Probability and Statistics. IDEA. Answers to List 5.

1. (a)

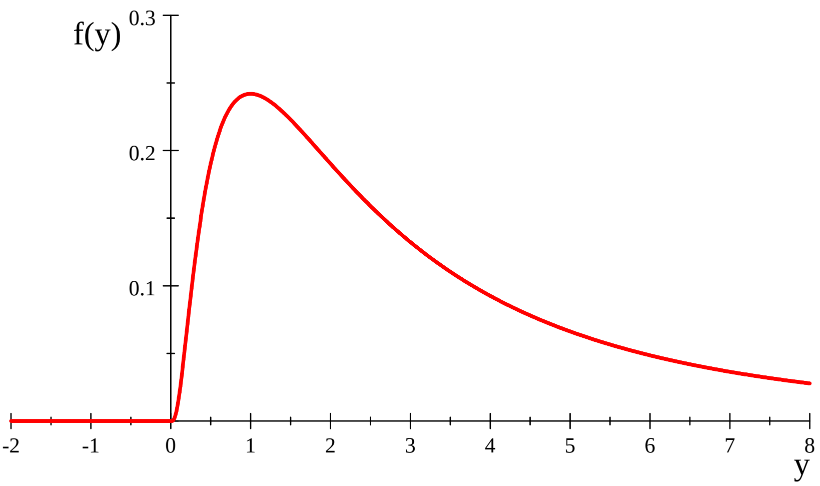
$$\tilde{x} \sim f_{\tilde{x}}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \text{for } -\infty < x < \infty$$

$$\tilde{x} = g^{-1}(\tilde{y}) = \ln \tilde{y}\ ; \qquad \tilde{y} = g(\tilde{x}) = e^{\tilde{x}} \Longrightarrow \tilde{y} > 0$$

$$\left|\frac{dx}{dy}\right| = \left|\frac{dg^{-1}(y)}{dy}\right| = \frac{1}{y} \quad \text{for } y > 0$$

$$\tilde{y} \sim f_{\tilde{y}}(y) = \begin{cases} f_{\tilde{x}}(\ln y) \cdot \left|\dfrac{dg^{-1}(y)}{dy}\right| = \dfrac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2}, & \text{for } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

The log-normal density:



Note that $\lim_{y\to 0^+} \underbrace{\frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2}}_{f_{\tilde{y}}(y)} = 0$ (Check it!). Hence, the log-normal density is continuous.

(b)

$$m = \mathrm{E}(\tilde{y}) = \mathrm{E}\left(e^{\tilde{x}}\right) = M_{\tilde{x}}(1) = e^{\mu + \frac{1}{2}\sigma^2} > 0.$$

$$s^2 = \mathrm{Var}\left(\tilde{y}\right) = \mathrm{Var}\left(e^{\tilde{x}}\right) = \mathrm{E}\left[\left(e^{\tilde{x}}\right)^2\right] - \left[\mathrm{E}\left(e^{\tilde{x}}\right)\right]^2 = \mathrm{E}\left[e^{2\tilde{x}}\right] - \left[e^{\mu+\frac{1}{2}\sigma^2}\right]^2$$

$$= M_{\tilde{x}}\left(2\right) - e^{2\mu+\sigma^2} = e^{2\mu+\frac{1}{2}\cdot 4\cdot\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\cdot\left(e^{\sigma^2}-1\right).$$

$$\mathrm{CV}_{\tilde{y}} = \frac{\left[\mathrm{Var}\left(\tilde{y}\right)\right]^{1/2}}{\left|\mathrm{E}\left(\tilde{y}\right)\right|} = \frac{e^{\mu+\frac{1}{2}\sigma^2}\left(e^{\sigma^2}-1\right)^{1/2}}{e^{\mu+\frac{1}{2}\sigma^2}} = \left(e^{\sigma^2}-1\right)^{1/2}.$$

(c) Note that $m = \mathrm{E}\left(\tilde{y}\right) > 0$ as $\tilde{y} > 0$.

$$\mathrm{CV}_{\tilde{y}} = \frac{s}{m} = \left(e^{\sigma^2}-1\right)^{1/2} \Longrightarrow \sigma^2 = \ln\left(1+\left(\mathrm{CV}_{\tilde{y}}\right)^2\right) = \ln\left(\frac{m^2+s^2}{m^2}\right).$$

$$m = e^{\mu+\frac{1}{2}\sigma^2} > 0 \Longrightarrow \ln m = \mu + \frac{1}{2}\sigma^2 \Longrightarrow$$

$$\mu = \ln m - \frac{1}{2}\sigma^2 = \ln m - \frac{1}{2}\ln\left(\frac{m^2+s^2}{m^2}\right)$$

$$= \ln m - \frac{1}{2}\ln\left(m^2+s^2\right) + \frac{1}{2}\ln m^2 = \ln m - \frac{1}{2}\ln\left(m^2+s^2\right) + \ln\left(m^2\right)^{1/2}$$

$$= 2\ln m - \frac{1}{2}\ln\left(m^2+s^2\right) = \ln\left(\frac{m^2}{\left[m^2+s^2\right]^{1/2}}\right).$$

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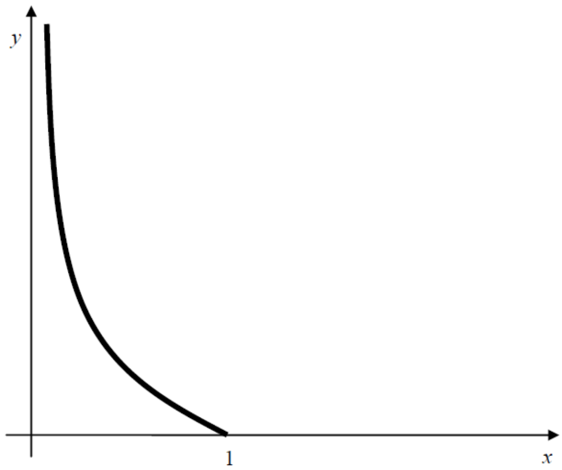
2.

$$\tilde{x} \sim f_{\tilde{x}}\left(x\right) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \tilde{y} &= g(\tilde{x}) = -2\ln\tilde{x} > 0, \text{ for } \tilde{x} \in (0,1) \\ \tilde{x} &= g^{-1}(\tilde{y}) = e^{-\tilde{y}/2} \end{aligned}$$



$$x = 0 \quad \rightarrow \quad y = \infty$$

$$x = 1 \quad \rightarrow \quad y = 0$$

$$\frac{dx}{dy} = \frac{dg^{-1}(y)}{dy} = \frac{e^{-y/2}}{(-2)} \longrightarrow \left| \frac{dg^{-1}(y)}{dy} \right| = \frac{1}{2} e^{-y/2}$$

Therefore

$$f_{\tilde{y}}(y) = \begin{cases} 1 \cdot \frac{1}{2} e^{-y/2} = \frac{1}{2} e^{-y/2} & \text{for} \quad 0 < y < \infty \\ 0 & \quad \text{elsewhere} \end{cases}$$

This is the density of an exponential distribution (or of the gamma distribution with $\alpha = 1$ and $\beta = 2$).

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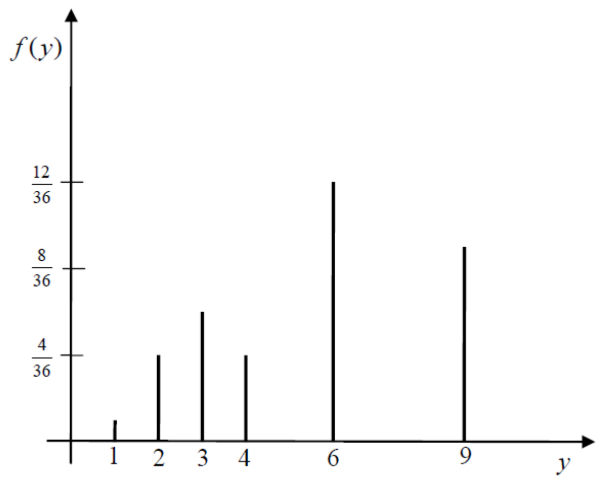
3.

$x_1 \backslash x_2$	1	2	3
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$
2	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{6}{36}$
3	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{9}{36}$

(a) $\tilde{y} = \tilde{x}_1 \cdot \tilde{x}_2 \sim f_{\tilde{y}}(y)$.

$$\begin{array}{llc}
\tilde{y} = 1 & (1,1) & \frac{1}{36} \\
\tilde{y} = 2 & (1,2)\,(2,1) & \frac{2}{36} + \frac{2}{36} = \frac{4}{36} \\
\tilde{y} = 3 & (1,3)\,(3,1) & \frac{3}{36} + \frac{3}{36} = \frac{6}{36} \\
\tilde{y} = 4 & (2,2) & \frac{4}{36} \\
\tilde{y} = 6 & (2,3)\,(3,2) & \frac{6}{36} + \frac{6}{36} = \frac{12}{36} \\
\tilde{y} = 9 & (3,3) & \frac{9}{36}
\end{array} \Longrightarrow$$

y	$f_{\tilde{y}}(y)$
1	$\frac{1}{36}$
2	$\frac{4}{36}$
3	$\frac{6}{36}$
4	$\frac{4}{36}$
6	$\frac{12}{36}$
9	$\frac{9}{36}$

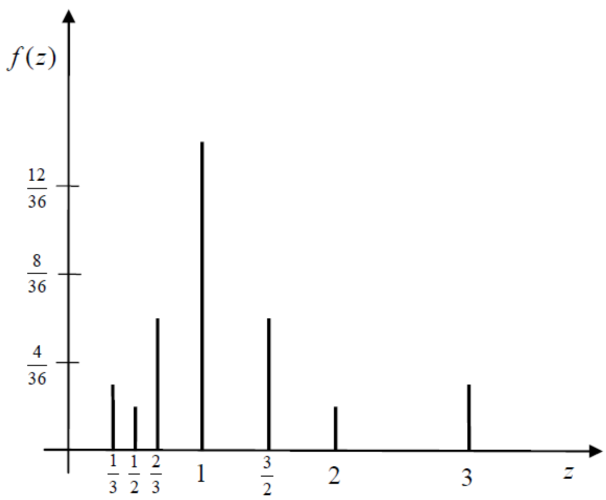


(b) $\tilde{z} = \dfrac{\tilde{x}_1}{\tilde{x}_2} \sim f_{\tilde{z}}(z)$

$\tilde{z}=1$	$(1,1)\,(2,2)\,(3,3)$	$\frac{1}{36}+\frac{4}{36}+\frac{9}{36}$
$\tilde{z}=\frac{1}{2}$	$(1,2)$	$\frac{2}{36}$
$\tilde{z}=\frac{1}{3}$	$(1,3)$	$\frac{3}{36}$
$\tilde{z}=\frac{2}{3}$	$(2,3)$	$\frac{6}{36}$
$\tilde{z}=2$	$(2,1)$	$\frac{2}{36}$
$\tilde{z}=3$	$(3,1)$	$\frac{3}{36}$
$\tilde{z}=\frac{3}{2}$	$(3,2)$	$\frac{6}{36}$

$\Longrightarrow$

z	$f_{\tilde{z}}(z)$
$\frac{1}{3}$	$\frac{3}{36}$
$\frac{1}{2}$	$\frac{2}{36}$
$\frac{2}{3}$	$\frac{6}{36}$
1	$\frac{14}{36}$
$\frac{3}{2}$	$\frac{6}{36}$
2	$\frac{2}{36}$
3	$\frac{3}{36}$



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4.

$$\tilde{x}_i \sim \Gamma(\alpha,\beta)\,,\quad i=1,...,n$$

$$M_{\tilde{x}_i}(t)=(1-\beta t)^{-\alpha}\,,\ i=1,...,n,\ \text{for } t<1/\beta$$

$$\tilde{y}=\sum_{i=1}^{n}\tilde{x}_i$$

Since the $\tilde{x}_i$'s are independent,

$$M_{\tilde{y}}(t) = \mathrm{E}\left(e^{t\tilde{y}}\right) = \mathrm{E}\left(\exp\left(t\sum_{i=1}^{n}\tilde{x}_i\right)\right) = \mathrm{E}\left(\exp\left(\sum_{i=1}^{n}t\tilde{x}_i\right)\right)$$

$$= \mathrm{E}\left(\prod_{i=1}^{n}e^{t\tilde{x}_i}\right) = \prod_{i=1}^{n}\mathrm{E}\left(e^{t\tilde{x}_i}\right) = \prod_{i=1}^{n}M_{\tilde{x}_i}(t) = \left[(1-\beta t)^{-\alpha}\right]^n = (1-\beta t)^{-(n\alpha)},$$

for $t < 1/\beta$. This has the same form as the MGF of a Γ variable. Since the exponent of $M_{\tilde{y}}(t)$ is $-n\alpha$, the parameter corresponding to α in the Γ is now $n\alpha$,

$$f_{\tilde{y}}(y; n\alpha, \beta) = \frac{1}{\beta^{n\alpha}\Gamma(n\alpha)}y^{n\alpha-1}e^{-y/\beta}; \quad \text{for } y > 0,$$

and $f_{\tilde{y}}(y; n\alpha, \beta) = 0$, otherwise. Thus, $\tilde{y} \sim \Gamma(n\alpha, \beta)$.

Since the exponential distribution is the gamma distribution with $\alpha = 1$ and $\beta = \theta$, the sum of n independent exponential random variables has the gamma distribution with parameters n and θ, $\Gamma(n, \theta)$.

Note that the gamma distribution when α is equal to a natural number n is used to model waiting time in a queue when the individual under consideration is in the nth position in that queue. In this case, we know that the expected waiting time will be $\alpha\beta = n\theta$.

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5. If $\tilde{\theta}$ is the angle that a "random spinner" makes with the positive side of the x axis, what is the density function of $\tan\tilde{\theta}$? Note that the distribution of the angle $\tilde{\theta}$ is obviously absolutely continuous and uniform on $(-\pi/2, \pi/2)$.

Thus, this exercise can be formulated as follows:

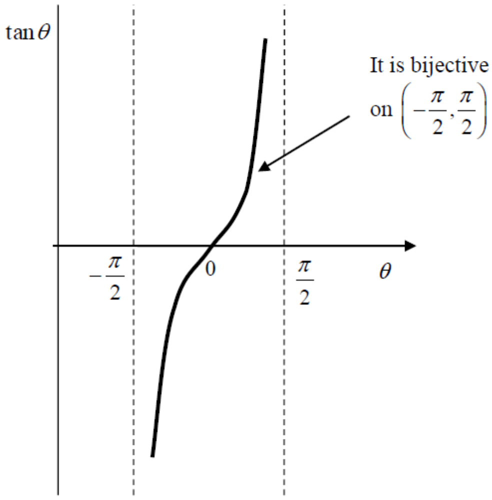
$$f_{\tilde{\theta}}(\theta) = \frac{1}{\pi}, \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

and $f_{\tilde{\theta}}(\theta) = 0$, otherwise.

$$g: \ t = g(\theta) = \tan\theta \in (-\infty, \infty) \ ,$$

$$g^{-1}: \ \theta = g^{-1}(t) = \arctan t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

$$f_{\tilde{t}}(t) = ?$$



$$J_{g^{-1}}(t) = \frac{d\theta}{dt} = \frac{1}{1+t^2} > 0$$

Hence,

$$f_{\tilde{t}}(t) = \frac{1}{\pi} \cdot |J_{g^{-1}}(t)| = \frac{1}{\pi} \cdot \frac{1}{1+t^2} \text{ for } -\infty < t < \infty \quad \text{(Density of the Cauchy distribution)}$$

Check (for free!):

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{1}{1+t^2} dt = \frac{1}{\pi} [\arctan t]_{-\infty}^{\infty} = \frac{1}{\pi} \left[\lim_{t \to \infty} \arctan(t) - \lim_{t \to -\infty} \arctan(t) \right]$$
$$= \frac{1}{\pi} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = 1$$

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6.

$$f(x_1, x_2) = 1, \quad \text{for } 0 < x_1 < 1, \ 0 < x_2 < 1$$

$$g : \begin{cases} u_1 = (-2 \ln x_1)^{\frac{1}{2}} \sin(2\pi x_2), & -\infty < u_1 < \infty \\ \\ u_2 = (-2 \ln x_1)^{\frac{1}{2}} \cos(2\pi x_2), & -\infty < u_2 < \infty \end{cases}$$

$f_{\tilde{u}_1, \tilde{u}_2}(u_1, u_2) = ?$

$$u_1^2 = -2 \ln x_1 \sin^2(2\pi x_2)$$
$$u_2^2 = -2 \ln x_1 \cos^2(2\pi x_2)$$

Since

$$u_1^2 + u_2^2 = -2 \ln x_1 \left[\sin^2(2\pi x_2) + \cos^2(2\pi x_2) \right] = -2 \ln x_1,$$

then

$$x_1 = e^{-\frac{1}{2}\left(u_1^2 + u_2^2\right)} \in (0, 1).$$

Since

$$
\begin{aligned}
\frac{u_1}{u_2} &= \tan\left(2\pi x_2\right), \\
2\pi x_2 &= \arctan\left(\frac{u_1}{u_2}\right),
\end{aligned}
$$

then

$$x_2 = \frac{1}{2\pi}\arctan\left(\frac{u_1}{u_2}\right) \in (0,1)$$

$$|J_{g^{-1}}(u_1,u_2)| = \left|\det\begin{bmatrix} \frac{dx_1}{du_1} & \frac{dx_1}{du_2} \\ \frac{dx_2}{du_1} & \frac{dx_2}{du_2} \end{bmatrix}\right| = \left|\det\begin{bmatrix} e^{-\frac{1}{2}\left(u_1^2+u_2^2\right)}\left(-u_1\right) & e^{-\frac{1}{2}\left(u_1^2+u_2^2\right)}\left(-u_2\right) \\ \frac{1}{2\pi}\frac{1}{1+\left(\frac{u_1}{u_2}\right)^2}\frac{1}{u_2} & \frac{1}{2\pi}\frac{1}{1+\left(\frac{u_1}{u_2}\right)^2}\left(-\frac{u_1}{u_2^2}\right) \end{bmatrix}\right|$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\frac{1}{1+\frac{u_1^2}{u_2^2}}e^{-\frac{1}{2}\left(u_1^2+u_2^2\right)}\left|\det\begin{bmatrix} -u_1 & -u_2 \\ \frac{1}{u_2} & -\frac{u_1}{u_2^2} \end{bmatrix}\right| = \frac{1}{2\pi}\frac{1}{1+\frac{u_1^2}{u_2^2}}e^{-\frac{1}{2}\left(u_1^2+u_2^2\right)}\left[1+\frac{u_1^2}{u_2^2}\right] \\
&= \frac{1}{2\pi}e^{-\frac{1}{2}\left(u_1^2+u_2^2\right)}
\end{aligned}
$$

$$f_{\tilde{u}_1,\tilde{u}_2}\left(u_1,u_2\right) = 1\cdot|J_{g^{-1}}(u_1,u_2)| = \frac{1}{2\pi}e^{-\frac{1}{2}\left(u_1^2+u_2^2\right)},\ \text{for } -\infty < u_1 < \infty,\ -\infty < u_2 < \infty.$$

Thus, $(\tilde{u}_1, \tilde{u}_2) \sim \mathrm{MN}\left(\underline{0}, \mathrm{I}\right)$, where I is the identity matrix. That is, $\tilde{u}_1$ and $\tilde{u}_2$ are independently distributed standard normal variables.

