Probability and Statistics. IDEA. Answers to List 6.

1. We have

$$\mathrm{E}\left[g\left(\tilde{x}_{t+1}\right)|\,\mathcal{F}_t\right] \geq g\left(\mathrm{E}\left[\tilde{x}_{t+1}|\,\mathcal{F}_t\right]\right) \qquad (\star)$$

by Jensen's inequality as g is convex.

(a)

$$\mathrm{E}\left[\tilde{x}_{t+1}|\,\mathcal{F}_t\right] \geq \tilde{x}_t \Longrightarrow g\left(\mathrm{E}\left[\tilde{x}_{t+1}|\,\mathcal{F}_t\right]\right) \geq g\left(\tilde{x}_t\right)$$

since g is increasing.

(b)

$$\mathrm{E}\left[\tilde{x}_{t+1}|\,\mathcal{F}_t\right] = \tilde{x}_t \Longrightarrow g\left(\mathrm{E}\left[\tilde{x}_{t+1}|\,\mathcal{F}_t\right]\right) = g\left(\tilde{x}_t\right).$$

Thus, using $(\star)$ in both (a) and (b) we get

$$\mathrm{E}\left[g\left(\tilde{x}_{t+1}\right)|\,\mathcal{F}_t\right] \geq g\left(\tilde{x}_t\right).$$

Note: $g(x) = |x|^r$ is a convex function when $r \geq 1$.

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2. $n = 16$, $\theta = 0.5$.

$$\mathrm{E}\left(\tilde{x}\right) = \mu = n\theta = 8$$

$$\mathrm{Var}\left(\tilde{x}\right) = \sigma^2 = n\theta\left(1-\theta\right) = 4$$

(a)

$$b\left(6;16,0.5\right) = \binom{16}{6}0.5^{16} = 0.1222.$$

$$P\{\tilde{x}=6\}=P\{5.5\leq\tilde{x}\leq 6.5\}=P\left\{\underbrace{\frac{5.5-8}{2}}_{-1.25}\leq\tilde{z}\leq\underbrace{\frac{6.5-8}{2}}_{-0.75}\right\}.$$

$$=N(-0.75)-N(-1.25)=N(1.25)-N(0.75)=0.1210.$$

The difference between the two probabilities is 0.0012.

(b)

$$P\{4<\tilde{x}\leq 7\}=P\{5\leq\tilde{x}\leq 7\}=\binom{16}{5}0.5^{16}+\binom{16}{6}0.5^{16}+\binom{16}{7}0.5^{16}=0.3634$$

$$P\{4<\tilde{x}\leq 7\}=P\{5\leq\tilde{x}\leq 7\}=P\{4.5\leq\tilde{x}\leq 7.5\}$$

$$=P\left\{\underbrace{\frac{4.5-8}{2}}_{-1.75}\leq\tilde{z}\leq\underbrace{\frac{7.5-8}{2}}_{-0.25}\right\}=N(-0.25)-N(-1.75)$$

$$=N(1.75)-N(0.25)=0.3612$$

The difference between the two probabilities is 0.0022.

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3. Binomial: $x=10, n=3000, \theta=0.005$.

$$b(10;3000,0.005)=\binom{3000}{10}(0.005)^{10}(0.995)^{2990}=0.0485.$$

Poisson: $x = 10, \lambda = n\theta = 15$

$$p\,(10; 15) = \frac{15^{10} e^{-15}}{10!} = 0.0486.$$

The difference between the two probabilities is 0.0001.

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4. (a) The stationary probabilistic vector $p^* = \begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix}$ must solve the following equation:

$$\begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/6 & 1/6 & 2/3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix}$$

The solution is a set of probabilistic vectors of dimension 1 with the functional form

$$p^* = \begin{pmatrix} \alpha, & 0, & 1-\alpha \end{pmatrix}, \quad \text{with } \alpha \in [0,1]$$

The eigenvalues of the transition matrix are $1, 1$, and $1/6$. Therefore, there is no ergodic probabilistic vector.

The state s_2 is transient and the states s_1 and s_3 are recurrent.

The absorbing states are s_1 and s_3. The ergodic sets are $\{s_1\}$ and $\{s_3\}$.

(b) The Markov chain of this exercise is absorbing. If

$$\lim_{t\to\infty} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/6 & 1/6 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}}_{\Pi}^{t} = \widehat{\Pi},$$

the matrix $\widehat{\Pi}$ must have this form:

$$\widehat{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 1-a \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\Pi \cdot \widehat{\Pi} = \lim_{t\to\infty} \Pi^{t+1} = \widehat{\Pi}$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/6 & 1/6 & 2/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 1-a \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 1-a \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ \dfrac{1+a}{6} & 0 & \dfrac{5-a}{6} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 1-a \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $a = 1/5$ so that

$$\widehat{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 0 & 4/5 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Hence, $\lim_{t\to\infty} P\{\tilde{x}_t = s_1 | \tilde{x}_0 = s_2\} = 1/5$, $\lim_{t\to\infty} P\{\tilde{x}_t = s_2 | \tilde{x}_0 = s_2\} = 0$, and

$\lim_{t\to\infty} P\{\tilde{x}_t = s_3 | \tilde{x}_0 = s_2\} = 4/5.$

Alternatively, make a permutation of states s_1 and s_2 so that the transition matrix Π has the following canonical from

$$\begin{pmatrix} Q & D \\ & \\ \widehat{0} & \mathrm{I} \end{pmatrix} = \begin{pmatrix} 1/6 & 1/6 & 2/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

whose limit when $t \to \infty$ is

$$\begin{pmatrix} \widehat{0} & B \\ & \\ \widehat{0} & \mathrm{I} \end{pmatrix} = \begin{pmatrix} 0 & a & 1-a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We know from Theorem 4 in the handout on Absorbing and Irreducible Markov Chains that $B = (\mathrm{I} - Q)^{-1} D$ so that

$$B = \begin{pmatrix} a \ , & 1-a \end{pmatrix} = \left(1 - \frac{1}{6}\right)^{-1} \cdot \begin{pmatrix} 1/6 \ , & 2/3 \end{pmatrix} = \begin{pmatrix} 1/5 \ , & 4/5 \end{pmatrix}.$$

In terms of the original numbering of states, this means that $\lim_{t\to\infty} P\{\tilde{x}_t = s_1 | \tilde{x}_0 = s_2\} = 1/5$ and $\lim_{t\to\infty} P\{\tilde{x}_t = s_3 | \tilde{x}_0 = s_2\} = 4/5$ and, thus, the limiting matrix $\widehat{\Pi}$ is the one given in (1).

