} \sum_{j=1}^{N} \mu^2}_{N(N-1)\mu^2}\right] \\
&= \frac{1}{N\left(N-1\right)}\left[\sum_{i=1}^{N} \sum_{j \neq i} c_i c_j - N\left(N-1\right)\mu^2\right] \\
&= \frac{1}{N\left(N-1\right)}\left[\underbrace{\left[\sum_{i=1}^{N} c_i\right]^2}_{N^2\mu^2} - \underbrace{\sum_{i=1}^{N} c_i^2}_{N(\sigma^2+\mu^2)}\right] - \mu^2 \\
&= \frac{1}{N\left(N-1\right)}\left[N\left(N-1\right)\mu^2 - N\sigma^2\right] - \mu^2 = -\frac{\sigma^2}{N-1}.
\end{aligned}
$$

(g)

$$
\begin{aligned}
\operatorname{Var}\left(\bar{\boldsymbol{x}}\right) &= \operatorname{Var}\left(\frac{\sum_{i=1}^{n} \tilde{x}_i}{n}\right) = \frac{1}{n^2} \operatorname{Var}\left(\sum_{i=1}^{n} \tilde{x}_i\right) \\
&= \frac{1}{n^2}\left[\sum_{i=1}^{n} \sigma^2 + \sum_{j \neq i} \sum_{i=1}^{n}\left(-\frac{\sigma^2}{N-1}\right)\right] \\
&= \frac{1}{n^2}\left[n\sigma^2 + n\left(n-1\right)\left(-\frac{\sigma^2}{N-1}\right)\right] \\
&= \frac{\sigma^2}{n} + \frac{1}{n}\left(n-1\right)\left(-\frac{\sigma^2}{N-1}\right) \\
&= \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right).
\end{aligned}
$$

(h)

$$\lim_{N\to\infty} \mathrm{Var}\left(\bar{\boldsymbol{x}}\right) = \frac{\sigma^2}{n}.$$

Note that, as the set C becomes larger, $\mathrm{Var}\left(\bar{\boldsymbol{x}}\right)$ tends to the variance of the mean of a "random" sample.

$- - - - - - - - - - - - - - -$

14. (a) If $\tilde{z} \sim \mathrm{N}(0,1)$ and $\tilde{x} = |\tilde{z}|$, then

$$f_{\tilde{x}}(x) = \begin{cases} 2n(x;\,0,1) = \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

From independence we have

$$f(x,y) = f_{\tilde{x}}(x) f_{\tilde{y}}(y) = \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \text{ for } x \in (0,\infty), y \in (-\infty,\infty).$$

Then,

$$f(x,y) = \begin{cases} \frac{1}{\pi}e^{-\frac{1}{2}(x^2+y^2)} & \text{for } x \in (0,\infty),\ y \in (-\infty,\infty) \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\mathrm{E}(\tilde{x}) = \int_0^\infty \int_{-\infty}^\infty x f(x,y) dy dx$$

$$= \underbrace{\left[\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy\right]}_{\text{Area under the standard normal density=1}} \cdot \left[\int_0^\infty x\frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\right]$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-\frac{1}{2}x^2}dx = \sqrt{\frac{2}{\pi}}\left[-e^{-\frac{1}{2}x^2}\right]_0^\infty = \sqrt{\frac{2}{\pi}}[0-(-1)] = \sqrt{\frac{2}{\pi}}$$

$$\tilde{y} \sim \mathrm{N}(0,1) \Rightarrow \mathrm{E}(\tilde{y}) = 0 \text{ and } \mathrm{Var}(\tilde{y}) = 1.$$

$$\mathrm{E}(\tilde{x}^2) = \underbrace{\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy\right]}_{\text{Area under the standard normal density} = 1} \cdot \left[\int_{0}^{\infty} \underbrace{x^2 \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}_{\text{symmetric function of } x \text{ with respect to } 0} dx\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{2}{2}\cdot \underbrace{\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx}_{\text{Variance of the standard normal distribution} = 1}$$

$$\Rightarrow \mathrm{E}(\tilde{x}^2) = 1, \quad \mathrm{Var}(\tilde{x}) = \mathrm{E}(\tilde{x}^2) - \mathrm{E}(\tilde{x})^2 = 1 - \frac{2}{\pi} = \frac{\pi - 2}{\pi}.$$

$\mathrm{Cov}(\tilde{x}, \tilde{y}) = 0$, since $\tilde{x}$ and $\tilde{y}$ are independent.

(c) Let

$$(v, s) = g(x, y) = \begin{cases} v = \dfrac{y}{x} \in (-\infty, \infty) \\ s = x \in (0, \infty) \end{cases}$$

$$(x, y) = g^{-1}(v, s) = \begin{cases} x = s \in (0, \infty) \\ y = vs \in (-\infty, \infty) \end{cases}$$

Then,

$$J_{g-1}(v, s) = \begin{pmatrix} 0 & 1 \\ s & v \end{pmatrix} \Rightarrow |J_{g-1}(v, s)| = |-s| = s$$

and

$$f_{(\tilde{v},\tilde{s})}(v, s) = \begin{cases} \frac{1}{\pi} e^{-\frac{1}{2}\left(s^2 + v^2 s^2\right)} s & \text{for } v \in (-\infty, \infty),\ s \in (0, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$f_{\tilde{v}}(v) = \int_0^{\infty} \frac{1}{\pi} e^{-\frac{1}{2}s^2(1+v^2)} s ds = \frac{1}{\pi}\left[-\frac{1}{1+v^2} e^{-\frac{1}{2}s^2(1+v^2)}\right]_0^{\infty}$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+v^2}, \quad \text{for } v \in (-\infty, \infty).$$