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7.

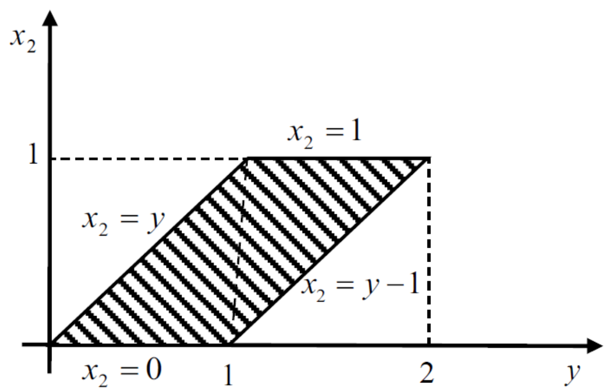
$$g : \begin{cases} y = x_1 + x_2 \in (x_2, x_2+1) \\ x_2 = x_2 \in (0,1) \end{cases} \Longrightarrow g^{-1} : \begin{cases} x_1 = y - x_2 \in (0,1) \\ x_2 = x_2 \in (0,1) \end{cases}$$

$$|J_{g^{-1}}(y,x_2)| = \left| \det \begin{bmatrix} \frac{dx_1}{dy} & \frac{dx_1}{dx_2} \\ \frac{dx_2}{dy} & \frac{dx_2}{dx_2} \end{bmatrix} \right| = \left| \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right| = 1.$$

Then,

$$f_{\tilde{y},\tilde{x}_2}(y,x_2) = 1 \cdot 1 = 1 \text{ for } 0 < x_2 < 1 \text{ and } x_2 < y < x_2 + 1$$

$$f_{\tilde{y},\tilde{x}_2}(y,x_2) = \begin{cases} 1 & \text{for} \quad 0 < x_2 < 1 \text{ and } x_2 < y < x_2 + 1 \\ 0 & \qquad \text{elsewhere} \end{cases}$$



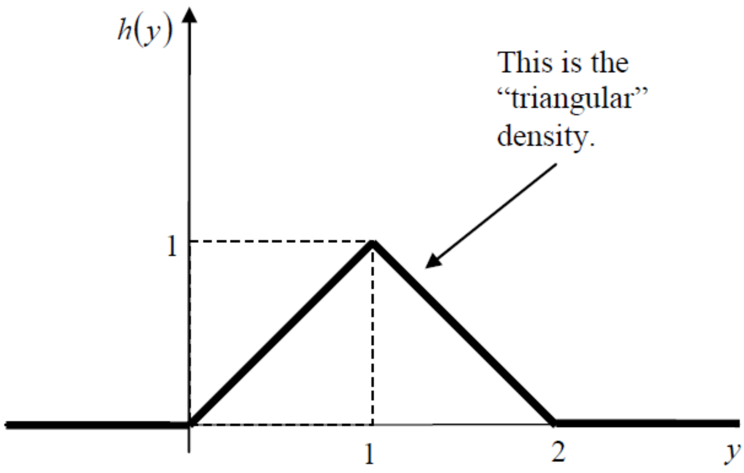
The density of y for $y \in (0,2)$ is

$$h(y) = \int_{-\infty}^{\infty} f_{\tilde{y},\tilde{x}_2}(y,x_2)\,dx_2 = \int_{\mathbb{R}} \mathbb{I}_C(y,x_2) \cdot 1dx_2 = \int_{C(y)} 1dx_2,$$

where the set C is the interior of the parallelogram in the previous figure.

Then,

$$h(y) = \begin{cases} 0 & \text{for} \quad y \leq 0 \\ \int_0^y 1dx_2 = y & \text{for} \quad 0 < y \leq 1 \\ \int_{y-1}^1 1dx_2 = 2 - y & \text{for} \quad 1 < y < 2 \\ 0 & \text{for} \quad y \geq 2 \end{cases}$$



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8. (a) The joint distribution of the random vector $(\tilde{x}, \tilde{y})$ is

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}, \quad \text{for } x \in (-\infty, +\infty) \text{ and } y \in (-\infty, +\infty).$$

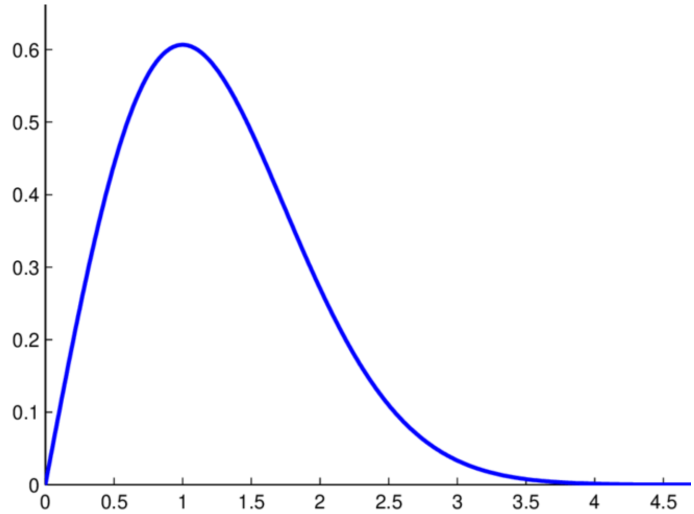
Let $(r, \theta) = g(x, y)$ with $r = (x^2 + y^2)^{1/2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. Thus, $(x, y) = g^{-1}(r, \theta)$ is given by $x = r\cos\theta$ and $y = r\sin\theta$. Therefore, $|J_{g^{-1}}(r, \theta)| = r > 0$. Then,

$$f_{\tilde{r},\tilde{\theta}}(r,\theta) = \begin{cases} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r & \text{for } r \in (0,\infty) \text{ and } \theta \in [0, 2\pi) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, computing the marginal density of $\tilde{r}$,

$$f_{\tilde{r}}(r) = \begin{cases} \int_0^{2\pi} f_{\tilde{r},\tilde{\theta}}(r,\theta)d\theta = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r \,[\theta]_0^{2\pi} = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & \text{for } r \in (0,\infty) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the distribution of the modulus $\tilde{r}$ of the vector $(\tilde{x}, \tilde{y})$ with random coordinates that are independent and normally distributed with zero mean and standard deviation σ is the Rayleigh distribution with parameter σ. Its density looks like this:



Moreover, computing the marginal density of $\tilde{\theta}$,

$$f_{\tilde{\theta}}(\theta) = \begin{cases} \int_0^\infty f_{\tilde{r},\tilde{\theta}}(r,\theta) dr = \dfrac{1}{2\pi}\left[-e^{-\frac{r^2}{2\sigma^2}}\right]_0^\infty = \dfrac{1}{2\pi} & \text{for } \theta \in [0, 2\pi) \\ & \text{or for } \theta \in (0, 2\pi) \\ 0 & \text{otherwise.} \end{cases}$$

(b) The random variables $\tilde{r}$ and $\tilde{\theta}$ are independent since we see that

$$f_{\tilde{r},\tilde{\theta}}(r,\theta) = f_{\tilde{z}}(z) f_{\tilde{r}}(r) \quad \text{for all } r \text{ and } \theta.$$

(c) To find the density of the square of the modulus $\tilde{w} = \tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2$ with

$r \in (0, \infty)$, we note that $\tilde{r} = \tilde{w}^{1/2} > 0$ so that

$$\left| \frac{dr}{dw} \right| = \frac{1}{2w^{1/2}} > 0.$$

Then,

$$f_{\tilde{w}}(w) = f_{\tilde{r}}(w^{1/2}) \left| \frac{dr}{dw} \right| = \frac{w^{1/2}}{\sigma^2} e^{-\frac{w}{2\sigma^2}} \frac{1}{2w^{1/2}} = \frac{1}{2\sigma^2} e^{-\frac{w}{2\sigma^2}}, \text{ for } w \in (0, \infty),$$

and $f_{\tilde{w}}(w) = 0$, otherwise. This is the density of an exponential distribution with parameter $2\sigma^2$.

The previous result also tells us that, if $\tilde{w}$ is exponential with parameter λ,

$$f_{\tilde{w}}(w) = \begin{cases} \frac{1}{\lambda} e^{-w/\lambda} & \text{for } w > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

then the random variable $\tilde{r} = \tilde{w}^{1/2} = \sqrt{\tilde{w}} > 0$ is Rayleigh with the scale parameter $\sigma = \sqrt{\lambda/2} > 0$.

Since the density of a χ^2_ν distribution is

$$f(w; \nu) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})} w^{\frac{\nu-2}{2}} e^{-w/2} & \text{for } w > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

it is immediate to see that

$$f_{\tilde{w}}(w) = f(w;2) = \begin{cases} \dfrac{1}{2}e^{-w/2} & \text{for } w > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

when $\sigma = 1$ so that $\tilde{w} \sim \chi_2^2$.

(d) We integrate by parts.

$$\mathrm{E}(\tilde{r}) = \int_0^\infty r f_{\tilde{r}}(r) dr = \int_0^\infty \underbrace{r}_{F(r)} \underbrace{\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}}_{G'(r)} dr =$$

$$\left[-re^{-\frac{r^2}{2\sigma^2}} \right]_0^\infty - \int_0^\infty \underbrace{1}_{F'(r)} \underbrace{\left(-e^{-\frac{r^2}{2\sigma^2}} \right)}_{G(r)} dr = 0 + \int_0^\infty e^{-\frac{r^2}{2\sigma^2}} dr = ...$$

Make the change of variable $z = r/\sigma$ so that $\left| \dfrac{dr}{dz} \right| = \sigma > 0$.

$$... = \int_0^\infty e^{-\frac{z^2}{2}} \sigma dr = \sigma \int_0^\infty e^{-\frac{z^2}{2}} dr = \sigma \sqrt{\frac{\pi}{2}}.$$

$$\mathrm{E}(\tilde{r}^2) = \int_0^\infty r^2 f_{\tilde{r}}(r) dr = \int_0^\infty \underbrace{r^2}_{F(r)} \underbrace{\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}}_{G'(r)} dr =$$

$$\left[-r^2 e^{-\frac{r^2}{2\sigma^2}} \right]_0^\infty - \int_0^\infty \underbrace{2r}_{F'(r)} \underbrace{\left(-e^{-\frac{r^2}{2\sigma^2}} \right)}_{G(r)} dr = 0 + 2\int_0^\infty re^{-\frac{r^2}{2\sigma^2}} dr = ...$$

$$2\left[-\sigma^2 e^{-\frac{r^2}{2\sigma^2}} \right]_0^\infty = 2\sigma^2.$$

Thus,

$$\mathrm{Var}(\tilde{r}) = \mathrm{E}(\tilde{r}^2) - [\mathrm{E}(\tilde{r})]^2 = 2\sigma^2 - \left(\frac{\pi}{2}\right)\sigma^2 = \left(2 - \frac{\pi}{2}\right)\sigma^2.$$

(e) As $\tilde{\theta}$ is uniform on $[0, 2\pi)$ or, without loss of generality, on $(0, 2\pi)$,

$$\mathrm{E}(\tilde{\theta}) = \pi \quad \text{and} \quad \mathrm{Var}(\tilde{\theta}) = \frac{(2\pi)^2}{12} = \frac{\pi^2}{3}.$$

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9. (a)

$$F(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ \int_0^y \int_0^x \frac{r+2t}{12} dr dt = \frac{xy(x+2y)}{24} & \text{if } 0 \le x < 2,\ 0 \le y < 2 \\ \int_0^2 \int_0^x \frac{r+2t}{12} dr dt = \frac{x(x+4)}{12} & \text{if } 0 \le x < 2,\ y \ge 2 \\ \int_0^y \int_0^2 \frac{r+2t}{12} dr dt = \frac{y(y+1)}{6} & \text{if } x \ge 2,\ 0 \le y < 2, \\ 1 & \text{if } x \ge 2,\ y \ge 2. \end{cases}$$

(b)

$$f_{\tilde{x}}(x) = \begin{cases} \int_0^2 \frac{x+2y}{12} dy = \frac{x+2}{6} & \text{if } x \in (0,2) \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{\tilde{y}}(y) = \begin{cases} \int_0^2 \frac{x+2y}{12} dx = \frac{1+2y}{6} & \text{if } y \in (0,2) \\ 0 & \text{otherwise.} \end{cases}$$

(c)

$$
f_{\tilde{y}|\tilde{x}}(y\,|x) = \begin{cases} \dfrac{f(x,y)}{f_{\tilde{x}}(x)} = \dfrac{\frac{1}{12}(x+2y)}{\frac{x+2}{6}} = \dfrac{x+2y}{2x+4} & \text{if } y \in (0,2) \\ \\ 0 & \text{otherwise.} \end{cases}
$$

for $x \in (0,2)$.

$$
f_{\tilde{y}|\tilde{x}}\left(y \left| \frac{1}{2}\right.\right) = \begin{cases} \dfrac{f\left(\frac{1}{2},y\right)}{f_{\tilde{x}}\left(\frac{1}{2}\right)} = \dfrac{1+4y}{10} & \text{if } y \in (0,2) \\ \\ 0 & \text{otherwise.} \end{cases}
$$

(d)

$$
\mathrm{E}(\tilde{y}) = \int_{-\infty}^{\infty} y f_{\tilde{y}}(y) dy = \int_0^2 y \frac{1+2y}{6} dy = \frac{11}{9},
$$

or, equivalently,

$$
\mathrm{E}(\tilde{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y) dx dy = \int_0^2\int_0^2 y \frac{x+2y}{12} dx dy = \frac{11}{9}.
$$

$$
\mathrm{E}(\tilde{y}|\,\tilde{x} = 1/2) = \int_{-\infty}^{\infty} y f_{\tilde{y}|\tilde{x}}\left(y \left| \frac{1}{2}\right.\right) dy = \int_0^2 y \frac{1+4y}{10} dy = \frac{19}{15}.
$$

(e)

$$
P\left(\{0 < \tilde{x} < 1\} \cap \{1 < \tilde{y} < 3/2\}\right) = \int_0^1 \int_1^{3/2} \frac{x+2y}{12} dy dx = \frac{1}{8}.
$$

(f) Note that $f_{\tilde{y}|\tilde{x}}(y\,|x) = f_{\tilde{y}}(y)$ for all $y \in (-\infty, \infty)$ if and only if

$$\frac{x+2y}{2(x+2)} = \frac{1+2y}{6}.$$

for all $y \in (0,2)$. Hence, $x = 1$.

The random variables $\tilde{x}$ and $\tilde{y}$ are not independent since

$$f_{\tilde{x}}(x) \cdot f_{\tilde{y}}(y) = \frac{x+2}{6} \cdot \frac{1+2y}{6} = \frac{(x+2)\cdot(1+2y)}{36} \neq f(x,y) = \frac{1}{12}\,(x+2y)\,,$$

for all $(x,y) \in (0,2) \times (0,2)$, except when $(x,y) = (1,1)$. Note that, for $(x,y) \in (0,2) \times (0,2)$,

$$f_{\tilde{x}}(x) \cdot f_{\tilde{y}}(y) - f(x,y) = \frac{(x+2)\cdot(1+2y)}{36} - \frac{1}{12}\,(x+2y)$$

$$= \frac{1}{18}\,(x-1)\,(y-1) = 0 \text{ if and only if } (x,y) = (1,1)\,.$$

(g) Let $y \in (0,2)$. Then, $w = g(y) = 2 - y^{1/3} \Longleftrightarrow y = g^{-1}(w) = (2-w)^3$ for $g^{-1}\{(0,2)\} = \left(2 - 2^{1/3}, 2\right) = (0.74, 2)\,.$

Moreover,

$$\frac{dy}{dw} = \frac{dg^{-1}(w)}{dw} = -3\,(2-w)^2 < 0$$

for $w \in \left(2 - 2^{1/3}, 2\right) \Longleftrightarrow y \in (0,2)$.