(c)

$$P\{0 \text{ heads}\} = \binom{3}{0}\left(\frac{1}{2}\right)^3 = \frac{1}{8},$$

$$P\{1 \text{ head}\} = \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{8},$$

$$P\{2 \text{ heads}\} = \binom{3}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{8},$$

$$P\{3 \text{ heads}\} = \binom{3}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

$$P(\{1 \text{ heads}\} \cup \{3 \text{ heads}\}) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}.$$

$$\begin{pmatrix} 1/2, & 3/8, & 1/8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 0 & 4/5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{23}{40}, & 0, & \frac{17}{40} \end{pmatrix}.$$

(d) From Theorem 3 in the handout we know that the expected number n_2 of periods before the chain is absorbed, given that it starts at s_2 is the first (and unique) entry of the 1×1 vector $(\mathrm{I} - Q)^{-1}\,\underline{1}$ so that

$$n_2 = \left(1 - \frac{1}{6}\right)^{-1} \cdot \underline{1} = \frac{6}{5} \cdot 1 = \frac{6}{5} = 1.2$$

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5. (i) To find the stationary probabilistic vectors, we solve the system of equations given by $p^*\Pi = p^*$.

(a) Any non-negative vector $p^* = (\alpha,\ \beta,\ 1 - \alpha - \beta)$ is a stationary probabilistic vector. There is no ergodic probabilistic vector since there are three eigenvalues equal to 1. In fact,

$$\lim_{t\to\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \widehat{\Pi}.$$

The states s_1, s_2, and s_3 are recurrent and absorbing. The Markov chain has three ergodic sets: $\{s_1\}$, $\{s_2\}$, and $\{s_3\}$.

(b) This Markov chain is irreducible but it is not regular. The unique stationary probabilistic vector is $p^* = (1/3,\ 1/3,\ 1/3)$. There is no ergodic probabilistic vector since the eigenvalues of Π are 1, $\frac{-1+\sqrt{3}i}{2}$, and $\frac{-1-\sqrt{3}i}{2}$; and all three eigenvalues have modulus equal to 1. In fact, the matrix Π^t is equal to

$$\Pi^t = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

when $t = 1 + 3n$, for $n = 0, 1, 2, ...,$

$$\Pi^t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

when $t = 2 + 3n$, for $n = 0, 1, 2, ...,$ and

$$\Pi^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

when $t = 3 + 3n$, for $n = 0, 1, 2, ...,$ and, thus, $\lim_{t\to\infty} \Pi^t$ does not exist.

All three states are recurrent so that there are no transient states. There are no absorbing states. The Markov chain has a single ergodic set: $\{s_1, s_2, s_3\}$.

(c) Any non-negative vector with the form $p^* = (\alpha,\ 1 - 2\alpha,\ \alpha)$ is a stationary probabilistic vector. Note that we must have $\alpha \in [0, 1/2]$. There is no ergodic probabilistic vector since the eigenvalues of Π are 1, 1, and -1; and all three eigenvalues have modulus equal to 1. In fact, the matrix Π^t is equal to

$$\Pi^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

when $t = 1 + 2n$, for $n = 0, 1, 2, ...$, and

$$\Pi^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

when $t = 2 + 2n$, for $n = 0, 1, 2, ...$, and, thus, $\lim_{t \to \infty} \Pi^t$ does not exist.

All three states are recurrent so that there are no transient states. The state s_2 is absorbing. The Markov chain has two ergodic sets: $\{s_2\}$ and $\{s_1, s_3\}$.

(d) The eigenvalues of Π are 1 and $-\frac{1}{4}$. This Markov chain is irreducible (in fact, it is regular). Therefore, there exists an ergodic probabilistic vector $\widehat{p}$, which coincides with the stationary probabilistic vector p^*.

$$\lim_{t \to \infty} \Pi^t = \begin{pmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{pmatrix} = \widehat{\Pi}.$$

Therefore, $\widehat{p} = p^* = (3/5,\ 2/5)$.

There are no absorbing states. Both states are recurrent so that none of them is transient. The Markov chain has a single ergodic set, which is the state space: $S = \{s_1, s_2\}$.

(ii) From Theorem 5 in the handout on Absorbing and Irreducible Markov

Chains we know that the mean recurrence time for states s_1 and s_2 are $1/p_1^* = 5/3 = 1.6667$ and $1/p_2^* = 5/2 = 2.5$, respectively.

From Theorem 6 in the handout we know that the mean first passage time m_{ij} from state s_i to state s_j can be computed as follows: find the fundamental matrix for the irreducible chain,

$$(\mathrm{I} - \Pi + \Pi^*)^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix} + \begin{pmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{pmatrix} \right]^{-1}$$

$$= \begin{pmatrix} 11/10 & -1/10 \\ -3/20 & 23/20 \end{pmatrix}^{-1} = \begin{pmatrix} 23/25 & 2/25 \\ 3/25 & 22/25 \end{pmatrix},$$

so that

$$m_{ij} = \frac{z_{jj} - z_{ij}}{p_j^*},$$

where z_{ij} is the (i,j)-entry of the fundamental matrix $(\mathrm{I} - \Pi + \Pi^*)^{-1}$ and p_j^* is the jth entry of the stationary probabilistic row vector p^*. Thus,

$$m_{12} = \frac{\frac{22}{25} - \frac{2}{25}}{\frac{2}{5}} = 2,$$

$$m_{21} = \frac{\frac{23}{25} - \frac{3}{25}}{\frac{3}{5}} = \frac{4}{3}.$$

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6. (a) This Markov chain is absorbing. We have the following transition matrix:

$$\Pi = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The states $s_{1,}s_2, s_4,$ and s_5 are transient. The states s_3 and s_6 are recurrent and absorbing. The Markov chain has has two ergodic sets: $\{s_3\}$ and $\{s_6\}$. It can be proved that the 6 eigenvalues of Π are $1, 1, 0, 0, -5/6$ and $5/6$; and

$$\lim_{t\to\infty} \Pi^t = \begin{pmatrix} 0 & 0 & 6/11 & 0 & 0 & 5/11 \\ 0 & 0 & 7/11 & 0 & 0 & 4/11 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5/11 & 0 & 0 & 6/11 \\ 0 & 0 & 4/11 & 0 & 0 & 7/11 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \widehat{\Pi}. \quad (2)$$

Note that, given the characteristics of this Markov chain, the long-run matrix $\widehat{\Pi}$ must have the following form (think about it!):

$$\widehat{\Pi} = \begin{pmatrix} 0 & 0 & a & 0 & 0 & 1-a \\ 0 & 0 & b & 0 & 0 & 1-b \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 1-c \\ 0 & 0 & d & 0 & 0 & 1-d \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the columns 3 and 6 of $\widehat{\Pi}$ have to belong to the eigenspace associated with the eigenvalue 1 of Π. Then,

$$\begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \\ c \\ d \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 1 \\ c \\ d \\ 0 \end{pmatrix}.$$

We solve for a, b, c, and d to obtain the columns 3 and 6. Therefore, the value $\widehat{\pi}_{16} = 1 - a = 5/11$ of the matrix $\widehat{\Pi}$ gives us the probability of entering into cell 6 if the initial cell was 1.