(d) We know that $\tilde{y} \sim \mathrm{N}(0,1)$. Let $\tilde{z} \sim \mathrm{N}(0,1)$. Then $\tilde{z}^2 \sim \chi_1^2$. Moreover, $\tilde{x} = \sqrt{\tilde{z}^2} = |\tilde{z}|$. Therefore, since $\tilde{x}$ and $\tilde{y}$ are independent,

$$\tilde{v} = \frac{\tilde{y}}{\tilde{x}} = \frac{\tilde{y}}{|\tilde{z}|} = \frac{\tilde{y}}{\sqrt{\tilde{z}^2}} \sim \mathrm{t}_1, \text{ as } \tilde{y} \sim \mathrm{N}(0,1) \text{ and } \tilde{z}^2 \sim \chi_1^2.$$

– – – – – – – – – – – – – – – – – –

15. Recall that

$$\mathrm{E}\left(\bar{\boldsymbol{x}}\right) = \mu$$

$$\mathrm{Var}\left(\bar{\boldsymbol{x}}\right) = \frac{\sigma^2}{n} \qquad (\star)$$

(a) Obviously, $\mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_j\right) = 0$ for $i \neq j$ since the random variables in the random sample $\{\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n\}$ are i.i.d.

(b)

$$\mathrm{Cov}\left(\bar{\boldsymbol{x}}, \tilde{x}_i\right) = \mathrm{Cov}\left(\frac{\sum_{j=1}^{n} \tilde{x}_j}{n}, \tilde{x}_i\right) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{Cov}\left(\tilde{x}_j, \tilde{x}_i\right) \ \text{(from Ex. 29(a) of List 3)}$$

$$= \frac{1}{n}\left[\mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_i\right) + \sum_{j \neq i} \mathrm{Cov}\left(\tilde{x}_j, \tilde{x}_i\right)\right] = \frac{1}{n}\left[\mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_i\right) + 0\right] \qquad \text{(from (a))}$$

$$= \frac{1}{n}\mathrm{Var}\left(\tilde{x}_i\right) = \frac{\sigma^2}{n}.$$

(c)

$$\mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_i - \bar{\boldsymbol{x}}\right) = \mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_i\right) - \mathrm{Cov}\left(\tilde{x}_i, \bar{\boldsymbol{x}}\right) \qquad \text{(from Ex. 29(a) of List 3)}$$

$$= \mathrm{Var}\left(\tilde{x}_i\right) - \frac{\sigma^2}{n} \qquad \text{(from (b))}$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \left(\frac{n-1}{n}\right)\sigma^2.$$

(d)

$$\mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_j - \bar{\boldsymbol{x}}\right) = \mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_j\right) - \mathrm{Cov}\left(\tilde{x}_i, \bar{\boldsymbol{x}}\right) \qquad \text{(from Ex. 29(a) of List 3)}$$

$$= 0 - \mathrm{Cov}\left(\tilde{x}_i, \bar{\boldsymbol{x}}\right) = -\frac{\sigma^2}{n}. \qquad \text{(from (a) and (b))}$$

(e)

$$\mathrm{Var}\left(\tilde{x}_i - \bar{\boldsymbol{x}}\right) = \mathrm{Var}\left(\tilde{x}_i\right) + \mathrm{Var}\left(\bar{\boldsymbol{x}}\right) - 2\mathrm{Cov}\left(\bar{\boldsymbol{x}}, \tilde{x}_i\right)$$

$$= \sigma^2 + \frac{\sigma^2}{n} - 2\left(\frac{\sigma^2}{n}\right) = \sigma^2 - \frac{\sigma^2}{n} = \left(\frac{n-1}{n}\right)\sigma^2. \qquad \text{(from } (\star) \text{ and (b))}$$

(f)

$$\mathrm{Cov}\left(\tilde{x}_i - \bar{\boldsymbol{x}}, \tilde{x}_j - \bar{\boldsymbol{x}}\right) =$$

$$\mathrm{Cov}\left(\tilde{x}_i, \tilde{x}_j\right) - \mathrm{Cov}\left(\bar{\boldsymbol{x}}, \tilde{x}_i\right) + \underbrace{\mathrm{Cov}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}\right)}_{\mathrm{Var}(\bar{\boldsymbol{x}})} - \mathrm{Cov}\left(\bar{\boldsymbol{x}}, \tilde{x}_j\right) \text{ (from Ex. 29(b) of List3)}$$

$$= 0 - \frac{\sigma^2}{n} + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = -\frac{\sigma^2}{n}. \qquad \text{(from } (\star)\text{, (a), and (b))}$$

(g)

$$\mathrm{Cov}\left(\bar{\boldsymbol{x}}, \tilde{x}_i - \bar{\boldsymbol{x}}\right) = \mathrm{Cov}\left(\bar{\boldsymbol{x}}, \tilde{x}_i\right) - \mathrm{Cov}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}\right) \qquad \text{(from Ex. 29(a) of List 3)}$$

$$= \mathrm{Cov}\left(\bar{\boldsymbol{x}}, \tilde{x}_i\right) - \mathrm{Var}\left(\bar{\boldsymbol{x}}\right) = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0. \qquad \text{(from } (\star) \text{ and (b))}$$

– – – – – – – – – – – – – – – – – –

16. (a)

$$\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \left(1+\frac{1}{\frac{\nu}{x^2}}\right)^{\frac{\nu}{x^2}\cdot\left(-\frac{x^2}{2}\right)} \cdot \left(1+\frac{1}{\frac{\nu}{x^2}}\right)^{-\frac{1}{2}}.$$

Let $y = \dfrac{\nu}{x^2}$ so that $y \to \infty$ if $\nu \to \infty$ for $-\infty < x < \infty$. Therefore,

$$\lim_{\nu\to\infty}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^{y\cdot\left(-\frac{x^2}{2}\right)} \cdot \left(1+\frac{1}{y}\right)^{-\frac{1}{2}}\right]$$

$$= \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y\cdot\left(-\frac{x^2}{2}\right)} \cdot \underbrace{\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{-\frac{1}{2}}}_{1}$$

$$= \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^{-\frac{x^2}{2}} = e^{-\frac{1}{2}x^2} \quad \text{for} \quad -\infty < x < \infty.$$