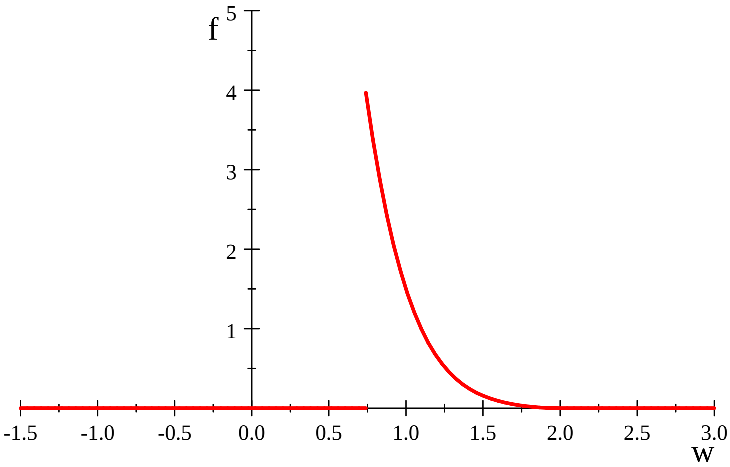
Since $f_{\tilde{y}}(y) = \dfrac{1+2y}{6}$ for $y \in (0,2)$, then

$$f_{\tilde{w}}(w) = f_{\tilde{y}}(g^{-1}(w)) \left|\frac{dg^{-1}(w)}{dw}\right| = \frac{1+2\,(2-w)^3}{6} 3\,(2-w)^2$$

$$= \left(\frac{1}{2} + (2-w)^3\right)(2-w)^2\,,\text{ for } w \in \left(2 - 2^{1/3}, 2\right).$$

Thus,

$$f_{\tilde{w}}(w) = \begin{cases} \left(\dfrac{1}{2} + (2-w)^3\right)(2-w)^2 & \text{for } 2 - 2^{1/3} < w < 2 \\ \\ 0 & \text{otherwise.} \end{cases}$$



$\mathrm{E}(\tilde{w}) = \int_{-\infty}^{\infty} g(y) f_{\tilde{y}}(y) dy = \int_0^2 \left(2 - y^{1/3}\right) \dfrac{1+2y}{6} dy$
$= \dfrac{1}{6} \int_0^2 \left(2 + 4y - y^{1/3} - 2y^{4/3}\right) dy = \dfrac{1}{6}\left[2y + 2y^2 - \dfrac{3}{4}y^{4/3} - \dfrac{6}{7}y^{7/3}\right]_0^2$
$= \dfrac{1}{6}\left[12 - \left(\dfrac{69}{14} \cdot 2^{1/3}\right)\right] = 2 - \left(\dfrac{23}{28} \cdot 2^{1/3}\right) = 0.965.$
Alternatively (and much more difficult and inefficient), we can compute

$$\mathrm{E}(\tilde{w}) = \int_{2-2^{1/3}}^{2} w \underbrace{\left(\frac{1}{2} + (2-w)^3\right)(2-w)^2}_{f_{\tilde{w}}(w)} dw = 2 - \left(\frac{23}{28} \cdot 2^{1/3}\right) = 0.965.$$

(h) *Density function method*: Let $(z, t) = g(x, y)$ be given by $z = xy$ and

$t = y$. Therefore, $(x, y) = g^{-1}(z, t)$ is given by $x = z/t$ and $y = t$. Therefore,

$$|J_{g^{-1}}(z,t)| = \left| \det \begin{pmatrix} 1/t & -z/t^2 \\ 0 & 1 \end{pmatrix} \right| = 1/t$$

and

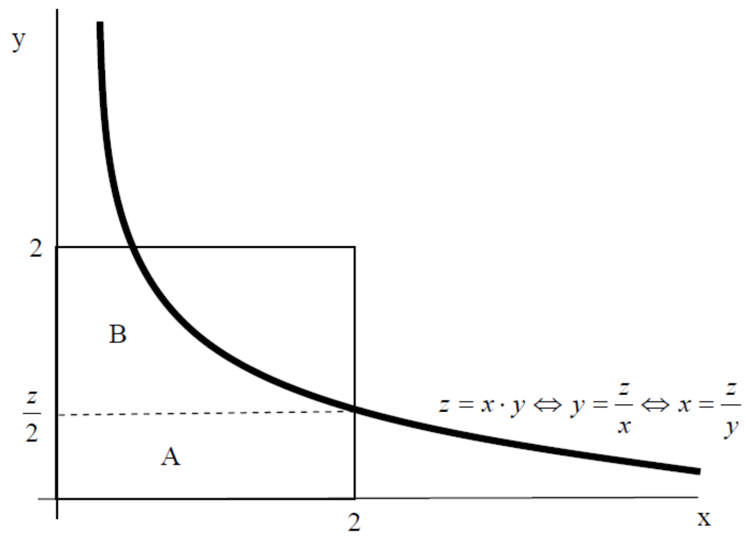
$$f_{\tilde{z},\tilde{t}}(z,t) = \begin{cases} f_{\tilde{x},\tilde{y}}\left(\frac{z}{t}, t\right) |J_{g^{-1}}(z,t)| = \frac{1}{12}\left(\frac{z}{t} + 2t\right)\frac{1}{t} = \frac{1}{12}\left(\frac{z}{t^2} + 2\right) & \text{if } \frac{z}{t} \in (0,2) \text{ and } t \in (0,2) \\ 0 & \text{otherwise.} \end{cases}$$

The set of positive density, $C = \{z/t \in (0,2) \text{ and } t \in (0,2)\}$, can also be written as $\{z \in (0, 2t) \text{ and } t \in (0,2)\}$ or as $\{z \in (0,4) \text{ and } t \in (z/2, 2)\}$. Make a draw in the plain (z,t) to see this!

Then, computing the marginal density of $\tilde{z}$,

$$f_{\tilde{z}}(z) = \begin{cases} \int_{\mathbb{R}} \mathbb{I}_C(z,t)\, f_{\tilde{z},\tilde{t}}(z,t)dt = \int_{z/2}^{2} \frac{1}{12}\left(\frac{z}{t^2} + 2\right) dt = \dfrac{4-z}{8} & \text{if } z \in (0,4) \\ 0 & \text{otherwise.} \end{cases}$$

Distribution function method:



$$F_{\tilde{z}}(z) = \begin{cases} 0 & \text{if } z < 0 \\ \int\limits_A \frac{1}{12}(x+2y)\,d(x,y) + \int\limits_B \frac{1}{12}(x+2y)\,d(x,y) = \frac{1}{2}z - \frac{1}{16}z^2 & \text{if } z \in [0,4] \\ 1 & \text{if } z > 4 \end{cases}$$

Note that

$$\int\limits_A \frac{1}{12}(x+2y)\,d(x,y) + \int\limits_B \frac{1}{12}(x+2y)\,d(x,y) =$$

$$\int_0^{\frac{z}{2}} \int_0^2 \frac{1}{12}(x+2y)\,dxdy + \int_{\frac{z}{2}}^2 \int_0^{\frac{z}{y}} \frac{1}{12}(x+2y)\,dxdy = \frac{z}{2} - \frac{z^2}{16}.$$

Therefore, by just differentiating,

$$f_{\tilde{z}}(z) = \begin{cases} F'(z) = \dfrac{4-z}{8} & \text{if } z \in (0,4) \\ \\ 0 & \text{otherwise.} \end{cases}$$

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10. If $x < 0$, then

$$F_{\tilde{x}}(x) = \int_{-\infty}^{x} \frac{1}{2} e^t dt = \frac{1}{2} e^x.$$

If $x \geq 0$, then

$$\int_{-\infty}^{x} \frac{1}{2} e^{-|t|} dt = \underbrace{\int_{-\infty}^{0} \frac{1}{2} e^t dt}_{1/2} + \int_{0}^{x} \frac{1}{2} e^{-t} dt = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} e^{-x} = 1 - \frac{1}{2} e^{-x}.$$

Therefore,

$$F_{\tilde{x}}(x) = \begin{cases} \frac{1}{2} e^x & \text{for } x < 0 \\ \\ 1 - \frac{1}{2} e^{-x} & \text{for } x \geq 0. \end{cases}$$

Note that $y = g(x) = x^3$ is a one-to-one correspondence (or bijection) and $x = g^{-1}(y) = y^{1/3}$ so that

$$f_{\tilde{y}}(y) = \frac{1}{6} e^{-|y^{1/3}|} y^{-2/3} \text{ for } x \in (-\infty, \infty)$$

Note that $z = g(x) = \ln(|x| + 1) \geq 0$ is not a bijection. Then, we use the distribution function method. Note that for $z \geq 0$ we have

$$F_{\tilde{z}}(z) = P\left\{\tilde{z} \leq z\right\} = P\left\{\ln\left(|\tilde{x}| + 1\right) \leq z\right\} = P\left\{1 - e^z \leq \tilde{x} \leq e^z - 1\right\} =$$

$$F_{\tilde{x}}(\underbrace{e^z - 1}_{\geq 0}) - F_{\tilde{x}}(\underbrace{1 - e^z}_{\leq 0}) = 1 - \frac{1}{2}e^{-(e^z - 1)} - \frac{1}{2}e^{(1-e^z)} = 1 - e^{(1-e^z)}.$$

Therefore,

$$F_{\tilde{z}}(z) = \begin{cases} 0 & \text{for } z < 0 \\ \\ 1 - e^{(1-e^z)} & \text{for } z \geq 0. \end{cases}$$

Differentiating $F_{\tilde{z}}$ one gets the density function,

$$f_{\tilde{z}}(z) = \begin{cases} e^{1-e^z}e^z = e^{1+z-e^{-z}} & \text{for } z > 0. \\ \\ 0 & \text{for } z \leq 0. \end{cases}$$

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11. $y = g(x) = 9 - x^2$ is a one-to-one correspondence (or bijective function) on $(0, 3)$. When $x \in (0, 3)$ then $y \in (0, 9)$. Therefore, $x = g^{-1}(y) = (9 - y)^{1/2}$ with $y \in (0, 9)$. Moreover,

$$\left|\frac{dg^{-1}(y)}{dy}\right| = \frac{1}{2}\left(9 - y\right)^{-1/2}.$$

$$f_{\tilde{y}}(y) = \begin{cases} f_{\tilde{x}}(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right| = \frac{1}{6}(9 - y)^{-1/2} & \text{for } y \in (0, 9) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Integrating the density $f_{\tilde{y}}$ we get the distribution function:

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \\ \int_0^y \frac{1}{6}(9-t)^{-1/2}dt = 1 - \dfrac{(9-y)^{1/2}}{3} & \text{for } y \in (0,9) \\ \\ 1 & \text{for } y \geq 9. \end{cases}$$

We could have used instead the distribution function method:

$F_{\tilde{y}}(y) = 0$ for $y \leq 0$, $F_{\tilde{y}}(y) = 1$ for $y \geq 9$, and, for $y \in (0,9)$, we have,

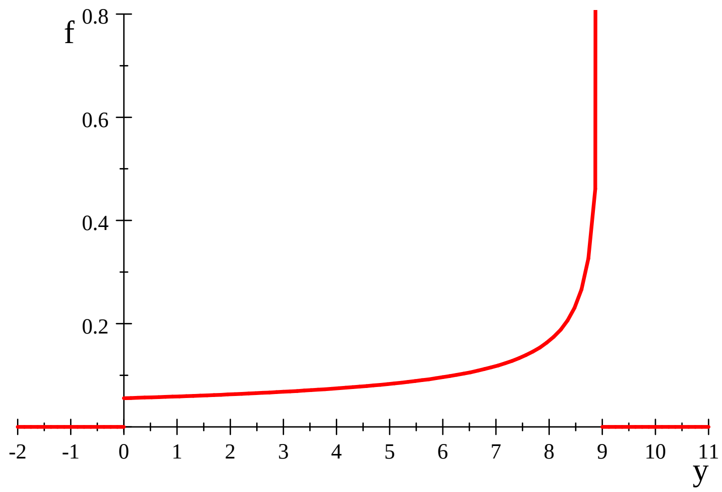
$$\begin{aligned} F_{\tilde{y}}(y) &= P\{\tilde{y} \leq y\} = P\left\{9 - \tilde{x}^2 \leq y\right\} = P\left\{9 - y \leq \tilde{x}^2\right\} \\ &= P(\left\{(9-y)^{1/2} \leq \tilde{x}\right\} \cup \left\{\tilde{x} \leq -(9-y)^{1/2}\right\}) \\ &= P\left\{(9-y)^{1/2} \leq \tilde{x}\right\} + P\left\{\tilde{x} \leq -(9-y)^{1/2}\right\} = P\left\{(9-y)^{1/2} \leq \tilde{x}\right\} + 0. \end{aligned}$$

Note that $P\left\{\tilde{x} \leq \underbrace{-(9-y)^{1/2}}_{\text{negative}}\right\} = 0$ for $y \in (0,9)$ since the density of $\tilde{x}$ is positive only on $(0,3)$.

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \\ P\left\{(9-y)^{1/2} \leq \tilde{x}\right\} = \int_{(9-y)^{1/2}}^3 \frac{1}{3}dx = 1 - \dfrac{(9-y)^{1/2}}{3} & \text{for } y \in (0,9) \\ \\ 1 & \text{for } y \geq 9, \end{cases}$$

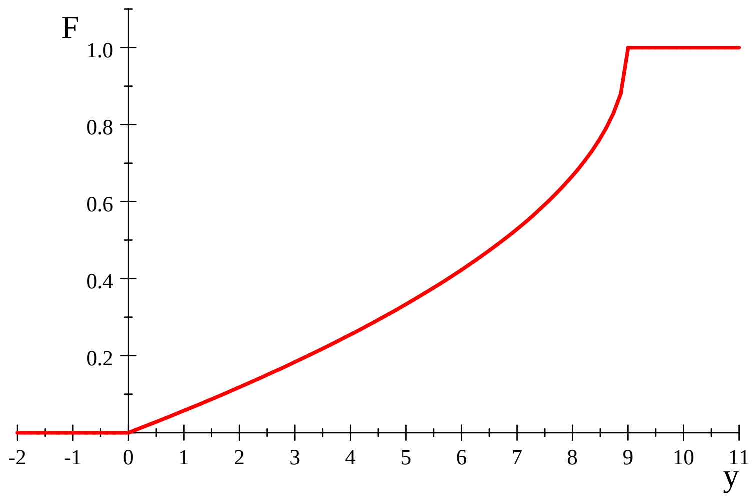
which is the same as before. Differentiating $F_{\tilde{y}}(y)$ we obtain $f_{\tilde{y}}(y)$ a.s.

Density function $f_{\tilde{y}}$:



Note that $\lim_{y\to 0^+} f_{\tilde{y}}(y) = 1/18$ and $\lim_{y\to 9^-} f_{\tilde{y}}(y) = \infty$.

Distribution function $F_{\tilde{y}}$:



Note that $F_{\tilde{y}}$ is continuous and differentiable except at $y = 0$ and $y = 9$ since $\lim_{y\to 0^-} F_{\tilde{y}}(y) = \lim_{y\to 0^+} F_{\tilde{y}}(y) = 0$, $\lim_{y\to 9^-} F_{\tilde{y}}(y) = \lim_{y\to 9^+} F_{\tilde{y}}(y) = 1$, $\lim_{y\to 0^-} F'_{\tilde{y}}(y) = 0 \neq \lim_{y\to 0^+} F'_{\tilde{y}}(y) = 1/18$ and $\lim_{y\to 9^-} F'_{\tilde{y}}(y) = \infty \neq \lim_{y\to 9^+} F'_{\tilde{y}}(y) = 0$.

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12. (a)

$$F_{\tilde{y}}(y) = P\{\tilde{y} \le y\} = P\{|\tilde{x}| \le y\} = P\{-y \le \tilde{x} \le y\} = F_{\tilde{x}}(y) - \lim_{z \to -y^-} F_{\tilde{x}}(z) \quad \text{for } y \ge 0$$

and $F_{\tilde{y}}(y) = 0$ for $y < 0$.

(b)

$$F_{\tilde{y}}(y) = F_{\tilde{x}}(y) - F_{\tilde{x}}(-y) \quad \text{for } y \ge 0$$

and $F_{\tilde{y}}(y) = 0$ for $y < 0$.

(c)

$$f_{\tilde{y}}(y) = f_{\tilde{x}}(y) + f_{\tilde{x}}(-y) \quad \text{for } y > 0$$

and $f_{\tilde{y}}(y) = 0$ for $y \le 0$.

(d)

$$f_{\tilde{y}}(y) = 2 f_{\tilde{x}}(y) \quad \text{for } y > 0$$

and $f_{\tilde{y}}(y) = 0$ for $y \le 0$.

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13. (a) $y = g(x) = x^2 - 8$ is bijective on $(0, 4)$. When $x \in (0, 4)$ then $y \in (-8, 8)$. Therefore, $x = g^{-1}(y) = (y + 8)^{1/2}$ with $y \in (-8, 8)$. Moreover,

$$\left| \frac{dg^{-1}(y)}{dy} \right| = \frac{1}{2}(y + 8)^{-1/2}.$$

Therefore,

$$f_{\tilde{y}}(y) = \begin{cases} f_{\tilde{x}}(g^{-1}(y)) \left| \dfrac{dg^{-1}(y)}{dy} \right| = \dfrac{(y+8)^{1/2}}{8} \cdot \dfrac{1}{2}\,(y+8)^{-1/2} = \dfrac{1}{16} & \text{for } y \in (-8,8) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Integrating the density $f_{\tilde{y}}$ we get the distribution function:

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{for } y \le -8 \\ \\ \int_{-8}^{y} \dfrac{1}{16} dy = \dfrac{1}{2}\left(\dfrac{y}{8} + 1\right) & \text{for } y \in (-8,8) \\ \\ 1 & \text{for } y \ge 8. \end{cases}$$

We could have used instead the distribution function method:

$F_{\tilde{y}}(y) = 0$ for $y \le -8$, $F_{\tilde{y}}(y) = 1$ for $y \ge 8$, and, for $y \in (-8,8)$, we have,

$$\begin{aligned} F_{\tilde{y}}(y) &= P\{\tilde{y} \le y\} = P\left\{\tilde{x}^2 - 8 \le y\right\} = P\left\{\tilde{x}^2 \le y + 8\right\} \\ &= P\left\{-(y+8)^{1/2} \le \tilde{x} \le (y+8)^{1/2}\right\} = P\left\{\tilde{x} \le (y+8)^{1/2}\right\} \end{aligned}$$

since $-(y+8)^{1/2}$ is negative for $y \in (-8,8)$ and the density of $\tilde{x}$ is positive only on $(0,4)$.

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{for } y \le -8 \\ P\left\{\tilde{x} \le (y+8)^{1/2}\right\} = \int_0^{(y+8)^{1/2}} \frac{x}{8} dx = \frac{1}{2}\left(\frac{y}{8}+1\right) & \text{for } y \in (-8, 8) \\ 1 & \text{for } y \ge 8, \end{cases}$$

which is the same as before. Computing the derivative of $F_{\tilde{y}}(y)$ we obtain the density $f_{\tilde{y}}(y)$.

(b) $P\{1 < \tilde{y} < 3\} = F_{\tilde{y}}(3) - F_{\tilde{y}}(1) = \frac{1}{2}\left(\frac{3}{8}+1\right) - \frac{1}{2}\left(\frac{1}{8}+1\right) = \frac{1}{8} = 0.125.$

$P\{\tilde{y} \ge -2\} = 1 - F_{\tilde{y}}(-2) = 1 - \frac{1}{2}\left(\frac{-2}{8}+1\right) = \frac{5}{8} = 0.625.$

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14. (a)

$$\mathrm{E}(\tilde{y}) = \int_0^1 x^3 6x(1-x)dx = \frac{1}{5},$$

$$\mathrm{E}\left(\tilde{y}^2\right) = \int_0^1 x^6 6x(1-x)dx = \frac{1}{12},$$

and, thus,

$$\mathrm{Var}(\tilde{y}) = \frac{1}{12} - \left(\frac{1}{5}\right)^2 = \frac{13}{300}.$$

(b)

$$f_{\tilde{y}}(y) = \begin{cases} f_{\tilde{x}}(g^{-1}(y)) \left|\frac{dg^{-1}(y)}{dy}\right| = 6y^{1/3}(1-y^{1/3})\frac{1}{3}y^{-2/3} = 2(y^{-1/3}-1) & \text{for } y \in (0,1) \\ 0 & \text{otherwise,} \end{cases}$$

and, integrating, we get

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \\ 3y^{2/3} - 2y & \text{for } y \in (0,1) \\ \\ 1 & \text{for } y \geq 1. \end{cases}$$

Note that we could also compute the mean of $\tilde{y}$ by doing

$$\mathrm{E}\left(\tilde{y}\right) = \int_0^1 y f_{\tilde{y}}(y) dy = \int_0^1 y \cdot 2(y^{-1/3} - 1) dy = \frac{1}{5},$$

and similarly for the variance.