Alternatively, if we renumber the states so that the first four states are transient and the last two absorbing, the transition matrix has the following canonical from:

$$\begin{pmatrix} Q & D \\ \widehat{0} & \mathrm{I} \end{pmatrix} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 4 & 5 & 3 & 6 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 4 \\ 5 \\ 3 \\ 6 \end{array},$$

whose limit when $t \to \infty$ is

$$\begin{pmatrix} \widehat{0} & B \\ & \\ \widehat{0} & \mathrm{I} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & a & 1-a \\ 0 & 0 & 0 & 0 & b & 1-b \\ 0 & 0 & 0 & 0 & c & 1-c \\ 0 & 0 & 0 & 0 & d & 1-d \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We know from Theorem 4 in the handout on Absorbing and Irreducible Markov Chains that $B = (\mathrm{I} - Q)^{-1} D$ so that

$$B = \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ -1/3 & 1 & 0 & -1/3 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & -1/3 & -1/3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 0 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 6/11 & 5/11 \\ 7/11 & 4/11 \\ 5/11 & 6/11 \\ 4/11 & 7/11 \end{pmatrix}.$$

In terms of the original numbering of cells, this means that $\lim_{t\to\infty} P\left\{\tilde{x}_t = s_6 | \tilde{x}_0 = s_1\right\} = 5/11$ and the limiting matrix $\widehat{\Pi}$ is the one given in (2).

(b) From Theorem 2, the expected number n_{52} of times the mouse will be in cell 2 given that it was initially in cell 5 is given by the value of the $(4,2)$-entry (after renumbering) of the fundamental matrix $(\mathrm{I} - Q)^{-1}$,

$$(\mathrm{I} - Q)^{-1} = \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ -1/3 & 1 & 0 & -1/3 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & -1/3 & -1/3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 26/11 & 18/11 & 18/11 & 15/11 \\ 12/11 & 21/11 & 10/11 & 12/11 \\ 18/11 & 15/11 & 26/11 & 18/11 \\ 10/11 & 12/11 & 12/11 & 21/11 \end{pmatrix}$$

so that $n_{42} = 12/11 = 1.0909$.

(c) From Theorem 3 in the handout we know that the expected number n_4 of periods before the chain is absorbed, given that it starts in cell 4 is given by the 3rd entry (after renumbering) of the column vector $n = (\mathrm{I} - Q)^{-1}\,\underline{1}$ so that

$$n = \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ -1/3 & 1 & 0 & -1/3 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & -1/3 & -1/3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \\ 5 \end{pmatrix}$$

and, thus, $n_4 = 7$.

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7. (a) The state s_i occurs when player a has $i - 1$ dollars, $i = 1, 2, 3, 4, 5$. Player a is ruined in state s_1 (when a has 0 dollars) and player b is ruined in state s_5 (when a has 4 dollars). We have thus the following transition matrix:

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This Markov chain is absorbing. The states s_1 and s_5 are absorbing. If we renumber the states so that the first three states $\{s_2, s_3, s_4\}$ are transient and the last two states $\{s_1, s_5\}$ are absorbing, the transition matrix has the following canonical from:

$$\begin{pmatrix} Q & D \\ & \\ \widehat{0} & \mathrm{I} \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

We know from Theorem 3 in the handout on Absorbing and Irreducible Markov Chains that the expected number n_3 of flips before the game ends, given that it starts in state s_3 (now in row/column 2 after renumbering) is given by the second entry (after renumbering) of the column vector $n = (\mathrm{I} - Q)^{-1}\,\underline{1}$ so that

$$\begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$$

and, thus, $n_3 = 4$.

(b) Since

$$\Pi^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5/8 & 1/8 & 0 & 1/8 & 1/8 \\ 3/8 & 0 & 1/4 & 0 & 3/8 \\ 1/8 & 1/8 & 0 & 1/8 & 5/8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the initial state was s_3 (a initially had 2 dollars), the probability of player a earning 4 dollars (state s_5) after four tosses is 3/8.

(c) We have

$$\lim_{t\to\infty} \Pi^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \widehat{\Pi}. \tag{4}$$

Note that columns 2, 3, and 4 of $\widehat{\Pi}$ have to be zero vectors. The columns 1 and 5 of $\widehat{\Pi}$ have to belong to the eigenspace associated with the eigenvalue 1 of Π and must have the following form:

$$\overline{\pi}_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{\pi}_5 = \begin{pmatrix} 0 \\ 1-\alpha \\ 1-\beta \\ 1-\gamma \\ 1 \end{pmatrix},$$

respectively. We solve for α, β, and γ in the equations $\Pi\overline{\pi}_1 = \overline{\pi}_1$ or $\Pi\overline{\pi}_5 = \overline{\pi}_5$ to obtain the columns 1 and 5 of $\widehat{\Pi}$. Therefore, the entry $\widehat{\pi}_{41} = \gamma = 1/4$ of the matrix $\widehat{\Pi}$ gives us the probability of player b getting 4 dollars (state s_1) if player a initially had three dollars (state s_4).

Alternatively, we can use the fact that the canonical form (3) has the following limit when $t \to \infty$:

$$\begin{pmatrix} \widehat{0} & B \\ & \\ \widehat{0} & \mathrm{I} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \alpha & 1-\alpha \\ 0 & 0 & 0 & \beta & 1-\beta \\ 0 & 0 & 0 & \gamma & 1-\gamma \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We know from Theorem 4 in the handout on Absorbing and Irreducible Markov Chains that $B = (\mathrm{I} - Q)^{-1} D$ so that

$$B = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

This means that the probability of player b getting 4 dollars (state s_1, now in row/column 4) if player a initially had three dollars (state s_4, now in row/column 3) is $\lim_{t\to\infty} P\{\tilde{x}_t = s_1 | \tilde{x}_0 = s_4\} = 1/4$.

From the expression for matrix B we also see that the limiting matrix $\widehat{\Pi}$ is the one given in (4).

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8. Since $\sum_{i=1}^{\infty} \frac{1}{i} = \infty$, every sample point $\omega \in \Omega$ satisfies $\limsup_{n\to\infty} \tilde{x}_n(\omega) = 1$ and $\liminf_{n\to\infty} \tilde{x}_n(\omega) = 0$ as, for every ω, the value $\tilde{x}_n(\omega)$ alternates between the values 1 and 0 infinitely often. In other words, $\tilde{x}_n(\omega)$ does not converge to 0 for all $\omega \in \Omega$. Therefore, $P\left\{\lim_{n\to\infty} \tilde{x}_n = 0\right\} = 0$. Hence, $\tilde{x}_n$ does not converges almost surely to the constant 0 as $n \to \infty$.