Hence,

$$\lim_{\nu\to\infty}\underbrace{\frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\cdot\Gamma\left(\dfrac{\nu}{2}\right)}}_{K(\nu)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \left[\lim_{\nu\to\infty}K(\nu)\right]e^{-\frac{1}{2}x^2} \quad \text{for} \quad -\infty < x < \infty.$$

Thus, $\lim\limits_{\nu\to\infty}K(\nu)$ has to be a constant such that $\int_{-\infty}^{\infty}\left[\lim\limits_{\nu\to\infty}K(\nu)\right]e^{-\frac{1}{2}x^2}dx = 1$. Therefore, from the standard normal density, we must have

$$\lim_{\nu\to\infty}\frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\cdot\Gamma\left(\dfrac{\nu}{2}\right)} = \frac{1}{\sqrt{2\pi}} \quad \text{or} \quad \lim_{\nu\to\infty}\frac{\sqrt{2}\cdot\Gamma\left(\dfrac{\nu+1}{2}\right)}{\sqrt{\nu}\cdot\Gamma\left(\dfrac{\nu}{2}\right)} = 1.$$

(b)

$$\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \left(1+x^2\right)^{-1} = \frac{1}{1+x^2} \quad \text{when } \nu = 1.$$

Therefore,

$$K(\nu)\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} = K(1)\cdot\frac{1}{1+x^2} \quad \text{when} \quad \nu = 1.$$

Note that $K(1)$ has to be a constant such that $\int_{-\infty}^{\infty} K(1)\dfrac{1}{1+x^2}dx = 1$. The Cauchy density with $\alpha = 0$ and $\beta = 1$ is

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \quad \text{for} \ -\infty < x < \infty,$$

so that $K(1) = \dfrac{1}{\pi}$. Alternatively, note also that

$$K(1) = \frac{\Gamma(1)}{\sqrt{\pi}\cdot\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}\cdot\sqrt{\pi}} = \frac{1}{\pi}.$$

(c) Since

$$P(C|\mathbb{I}_A = 1) = \frac{P(C\cap A)}{P(A)}, \quad \text{for all } C \in \mathcal{F},$$

we have

$$P(C|\mathbb{I}_A = 1) = \frac{P(C)}{P(A)}, \quad \text{for all } C \in \mathcal{F} \ \text{ with } C \subset A,$$

so that

$$\mathrm{E}(\tilde{x}|\mathbb{I}_A = 1) = \int_A \frac{1}{P(A)}\tilde{x}dP = \int_{[0,1]} \frac{1}{P(A)}xf(x)dx.$$

In fact, the conditional density of $\tilde{x}$ given $A = \{\tilde{x} \in [0,1]\}$ is

$$f(x\,|A) = \begin{cases} \dfrac{f(x)}{P(A)} & \text{if } x \in (0,1) \\ \\ 0 & \text{otherwise,} \end{cases}$$

as follows from Exercise 14 of List 2.

$$P(A) = P_{\tilde{x}}\left[0,1\right] = \int_0^1 \frac{1}{\pi}\frac{1}{1+x^2}dx = \frac{1}{\pi}\left[\arctan x\right]_0^1 = \frac{1}{\pi}\frac{\pi}{4} = \frac{1}{4}.$$

Then,

$$\mathrm{E}\left(\tilde{x} \mid \tilde{x} \in [0,1]\right) = \int_0^1 \frac{1}{1/4}x\frac{1}{\pi}\frac{1}{1+x^2}dx = \frac{4}{\pi}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{2}{\pi}\ln 2.$$

– – – – – – – – – – – – – – – –

17. (a) The uniform density on $(0,1)$ is

$$f(x) = \begin{cases} 1 & \text{for } x \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

and its distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } x \in (0,1) \\ 1 & \text{for } x \geq 1. \end{cases}$$

Then,

$$f_{\tilde{x}_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}f(x)\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j}$$

$$= \begin{cases} \dfrac{n!}{(j-1)!(n-j)!} x^{j-1} (1-x)^{n-j} & \text{for } x \in (0,1) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Note that the previous density $f_{\tilde{x}_{(j)}}$ is the density of a beta distribution with the parameters $\alpha = j$ and $\beta = n - j + 1$. Observe that, since n and j are strictly positive integers,

$$\frac{\Gamma(a+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} = \frac{n!}{(j-1)!(n-j)!}.$$

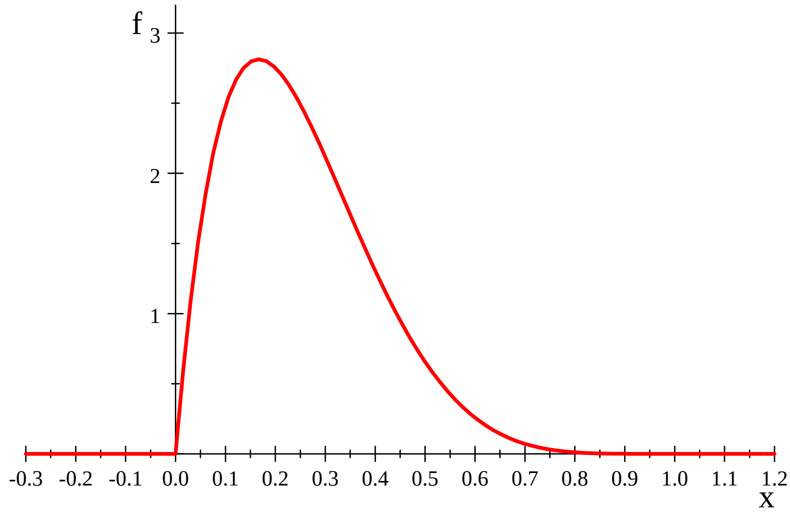
Therefore,

$$\mathrm{E}\left(\tilde{x}_{(j)}\right) = \frac{\alpha}{\alpha+\beta} = \frac{j}{j+(n-j+1)} = \frac{j}{n+1}$$

If $j = 2$ and $n = 7$,

$$f_{\tilde{x}_{(2)}}(x) = \begin{cases} \dfrac{7!}{1!5!} x^{1} (1-x)^{5} = 42x(1-x)^{5} & \text{for } x \in (0,1) \\ \\ 0 & \text{otherwise.} \end{cases}$$

The plot of the density of the 2nd order statistic with $n = 7$ is



Note that the density of $\tilde{x}_{(2)}$ reaches its maximum at $x = 1/6 = 0.16667$.