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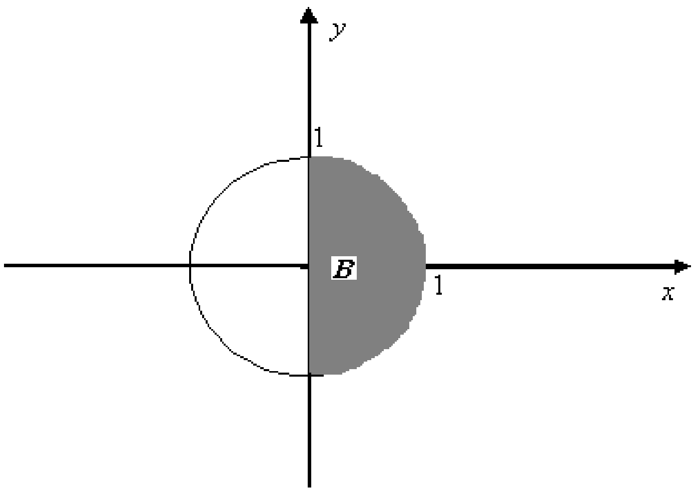
15. (a)



$$B = \{(x,y) \in \mathbb{R}^2 \left| x > 0, x^2 + y^2 < 1 \right\},$$

$$\int_B k\left(x+y\right) d\left(x,y\right) = ?$$

Change of variable to polar coordinates:

$$g^{-1}: \begin{cases} x = r\cos\theta \\ \\ y = r\sin\theta \end{cases} \Rightarrow J_{g^{-1}}\left(r,\theta\right) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and

$$\left|J_{g^{-1}}\left(r,\theta\right)\right| = \left|r\cos^2\theta + r\sin^2\theta\right| = r > 0.$$

Thus,

$$\int_B k\left(x+y\right) d\left(x,y\right) = \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k\left(r\cos\theta + r\sin\theta\right) r d\theta dr$$

$$= k\left[\int_0^1 r^2 dr\right]\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\theta + \sin\theta\right) d\theta\right] = k \cdot \frac{1}{3} \cdot 2 = \frac{2k}{3} \Rightarrow k = \frac{3}{2}$$

(b) Note that

$$g: \begin{cases} \tilde{z} \overset{d}{=} \widetilde{r} = \left(\tilde{x}^2 + \tilde{y}^2\right)^{\frac{1}{2}} \\ \\ \tilde{\theta} = \arctan\left(\dfrac{\tilde{x}}{\tilde{y}}\right) \end{cases}$$

so that

$$f_{\tilde{z},\tilde{\theta}}\left(z,\theta\right) = \begin{cases} \dfrac{3}{2}\left(z\cos\theta + z\sin\theta\right) z, \text{ for } z \in \left(0,1\right), \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \\ \\ 0, \quad \text{otherwise,} \end{cases}$$

which implies that

$$f_{\tilde{z}}(z) = \begin{cases} \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2} z^2 (\cos\theta + \sin\theta)\, d\theta = 3z^2 \text{ for } z \in (0,1) \\ \\ 0, \quad \text{otherwise.} \end{cases}$$

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16. (a)

$$f_{\tilde{x}_1}(x_1) = \int_0^\infty 6e^{-3x_1 - 2x_2} dx_2 = 3e^{-3x_1} \text{ for } x_1 > 0 \text{ and } f_{\tilde{x}_1}(x_1) = 0 \text{ otherwise.}$$

Then

$$f_{\tilde{x}_2 \mid \tilde{x}_1}(x_2 \mid x_1) = \frac{f_{\tilde{x}_1, \tilde{x}_2}(x_1, x_2)}{f_{\tilde{x}_1}(x_1)} = \frac{6e^{-3x_1 - 2x_2}}{3e^{-3x_1}} = 2e^{-2x_2} \text{ for } x_2 > 0$$

and $f_{\tilde{x}_2 \mid \tilde{x}_1}(x_2 \mid x_1) = 0$ otherwise, with $x_1 > 0$. Therefore,

$$f_{\tilde{x}_2 \mid \tilde{x}_1}(x_2 \mid 5) = \frac{f_{\tilde{x}_1, \tilde{x}_2}(5, x_2)}{f_{\tilde{x}_1}(5)} = 2e^{-2x_2} \text{ for } x_2 > 0.$$

$\Longrightarrow$

$$\mathrm{E}(\tilde{x}_2 \mid \tilde{x}_1 = 5) = \int_0^\infty x_2 \cdot 2e^{-2x_2} dx_2 = \frac{1}{2}.$$

The random variables $\tilde{x}_1$ and $\tilde{x}_2$ are independent since

$$f_{\tilde{x}_2}(x_2) = \int_0^\infty 6e^{-3x_1 - 2x_2} dx_1 = 2e^{-2x_2} \text{ for } x_2 > 0 \text{ and } f_{\tilde{x}_2}(x_2) = 0 \text{ otherwise.}$$

Therefore,

$$f_{\tilde{x}_1,\tilde{x}_2}(x_1, x_2) = 6e^{-3x_1-2x_2} = f_{\tilde{x}_1}(x_1)\cdot f_{\tilde{x}_2}(x_2) = 3e^{-3x_1}\cdot 2e^{-2x_2} \text{ for } x_1 > 0,\ x_2 > 0,$$

and, obviously, $f_{\tilde{x}_1,\tilde{x}_2}(x_1, x_2) = 0 = f_{\tilde{x}_1}(x_1) \cdot f_{\tilde{x}_2}(x_2)$, otherwise.

Note that independence implies that $\mathrm{E}(\tilde{x}_2 \mid \tilde{x}_1 = x_1) = \mathrm{E}(\tilde{x}_2) = 1/2$ for all $x_1 > 0$.

Similarly, $\mathrm{E}(\tilde{x}_1 \mid \tilde{x}_2 = x_2) = \mathrm{E}(\tilde{x}_1) = \int_0^\infty x_1 \cdot 3e^{-3x_1} dx_1 = 1/3$ for all $x_2 > 0$.

(b) If $f_{\tilde{x}}(x) = \theta e^{-\theta x}$ for $x > 0$ and $f_{\tilde{x}}(x) = 0$ otherwise, with $\theta > 0$, then

$$M_{\tilde{x}}(t) = \int_0^\infty \theta e^{-\theta x} e^{tx} dx = \theta \int_0^\infty e^{-(\theta-t)x} dx = -\theta \left[\frac{e^{-(\theta-t)x}}{(\theta-t)}\right]_0^\infty$$

$$= -\theta\left(0 - \frac{1}{(\theta-t)}\right) = \frac{\theta}{\theta-t} \quad \text{for } t < \theta.$$

$$\mathrm{E}(\tilde{x}) = M'_{\tilde{x}}(t)|_{t=0} = \left.\frac{\theta}{(\theta-t)^2}\right|_{t=0} = \frac{\theta}{\theta^2} = \frac{1}{\theta},$$

$$\mathrm{E}(\tilde{x}^2) = M''_{\tilde{x}}(t) = \left.\frac{2\theta}{(\theta-t)^3}\right|_{t=0} = \frac{2\theta}{\theta^3} = \frac{2}{\theta^2},$$

$$\mathrm{Var}(\tilde{x}) = \mathrm{E}\left(\tilde{x}^2\right) - [\mathrm{E}(\tilde{x})]^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2},$$

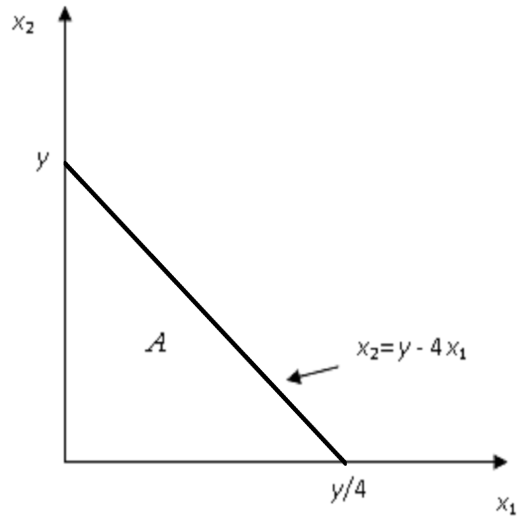
$$\mathrm{CV}_{\tilde{x}} = \frac{\sqrt{\mathrm{Var}(\tilde{x})}}{|\mathrm{E}(\tilde{x})|} = \frac{1/\theta}{1/\theta} = 1.$$

Thus,

$$M_{\tilde{x}_1}(t) = \frac{3}{3-t} \quad \text{for } t < 3, \quad \mathrm{E}(\tilde{x}_1) = \frac{1}{3}, \quad \mathrm{Var}(\tilde{x}_1) = \frac{1}{9}, \quad \mathrm{CV}_{\tilde{x}_1} = 1,$$

$$M_{\tilde{x}_2}(t) = \frac{2}{2-t}, \quad \text{for } t < 2, \quad \mathrm{E}(\tilde{x}_2) = \frac{1}{2}, \quad \mathrm{Var}(\tilde{x}_2) = \frac{1}{4}, \quad \mathrm{CV}_{\tilde{x}_2} = 1.$$

(c)



For $y > 0$,

$$F_{\tilde{y}}(y) = P\{\tilde{y} \leq y\} = P\{4\tilde{x}_1 + \tilde{x}_2 \leq y\} = P\{(\tilde{x}_1, \tilde{x}_2) \in A\}$$

$$= \int_0^{y/4} \int_0^{y-4x_1} 6e^{-3x_1-2x_2} dx_2 dx_1 = 1 - \frac{8}{5}e^{-\frac{3}{4}y} + \frac{3}{5}e^{-2y}.$$

Therefore,

$$F_{\tilde{y}}(y) = \begin{cases} 1 - \frac{8}{5}e^{-\frac{3}{4}y} + \frac{3}{5}e^{-2y} & \text{if } y > 0 \\ \\ 0 & \text{if } y \leq 0 \end{cases}$$

(d) Consider the following change of variable for $x_1 > 0$ and $x_2 > 0$:

$$(y, x_2) = g(x_1, x_2) : \begin{cases} y = 4x_1 + x_2 \in (x_2, \infty) \\ \\ x_2 = x_2 \in (0, \infty) \end{cases}$$

so that

$$(x_1, x_2) = g^{-1}(y, x_2) : \begin{cases} x_1 = \dfrac{1}{4}(y - x_2) \in (0, \infty) \\ \\ x_2 = x_2 \in (0, \infty) \end{cases}$$

$$J_{g^{-1}}(y, x_2) = \begin{pmatrix} 1/4 & -1/4 \\ 0 & 1 \end{pmatrix} \Longrightarrow |J_{g^{-1}}(y, x_2)| = \frac{1}{4}.$$

Then,

$$f_{\tilde{y}, \tilde{x}_2}(y, x_2) = \begin{cases} 6e^{-3\overbrace{\left[\frac{1}{4}(y - x_2)\right]}^{x_1} - 2x_2} \cdot \dfrac{1}{4} = \dfrac{3}{2} e^{-\frac{3}{4}y - \frac{5}{4}x_2} & \text{if } y > x_2 \text{ and } x_2 > 0 \\ & \Longleftrightarrow x_2 \in (0, y) \\ \\ 0 & \text{otherwise} \end{cases}$$

(e) For $y > 0$,

$$f_{\tilde{y}}(y) = \int_0^y \frac{3}{2} e^{-\frac{3}{4}y - \frac{5}{4}x_2} dx_2 = \frac{6}{5}\left(e^{-\frac{3}{4}y} - e^{-2y}\right).$$

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{6}{5}\left(e^{-\frac{3}{4}y} - e^{-2y}\right) & \text{if } y > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

(f) $F_{\tilde{y}}(y) = \int_0^y \dfrac{6}{5}\left(e^{-\frac{3}{4}t} - e^{-2t}\right) dt = 1 - \frac{8}{5}e^{-\frac{3}{4}y} + \frac{3}{5}e^{-2y}$ if $y > 0$, and $F_{\tilde{y}}(y) = 0$ if $y \leq 0$.

$f_{\tilde{y}}(y) = F'_{\tilde{y}}(y) = \dfrac{d\left[1 - \frac{8}{5}e^{-\frac{3}{4}y} + \frac{3}{5}e^{-2y}\right]}{dy} = \dfrac{6}{5}\left(e^{-\frac{3}{4}y} - e^{-2y}\right)$ if $y > 0$, and $f_{\tilde{y}}(y) = 0$ if $y \leq 0$.

(g)

$$f_{\tilde{y}\,|\,\tilde{x}_2}\left(y\,|\,x_2\right) = \frac{f_{\tilde{y},\tilde{x}_2}\left(x_1, x_2\right)}{f_{\tilde{x}_2}\left(x_2\right)} = \frac{\frac{3}{2}e^{-\frac{3}{4}y-\frac{5}{4}x_2}}{2e^{-2x_2}} = \frac{3}{4}e^{-\frac{3}{4}y+\frac{3}{4}x_2}$$

for $y > x_2$ and $f_{\tilde{y}\,|\,\tilde{x}_2}\left(y\,|\,x_2\right) = 0$ otherwise, with $x_2 > 0$.

Thus,

$$f_{\tilde{y}\,|\,\tilde{x}_2}\left(y\,|\,3\right) = \begin{cases} \frac{3}{4}e^{-\frac{3}{4}y+\frac{9}{4}} & \text{if } y > 3 \\ \\ 0 & \text{otherwise} \end{cases}$$

(h)

$$\begin{aligned} \mathrm{E}\left(\tilde{y}\right) &= \mathrm{E}\left(4\tilde{x}_1 + \tilde{x}_2\right) = \mathrm{E}\left(4\tilde{x}_1\right) + \mathrm{E}\left(\tilde{x}_2\right) = 4\mathrm{E}\left(\tilde{x}_1\right) + \mathrm{E}\left(\tilde{x}_2\right) \\ &= \left(4 \cdot \frac{1}{3}\right) + \frac{1}{2} = \frac{11}{6}. \end{aligned}$$

Alternatively, we can compute

$$\mathrm{E}\left(\tilde{y}\right) = \int_0^\infty y\frac{6}{5}\left(e^{-\frac{3}{4}y} - e^{-2y}\right)dy = \frac{11}{6}.$$

$$\begin{aligned} \mathrm{E}\left(\tilde{y} \mid \tilde{x}_2 = 3\right) &= \mathrm{E}\left(4\tilde{x}_1 + \tilde{x}_2 \mid \tilde{x}_2 = 3\right) = \mathrm{E}\left(4\tilde{x}_1 \mid \tilde{x}_2 = 3\right) + \mathrm{E}\left(\tilde{x}_2 \mid \tilde{x}_2 = 3\right) \\ &= \mathrm{E}\left(4\tilde{x}_1\right) + 3 = 4\mathrm{E}\left(\tilde{x}_1\right) + 3 = \frac{4}{3} + 3 = \frac{13}{3} \end{aligned}$$

since $\tilde{x}_1$ and $\tilde{x}_2$ are independent.

Alternatively, we can compute

$$\mathrm{E}\left(\tilde{y} \mid \tilde{x}_2 = 3\right) = \int_3^\infty y\frac{3}{4}e^{-\frac{3}{4}y+\frac{9}{4}}dy = \frac{13}{3}.$$

(i)

$$\begin{aligned}\mathrm{Cov}\left(\tilde{y},\tilde{x}_2\right) &= \mathrm{Cov}\left(4\tilde{x}_1+\tilde{x}_2,\tilde{x}_2\right) = \mathrm{Cov}\left(4\tilde{x}_1,\tilde{x}_2\right)+\mathrm{Cov}\left(\tilde{x}_2,\tilde{x}_2\right)\\ &= 4\mathrm{Cov}\left(\tilde{x}_1,\tilde{x}_2\right)+\mathrm{Var}\left(\tilde{x}_2\right) = 0+\mathrm{Var}\left(\tilde{x}_2\right)=\frac{1}{4}.\end{aligned}$$

as $\mathrm{Cov}\left(\tilde{x}_1,\tilde{x}_2\right)=0$ due to the independence between $\tilde{x}_1$ and $\tilde{x}_2$.