Observe that $P\{|\tilde{x}_n - 0| < \varepsilon\} \geq 1 - \frac{1}{n}$, for all $\varepsilon > 0$. Therefore,

$$\lim_{n\to\infty} P\{|\tilde{x}_n - 0| < \varepsilon\} = 1$$

so that $\tilde{x}_n$ converges in probability to the constant 0 as $n \to \infty$.

Finally, if $\tilde{x}_n \xrightarrow{p} 0$, then $\tilde{x}_n \xrightarrow{d} 0$.

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9. The Markov chain of this exercise is neither absorbing nor irreducible. It is not absorbing since, even if there is an absorbing state s_2, it is impossible to go to this absorbing state if we start in any of the other two states.

(a) The stationary probabilistic vector $p^* = \begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix}$ must solve the following equation:

$$\begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 3/10 & 0 & 7/10 \end{pmatrix} = \begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix}.$$

The solution is a set of probabilistic vectors of dimension 1 with the functional form

$$p^* = \begin{pmatrix} \frac{3(1-\beta)}{8}, & \beta, & \frac{5(1-\beta)}{8} \end{pmatrix}, \text{ with } \beta \in [0,1]$$

or

$$p^* = \begin{pmatrix} \alpha, & 1 - \frac{8\alpha}{3}, & \frac{5\alpha}{3} \end{pmatrix}, \text{ with } \alpha \in [0, 3/8].$$

The eigenvalues of the transition matrix are $1, 1$, and $1/5$. Therefore, there is no ergodic probabilistic vector. All three states are recurrent so that there are no transient states. The single absorbing state is s_2. There are two ergodic sets: $\{s_1, s_3\}$ and $\{s_2\}$.

(b)

$$\lim_{t\to\infty} \underbrace{\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 3/10 & 0 & 7/10 \end{pmatrix}}_{\Pi}^{t} = \widehat{\Pi}.$$

The matrix $\widehat{\Pi}$ must have this form:

$$\widehat{\Pi} = \begin{pmatrix} a & 0 & 1-a \\ 0 & 1 & 0 \\ b & 0 & 1-b \end{pmatrix}.$$

Then, using the fact that $\widehat{\Pi} \cdot \Pi = \lim_{t\to\infty} \Pi^{t+1} = \widehat{\Pi}$, we have

$$\begin{pmatrix} a & 0 & 1-a \\ 0 & 1 & 0 \\ b & 0 & 1-b \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 3/10 & 0 & 7/10 \end{pmatrix} = \begin{pmatrix} a & 0 & 1-a \\ 0 & 1 & 0 \\ b & 0 & 1-b \end{pmatrix},$$

which becomes

$$\begin{pmatrix} \frac{1}{5}a + \frac{3}{10} & 0 & \frac{7}{10} - \frac{1}{5}a \\ 0 & 1 & 0 \\ \frac{1}{5}b + \frac{3}{10} & 0 & \frac{7}{10} - \frac{1}{5}b \end{pmatrix} = \begin{pmatrix} a & 0 & 1-a \\ 0 & 1 & 0 \\ b & 0 & 1-b \end{pmatrix}.$$

Therefore, we get $a = 3/8$ and $b = 3/8$ so that

$$\widehat{\Pi} = \begin{pmatrix} 3/8 & 0 & 5/8 \\ 0 & 1 & 0 \\ 3/8 & 0 & 5/8 \end{pmatrix}.$$

Alternatively, we could use the fact that, if we start in states s_1 or s_3, we never go to the state s_2. Therefore, if we start in s_1 or s_3, the relevant transition matrix is

$$\begin{pmatrix} 1/2 & 1/2 \\ 3/10 & 7/10 \end{pmatrix},$$

where the row/column 1 refers to state s_1 and the row/column 2 refers to state s_3. The eigenvalues of this matrix are 1 and 1/5. This Markov chain has a unique stationary probabilistic vector that coincides with the ergodic probabilistic vector, which is $(3/8,\ 5/8)$. Note that this vector can also be obtained by making $\beta = 0$ in the stationary probabilistic vector p^* of the original 3-state Markov chain.

(c)

$$\begin{pmatrix} 1/3, & 1/2, & 1/6 \end{pmatrix} \begin{pmatrix} 3/8 & 0 & 5/8 \\ 0 & 1 & 0 \\ 3/8 & 0 & 5/8 \end{pmatrix} = \begin{pmatrix} 3/16, & 1/2, & 5/16 \end{pmatrix}.$$

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10. (a) This Markov chain is neither irreducible nor absorbing. The stationary probabilistic vector

$$p^* = \left(\begin{array}{cccc} \alpha, & \beta, & \gamma, & 1-\alpha-\beta-\gamma \end{array} \right)$$

must solve the following equation:

$$\left(\begin{array}{cccc} \alpha, & \beta, & \gamma, & 1-\alpha-\beta-\gamma \end{array} \right) \cdot \Pi =$$

$$\left(\begin{array}{cccc} \frac{\alpha}{5}+\frac{7\beta}{10}, & \frac{4\alpha}{5}+\frac{3\beta}{10}, & \frac{1}{2}-\frac{\alpha}{2}-\frac{\beta}{2}+\frac{\gamma}{10}, & \frac{1}{2}-\frac{\alpha}{2}-\frac{\beta}{2}-\frac{\gamma}{10} \end{array} \right)$$

$$= \left(\begin{array}{cccc} \alpha, & \beta, & \gamma, & 1-\alpha-\beta-\gamma \end{array} \right).$$

Solving for α, β, and γ we get that the set of probabilistic vectors is a subset of R^4 of dimension 1 with the following functional form:

$$p^* = \left(\alpha, \ \frac{8\alpha}{7}, \ \frac{5}{9} - \frac{25\alpha}{21}, \ \frac{4}{9} - \frac{20\alpha}{21} \right) \quad \text{for all } \alpha \in [0, 7/15] \, .$$

There are two ergodic sets: $\{s_1, s_2\}$ and $\{s_3, s_4\}$. Thus, there is no ergodic probabilistic vector. All four states are recurrent so that there are no transient states, It can be proved that the eigenvalues of the transition matrix Π are $1, 1, 0.1$, and -0.5 (but we do not need to compute them!).