Moreover, in this case,

$$\mathrm{E}\left(\tilde{x}_{(2)}\right) = \frac{j}{n+1} = \frac{2}{7+1} = \frac{1}{4}.$$

(b)

$$\mathrm{E}\left(\tilde{x}_{\text{med}}\right) = \mathrm{E}\left(\tilde{x}_{\left(\frac{n+1}{2}\right)}\right) = \frac{j}{n+1} = \frac{\left(\frac{n+1}{2}\right)}{n+1} = \frac{1}{2} \quad \text{if } n \text{ is odd,}$$

$$\mathrm{E}\left(\tilde{x}_{\text{med}}\right) = \mathrm{E}\left(\frac{\tilde{x}_{\left(\frac{n}{2}\right)} + \tilde{x}_{\left(\frac{n}{2}+1\right)}}{2}\right) = \frac{1}{2}\left[\mathrm{E}\left(\tilde{x}_{\left(\frac{n}{2}\right)}\right) + \mathrm{E}\left(\tilde{x}_{\left(\frac{n}{2}+1\right)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\left(\frac{n}{2}\right)}{n+1} + \frac{\left(\frac{n}{2}+1\right)}{n+1}\right] = \frac{1}{2} \quad \text{if } n \text{ is even.}$$

Thus, $\mathrm{E}\left(\tilde{x}_{\text{med}}\right) = 1/2$ for all sample sizes.

Since $n = 7$ is odd, the density of the median of the sample is

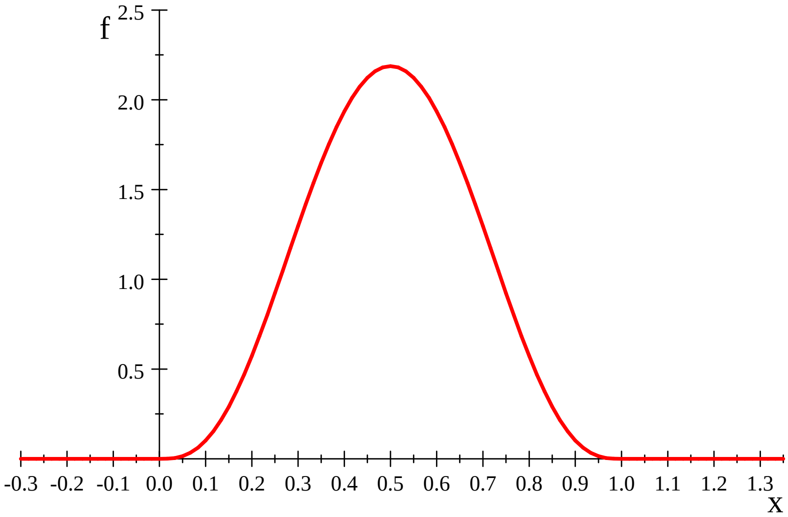
$$f_{\tilde{x}_{\text{med}}}(x) = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} f(x)\left[F(x)\right]^{(n-1)/2}\left[1 - F(x)\right]^{(n-1)/2}$$

$$
= \begin{cases} \dfrac{7!}{(3!)^2} x^3 (1-x)^3 = 140 x^3 (1-x)^3 & \text{for } x \in (0,1) \\ \\ 0 & \text{otherwise.} \end{cases}
$$

Note that the previous density $f_{\tilde{x}_{\text{med}}}$ is the density of a beta distribution with the parameters $\alpha = 4$ and $\beta = 4$ since

$$
\frac{\Gamma(a+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\Gamma(8)}{\Gamma(4)\Gamma 4)} = 140
$$

The plot of the density of the sample median with $n = 7$ is



Note that this density reaches its maximum at $x = 1/2$ and is symmetric with respect to $1/2$.

— — — — — — — — — — — — — — —

18. (a) The exponential density with parameter θ is

$$f(x) = \begin{cases} \dfrac{1}{\theta} e^{-x/\theta} & \text{for } x > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

and its distribution function is

$$F(x) = \begin{cases} 1 - e^{-x/\theta} & \text{for } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The distribution function of the sample minimum is

$$F_{\tilde{x}_{\min}}(x) = 1 - [1 - F(x)]^n = \begin{cases} 1 - e^{-nx/\theta} & \text{for } x > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

so that the density of the sample minimum is

$$f_{\tilde{x}_{\min}}(x) = \frac{dF_{\tilde{x}_{\min}}(x)}{dx} = \begin{cases} \dfrac{n}{\theta} e^{-nx/\theta} & \text{for } x > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

which is the density of the exponential distribution with the parameter θ/n. This implies that $\mathrm{E}\left(\tilde{x}_{\min}\right) = \theta/n$.