Alternatively, we can compute

$$\mathrm{E}\left(\tilde{y}\cdot\tilde{x}_2\right)=\int_0^\infty\int_0^y yx_2\frac{3}{2}e^{-\frac{3}{4}y-\frac{5}{4}x_2}dx_2dy=\frac{7}{6}$$

or

$$\mathrm{E}\left(\tilde{y}\cdot\tilde{x}_2\right)=\int_0^\infty\int_{x_2}^\infty yx_2\frac{3}{2}e^{-\frac{3}{4}y-\frac{5}{4}x_2}dydx_2=\frac{7}{6}.$$

Then,

$$\mathrm{Cov}\left(\tilde{y},\tilde{x}_2\right)=\mathrm{E}\left(\tilde{y}\cdot\tilde{x}_2\right)-\mathrm{E}\left(\tilde{y}\right)\mathrm{E}\left(\tilde{x}_2\right)=\frac{7}{6}-\left(\frac{11}{6}\cdot\frac{1}{2}\right)=\frac{1}{4}.$$

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17. (a)

$$\begin{aligned}F_{\tilde{y}}(y) &= P\left\{\tilde{y}\le y\right\}=P\left\{\tilde{x}^{1/2}\le y\right\}\\ &= P\left\{\tilde{x}\le y^2\right\}=\begin{cases}\displaystyle\int_0^{y^2}e^{-x}dx=1-e^{-y^2} & \text{for } y>0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

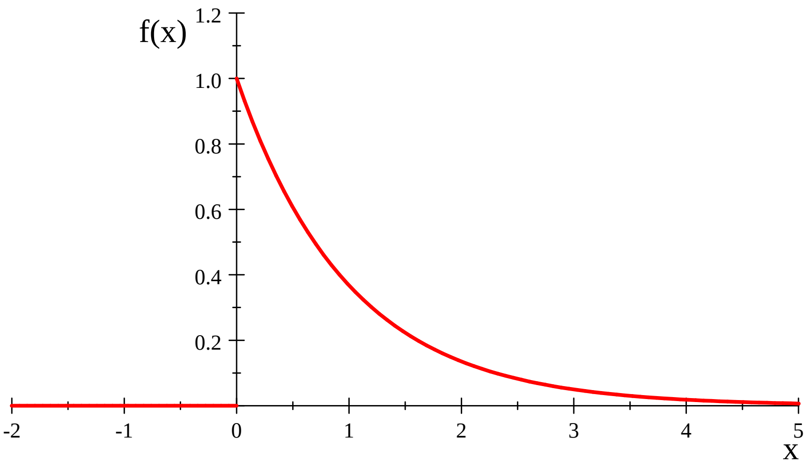
(b) $y=g(x)=x^{1/2}$ is bijective for $x>0$. Then, $x=g^{-1}(y)=y^2$ for $y>0$

and $\frac{dg^{-1}(y)}{dy} = 2y > 0$. Thus,

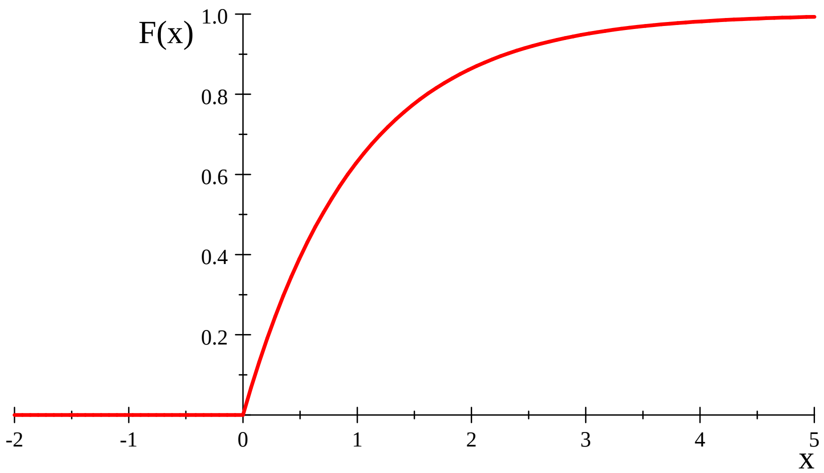
$$f_{\tilde{y}}(y) = \begin{cases} e^{-y^2} \cdot (2y) = 2ye^{-y^2} & \text{for } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(c) It is immediate to see that $F_{\tilde{y}}(y) = \int_{-\infty}^{y} f_{\tilde{y}}(s)ds$ and $f_{\tilde{y}}(y) = F'_{\tilde{y}}(y)$ for all $y \in \mathbb{R}$.

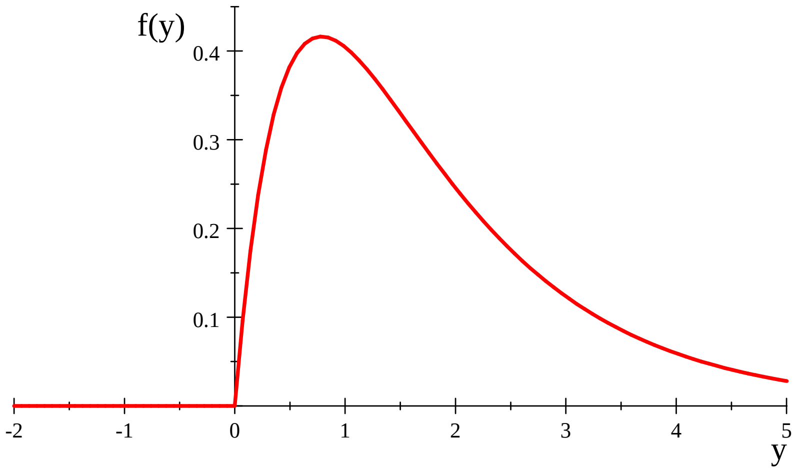
(d) $F_{\tilde{x}}$ is not differentiable at $x = 0$ since the density $f_{\tilde{x}}$ is discontinuous at $x = 0$. However, $f_{\tilde{y}}(0) = 0$ so that the density function $f_{\tilde{y}}$ is continuous everywhere. Hence, the distribution function $F_{\tilde{y}}$ is differentiable everywhere. In particular, the derivative of $F_{\tilde{y}}$ at $y = 0$ is $F'_{\tilde{y}}(0) = 0$.

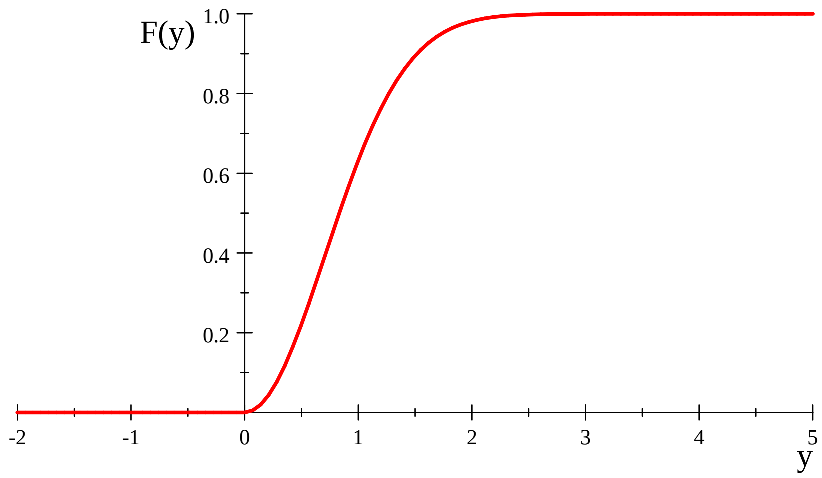


$$F_{\tilde{x}}(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$



Note that $\underset{y \in \mathbb{R}}{\arg\max}\ f_{\tilde{y}}(y) = \dfrac{\sqrt{2}}{2}$.





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18. (a)

$$F(x,y) = \begin{cases} \int_0^x \int_0^y e^{-y}e^{-x}dydx = (1-e^{-x})(1-e^{-y}) & \text{if } x>0, y>0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$f_{\tilde{x}}(x) = \begin{cases} \int_0^\infty e^{-(x+y)}dy = e^{-x} & \text{if } x>0 \\ \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\tilde{y}}(y) = \begin{cases} \int_0^\infty e^{-(x+y)}dx = e^{-y} & \text{if } y>0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

$\tilde{x}$ and $\tilde{y}$ are independent because $f(x,y) = f_{\tilde{x}}(x) \cdot f_{\tilde{y}}(y)$ for all $(x,y) \in \mathbb{R}^2$.

(c)

$$f_{\tilde{y}|\tilde{x}}(y\,|3) = \begin{cases} \dfrac{f(3,y)}{f_{\tilde{x}}(3)} = \dfrac{e^{-3-y}}{e^{-3}} = e^{-y} & \text{if } y>0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

In fact, from independence we have that $f_{\tilde{y}|\tilde{x}}(y\,|x) = f_{\tilde{y}}(y)$ for all $x>0$.

(d)

$$\int_0^1 \int_x^1 e^{-(x+y)}dydx = \frac{1}{2}e^{-2} - e^{-1} + \frac{1}{2} = 0.1998.$$

(e) $\tilde{z} = \tilde{x} - 3\tilde{y} \Longleftrightarrow \tilde{x} = \tilde{z} + 3\tilde{y} \Longleftrightarrow \tilde{y} = \dfrac{1}{3}\left(\tilde{x} - \tilde{z}\right).$

$F(z) = P\left\{\tilde{z} \le z\right\} = P\left\{\tilde{x} - 3\tilde{y} \le z\right\} = P_{\tilde{x},\tilde{y}}\left(C\right),$

where $C = \{(x, y) \in \mathbb{R}^2 | \, x - 3y \leq z\}$.

Note that $x = z + 3y > 0$ and $y > 0$ in the set A where $f(x, y) \neq 0$. Therefore, $y > \max\{0, -z/3\}$ in the set A. Hence, if $z > 0$ then $y > 0$, whereas if $z \leq 0$, then $y > -z/3$.

Therefore, if $z \leq 0$ then

$F(z) = P_{\tilde{x},\tilde{y}}(C) = \int_{-z/3}^{\infty} \left(\int_0^{z+3y} e^{-(x+y)} dx \right) dy = \frac{3}{4} e^{z/3}$,

whereas, if $z > 0$,

$F(z) = P_{\tilde{x},\tilde{y}}(C) = \int_0^{\infty} \left(\int_0^{z+3y} e^{-(x+y)} dx \right) dy = 1 - \frac{1}{4} e^{-z}$.

Thus,

$$F(z) = \begin{cases} \frac{3}{4} e^{z/3} & \text{for } z \leq 0 \\ \\ 1 - \frac{1}{4} e^{-z} & \text{for } z > 0. \end{cases}$$

(f) Let us consider the following one-to-correspondence $g : A \longrightarrow g(A)$, where A is the set for which $f(x, y) \neq 0$:

$$(z, x) = g(x, y) : \begin{cases} z = x - 3y \in (-\infty, \infty) \\ \\ x = x > \max\{0, z\} \end{cases}$$

$$(x, y) = g^{-1}(z, x) : \begin{cases} x = x > 0 \\ \\ y = \frac{1}{3}(x - z) > 0. \end{cases}$$

Note that $x > 0$ and $y = \frac{1}{3}(x - z) > 0$ on the set A where $f(x, y) \neq 0$. Therefore, $x > \max\{0, z\}$ in the set A. Hence, if $z \leq 0$ then $x > 0$, whereas if

$z > 0$ then $x > z$.

$$J_{g^{-1}}(z,x) = \begin{bmatrix} 0 & 1 \\ -1/3 & 1/3 \end{bmatrix} \Longrightarrow |J_{g^{-1}}| = \frac{1}{3}.$$

Note that $-(x+y) = -\left(x + \frac{1}{3}(x-z)\right) = -\frac{1}{3}(4x-z)$. Therefore,

$$f_{\tilde{z},\tilde{x}}(z,x) = \begin{cases} f(g^{-1}(z,x))\,|J_{g^{-1}}(z,x)| = e^{-\frac{1}{3}(4x-z)} \cdot \frac{1}{3} & \text{for } z \in (-\infty,\infty),\ x > \max\{0,z\} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, computing the marginal density of $\tilde{z}$, we get

$$f_{\tilde{z}}(z) = \int_z^\infty e^{-\frac{1}{3}(4x-z)} \cdot \frac{1}{3} dx = \frac{1}{4} e^{-z} \text{ for } z > 0,$$

and

$$f_{\tilde{z}}(z) = \int_0^\infty e^{-\frac{1}{3}(4x-z)} \cdot \frac{1}{3} dx = \frac{1}{4} e^{z/3} \text{ for } z \leq 0.$$

so that

$$f_{\tilde{z}}(z) = \begin{cases} \frac{1}{4} e^{z/3} & \text{for } z \leq 0 \\ \frac{1}{4} e^{-z} & \text{for } z > 0. \end{cases}$$

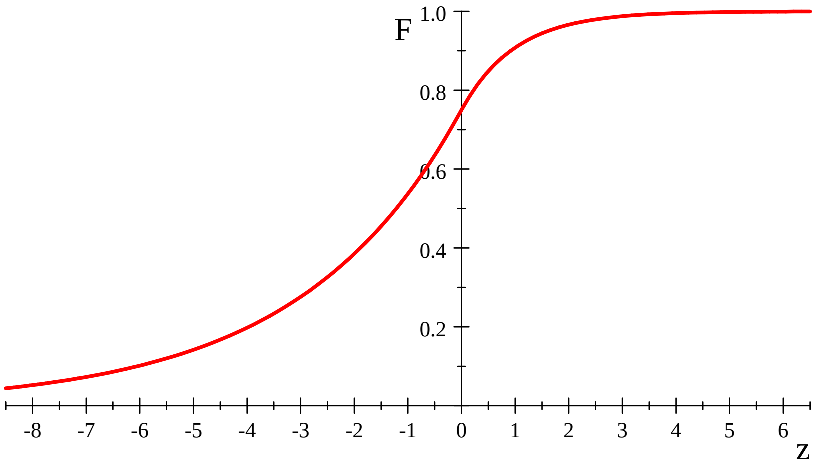
(g) Obviously $F_{\tilde{z}}(z) = \int_{-\infty}^z f_{\tilde{z}}(z) dz$

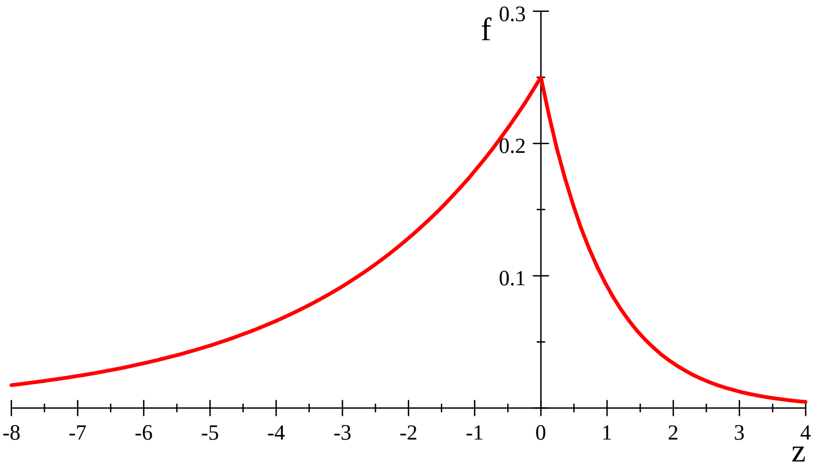
$$F_{\tilde{z}}(z) = \begin{cases} \int_{-\infty}^z \frac{1}{4} e^{z/3} dz = \frac{3}{4} e^{z/3} & \text{for } z \leq 0 \\ \int_{-\infty}^0 \frac{1}{4} e^{z/3} dz + \int_0^z \frac{1}{4} e^{-z} dz = \frac{3}{4} + \left(\frac{1}{4} - \frac{1}{4} e^{-z}\right) = 1 - \frac{1}{4} e^{-z} & \text{for } z > 0, \end{cases}$$

while $f_{\tilde{z}}(z) = F'_{\tilde{z}}(z)$ a.e. since

$$f_{\tilde{z}}(z) = \begin{cases} \dfrac{d\left(\frac{3}{4}e^{z/3}\right)}{dz} = \dfrac{1}{4}e^{z/3} & \text{for } z < 0 \\ \\ \dfrac{d\left(1-\frac{1}{4}e^{-z}\right)}{dz} = \dfrac{1}{4}e^{-z} & \text{for } z > 0. \end{cases}$$

(h)





The density function $f_{\tilde{z}}$ is not differentiable at $z = 0$. However, the density $f_{\tilde{z}}$ is continuous everywhere. This implies that the distribution function $F_{\tilde{z}}$ is differentiable everywhere. In particular, note that

$$\lim_{z\to 0^-}\frac{d\left(\frac{3}{4}e^{z/3}\right)}{dz}=\lim_{z\to 0^-}\left(\frac{1}{4}e^{z/3}\right)=\lim_{z\to 0^+}\frac{d\left(1-\frac{1}{4}e^{-z}\right)}{dz}=\lim_{z\to 0^+}\left(\frac{1}{4}e^{-z}\right)=\frac{1}{4}.$$

Another implication of the previous facts is that $F_{\tilde{z}}$ is not twice differentiable at $z=0$.

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19. (a)

$$g:\begin{cases} y=x_1+x_2\in(x_2,x_2+2) \\ x_2=x_2\in(0,1)\end{cases} \Longrightarrow g^{-1}:\begin{cases} x_1=y-x_2\in(0,2) \\ x_2=x_2\in(0,1)\end{cases}$$

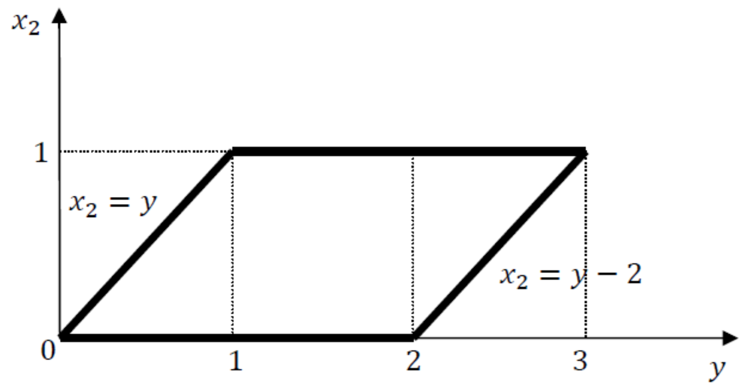
$$|J_{g^{-1}}(y,x_2)|=\left|\det\begin{bmatrix}\frac{dx_1}{dy} & \frac{dx_1}{dx_2}\\ \frac{dx_2}{dy} & \frac{dx_2}{dx_2}\end{bmatrix}\right|=\left|\det\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\right|=1.$$

Then,

$$f_{\tilde{y},\tilde{x}_2}(y,x_2)=\frac{1}{2}\cdot 1=\frac{1}{2}\quad\text{for } 0<x_2<1 \text{ and } x_2<y<x_2+2,$$

so that

$$f_{\tilde{y},\tilde{x}_2}(y,x_2)=\begin{cases}\frac{1}{2} & \text{for}\quad 0<x_2<1 \text{ and } x_2<y<x_2+2\\ 0 & \text{elsewhere.}\end{cases}$$

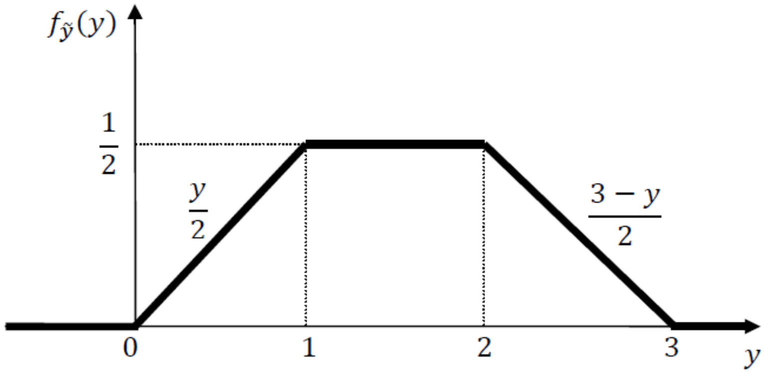


The density of y for $y \in (0,3)$ is

$$f_{\tilde{y}}(y) = \int_{-\infty}^{\infty} f_{\tilde{y},\tilde{x}_2}(y, x_2)\, dx_2 = \int_{\mathbb{R}} \mathbb{I}_C(y, x_2)\, \frac{1}{2} dx_2 = \int_{C(y)} \frac{1}{2} dx_2,$$

where the set C is the interior of the parallelogram in the previous figure.