(b) Note that, if we start in the ergodic set $\{s_1, s_2\}$, the relevant transition matrix is $\left(\begin{array}{cc} 1/5 & 4/5 \\ 7/10 & 3/10 \end{array} \right)$. This matrix has a unique stationary probabilistic vector, $p_a^* = \left(\begin{array}{cc} 7/15, & 8/15 \end{array} \right)$, which coincides with its ergodic probabilistic vector. If we start in the ergodic set $\{s_3, s_4\}$, the relevant transition matrix is

$\begin{pmatrix} 3/5 & 2/5 \\ 1/2 & 1/2 \end{pmatrix}$. This matrix has a unique stationary probabilistic vector, $p_b^* = \begin{pmatrix} 5/9, & 4/9 \end{pmatrix}$, which coincides with its ergodic probabilistic vector. Note that p_a^* coincides with the first two components of the stationary probabilistic vector p^* when $\alpha = 7/15$, while p_b^* coincides with the last two components of p^* when $\alpha = 0$. Therefore,

$$\lim_{t\to\infty} \Pi^t = \begin{pmatrix} 7/15 & 8/15 & 0 & 0 \\ 7/15 & 8/15 & 0 & 0 \\ 0 & 0 & 5/9 & 4/9 \\ 0 & 0 & 5/9 & 4/9 \end{pmatrix} = \widehat{\Pi}.$$

Alternatively, we know that the matrix $\widehat{\Pi}$ should have the following functional form:

$$\widehat{\Pi} = \begin{pmatrix} a & 1-a & 0 & 0 \\ b & 1-b & 0 & 0 \\ 0 & 0 & c & 1-c \\ 0 & 0 & d & 1-d \end{pmatrix},$$

Then, using the fact that $\widehat{\Pi} \cdot \Pi = \widehat{\Pi}$, we get

$$\begin{pmatrix} a & 1-a & 0 & 0 \\ b & 1-b & 0 & 0 \\ 0 & 0 & c & 1-c \\ 0 & 0 & d & 1-d \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 \\ \frac{7}{10} & \frac{3}{10} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{10} - \frac{a}{2} & \frac{a}{2} + \frac{3}{10} & 0 & 0 \\ \frac{7}{10} - \frac{b}{2} & \frac{b}{2} + \frac{3}{10} & 0 & 0 \\ 0 & 0 & \frac{c}{10} + \frac{1}{2} & \frac{1}{2} - \frac{c}{10} \\ 0 & 0 & \frac{d}{10} + \frac{1}{2} & \frac{1}{2} - \frac{d}{10} \end{pmatrix}$$

so that we can immediately solve for the values of a, b, c and d.

Hence,

$\lim_{t\to\infty} P\{\tilde{x}_t = s_2 | \tilde{x}_0 = s_2\} = \hat{\pi}_{22} = 8/15$, $\lim_{t\to\infty} P\{\tilde{x}_t = s_1 | \tilde{x}_0 = s_3\} = \hat{\pi}_{31} = 0$, and $\lim_{t\to\infty} P\{\tilde{x}_t = s_3 | \tilde{x}_0 = s_4\} = \hat{\pi}_{43} = 5/9$.

If all the states are equally likely at $t = 0$, then the long-run distribution of states is given by

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \cdot \widehat{\Pi} = \left(\frac{7}{30}, \frac{4}{15}, \frac{5}{18}, \frac{2}{9}\right).$$

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11. The Markov chain with the transition matrix Π_1 is regular and, thus, irreducible. The Markov chains with the transition matrices Π_2 and Π_4 are absorbing. The Markov chain with the transition matrix Π_3 is irreducible but not regular.

(a) Stationary probabilistic vectors:

$$p^* = \begin{pmatrix} p_1^* , & p_2^* \end{pmatrix} \text{ such that } p^* \Pi_k = p^*.$$

We get

$$p_1^* = \begin{pmatrix} 2/11 , & 9/11 \end{pmatrix}, \quad p_2^* = \begin{pmatrix} 0 , & 1 \end{pmatrix},$$

$$p_3^* = \begin{pmatrix} 1/2 , & 1/2 \end{pmatrix} \quad \text{and} \quad p_4^* = \begin{pmatrix} 0 , & 1 \end{pmatrix}.$$

The Markov chain 1 has no absorbing states, has one ergodic set, $S = \{s_1, s_2\}$, and both states are recurrent so that none of them is transient. The Markov chains 2 and 4 have one absorbing state, s_2, one ergodic set, $\{s_2\}$, and state s_1 is transient while state s_2 is recurrent. The Markov chain 3 has no absorbing states, has one ergodic set, $S = \{s_1, s_2\}$, and both states are recurrent so that none of them is transient.

(b) As the number of stationary states is one for all four Markov chains, their corresponding matrices have only one eigenvalue equal to one. This means that, if the matrix $\widehat{\Pi}_k$ exists, it has all the rows identical and equal to the stationary probabilistic vector.

The eigenvalues of the matrix Π_1 are 1 and 7/18 so that the second eigenvalue is strictly smaller (in modulus) than 1 and, thus,

$$\widehat{\Pi}_1 = \begin{pmatrix} 2/11 & 9/11 \\ 2/11 & 9/11 \end{pmatrix}$$

and the ergodic probabilistic vector is $\widehat{p}_1 = p_1^* = \begin{pmatrix} 2/11 \ , & 9/11 \end{pmatrix}$.

The eigenvalues of the matrix Π_2 are 1 and 1/2 so that the second eigenvalue is strictly smaller (in modulus) than 1 and, thus,

$$\widehat{\Pi}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

and the ergodic probabilistic vector is $\widehat{p}_2 = p_2^* = \begin{pmatrix} 0 \ , & 1 \end{pmatrix}$.

The limit matrix $\widehat{\Pi}_3$ does not exist since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

when t is odd and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

when t is even. Note that the eigenvalues of the matrix Π_3 are 1 and -1 (Case

III in the class notes).

The eigenvalues of the matrix Π_4 are 1 and 0 so that the second eigenvalue is strictly smaller (in modulus) than 1 and, thus,

$$\widehat{\Pi}_4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

and the ergodic probabilistic vector is $\widehat{p}_2 = p_2^* = \begin{pmatrix} 0 \ , & 1 \end{pmatrix}$.

(c) Since the ergodic distribution is the distribution of states in the long run for all initial distributions, we have that the distributions of states in the long run for the Markov chains 1, 2, and 4 are given by the probabilistic vectors $\begin{pmatrix} 2/11 \ , & 9/11 \end{pmatrix}$, $\begin{pmatrix} 0 \ , & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \ , & 1 \end{pmatrix}$, respectively. These three Markov chains belong to the Case I in the class notes.

Since the matrix $\widehat{\Pi}_3$ does not exist for the Markov chain 3, there is no long-run distribution of states.

(d) From Theorem 5 in the handout on Absorbing and Irreducible Markov Chains we know that the mean recurrence time for states s_1 and s_2 are $1/p_{11}^* = 11/2 = 5.5$ and $1/p_{12}^* = 11/9 = 1.2222$, respectively.