The distribution function of the sample maximum is

$$F_{\tilde{x}_{\max}}(x) = [F(x)]^n = \begin{cases} \left(1-e^{-x/\theta}\right)^n & \text{for } x > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

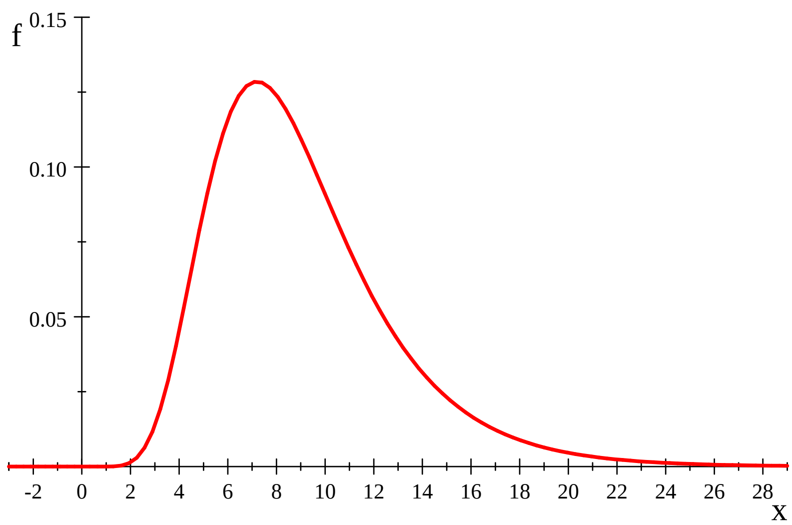
so that the density of the sample maximum is

$$f_{\tilde{x}_{\max}}(x) = \frac{dF_{\tilde{x}_{\max}}(x)}{dx} = \begin{cases} \dfrac{n}{\theta} e^{-x/\theta} \left(1-e^{-x/\theta}\right)^{n-1} & \text{for } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

When $\theta = 3$ and $n = 11$, the density of the sample maximum becomes

$$f_{\tilde{x}_{\max}}(x) = \begin{cases} \dfrac{11}{3} e^{-x/3} \left(1-e^{-x/3}\right)^{10} & \text{for } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

Its plot is



Note that this density reaches its maximum at $x = 3 \ln 11 = 7.1937$.

$$\mathrm{E}\left(\tilde{x}_{\min}\right) = \frac{\theta}{n} = \frac{3}{11} = 0.27273.$$

$$\mathrm{E}\left(\tilde{x}_{\max}\right) = \int_0^\infty x \frac{11}{3} e^{-x/3} \left(1 - e^{-x/3}\right)^{10} dx = 9.0596.$$

(b) The population median m satisfies

$$F(m) = 1 - e^{-m/\theta} = 1/2 \Longrightarrow m = \theta \ln 2$$

The mean of the exponential population with the parameter θ is $\mu = \theta$. Therefore, since $1 > \ln 2 = 0.69315$, the median of the population is smaller than the mean of the population, $\mu > m$.

By modifying the original exponential density function at the single point $x = 0$, we can get the following density:

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{for } x \geq 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

which characterizes exactly the same exponential distribution. The previous density reaches its maximum at $x = 0$ so that the mode of the exponential distribution is 0.

Therefore, we get the following comparison of the three measures of location of the population distribution:

$$\text{mean} = \theta > \text{median} = \theta \ln 2 > \text{mode} = 0.$$

The density of the sample median when n is odd is

$$f_{\tilde{x}_{\text{med}}}(x) = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} f(x) \left[F(x)\right]^{(n-1)/2} \left[1 - F(x)\right]^{(n-1)/2}$$

$\Longrightarrow$

$$f_{\tilde{x}_{\text{med}}}(x) = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} \frac{1}{\theta} e^{-x/\theta} \left[1-e^{-x/\theta}\right]^{(n-1)/2} \left[e^{-x/\theta}\right]^{(n-1)/2}$$

$$= \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} \frac{1}{\theta} \left(e^{-x/\theta}\right)^{(n+1)/2} \left(1-e^{-x/\theta}\right)^{(n-1)/2}, \quad \text{for } x > 0.$$

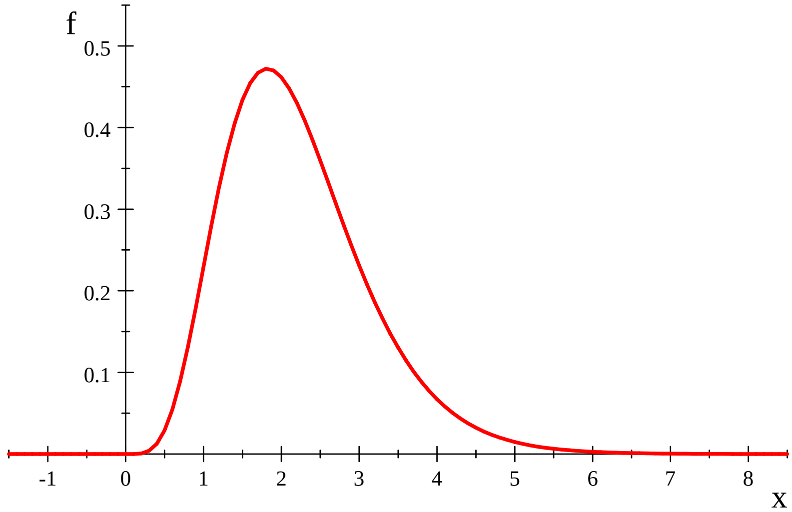
Thus,

$$f_{\tilde{x}_{\text{med}}}(x) = \begin{cases} \dfrac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} \dfrac{1}{\theta} \left(e^{-x/\theta}\right)^{(n+1)/2} \left(1-e^{-x/\theta}\right)^{(n-1)/2} & \text{for } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

If $\theta = 3$ and $n = 11$, then

$$f_{\tilde{x}_{\text{med}}}(x) = \begin{cases} \dfrac{11!}{(5!)^2} \dfrac{1}{3} \left(e^{-x/3}\right)^6 \left(1-e^{-x/3}\right)^5 = 924 e^{-2x} \left(1-e^{-x/3}\right)^5 & \text{for } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The plot of this density is



Note that this density reaches its maximum at $x = 3\ln\left(\frac{11}{6}\right) = 1.8184$. This is the mode of the distribution of $\tilde{x}_{\text{med}}$.