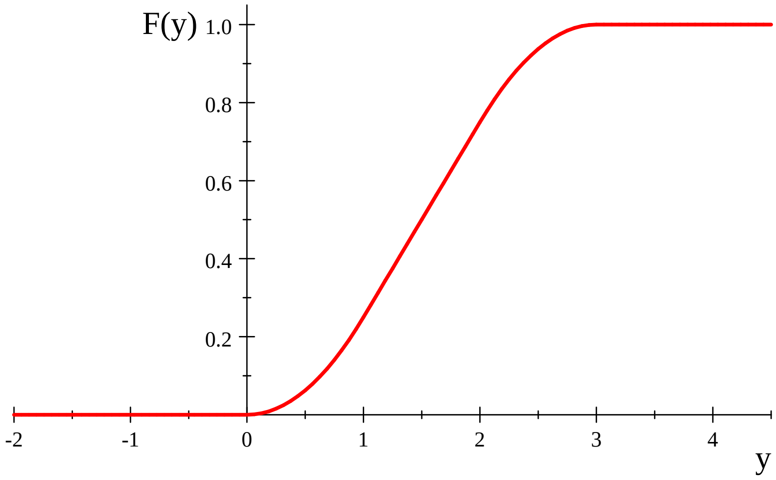
$$f_{\tilde{y}}(y) = \begin{cases} 0 & \text{for} \quad y \leq 0 \\ \int_0^y \frac{1}{2} dx_2 = \frac{y}{2} & \text{for} \quad 0 < y \leq 1 \\ \int_0^1 \frac{1}{2} dx_2 = \frac{1}{2} & \text{for} \quad 1 < y \leq 2 \\ \int_{y-2}^1 \frac{1}{2} dx_2 = \frac{3-y}{2} & \text{for} \quad 2 < y < 3 \\ 0 & \text{for} \quad y \geq 3. \end{cases}$$



$$\int_{-\infty}^{\infty} f_{\tilde{y}}\,(y) = 0 + \int_0^1 \frac{y}{2}dy + \int_1^2 \frac{1}{2}dy + \int_2^3 \frac{3-y}{2}dy + 0 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$

(b)

$$F_{\tilde{y}}\,(y) = \begin{cases} 0 & \text{for} \quad y \leq 0 \\ \int_0^y \frac{y}{2}dy = \frac{1}{4}y^2 & \text{for} \quad 0 < y \leq 1 \\ \frac{1}{4} + \int_1^y \frac{1}{2}dy = \frac{2y-1}{4} & \text{for} \quad 1 < y \leq 2 \\ \frac{3}{4} + \int_2^y \frac{3-y}{2}dy = -\frac{y^2}{4} + \frac{3y}{2} - \frac{5}{4} & \text{for} \quad 2 < y < 3 \\ 1 & \text{for} \quad y \geq 3. \end{cases}$$



(c) Note that

$$f_{\tilde{x}_2}(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 = \int_0^2 \frac{1}{2} dx_1 = 1 \quad \text{for } 0 < x_2 < 1$$

and $f_{\tilde{x}_2}(x_2) = 0$ otherwise. Therefore

$$f_{\tilde{y}|\tilde{x}_2}(y\,|x_2) = \frac{f_{\tilde{y},\tilde{x}_2}(y, x_2)}{f_{\tilde{x}_2}(x_2)} = \begin{cases} \frac{1}{2} & \text{for} \quad x_2 < y < x_2 + 2 \\ \\ 0 & \quad \text{elsewhere} \end{cases}$$

with $0 < x_2 < 1$. Hence,

$$f_{\tilde{y}|\tilde{x}_2}\left(y \middle| \frac{1}{4}\right) = \begin{cases} \frac{1}{2} & \text{for} \quad \frac{1}{4} < y < \frac{9}{4} \\ \\ 0 & \quad \text{elsewhere.} \end{cases}$$

and, thus,

$$\mathrm{E}(\tilde{y}\,|\tilde{x}_2 = 1/4) = \int_{1/4}^{9/4} y \frac{1}{2} dy = \frac{5}{4}.$$

We have

$$f_{\tilde{x}_2|\tilde{y}}\left(x_2\,|y\right) = \frac{f_{\tilde{y},\tilde{x}_2}\left(y,x_2\right)}{f_{\tilde{y}}\left(y\right)} = \begin{cases} \dfrac{1/2}{y/2} = \dfrac{1}{y} & \text{for} \quad 0 < x_2 < y,\ 0 < y \leq 1 \\ \\ \dfrac{1/2}{1/2} = 1 & \text{for} \quad 0 < x_2 < 1,\ 1 < y \leq 2 \\ \\ \dfrac{1/2}{\left(3-y\right)/2} = \dfrac{1}{3-y} & \text{for} \quad y-2 < x_2 < 1,\ 2 < y < 3 \\ \\ 0 & \qquad \text{otherwise,} \end{cases}$$

with $0 < y < 3$. Therefore

$$f_{\tilde{x}_2|\tilde{y}}\left(x_2 \left| \frac{5}{2}\right.\right) = \begin{cases} \dfrac{1}{3-\frac{5}{2}} = 2 & \text{for} \quad \dfrac{1}{2} < x_2 < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\mathrm{E}\left(\tilde{x}_2\,|\tilde{y} = 5/2\right) = \int_{1/2}^{1} x_2 \cdot 2dx_2 = \frac{3}{4}.$$

Moreover,

$$f_{\tilde{x}_2|\tilde{y}}\left(x_2 \left| \frac{3}{2}\right.\right) = \begin{cases} 1 & \text{for} \quad 0 < x_2 < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\mathrm{E}\left(\tilde{x}_2\,|\tilde{y} = 3/2\right) = \int_{0}^{1} x_2 \cdot 1dx_2 = \frac{1}{2}.$$

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20. (a)

$$f_{\tilde{x}}(x)=\begin{cases}\displaystyle\int_0^1 \frac{4}{3}(x+xy)dy = 2x \quad \text{for } 0<x<1\\[2em] 0 \quad \text{otherwise}\end{cases}$$

$$f_{\tilde{y}}(y)=\begin{cases}\displaystyle\int_0^1 \frac{4}{3}(x+xy)dx = \frac{2}{3}\,(1+y) \quad \text{for } 0<y<1\\[2em] 0 \quad \text{otherwise}\end{cases}$$

Note that $f_{\tilde{x}}(x)\cdot f_{\tilde{y}}(y) = 2x\cdot\frac{2}{3}\,(1+y) = \frac{4}{3}(x+xy)$ so that $f(x,y) = f_{\tilde{x}}(x)\cdot f_{\tilde{y}}(y)$ for $(x,y)\in(0,1)\times(0,1)$ and, obviously, $f(x,y) = f_{\tilde{x}}(x)\cdot f_{\tilde{y}}(y) = 0$ for $(x,y)\notin(0,1)\times(0,1)$. Therefore, $\tilde{x}$ and $\tilde{y}$ are independent.

(b) Since $\tilde{x}$ and $\tilde{y}$ are independent,

$$f_{\tilde{y}|\tilde{x}}\,(y\,|x\,) = \frac{f(x,y)}{f_{\tilde{x}}(x)} = \frac{f_{\tilde{x}}(x)\cdot f_{\tilde{y}}(y)}{f_{\tilde{x}}(x)} = f_{\tilde{y}}(y), \quad \text{for } 0<x<1.$$

Therefore, $\mathrm{E}\,(\tilde{y}\,|\tilde{x}=1/3\,) = \mathrm{E}\,(\tilde{y})\,,$

$$\mathrm{E}\,(\tilde{y}) = \int_0^1 y\frac{2}{3}\,(1+y)\,dy = \frac{5}{9} = \mathrm{E}\,(\tilde{y}\,|\tilde{x}=1/3\,)\,.$$

(c)

$$\mathrm{E}\,(\ln\tilde{x}) = \int_0^1 (\ln x\cdot 2x)\,dx \qquad\qquad \text{(integrating by parts)}$$

$$= \left[\ln x\cdot x^2\right]_0^1 - \int_0^1\left(\frac{1}{x}\cdot x^2\right)dx = 0 - \lim_{x\to 0}\left(\ln x\cdot x^2\right) - \int_0^1 x dx = -\frac{1}{2}$$

since

$$\lim_{x\to 0}\left(\ln x\cdot x^2\right)=\lim_{x\to 0}\left(\frac{\ln x}{x^{-2}}\right)=\frac{-\infty}{\infty},$$

and, thus, we can apply L'Hôpital's rule,

$$\lim_{x\to 0}\left(\frac{\ln x}{x^{-2}}\right)=\lim_{x\to 0}\left(\frac{\frac{1}{x}}{-2x^{-3}}\right)=\lim_{x\to 0}\left(\frac{x^2}{-2}\right)=0,$$

and

$$\int_0^1 x dx=\left[\frac{x^2}{2}\right]_0^1=\frac{1}{2}.$$

(d) If either $x<0$ or $y<0$, it follows immediately that $F(x,y)=0$. For $0<x<1$ and $0<y<1$ (Region I of the figure) we get

$$F(x,y)=\int_0^y\int_0^x\frac{4}{3}(x+xy)dxdy=\frac{1}{3}x^2y\left(y+2\right),$$

for $x>1$ and $0<y<1$ (Region II of the figure) we get

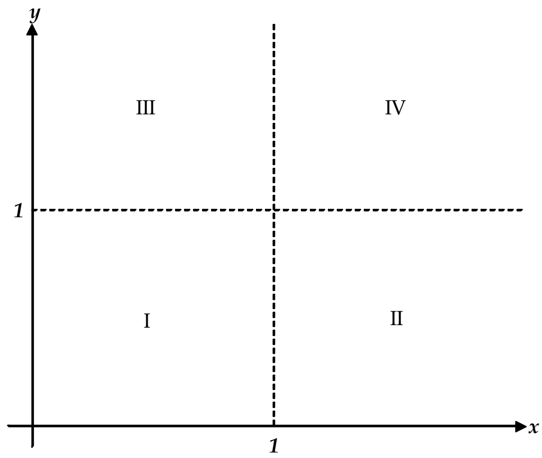
$$F(x,y)=\int_0^y\int_0^1\frac{4}{3}(x+xy)dxdy=\frac{1}{3}y\left(y+2\right),$$

for $0<x<1$ and $y>1$ (Region III of the figure) we get

$$F(x,y)=\int_0^1\int_0^x\frac{4}{3}(x+xy)dxdy=x^2,$$

and for $x>1$ and $y>1$ (Region IV of the figure) we get

$$F(x,y)=\int_0^1\int_0^1\frac{4}{3}(x+xy)dxdy=1.$$



Since the joint distribution function is everywhere continuous, the boundaries between any two of these regions can be included in either one, and we can write

$$F(x,y)=\begin{cases}0 & \text{for } x\le 0 \text{ or } y\le 0\\[2ex] \dfrac{1}{3}x^2y\,(y+2) & \text{for } 0<x<1,\, 0<y<1\\[2ex] \dfrac{1}{3}y\,(y+2) & \text{for } x\ge 1,\, 0<y<1\\[2ex] x^2 & \text{for } 0<x<1,\, y\ge 1\\[2ex] 1 & \text{for } x\ge 1,\, y\ge 1\end{cases}$$

It is immediate to see that

$$\frac{\partial^2 F(x,y)}{\partial x\partial y}=\frac{4}{3}(x+xy),\ \text{for } x\in(0,1)\,,\ y\in(0,1)$$

and

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = 0, \text{ for } (x,y) \in C,$$

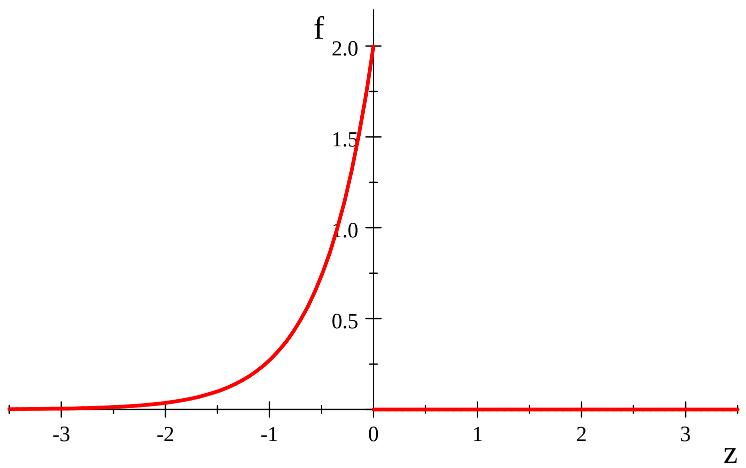
where C is the interior of the complement of Region I. Note that the boundary of Region I has zero Lebesgue measure. Therefore, $\dfrac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$ a.e. with respect to Lebesgue measure in $\mathbb{R}^2$. Note also that the discontinuities of the density function $f(x,y)$ occur only at the boundary of Region I.

(e)

$$z = g(x) = \ln x \in (-\infty, 0)\,, \quad x = g^{-1}(z) = e^z \in (0,1), \quad \frac{dg^{-1}(z)}{dz} = e^z > 0$$

Then,

$$f_{\tilde{z}}(z) = \begin{cases} 2e^z e^z = 2e^{2z} & \text{for } z \in (-\infty, 0) \\ \\ 0 & \text{otherwise} \end{cases}$$



(f)

$$
\begin{aligned}
M_{\tilde{z}}(t) &= \mathrm{E}\left(e^{t\tilde{z}}\right) = \int_{-\infty}^{0} 2e^{2z}e^{tz}dz \\
&= 2\int_{-\infty}^{0} e^{z(2+t)}dz = 2\left[\frac{e^{z(2+t)}}{2+t}\right]_{-\infty}^{0} = \frac{2}{t+2} = 2\left(t+2\right)^{-1}
\end{aligned}
$$

since $\lim_{z\to-\infty}\dfrac{e^{z(2+t)}}{2+t} = 0$ when t is around zero. Note that $M_{\tilde{z}}(t)$ is well defined for $t > -2$ so that it is well defined in a neighborhood of zero.

$$
M'_{\tilde{z}}(t) = -2(t+2)^{-2} \Longrightarrow \mathrm{E}\left(\tilde{z}\right) = M'_{\tilde{z}}(0) = -\frac{1}{2},
$$

which agrees with what we have obtained in part (c).

$$
M''_{\tilde{z}}(t) = 4(t+2)^{-3} \Longrightarrow \mathrm{E}\left(\tilde{z}^2\right) = M''_{\tilde{z}}(0) = \frac{1}{2}
$$

so that

$$
\mathrm{Var}\left(\tilde{z}\right) = \mathrm{E}\left(\tilde{z}^2\right) - \left[\mathrm{E}\left(\tilde{z}\right)\right]^2 = \frac{1}{2} - \left(-\frac{1}{2}\right)^2 = \frac{1}{4}.
$$

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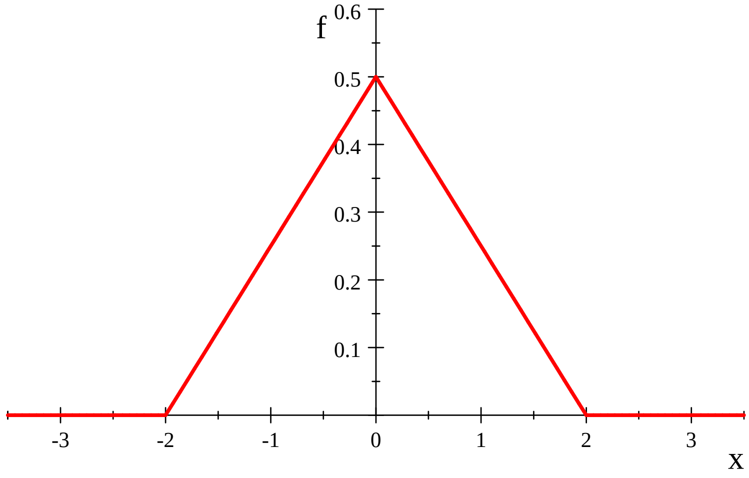
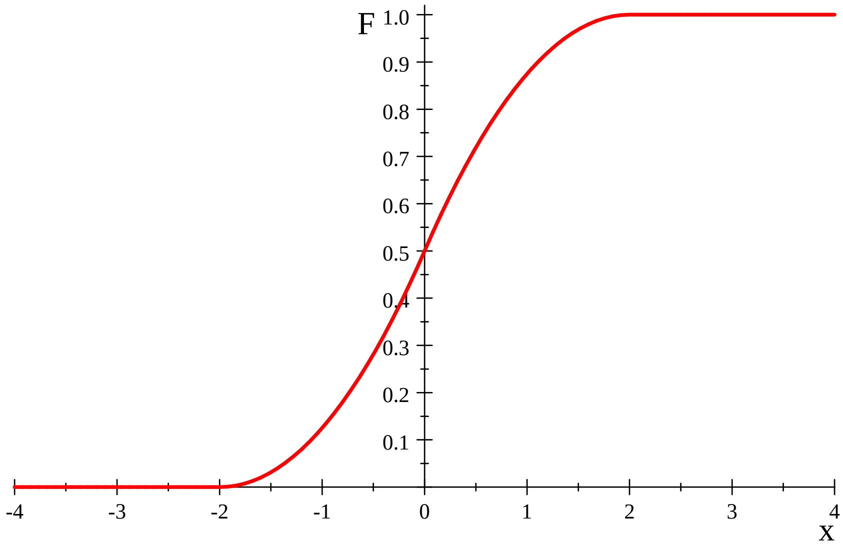
21. (a) $f_{\tilde{x}}(x)$



$$F_{\tilde{x}}(x) = \begin{cases} 0 & \text{for } x \leq -2 \\ \int\limits_{-2}^{x} \frac{x+2}{4} dx = \frac{x^2}{8} + \frac{x}{2} + \frac{1}{2} & \text{for } x \in (-2, 0] \\ \int\limits_{-2}^{0} \frac{x+2}{4} dx + \int\limits_{0}^{x} \frac{2-x}{4} dx = \frac{1}{2} + \int\limits_{0}^{x} \frac{2-x}{4} dx = -\frac{x^2}{8} + \frac{x}{2} + \frac{1}{2} & \text{for } x \in (0, 2) \\ 1 & \text{for } x \geq 2. \end{cases}$$