From Theorem 6 in the handout we know that the mean first passage time m_{ij} from state s_i to state s_j can be computed as follows: find the fundamental matrix for the irreducible chain,

$$(\mathrm{I} - \Pi_1 + \Pi_1^*)^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 1/2 \\ 1/9 & 8/9 \end{pmatrix} + \begin{pmatrix} 2/11 & 9/11 \\ 2/11 & 9/11 \end{pmatrix} \right]^{-1}$$

$$= \begin{pmatrix} 15/22 & 7/22 \\ 7/99 & 92/99 \end{pmatrix}^{-1} = \begin{pmatrix} 184/121 & -63/121 \\ -14/121 & 135/121 \end{pmatrix},$$

so that

$$m_{ij} = \frac{z_{jj} - z_{ij}}{p_j^*},$$

where z_{ij} is the (i, j)-entry of the fundamental matrix $(\mathrm{I} - \Pi + \Pi^*)^{-1}$ and p_j^* is the jth entry of the stationary probabilistic row vector p^*. Thus,

$$m_{12} = \frac{\frac{135}{121} - \left(-\frac{63}{121}\right)}{\frac{9}{11}} = 2,$$

$$m_{21} = \frac{\frac{184}{121} - \left(-\frac{14}{121}\right)}{\frac{2}{11}} = 9.$$

(e) Note that the matrices

$$\Pi_2 = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Pi_4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

correspond to absorbing Markov chains and they are already displayed in the canonical form,

$$\begin{pmatrix} Q & D \\ \widehat{0} & \mathrm{I} \end{pmatrix}.$$

We know from Theorem 2 in the handout on Absorbing and Irreducible Markov Chains that the expected number n_{11} of times the chain is in state s_1 given that it starts in state s_1 is given by the position $(1, 1)$ of the matrix $(\mathrm{I} - Q)^{-1}$, which in these two cases is just a scalar. Therefore, for matrix Π_2,

$$n_{11} = (1 - 1/2)^{-1} = 2,$$

whereas for the matrix Π_4,

$$n_{11} = (1 - 0)^{-1} = 1.$$

Note that, for the chain with the matrix Π_4, if the chain starts in state s_1 then it is immediately absorbed by state s_2 with probability one so that it only stays one period in state s_1.

Obviously, for these two chains with the transition matrices Π_2 and Π_4 having a single transient state, the expected number of periods before the chain is absorbed when the chain starts in state s_1 coincides with the expected number of times the chain is in the transient state s_1 given that it starts in s_1.

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12. (a) Probability function of the Poisson distribution:

$$p(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad \text{for } x = 0, 1, 2, ...$$

$P\{\tilde{x} = 0\} = e^{-\lambda} \Longrightarrow P\{\tilde{x} \geq 1\} = 1 - P\{\tilde{x} = 0\} = 1 - e^{-\lambda} = 0.9$

$\Longrightarrow e^{-\lambda} = 0.1$

Then, $-\lambda = \ln 0.1 = -2.3026$ so that $\lambda = 2.3026$

$\mathrm{E}(\tilde{x}) = \lambda = 2.3026 =$ expected number of accidents during a year.

(b) $\widehat{\lambda} = \dfrac{2.3026}{2} = 1.1513 =$ expected number of accidents in six months.

$\tilde{y} =$ number of accidents in six months

$P\{\tilde{y} = y\} = \dfrac{\widehat{\lambda}^y e^{-\widehat{\lambda}}}{y!} \Longrightarrow$

$P\{\tilde{y} = 0\} = e^{-\widehat{\lambda}} = e^{-1.1513} = 0.3162$

$P\{\tilde{y} = 1\} = \dfrac{\widehat{\lambda}^1 e^{-\widehat{\lambda}}}{1!} = 1.1513 \cdot e^{-1.1513} = 0.3641$

$P\{\tilde{y} = 2\} = \dfrac{\widehat{\lambda}^2 e^{-\widehat{\lambda}}}{2!} = \dfrac{(1.1513)^2 \cdot e^{-1.1513}}{2} = 0.2096$

$P\{\tilde{y} \geq 3\} = 1 - \sum_{n=0}^{2} P\{\tilde{y} = n\} = 1 - 0.3162 - 0.3641 - 0.2096 = 0.1101.$

(c)

$$P\{1 \leq \tilde{x} \leq 3 \,|\tilde{x} \geq 2\} = \frac{P(\{1 \leq \tilde{x} \leq 3\} \bigcap \{\tilde{x} \geq 2\})}{P(\tilde{x} \geq 2)} = \frac{P\{2 \leq \tilde{x} \leq 3\}}{P\{\tilde{x} \geq 2\}}$$

$$= \frac{P\{\tilde{x} = 2\} + P\{\tilde{x} = 3\}}{1 - P\{\tilde{x} = 0\} - P\{\tilde{x} = 1\}} = \frac{\dfrac{\lambda^2 e^{-\lambda}}{2!} + \dfrac{\lambda^3 e^{-\lambda}}{3!}}{1 - e^{-\lambda} - \lambda \cdot e^{-\lambda}}$$

$$= \frac{\dfrac{(2.3026)^2 \cdot e^{-2.3026}}{2!} + \dfrac{(2.3026)^3 \cdot e^{-2.3026}}{3!}}{1 - e^{-2.3026} - 2.3026 \cdot e^{-2.3026}}$$

$$= \frac{0.2651 + 0.2035}{1 - 0.1 - 0.2303} = 0.6996.$$

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13. Since $\tilde{x} \sim \mathrm{N}(0,1)$ has a symmetric density around 0, then $\tilde{x}_n = -\tilde{x} \sim \mathrm{N}(0,1)$ for all n. To see this, note that

$$n(x;0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \text{for } x \in (-\infty, \infty)$$

and

$$f_{\tilde{x}_n}(x_n) = n(-x_n;0,1) \cdot |-1| = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-x_n)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_n^2}{2}} = n(x_n;0,1), \quad \text{for } x_n \in (-\infty, \infty).$$

Therefore, trivially $\tilde{x}_n \xrightarrow{d} \tilde{x}$.

For convergence in probability we need to prove that

$$\lim_{n\to\infty} P\left\{|\tilde{x}_n - \tilde{x}| < \varepsilon\right\} = 1 \quad \text{for all } \varepsilon > 0.$$

Note that

$$\lim_{n\to\infty} P\left\{|\tilde{x}_n - \tilde{x}| < \varepsilon\right\} = \lim_{n\to\infty} P\left\{|-\tilde{x} - \tilde{x}| < \varepsilon\right\} = P\left\{|-2\tilde{x}| < \varepsilon\right\} = P\left\{|2\tilde{x}| < \varepsilon\right\}$$

$$= P\left\{|\tilde{x}| < \varepsilon/2\right\} = P\left\{-\varepsilon/2 < \tilde{x} < \varepsilon/2\right\} < 1, \quad \text{for all } \varepsilon > 0.$$

Therefore, $\tilde{x}_n$ does not converge to $\tilde{x}$ in probability.