Expectation of the sample median:

$$\mathrm{E}\left(\tilde{x}_{\text{med}}\right) = \int_{\mathbb{R}} x f_{\tilde{x}_{\text{med}}}(x)dx = \int_0^\infty x \cdot 924e^{-2x}\left(1-e^{-x/3}\right)^5 dx = 2.2096.$$

We know that the expectation of the sample mean is equal to the population mean:

$$\mathrm{E}\left(\bar{\boldsymbol{x}}\right) = \mu = \theta = 3.$$

The median of the population is

$$m = \theta\ln 2 = 3\ln 2 = 2.0794.$$

Therefore,

$$\begin{aligned} \mathrm{E}\left(\bar{\boldsymbol{x}}\right) &= \mu = 3 > \mathrm{E}\left(\tilde{x}_{\text{med}}\right) = 2.2096 > m = 2.0794 \\ &> \text{mode of } \tilde{x}_{\text{med}} = 1.8184 > \text{mode of } \tilde{x} = 0. \end{aligned}$$

— — — — — — — — — — — — — — —

19. (a) Assume that $n = 2$ and let $\tilde{S}_2 = \tilde{x}_1 + \tilde{x}_2$. Recall that the formula of the convolution pmf is

$$f_{\tilde{S}_2}(S_2) = \sum_{x_1 \in \tilde{x}_1(\Omega)} f_{\tilde{x}_2}(S_2 - x_1) f_{\tilde{x}_1}(x_1), \text{ for } S_2 \in \tilde{S}_2(\Omega), \text{ with } S_2 - x_1 \in \tilde{x}_2(\Omega).$$

Note that $S_2 \in \tilde{S}_2(\Omega) = \{0, 1, 2, ...\}$, $x_1 \in \tilde{x}_1(\Omega) = \{0, 1, 2, ...\}$ and $S_2 - x_1 \in \tilde{x}_2(\Omega) = \{0, 1, 2, ...\}$ so that $x_1 \leq S_2$. Therefore,

$$f_{\tilde{S}_2}(S_2) = \sum_{x_1=0}^{S_2} \frac{\lambda_2^{S_2 - x_1} e^{-\lambda_2}}{(S_2 - x_1)!} \cdot \frac{\lambda_1^{x_1} e^{-\lambda_1}}{x_1!} = \frac{e^{-(\lambda_1+\lambda_2)}}{S_2!} \sum_{x_1=0}^{S_2} \frac{S_2! \lambda_1^{x_1} \lambda_2^{S_2 - x_1}}{x_1!(S_2 - x_1)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{S_2!} \sum_{x_1=0}^{S_2} \binom{S_2}{x_1} \lambda_1^{x_1} \lambda_2^{S_2 - x_1} = \frac{e^{-(\lambda_1+\lambda_2)}}{S_2!} (\lambda_1 + \lambda_2)^{S_2}$$

$$= \frac{\left(\sum_{i=1}^{2} \lambda_i\right)^{S_2} e^{-\left(\sum_{i=1}^{2} \lambda_i\right)}}{S_2!}, \quad \text{for } S_2 = 0, 1, 2, ...$$

where the fourth equality follows from the Newton's binomial theorem. Therefore, the random variable $\tilde{S}_2 = \tilde{x}_1 + \tilde{x}_2$ has a Poisson distribution with the parameter $\sum_{i=1}^{2} \lambda_i$. Note that $\tilde{S}_{i+1} = \tilde{S}_i + \tilde{x}_{i+1}$ so that we can proceed inductively for $i = 3, 4, ...$ to get that $\tilde{S} \equiv \tilde{S}_n = \sum_{i=1}^{n} \tilde{x}_i$ has a Poisson distribution with the parameter $\sum_{i=1}^{n} \lambda_i$.

(b) Note that $\bar{x} = g(S) = \dfrac{S}{n}$, where g is a one-to-one correspondence. Therefore, $S = g^{-1}(\bar{x}) = n\bar{x}$. Since the pmf of $\tilde{S}$ is

$$f_{\tilde{S}}(S) = \frac{\left(\sum_{i=1}^{n} \lambda_i\right)^{S} e^{-\left(\sum_{i=1}^{n} \lambda_i\right)}}{S!}, \quad \text{for } S = 0, 1, 2, ...,$$

we can apply the change of variable to get

$$f_{\bar{\boldsymbol{x}}}(\bar{x}) = \frac{\left(\sum_{i=1}^{n} \lambda_i\right)^{n\bar{x}} e^{-\left(\sum_{i=1}^{n} \lambda_i\right)}}{(n\bar{x})!}, \quad \text{for } \bar{x} = 0, \frac{1}{n}, \frac{2}{n}, \ldots$$

(c) If $\lambda_i = \lambda$ for $i = 1, \ldots, n$, then the pmf obtained in (b) becomes

$$f_{\bar{\boldsymbol{x}}}(\bar{x}) = \frac{(n\lambda)^{n\bar{x}} e^{-n\lambda}}{(n\bar{x})!}, \quad \text{for } \bar{x} = 0, \frac{1}{n}, \frac{2}{n}, \ldots$$

(d) Using the Central Limit Theorem,

$$\frac{\bar{\boldsymbol{x}}_n - \mathrm{E}\left(\bar{\boldsymbol{x}}_n\right)}{\sqrt{\mathrm{Var}\left(\bar{\boldsymbol{x}}_n\right)}} = \frac{\bar{\boldsymbol{x}}_n - \lambda}{\sqrt{\lambda/n}} = \frac{\sqrt{n}\left(\bar{\boldsymbol{x}}_n - \lambda\right)}{\sqrt{\lambda}} \longrightarrow \mathrm{N}\left(0, 1\right), \quad \text{as } n \longrightarrow \infty.$$