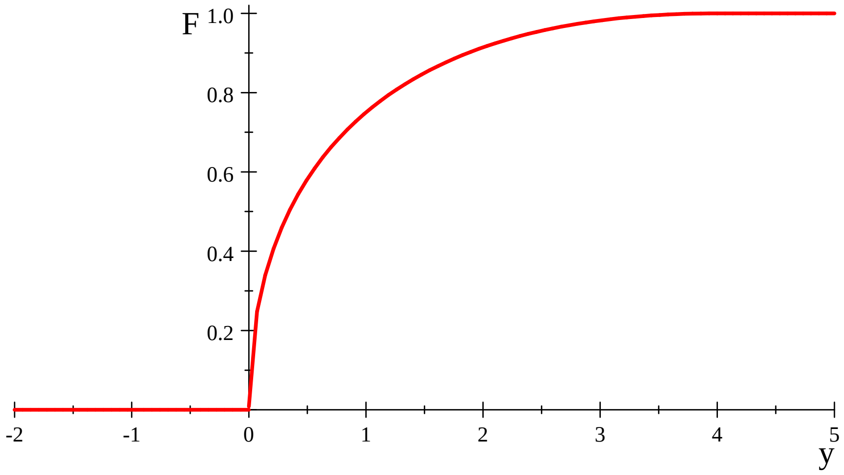
(b) Since

$$F_{\tilde{y}}(y) = P\left\{\tilde{y} \le y\right\} = P\left\{\tilde{x}^2 \le y\right\} = P\left\{-y^{1/2} \le \tilde{x} \le y^{1/2}\right\}$$

$$= \int_{-y^{1/2}}^{0} \frac{x+2}{4} dx + \int_{0}^{y^{1/2}} \frac{2-x}{4} dx = y^{1/2} - \frac{y}{4} \quad \text{for } x \in (-2, 2) \iff y \in (0, 4)\,.$$

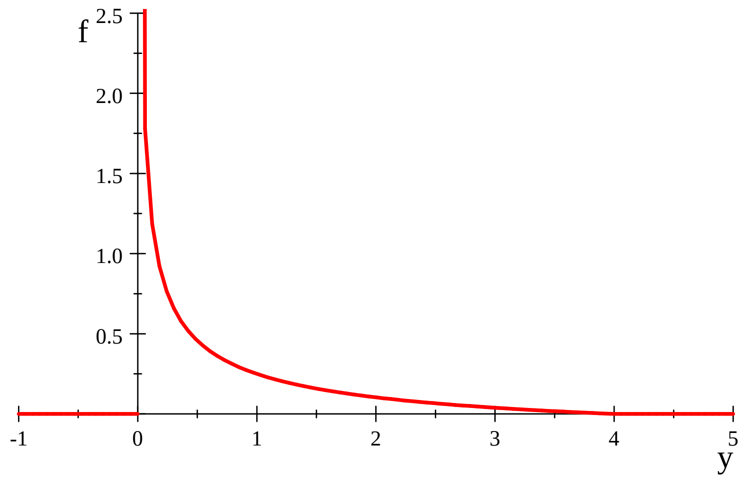
Therefore,

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{for } y \le 0 \\ y^{1/2} - \dfrac{y}{4} & \text{for } y \in (0, 4) \\ 1 & \text{for } y \ge 4. \end{cases}$$



We know that $f_{\tilde{y}}(y) = F'_{\tilde{y}}(y)$ for all y where the distribution function $F_{\tilde{y}}$ is differentiable. Thus,

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{y^{-1/2}}{2} - \dfrac{1}{4} & \text{for } y \in (0, 4) \\ \\ 0 & \text{otherwise} \end{cases}$$



Note that the density $f_{\tilde{y}}$ is continuous at $y = 4$ and, hence, the distribution function $F_{\tilde{y}}$ is differentiable at $y = 4$. However, the density $f_{\tilde{y}}$ is discontinuous at $y = 0$ and thus the distribution function $F_{\tilde{y}}$ is non-differentiable at $y = 0$. We see that

$$\lim_{y \to 0^-} f_{\tilde{y}}(y) = 0 \neq \infty = \lim_{y \to 0^+} f_{\tilde{y}}(y).$$

Note: This exercise could be also solved using the density function technique instead of the distribution function technique. To this end we first need to make the transformation $\tilde{z} = |\tilde{x}|$ and find the distribution function $F_{\tilde{z}}$ of $\tilde{z}$. Note that

$$F_{\tilde{z}}(z) = P\{\tilde{z} \leq z\} = P\{|\tilde{x}| \leq z\} = P\{-z \leq \tilde{x} \leq z\}$$

$$= \int_{-z}^{0} \frac{x+2}{4} dx + \int_{0}^{z} \frac{2-x}{4} dx = z - \frac{z^2}{4} \text{ for } x \in (-2, 2) \Longleftrightarrow z \in (0, 2).$$

Therefore,

$$F_{\tilde{z}}(z) = \begin{cases} 0 & \text{for } z \leq 0 \\ z - \dfrac{z^2}{4} & \text{for } z \in (0, 2) \\ 1 & \text{for } z \geq 2. \end{cases}$$

Therefore, the density of $\tilde{z}$ is

$$f_{\tilde{z}}(z) = \begin{cases} F'_{\tilde{z}}(z) = 1 - \dfrac{z}{2} & \text{for } z \in (0, 2) \\ 0 & \text{otherwise} \end{cases}$$

Then, define $\tilde{y} = g(\tilde{z}) = \tilde{z}^2$. Note that $g : (0, 2) \longrightarrow (0, 4)$ is a one-to-one correspondence. Thus, $z = g^{-1}(y) = y^{1/2}$ and $\dfrac{dz}{dy} = \dfrac{dg^{-1}(y)}{dy} = \dfrac{y^{-1/2}}{2} > 0$ for

$y \in (0,4)$. Therefore, the density of $\tilde{y}$ is

$$f_{\tilde{y}}(y) = \begin{cases} f_{\tilde{z}}(g^{-1}(y)) \left| \dfrac{g^{-1}(y)}{dy} \right| = \left(1 - \dfrac{y^{1/2}}{2}\right) \dfrac{y^{-1/2}}{2} = \dfrac{y^{-1/2}}{2} - \dfrac{1}{4} & \text{for } y \in (0,4) \\ \\ 0 & \text{otherwise.} \end{cases}$$

and, thus, the distribution function is

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \\ \int\limits_0^y \left(\dfrac{y^{-1/2}}{2} - \dfrac{1}{4} \right) dy = y^{1/2} - \dfrac{y}{4} & \text{for } y \in (0,4) \\ \\ 1 & \text{for } y \geq 4. \end{cases}$$

(c) The distribution of the random vector $(\tilde{x}, \tilde{y})$ does NOT have a density $f_{\tilde{x},\tilde{y}}(x,y)$. To see this, let us define the following subset C of $\mathbb{R}^2$:

$$C = \left\{ (x,y) \in \mathbb{R}^2 \middle| \, y = x^2 \right\}$$

The set C is the graph of a parabola on the plane. Then, on the one hand,

$$P\left\{(x,y) \in C\right\} = 1.$$

but, on the other hand, if the density $f_{\tilde{x},\tilde{y}}\left(x,y\right)$ exists, we should have

$$\int_C f_{\tilde{x},\tilde{y}}(x,y) d\left(x,y\right) = 1,$$

which is impossible since the set C has zero Lebesgue measure on $\mathbb{R}^2$ and, thus, the previous integral should be equal to zero.

(d)

$$\mathrm{E}(\tilde{x}) = \int_{-2}^{0} x \cdot \frac{x+2}{4} dx + \int_{0}^{2} x \cdot \frac{2-x}{4} dx = 0,$$

$$\mathrm{E}(\tilde{y}) = \mathrm{E}\left(\tilde{x}^2\right) = \int_{-2}^{0} x^2 \cdot \frac{x+2}{4} dx + \int_{0}^{2} x^2 \cdot \frac{2-x}{4} dx = \frac{2}{3}$$

or

$$\mathrm{E}(\tilde{y}) = \int_{0}^{4} y \left(\frac{y^{-1/2}}{2} - \frac{1}{4}\right) dy = \frac{2}{3},$$

$$\mathrm{E}(\tilde{x} \cdot \tilde{y}) = \mathrm{E}\left(\tilde{x}^3\right) = \int_{-2}^{0} x^3 \cdot \frac{x+2}{4} dx + \int_{0}^{2} x^3 \cdot \frac{2-x}{4} dx = 0,$$

$$\mathrm{Cov}(\tilde{x}, \tilde{y}) = \mathrm{E}(\tilde{x} \cdot \tilde{y}) - \mathrm{E}(\tilde{x}) \cdot \mathrm{E}(\tilde{y}) = 0 - 0 \cdot \frac{2}{3} = 0.$$

However, even if $\tilde{x}$ and $\tilde{y}$ are uncorrelated, they are not independent. It is obvious that the values taken by $\tilde{y}$ depend on the values taken by $\tilde{x}$. According to the definition of independence between two random variables, if $\tilde{x}$ and $\tilde{y}$ are independent we should have that

$$P\{\tilde{x} \in B_1, \tilde{y} \in B_2\} = P\{\tilde{x} \in B_1\} \cdot P\{\tilde{y} \in B_2\}$$

for all pairs B_1 and B_2 of Borel sets. Then, we can check that the random variables $\tilde{x}$ and $\tilde{y}$ are not independent. For instance,

$$P\{\tilde{x} \in (-1,0), \tilde{y} \in (2,4)\} = 0$$

as $\{\tilde{x} \in (-1,0)\} \cap \{\tilde{y} \in (2,4)\} = \emptyset$ because $\tilde{x} \in (-1,0) \Rightarrow \tilde{y} \in (0,1)$.

Moreover,

$$P\{\tilde{x} \in (-1,0)\} = \int_{-1}^{0} \frac{x+2}{4} dx = \frac{3}{8}$$

$$P\{\tilde{y} \in (2,4)\} = \int_{2}^{4} \left(\frac{y^{-1/2}}{2} - \frac{1}{4} \right) dy = \frac{3}{2} - \sqrt{2},$$

Therefore,

$$P\{\tilde{x} \in (-1,0), \tilde{y} \in (2,4)\} \neq P\{\tilde{x} \in (-1,0)\} \cdot P\{\tilde{y} \in (2,4)\}.$$

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22. Assume that the random variables are absolutely continuous. Define $\tilde{y} = \tilde{x}_1 + \tilde{x}_2$ so that $\tilde{z} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = \tilde{y} + \tilde{x}_3$. We know that the density of $\tilde{y} + \tilde{x}_3$ is

$$f_{\tilde{y}+\tilde{x}_3}(z) = f_{\tilde{x}_1+\tilde{x}_2+\tilde{x}_3}(z) = f_{\tilde{z}}(z) = \int_{\mathbb{R}} f_3(z-y) f_{\tilde{y}}(y) dy.$$

Moreover, the density of $\tilde{y} = \tilde{x}_1 + \tilde{x}_2$ is

$$f_{\tilde{y}}(y) = f_{\tilde{x}_1+\tilde{x}_2}(y) = \int_{\mathbb{R}} f_2(y-x) f_1(x) dx.$$

Therefore, combining the previous two equations, we get

$$f_{\tilde{x}_1+\tilde{x}_2+\tilde{x}_3}(z) = f_{\tilde{z}}(z) = \int_{\mathbb{R}} f_3(z-y) f_{\tilde{y}}(y) dy$$

$$= \int_{\mathbb{R}} f_3(z-y) \left[\int_{\mathbb{R}} f_2(y-x) f_1(x) dx \right] dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f_3(z-y) f_2(y-x) f_1(x) dx dy.$$

We can obtain an alternative formula for the convolution density. To do so, we define $\tilde{y} = \tilde{x}_2 + \tilde{x}_3$ so that $\tilde{z} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = \tilde{x}_1 + \tilde{y}$. We know that the density of $\tilde{x}_1 + \tilde{y}$ is

$$f_{\tilde{x}_1+\tilde{y}}(z) = f_{\tilde{x}_1+\tilde{x}_2+\tilde{x}_3}(z) = f_{\tilde{z}}(z) = \int_{\mathbb{R}} f_{\tilde{y}}(z - x_1) f_1(x_1) dx_1.$$

Moreover, the density of $\tilde{y} = \tilde{x}_2 + \tilde{x}_3$ is

$$f_{\tilde{y}}(y) = f_{\tilde{x}_2+\tilde{x}_3}(y) = \int_{\mathbb{R}} f_3(y - x_2) f_2(x_2) dx_2.$$

Therefore, combining the previous two equations, we get

$$f_{\tilde{x}_1+\tilde{x}_2+\tilde{x}_3}(z) = f_{\tilde{z}}(z) = \int_{\mathbb{R}} f_{\tilde{y}}(z - x_1) f_1(x_1) dx_1$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f_3(z - x_1 - x_2) f_2(x_2) dx_2 \right] f_1(x_1) dx_1$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f_3(z - x_1 - x_2) f_2(x_2) f_1(x_1) dx_2 dx_1.$$

Following the same steps, it is straightforward to prove that, if the random variables are discrete, then the probability function of the convolution is

$$f_{\tilde{x}_1+\tilde{x}_2+\tilde{x}_3}(z) = f_{\tilde{z}}(z) = \sum_{y \in \tilde{y}(\Omega)} \sum_{x \in \tilde{x}_1(\Omega)} f_3(z - y) f_2(y - x) f_1(x), \text{ for } z \in \tilde{z}(\Omega),$$

with $z - y \in \tilde{x}_3(\Omega)$ and $y - x \in \tilde{x}_2(\Omega)$.

Alternatively,

$$f_{\tilde{x}_1+\tilde{x}_2+\tilde{x}_3}(z) = f_{\tilde{z}}(z) = \sum_{x_2 \in \tilde{x}_2(\Omega)} \sum_{x_1 \in \tilde{x}_1(\Omega)} f_3(z - x_1 - x_2) f_2(x_2) f_1(x_1), \text{ for } z \in \tilde{z}(\Omega),$$

with $z - x_1 - x_2 \in \tilde{x}_3(\Omega)$.

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23. (a)

$$\mathrm{E}\left[(\tilde{x}-\mu)^k\right] = \sum_j p_j \mathrm{E}\left[(\tilde{x}-\mu)^k \left|\tilde{x}=\tilde{x}_j\right.\right]$$

$$= \sum_j p_j \mathrm{E}\left[(\tilde{x}_j-\mu)^k\right] = \sum_j p_j \mathrm{E}\left[\left[\left(\tilde{x}_j-\mu_j\right)+(\mu_j-\mu)\right]^k\right].$$

Making the binomial expansion of $\left[\left(\tilde{x}_j-\mu_j\right)+(\mu_j-\mu)\right]^k$, we get

$$\sum_j p_j \mathrm{E}\left[\sum_{n=0}^{k}\binom{k}{n}\left(\mu_j-\mu\right)^{k-n}\left(\tilde{x}_j-\mu_j\right)^n\right]$$

$$= \sum_j p_j \left(\sum_{n=0}^{k}\binom{k}{n}\left(\mu_j-\mu\right)^{k-n}\mathrm{E}\left[\left(\tilde{x}_j-\mu_j\right)^n\right]\right).$$

(b) We can evaluate the formula obtained in (a) for $k=2$ (please, do it!). Alternatively, we can also perform a direct computation,

$$\sigma^2 = \mathrm{E}\left[(\tilde{x}-\mu)^2\right] = \mathrm{E}\left(\tilde{x}^2\right) - \mu^2,$$

where

$$\mathrm{E}\left(\tilde{x}^2\right) = \sum_j p_j \mathrm{E}\left(\tilde{x}^2 \left|\tilde{x}=\tilde{x}_j\right.\right) = \sum_j p_j \mathrm{E}\left(\tilde{x}_j^2\right)$$

$$= \sum_j p_j \left[\mathrm{Var}\left(\tilde{x}_j\right) + \left[\mathrm{E}\left(\tilde{x}_j\right)\right]^2\right] = \sum_j p_j \left(\sigma_j^2+\mu_j^2\right).$$

Therefore,

$$\sigma^2 = \sum_j p_j \left(\sigma_j^2 + \mu_j^2\right) - \mu^2 = \sum_j p_j \left(\sigma_j^2 + \mu_j^2 - \mu^2\right),$$

where the last equality holds since $\sum_j p_j = 1$.

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24. (a) $f_{\tilde{x}}(x) = \int_0^1 \frac{4}{3}(x+xy)dy = 2x$ for $x \in (0,1)$ and $f_{\tilde{x}}(x) = 0$, otherwise. $f_{\tilde{y}}(y) = \int_0^1 \frac{4}{3}(x+xy)dx = \frac{2}{3}(1+y)$ for $y \in (0,1)$ and $f_{\tilde{y}}(y) = 0$, otherwise.

Clearly $f_{\tilde{x},\tilde{y}}(x,y) = f_{\tilde{x}}(x) \cdot f_{\tilde{y}}(y)$ so that $\tilde{x}$ and $\tilde{y}$ are independent and, thus, $\mathrm{Cov}(\tilde{x},\tilde{y}) = 0$.