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14. (a) Using the Taylor expansion at $y = 0$, we have

$$e^y = \sum_{x=0}^{\infty} \frac{y^x}{x!}. \tag{1}$$

$$M_{\tilde{x}}(t) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} e^{tx} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} \qquad \text{(from (1))}$$

$$= e^{\lambda\left(e^t - 1\right)}.$$

(b)

$$M'_{\tilde{x}}(t) = M_{\tilde{x}}(t) \cdot \lambda e^t \Longrightarrow \mu = M'_{\tilde{x}}(0) = \underbrace{M_{\tilde{x}}(0)}_{1} \cdot \lambda e^0 = \lambda$$

$$M''_{\tilde{x}}(t) = M'_{\tilde{x}}(t) \cdot \lambda e^t + M_{\tilde{x}}(t) \cdot \lambda e^t = M_{\tilde{x}}(t) \cdot \lambda^2 e^{2t} + M_{\tilde{x}}(t) \cdot \lambda e^t$$

$$= M_{\tilde{x}}(t) \lambda e^t \left(\lambda e^t + 1 \right)$$

$$\Longrightarrow \mathrm{E}\left(\tilde{x}^2\right) = M''_{\tilde{x}}(0) = \underbrace{M_{\tilde{x}}(0)}_{1} \cdot \lambda e^0 \left(\lambda e^0 + 1 \right) = \lambda \left(\lambda + 1 \right) = \lambda^2 + \lambda$$

$$\Longrightarrow \sigma^2 = M''_{\tilde{x}}(0) - \left[M_{\tilde{x}}(0) \right]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

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15. Consider the following 4×4 transition matrix for the number of dead firms, where the value in the cell (i, j) gives us the probability of having $j - 1$ dead firms in a given period if in the previous period we had $i - 1$ dead firms $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$:

$$\begin{pmatrix} \binom{3}{0}\left(\frac{4}{5}\right)^3 & \binom{3}{1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^2 & \binom{3}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^1 & \binom{3}{3}\left(\frac{1}{5}\right)^3 \\ 0 & \binom{2}{0}\left(\frac{4}{5}\right)^2 & \binom{2}{1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^1 & \binom{2}{2}\left(\frac{1}{5}\right)^2 \\ 0 & 0 & \binom{1}{0}\left(\frac{4}{5}\right)^1 & \binom{1}{1}\left(\frac{1}{5}\right)^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 64/125 & 48/125 & 12/125 & 1/125 \\ \\ 0 & 16/25 & 8/25 & 1/25 \\ \\ 0 & 0 & 4/5 & 1/5 \\ \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The previous matrix can be written as

$$\begin{pmatrix} Q & D \\ \underline{0} & 1 \end{pmatrix},$$

where Q is a 3×3 matrix, D is a 3×1 column, and $\underline{0}$ is a 1×3 row of zeroes. Note that

$$Q = \begin{pmatrix} 64/125 & 48/125 & 12/125 \\ \\ 0 & 16/25 & 8/25 \\ \\ 0 & 0 & 4/5 \end{pmatrix}.$$

According to Theorem 3 in the handout on Absorbing and Irreducible Markov Chains, we should compute the fundamental matrix $(\mathrm{I} - Q)^{-1}$ of this absorbing chain,

$$(\mathrm{I}-Q)^{-1} = \begin{pmatrix} 1-\dfrac{64}{125} & -\dfrac{48}{125} & -\dfrac{12}{125} \\ 0 & 1-\dfrac{16}{25} & -\dfrac{8}{25} \\ 0 & 0 & 1-\dfrac{4}{5} \end{pmatrix}^{-1} = \begin{pmatrix} 125/61 & 400/183 & 820/183 \\ 0 & 25/9 & 40/9 \\ 0 & 0 & 5 \end{pmatrix}.$$

Therefore, if we start from state 1 ($1-1=0$ dead firms), then the expected number of periods before being absorbed by state 4 ($4-1=3$ dead firms) is given by the sum of the numbers in the first row of $(\mathrm{I}-Q)^{-1}$,

$$\frac{125}{61}+\frac{400}{183}+\frac{820}{183}=\frac{1595}{183}=8.7158 \text{ periods.}$$

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16. (a) The stationary probabilistic vector $p^* = \begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix}$ must solve the following equation:

$$\begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 2/5 \\ 0 & 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \alpha, & \beta, & 1-\alpha-\beta \end{pmatrix}.$$

The solution is a set of probabilistic vectors of dimension 1 with the functional form

$$p^* = \begin{pmatrix} \alpha, & \dfrac{5(1-\alpha)}{9}, & \dfrac{4(1-\alpha)}{9} \end{pmatrix}, \text{ with } \alpha \in [0,1]\,.$$

The eigenvalues of the transition matrix are $1, 1$, and $1/10$. Therefore, there is no ergodic probabilistic vector. There are no transient states. All three states are recurrent. The single absorbing state is s_1. There are two ergodic sets: $\{s_1\}$ and $\{s_2, s_3\}$.

(b)

$$\lim_{t\to\infty} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 2/5 \\ 0 & 1/2 & 1/2 \end{pmatrix}}_{\Pi}^{t} = \widehat{\Pi}.$$

The matrix $\widehat{\Pi}$ must have this form:

$$\widehat{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1-a \\ 0 & b & 1-b \end{pmatrix}.$$

Then, using the fact that $\widehat{\Pi} \cdot \Pi = \lim_{t\to\infty} \Pi^{t+1} = \widehat{\Pi}$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1-a \\ 0 & b & 1-b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 2/5 \\ 0 & 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1-a \\ 0 & b & 1-b \end{pmatrix},$$

which becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10}a + \frac{1}{2} & \frac{1}{2} - \frac{1}{10}a \\ 0 & \frac{1}{10}b + \frac{1}{2} & \frac{1}{2} - \frac{1}{10}b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1-a \\ 0 & b & 1-b \end{pmatrix}.$$

Therefore, we get $a = 5/9$ and $b = 5/9$ so that

$$\widehat{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/9 & 4/9 \\ 0 & 5/9 & 4/9 \end{pmatrix}.$$

Alternatively, we could use the fact that, if we start in states s_2 or s_3, we never go to the state s_1. Therefore, if we start in s_2 or s_3, the relevant transition matrix is

$$\begin{pmatrix} 3/5 & 2/5 \\ 1/2 & 1/2 \end{pmatrix},$$

where the row/column 1 refers to state s_2 and the row/column 2 refers to state s_3. The eigenvalues of this matrix are 1 and 1/10. This Markov chain has a unique stationary probabilistic vector that coincides with the ergodic probabilistic vector, which is $(5/9,\ 4/9)$. Note that this vector can also be obtained by making $\alpha = 0$ in the stationary probabilistic vector p^* of the original 3-state Markov chain.