Thus, $\sqrt{n}\left(\bar{\boldsymbol{x}}_n - \lambda\right) \longrightarrow \mathrm{N}\left(0, \lambda\right).$

− − − − − − − − − − − − − − − −

20. The derivative of

$$F_{\tilde{x}_{(j)}}(x) = \sum_{k=j}^{n} \binom{n}{k} \left[F(x)\right]^k \left[1 - F(x)\right]^{n-k}$$

is the density

$$f_{\tilde{x}_{(j)}}(x) = \sum_{k=j}^{n} k \binom{n}{k} \left[F(x)\right]^{k-1} f(x) \left[1 - F(x)\right]^{n-k}$$

$$- \sum_{k=j}^{n} (n-k) \binom{n}{k} \left[F(x)\right]^k \left[1 - F(x)\right]^{n-k-1} f(x)$$

$$= f(x)\left\{\sum_{k=j}^{n} k\binom{n}{k}[F(x)]^{k-1}[1-F(x)]^{n-k}\right.$$

$$\left.-\sum_{k=j}^{n-1}(n-k)\binom{n}{k}[F(x)]^{k}[1-F(x)]^{n-k-1}\right\}.$$

where we have used the fact that $(n-k)\binom{n}{k}[F(x)]^{k}[1-F(x)]^{n-k-1}$ is equal to zero when $k=n,$ so that

$$\sum_{k=j}^{n}(n-k)\binom{n}{k}[F(x)]^{k}[1-F(x)]^{n-k-1} = \sum_{k=j}^{n-1}(n-k)\binom{n}{k}[F(x)]^{k}[1-F(x)]^{n-k-1}.$$

Note that

$$k\binom{n}{k} = \frac{n!k}{k!(n-k)!} = \frac{(n-1)!n}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$$

and

$$(n-k)\binom{n}{k} = \frac{n!(n-k)}{k!(n-k)!} = \frac{(n-1)!n}{k!(n-k-1)!} = n\binom{n-1}{k}.$$

Therefore,

$$f_{\tilde{x}_{(j)}}(x) = nf(x)\left\{\sum_{k=j}^{n}\binom{n-1}{k-1}[F(x)]^{k-1}[1-F(x)]^{n-k}\right.$$

$$\left.-\sum_{k=j}^{n-1}\binom{n-1}{k}[F(x)]^{k}[1-F(x)]^{n-k-1}\right\}.$$

Concerning the first sum inside the brackets, we have

$$\sum_{k=j}^{n}\binom{n-1}{k-1}[F(x)]^{k-1}[1-F(x)]^{n-k} =$$

$$\binom{n-1}{j-1}[F(x)]^{j-1}[1-F(x)]^{n-j}+\sum_{k=j+1}^{n}\binom{n-1}{k-1}[F(x)]^{k-1}[1-F(x)]^{n-k}.$$

Concerning the second sum inside the brackets, we have

$$\sum_{k=j}^{n-1}\binom{n-1}{k}[F(x)]^{k}[1-F(x)]^{n-k-1}$$

$$=\sum_{k=j+1}^{n}\binom{n-1}{k-1}[F(x)]^{k-1}[1-F(x)]^{\overbrace{n-(k-1)-1}^{n-k}}. \qquad \text{(Check it!)}$$

Thus, the expression inside the brackets becomes

$$\sum_{k=j}^{n}\binom{n-1}{k-1}[F(x)]^{k-1}[1-F(x)]^{n-k}-\sum_{k=j}^{n-1}\binom{n-1}{k}[F(x)]^{k}[1-F(x)]^{n-k-1}$$

$$=\binom{n-1}{j-1}[F(x)]^{j-1}[1-F(x)]^{n-j},$$

which implies that

$$f_{\tilde{x}_{(j)}}(x)=nf(x)\binom{n-1}{j-1}[F(x)]^{j-1}[1-F(x)]^{n-j}$$

$$=\frac{(n-1)!n}{(j-1)!(n-j)!}f(x)[F(x)]^{j-1}[1-F(x)]^{n-j}$$

$$=\frac{n!}{(j-1)!(n-j)!}f(x)[F(x)]^{j-1}[1-F(x)]^{n-j}.$$

– – – – – – – – – – – – – – – –

21. (a) Note that $P\{\tilde{x}\leq x^0-z\}=P\{\tilde{x}\geq x^0+z\}$ for all $z\in\mathbb{R}$ is equivalent to $P\{\tilde{x}-x^0\leq -z\} = P\{\tilde{x}-x^0\geq z\}$ or $P\{\tilde{y}\leq -z\} = P\{\tilde{y}\geq z\}$ for all $z\in\mathbb{R}$, which means that the distribution of $\tilde{y}$ is symmetric with respect to zero.

(b) Note that the random variables $\tilde{y}$ and $-\tilde{y}$ have the same distribution if and only if $F_{\tilde{y}}(y) = F_{-\tilde{y}}(y)$ for all $y \in \mathbb{R}$, which is in turn equivalent to $F_{\tilde{y}}(-z) = F_{-\tilde{y}}(-z)$ for all $z \in \mathbb{R}$.

On the one hand,

$$F_{-\tilde{y}}(-z) = P\{-\tilde{y} \leq -z\} = P\{\tilde{y} \geq z\}.$$

On the other hand,

$$F_{\tilde{y}}(-z) = P\{\tilde{y} \leq -z\}.$$

Therefore, $F_{-\tilde{y}}(-z) = F_{\tilde{y}}(-z)$ if and only $P\{\tilde{y} \geq z\} = P\{\tilde{y} \leq -z\}$ for all $z \in \mathbb{R}$, which means that the distribution of $\tilde{y}$ is symmetric with respect to zero.

(c) If $\tilde{y}$ is discrete its probability function satisfies $f_{\tilde{y}}(y) = P\{\tilde{y} \geq y\} - P\{\tilde{y} \geq y^*\}$, where y^* is the smallest value in the range of $\tilde{y}$ that is strictly greater than y, and $f_{\tilde{y}}(-y) = P\{\tilde{y} \leq -y\} - P\{\tilde{y} \leq -y^*\}$. Note that $-y^*$ is the largest value in the range of $\tilde{y}$ that is strictly smaller than $-y$. If the distribution of $\tilde{y}$ is symmetric with respect to zero then $P\{\tilde{y} \leq -y\} = P\{\tilde{y} \geq y\}$ and $P\{\tilde{y} \leq -y^*\} = P\{\tilde{y} \geq y^*\}$ which implies that $f_{\tilde{y}}(y) = f_{\tilde{y}}(-y)$ for all $y \in \tilde{y}(\Omega)$.

If $f_{\tilde{y}}(y) = f_{\tilde{y}}(-y)$ for all $y \in \tilde{y}(\Omega)$, then $P\{\tilde{y} \geq y\} = \sum_{z \geq y} f_{\tilde{y}}(z)$ and $P\{\tilde{y} \leq -y\} = \sum_{z \leq -y} f_{\tilde{y}}(z) = \sum_{-z \geq y} f_{\tilde{y}}(-z) = \sum_{s \geq y} f_{\tilde{y}}(s)$. Thus, $P\{\tilde{y} \geq y\} = P\{\tilde{y} \leq -y\}$ so that the distribution of $\tilde{y}$ is symmetric with respect to zero.

If $\tilde{y}$ is absolutely continuous and its distribution is symmetric with respect to zero, $P\{\tilde{y} \leq -y\} = P\{\tilde{y} \geq y\}$ for all $y \in \mathbb{R}$, then

$$\int_{-\infty}^{-y} f_{\tilde{y}}(z)dz = \int_{y}^{\infty} f_{\tilde{y}}(z)dz, \quad \text{for all } y \in \mathbb{R}.$$

Thus, the derivatives with respect to y of the integrals in both sides of the previous equality must coincide. We compute these derivatives using Leibniz rule to obtain

$$-f_{\tilde{y}}(-y) = -f_{\tilde{y}}(y) \quad \text{or} \quad f_{\tilde{y}}(-y) = f_{\tilde{y}}(y).$$

If the density of $\tilde{y}$ satisfies that $f_{\tilde{y}}(-y) = f_{\tilde{y}}(y)$ for all $y \in \mathbb{R}$, then

$$P\{\tilde{y} \leq -y\} = \int_{-\infty}^{-y} f_{\tilde{y}}(z)dz = \int_{-\infty}^{-y} f_{\tilde{y}}(-z)dz = |-F_{\tilde{y}}(-z)|_{-\infty}^{-y} = 1-F_{\tilde{y}}(y) = P\{\tilde{y} \geq y\},$$

so that the distribution of $\tilde{y}$ is symmetric with respect to zero.

(d) Define the random variable $\tilde{y} = \tilde{x} - x^0$. Note that, if the probability (density) function of $\tilde{x}$ is $f_{\tilde{x}}$, then

$$f_{\tilde{y}}(y) = f_{\tilde{x}}(x^0 + y),$$

where the equality follows from finding the probability (density) function of $\tilde{y}$ from the probability (density) function of $\tilde{x} = x^0 + y$. Thus,

$$f_{\tilde{y}}(-y) = f_{\tilde{x}}(x^0 - y).$$

Acording to part (a), the random variable $\tilde{y}$ has a distribution that is symmetric with respect to zero. Thus, $f_{\tilde{y}}(y) = f_{\tilde{y}}(-y)$ for all y (see part (c)), which is equivalent to $f_{\tilde{x}}(x^0 + y) = f_{\tilde{x}}(x^0 - y)$ for all y so that the function $f_{\tilde{x}}$ is symmetric with respect to x^0.

(e) Define the random variable $\tilde{y} = \tilde{x} - x^0$ so that $\mathrm{E}(\tilde{y}) = \mathrm{E}(\tilde{x}) - x^0$. Note that, from parts (a) and (b), $\tilde{y}$ has the same distribution as the random variable $-\tilde{y}$, which implies that $\mathrm{E}(\tilde{y}) = \mathrm{E}(-\tilde{y}) = -\mathrm{E}(\tilde{y})$. Therefore, we must have $\mathrm{E}(\tilde{y}) = 0$ and thus $\mathrm{E}(\tilde{x}) = x^0$.

(f) Since $\mathrm{E}(\tilde{x}) = x^0$, the coefficient of asymmetry (or skewness) is given by

$$\mathrm{CA} = \frac{\mu_3}{\sigma^3} = \frac{\mathrm{E}\left[(\tilde{x} - x^0)^3\right]}{\left(\mathrm{E}\left[(\tilde{x} - x^0)^2\right]\right)^{3/2}}.$$

Note that

$$\mathrm{E}\left[(\tilde{x} - x^0)^3\right] = \mathrm{E}\left(\tilde{y}^3\right) = \mathrm{E}\left[(-\tilde{y})^3\right] = \mathrm{E}\left[-(\tilde{y})^3\right] = -\mathrm{E}\left(\tilde{y}^3\right),$$

where the first inequality comes from the definition of $\tilde{y}$ given in part (a) and the second equality follows from the fact that $\tilde{y}$ and $-\tilde{y}$ have the same distribution (see part (b)). Thus, $\mathrm{E}\left(\tilde{y}^3\right) = -\mathrm{E}\left(\tilde{y}^3\right)$ implies that $\mathrm{E}\left[(\tilde{x} - x^0)^3\right] = \mathrm{E}\left(\tilde{y}^3\right) = 0$, which implies in turn that $\mathrm{CA} = 0$.

– – – – – – – – – – – – – – –