Therefore,

$$\lim_{t\to\infty} P\left\{\tilde{x}_t = s_3 | \tilde{x}_0 = s_2\right\} = \widehat{\pi}_{23} = \frac{4}{9}.$$

(c)

$$\begin{pmatrix} 1/3, & 1/2, & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/9 & 4/9 \\ 0 & 5/9 & 4/9 \end{pmatrix} = \begin{pmatrix} 1/3, & 10/27, & 8/27 \end{pmatrix}.$$

Thus,

$$\lim_{t\to\infty} P\left\{\tilde{x}_t = s_2\right\} = \frac{10}{27}.$$

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17. $\tilde{x} \sim \Gamma(\alpha, \beta)$, so that $\mathrm{E}(\tilde{x}) = \alpha\beta$, $\mathrm{Var}(\tilde{x}) = \alpha\beta^2$, and $M_{\tilde{x}}(t) = (1-\beta t)^{-\alpha}$ for $t < 1/\beta$.

$$\tilde{z} = \frac{\tilde{x} - \alpha\beta}{\sqrt{\alpha}\beta} = \frac{\tilde{x}}{\sqrt{\alpha}\beta} - \sqrt{\alpha}$$

$$\begin{aligned} M_{\tilde{z}}(t) &= \mathrm{E}\left(e^{t\tilde{z}}\right) = \mathrm{E}\left[\exp\left(t\left(\frac{\tilde{x}}{\sqrt{\alpha}\beta} - \sqrt{\alpha}\right)\right)\right] \\ &= e^{-\sqrt{\alpha}t} \cdot \mathrm{E}\left[\exp\left(\frac{t\tilde{x}}{\sqrt{\alpha}\beta}\right)\right] = e^{-\sqrt{\alpha}t} \cdot M_{\tilde{x}}\left(\frac{t}{\sqrt{\alpha}\beta}\right) \\ &= e^{-\sqrt{\alpha}t} \cdot \left(1 - \beta\frac{t}{\sqrt{\alpha}\beta}\right)^{-\alpha} = e^{-\sqrt{\alpha}t} \cdot \left(1 - \frac{t}{\sqrt{\alpha}}\right)^{-\alpha} \quad \text{for } t < \sqrt{\alpha}. \end{aligned}$$

Taking logarithms,

$$\begin{aligned} \ln M_{\tilde{z}}(t) &= -\sqrt{\alpha}t - \alpha\ln\left(1 - \frac{t}{\sqrt{\alpha}}\right) \\ &= -\sqrt{\alpha}t - \alpha \cdot \underbrace{\left[0 - \frac{t}{\sqrt{\alpha}} - \frac{t^2}{2\alpha} - \frac{t^3}{3\alpha^{3/2}} - \frac{t^4}{4\alpha^2} - \ldots\right]}_{\text{this is the Taylor expansion of } \ln\left(1-\frac{t}{\sqrt{\alpha}}\right) \text{ around } t=0} \\ &= \underbrace{-\sqrt{\alpha}t + \sqrt{\alpha}t}_{=0} + \frac{t^2}{2} + \underbrace{\frac{t^3}{3\sqrt{\alpha}} + \frac{t^4}{4\alpha} + \frac{t^5}{5\alpha^{3/2}} - \ldots}_{\text{all this terms tend to 0 as } \alpha\to\infty} \end{aligned}$$

Therefore,

$$\lim_{\alpha\to\infty}\left[\ln M_{\tilde{z}}(t)\right] = \ln\left[\lim_{\alpha\to\infty} M_{\tilde{z}}(t)\right] = \frac{t^2}{2} \text{ as } \ln(\cdot) \text{ is a continuous function.}$$

Then,

$$\lim_{\alpha\to\infty} M_{\tilde{z}}(t) = e^{t^2/2}, \text{ which is the MGF of the standard normal distribution.}$$

This proves that $\tilde{z} = \dfrac{\tilde{x} - \alpha\beta}{\sqrt{\alpha}\beta} \longrightarrow \mathrm{N}(0,1)$ when $\alpha \to \infty$ and β remains constant thanks to Levy's theorem.

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18. Note that independence implies that

$$\mathrm{Var}\left(\overline{\boldsymbol{x}}_n\right) = \mathrm{Var}\left(\frac{\sum_{i=1}^n \tilde{x}_i}{n}\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^n \tilde{x}_i\right) = \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}\left(\tilde{x}_i\right) = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n} < \infty.$$

Chevyshev's inequality says that

$$P\left\{\left|\tilde{y} - \mathrm{E}\left(\tilde{y}\right)\right| \geq \varepsilon\right\} \leq \frac{\mathrm{Var}\left(\tilde{y}\right)}{\varepsilon^2}, \quad \text{for all } \varepsilon > 0,$$

for any random variable $\tilde{y}$ with $\mathrm{Var}\left(\tilde{y}\right) < \infty$. Making $\tilde{y} = \overline{\boldsymbol{x}}_n$, we get

$$P\left\{\left|\overline{\boldsymbol{x}}_n - \mathrm{E}\left(\overline{\boldsymbol{x}}_n\right)\right| \geq \varepsilon\right\} \leq \frac{\mathrm{Var}\left(\overline{\boldsymbol{x}}_n\right)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}, \quad \text{for all } \varepsilon > 0.$$

Taking limits in both sides,

$$\lim_{n\to\infty} \underbrace{P\left\{\left|\overline{\boldsymbol{x}}_n - \mathrm{E}\left(\overline{\boldsymbol{x}}_n\right)\right| \geq \varepsilon\right\}}_{\geq 0} \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0,$$

so that

$$\lim_{n\to\infty} P\left\{\left|\overline{\boldsymbol{x}}_n - \mathrm{E}\left(\overline{\boldsymbol{x}}_n\right)\right| \geq \varepsilon\right\} = 0,$$

which is the weak law of large numbers, $\overline{\boldsymbol{x}}_n - \mathrm{E}\left(\overline{\boldsymbol{x}}_n\right) \xrightarrow{p} 0$.

Note that the random variables of the sequence $\{\tilde{x}_i\}_{i=1}^{\infty}$ do not need to have the same mean for the proof. Only the assumptions of independency and common finite variance are used. If the random variables of the sequence $\{\tilde{x}_i\}_{i=1}^{\infty}$ had the same mean μ, then $\mathrm{E}(\overline{\boldsymbol{x}}_n) = \mathrm{E}(\tilde{x}_i) = \mu$ for all i and n, and we will get

$$\lim_{n\to\infty} P\{|\overline{\boldsymbol{x}}_n - \mu| \geq \varepsilon\} = 0,$$

so that $\overline{\boldsymbol{x}}_n \xrightarrow{p} \mu$.

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