Alternatively, you can compute

$$\mathrm{E}\left(\tilde{x}\cdot\tilde{y}\right) = \int_0^1\int_0^1 xy\frac{4}{3}(x+xy)dxdy = \frac{10}{27},$$

$$\mathrm{E}\left(\tilde{x}\right) = \int_0^1\int_0^1 x\frac{4}{3}(x+xy)dxdy = \int_0^1 x\cdot 2xdx = \frac{2}{3},$$

$$\mathrm{E}\left(\tilde{y}\right) = \int_0^1\int_0^1 y\frac{4}{3}(x+xy)dxdy = \int_0^1 y\cdot\frac{2}{3}(1+y)dy = \frac{5}{9}.$$

Thus,

$$\mathrm{Cov}\left(\tilde{x},\tilde{y}\right) = \mathrm{E}\left(\tilde{x}\cdot\tilde{y}\right) - \mathrm{E}\left(\tilde{x}\right)\mathrm{E}\left(\tilde{y}\right) = \frac{10}{27} - \left(\frac{2}{3}\cdot\frac{5}{9}\right) = 0.$$

(b)

$$M_{\tilde{x}}(t) = \mathrm{E}\left(e^{t\tilde{x}}\right) = \int_0^1 e^{tx}f_{\tilde{x}}(x)dx = \int_0^1 e^{tx}\cdot 2xdx.$$

The previous integral can be solved by parts by making $F(x) = 2x$ and

$g(x) = G'(x) = e^{tx}$ so that $f(x) = F'(x) = 2$ and $G(x) = \dfrac{e^{tx}}{t}$. Then,

$$M_{\tilde{x}}(t) = \int_0^1 e^{tx} \cdot 2x dx = \left[2x\frac{e^{tx}}{t}\right]_0^1 - \int_0^1 2\frac{e^{tx}}{t} dx = \left[2x\frac{e^{tx}}{t}\right]_0^1 - \left[2\frac{e^{tx}}{t^2}\right]_0^1$$

$$= \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2} = \frac{2te^t - 2e^t + 2}{t^2}, \text{ for } t \neq 0,$$

and $M_{\tilde{x}}(0) = 1$. Note that

$$\lim_{t\to 0} M_{\tilde{x}}(t) = \frac{0}{0}$$

and, from l'Hôpital's rule, we get

$$\lim_{t\to 0} M_{\tilde{x}}(t) = \lim_{t\to 0} \frac{2te^t}{2t} = \lim_{t\to 0} e^t = 1.$$

The derivative of $M_{\tilde{x}}(t)$ is

$$M'_{\tilde{x}}(t) = \frac{2e^t}{t} - \frac{4e^t}{t^2} + \frac{4e^t}{t^3} - \frac{4}{t^3} = \frac{2t^2e^t - 4te^t + 4e^t - 4}{t^3} \text{ for } t \neq 0.$$

Therefore,

$$\lim_{t\to 0} M'_{\tilde{x}}(t) = \frac{0}{0}.$$

Let us apply l'Hôpital's rule,

$$\lim_{t\to 0} M'_{\tilde{x}}(t) = \lim_{t\to 0} \frac{2t^2e^t}{3t^2} = \lim_{t\to 0} \frac{2e^t}{3} = \frac{2}{3}.$$

(c)

$$(w, z) = g(x, y) : \begin{cases} w = \ln x \in (-\infty, 0) \\ \\ z = \ln\left(\dfrac{x}{y}\right) \in (w, \infty) \end{cases}$$

$$(x,y)=g^{-1}(w,z):\begin{cases} x=e^{w}\in(0,1) \\ \\ y=e^{w-z}\in(0,1) \end{cases}$$

Note that $z=\ln x-\ln y=w-\ln y$. Since $y\in(0,1)$, then $\ln y\in(-\infty,0)$ and $-\ln y\in(0,\infty)$. Therefore,

$$z=\ln x-\ln y=w-\ln y\in(w,\infty)\,.$$

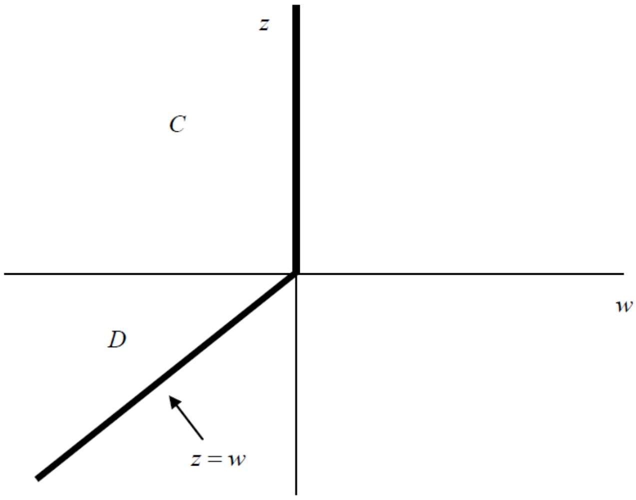
$$J_{g^{-1}}(w,z)=\begin{pmatrix} e^{w} & 0 \\ \\ e^{w-z} & -e^{w-z} \end{pmatrix}\Longrightarrow \det J_{g^{-1}}(w,z)=-e^{2w-z}<0$$

$$\Longrightarrow |\det J_{g^{-1}}|=e^{2w-z}$$

Then,

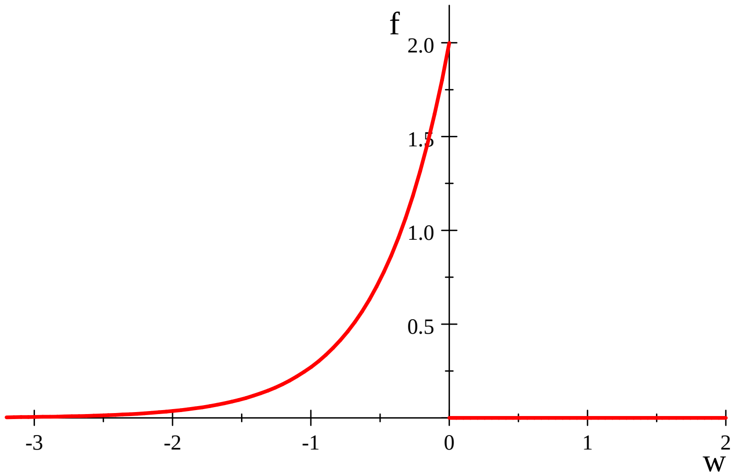
$$f_{\tilde{w},\tilde{z}}(w,z)=\begin{cases} \dfrac{4}{3}(e^{w}+e^{w}e^{w-z})e^{2w-z}=\dfrac{4}{3}\left(e^{3w-z}+e^{4w-2z}\right), \text{ for } w\in(-\infty,0)\,,\ z\in(w,\infty) \\ \\ 0, \quad \text{otherwise.} \end{cases}$$

(d) The region of positive density for $f_{\tilde{w},\tilde{z}}(w,z)$ is $C\cup D$:



Marginal density $f_{\tilde{w}}(w)$:

$$f_{\tilde{w}}(w) = \begin{cases} \displaystyle\int_w^\infty \frac{4}{3}\left(e^{3w-z} + e^{4w-2z}\right) dz = 2e^{2w} \text{ for } w \in (-\infty, 0) \\ \\ 0, \quad \text{otherwise} \end{cases}$$



Marginal density $f_{\tilde{z}}(z)$:

If $z \le 0$, then

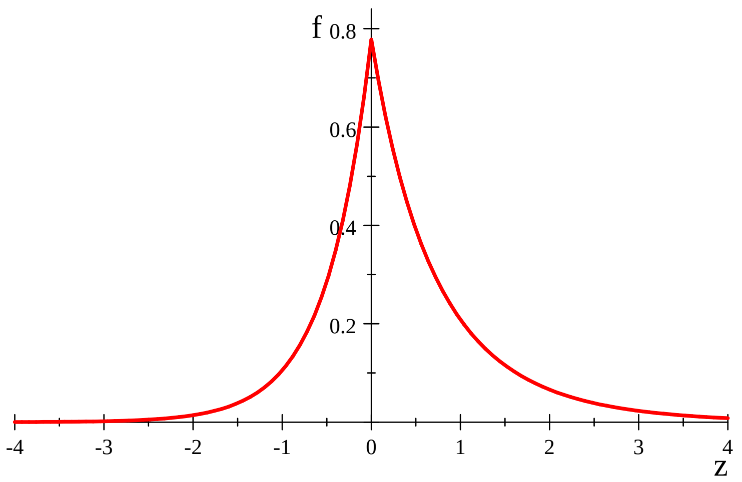
$$f_{\tilde{z}}(z) = \int\limits_{D} f_{\tilde{w},\tilde{z}}(w,z)dw$$
$$= \int_{-\infty}^{z} \frac{4}{3}\left(e^{3w-z} + e^{4w-2z}\right) dw = \frac{7}{9}e^{2z}.$$

If $z > 0$, then

$$f_{\tilde{z}}(z) = \int\limits_{C} f_{\tilde{w},\tilde{z}}(w,z)dw$$
$$= \int_{-\infty}^{0} \frac{4}{3}\left(e^{3w-z} + e^{4w-2z}\right) dw = \frac{1}{9}\left(3e^{-2z} + 4e^{-z}\right).$$

Note that $f_{\tilde{z}}(0) = \dfrac{7}{9}$ and that the density is continuous (but not differentiable) at $z = 0$. Thus,

$$f_{\tilde{z}}(z) = \begin{cases} \dfrac{7}{9}e^{2z} & \text{for } z \le 0 \\ \\ \dfrac{1}{9}\left(3e^{-2z} + 4e^{-z}\right) & \text{for } z > 0. \end{cases}$$



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25. (a) Marginal densities:

$$f_{\tilde{x}}(x) = \begin{cases} \displaystyle\int_0^2 \frac{2x+y}{12} dy = \frac{1+2x}{6} & \text{if } x \in (0,2) \\ \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{\tilde{y}}(y) = \begin{cases} \displaystyle\int_0^2 \frac{2x+y}{12} dx = \frac{y+2}{6} & \text{if } y \in (0,2) \\ \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{E}(\tilde{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y) dx dy = \int_{-\infty}^{\infty} x f_{\tilde{x}}(x) dx = \int_0^2 x \frac{1+2x}{6} dx = \frac{11}{9}.$$

$$\mathrm{E}(\tilde{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y) dx dy = \int_{-\infty}^{\infty} y f_{\tilde{y}}(y) dy = \int_0^2 y \frac{y+2}{6} dy = \frac{10}{9}.$$

$$\mathrm{E}(\tilde{x} \cdot \tilde{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y) dx dy = \int_0^2 \int_0^2 xy \frac{2x+y}{12} dx dy = \frac{4}{3}.$$

$$\mathrm{Cov}(\tilde{x} \cdot \tilde{y}) = \mathrm{E}(\tilde{x} \cdot \tilde{y}) - \mathrm{E}(\tilde{x})\mathrm{E}(\tilde{x}) = \frac{4}{3} - \left(\frac{11}{9} \cdot \frac{10}{9}\right) = -\frac{2}{81}.$$

Thus, since $\mathrm{Cov}(\tilde{x} \cdot \tilde{y}) \neq 0$, the random variable $\tilde{x}$ and $\tilde{y}$ are not independent.

(b) If $y \in (0,2)$, then

$$f_{\tilde{x}|\tilde{y}}(x\,|y) = \frac{f(x,y)}{f_{\tilde{y}}(y)} = \frac{\frac{2x+y}{12}}{\frac{y+2}{6}} = \frac{2x+y}{2\,(y+2)} \text{ for } x \in (0,2)$$

and $f_{\tilde{x}|\tilde{y}}(x\,|y) = 0$ otherwise.

Thus,

$$f_{\tilde{x}|\tilde{y}}(x\,|1/2) = \frac{2x+\dfrac{1}{2}}{2\left(\dfrac{1}{2}+2\right)} = \frac{4x+1}{10} \text{ for } x \in (0,2)$$

and $f_{\tilde{x}|\tilde{y}}(x\,|1/2) = 0$ otherwise.

$$\mathrm{E}(\tilde{x}\,|\tilde{y} = 1/2) = \int_0^2 x\frac{4x+1}{10}dx = \frac{19}{15}.$$

(c)

$$g : (x,y) \longmapsto (z,x) : \begin{cases} z = 3y - \dfrac{x}{2} \\ \\ x = x \in (0,2), \end{cases}$$

$$g^{-1} : (z,x) \longmapsto (x,y) : \begin{cases} x = x \in (0,2) \\ \\ y = \dfrac{1}{3}\left(z+\dfrac{x}{2}\right) \in (0,2), \end{cases}$$

$$|J_{g^{-1}}(z,x)| = \det\begin{pmatrix} 0 & 1 \\ \\ 1/3 & 1/6 \end{pmatrix} = \left|-\frac{1}{3}\right| = \frac{1}{3}.$$

Note that

$$\frac{1}{3}\left(z+\frac{x}{2}\right) \in (0,2) \Longleftrightarrow x \in (-2z, 12-2z) \Longleftrightarrow z \in \left(-\frac{x}{2}, 6-\frac{x}{2}\right).$$

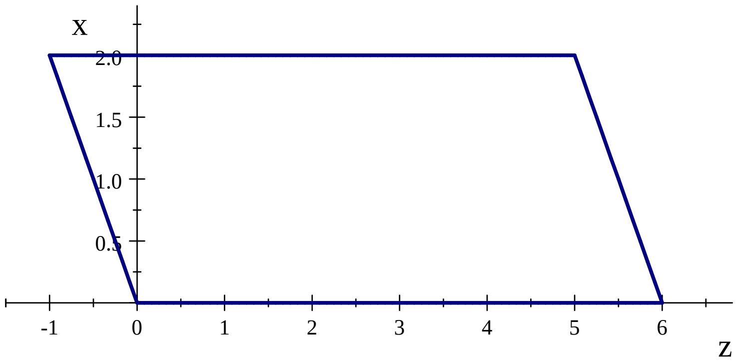
Thus,

$$f_{\tilde{z},\tilde{x}}(z,x) = \left[\frac{2x+\dfrac{1}{3}\left(z+\dfrac{x}{2}\right)}{12}\right]\cdot\frac{1}{3} \text{ if } x \in (0,2) \text{ and } x \in (-2z, 12-2z)\,,$$

which, after simplifying, becomes

$$f_{\tilde{z},\tilde{x}}(z,x) = \begin{cases} \dfrac{1}{108}z + \dfrac{13}{216}x & \text{if } x \in (0,2) \text{ and } x \in (-2z, 12-2z) \\ \\ 0 & \text{otherwise.} \end{cases}$$

The region where the density of the random vector $(\tilde{z},\tilde{x})$ is different from zero is the interior of the following parallelogram, which is $g((0,2)\times(0,2))$:



To compute the marginal density $f_{\tilde{z}}(z)$, we consider the following intervals for z :

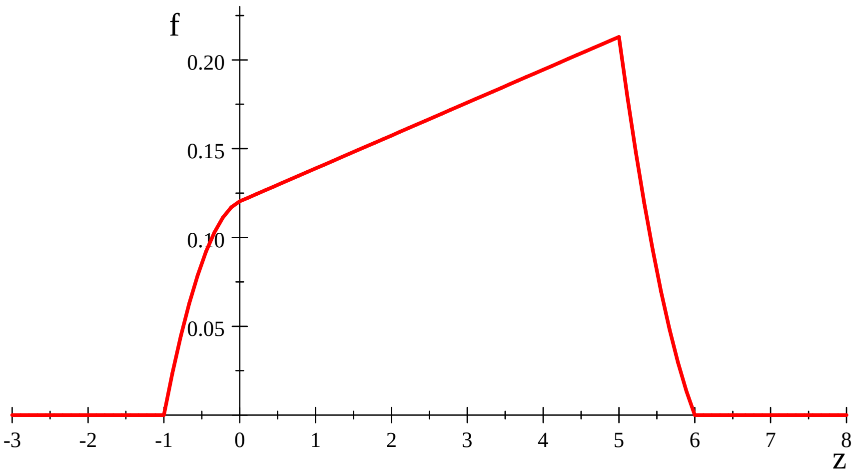
$$\text{for } z \in (-1,0) \Rightarrow \int_{-2z}^{2} \left(\frac{1}{108}z + \frac{13}{216}x\right) dx = -\frac{11}{108}z^2 + \frac{1}{54}z + \frac{13}{108}$$

$$\text{for } z \in (0,5) \Rightarrow \int_{0}^{2} \left(\frac{1}{108}z + \frac{13}{216}x\right) dx = \frac{1}{54}z + \frac{13}{108}$$

$$\text{for } z \in (5,6) \Rightarrow \int_{0}^{12-2z} \left(\frac{1}{108}z + \frac{13}{216}x\right) dx = \frac{11}{108}z^2 - \frac{4}{3}z + \frac{13}{3}$$

Then,

$$f_{\tilde{z}}(z) = \begin{cases} -\dfrac{11}{108}z^2 + \dfrac{1}{54}z + \dfrac{13}{108} & \text{if } -1 \le z < 0 \\[2ex] \dfrac{1}{54}z + \dfrac{13}{108} & \text{if } 0 \le z < 5 \\[2ex] \dfrac{11}{108}z^2 - \dfrac{4}{3}z + \dfrac{13}{3} & \text{if } 5 \le z < 6 \\[2ex] 0 & \text{otherwise.} \end{cases}$$



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26.

$$z = cx + b \Longleftrightarrow x = \frac{z-b}{c} \text{ so that } \left|\frac{dx}{dz}\right| = \frac{1}{c}.$$

Note that the logistic and the three types of generalized extreme value distributions have a density that can be written as

$$f_{\tilde{x}}(x; m, s, \alpha) = \frac{\alpha}{s} h\left(\left[\frac{x-m}{s}\right]^k, \left[\frac{x-m}{s}\right]^{k-1}\right),$$

where $k = \alpha = 1$ for the logistic and generalized extreme value type I distributions, $k = -\alpha < 0$ for the generalized extreme value type II distribution, and $k = \alpha > 0$ for the generalized extreme value type III distribution. Thus, if

$$\tilde{x} \sim f_{\tilde{x}}(x; m, s, \alpha),$$

then the density of $\tilde{z}$ will be

$$f_{\tilde{z}}(z; m, s, \alpha) = f_{\tilde{x}}\left(\frac{z-b}{c}; m, s, \alpha\right)\frac{1}{c} = \frac{\alpha}{s}h\left(\left[\frac{\frac{z-b}{c} - m}{s}\right]^{k}, \left[\frac{\frac{z-b}{c} - m}{s}\right]^{k-1}\right)\frac{1}{c}$$

$$= \frac{\alpha}{cs}h\left(\left[\frac{z-(cm+b)}{cs}\right]^{k}, \left[\frac{z-(cm+b)}{cs}\right]^{k-1}\right).$$

Therefore, the random variable $\tilde{z}$ has the logistic, generalized extreme value type I, type II or type III distributions with location parameter $cm + b$ and scale parameter $cs > 0$ and the shape parameter α remains the same.

Obviously, if $c = 1/s$ and $b = -m/s$, then the distributions become standard as the density of $\tilde{z}$ would be

$$\alpha h\left(z^{k}, z^{k-1}\right) = f_{\tilde{z}}(z; 0, 1, \alpha).$$

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27. (a) The density of $\tilde{w}$ is exponential with parameter equal to one,

$$f_{\tilde{w}}(w) = \begin{cases} e^{-w} & \text{for } w > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x = m - s\ln w \in (-\infty, \infty) \iff w = e^{-\left(\frac{x-m}{s}\right)} > 0$. Then, $\left|\dfrac{dw}{dx}\right| = \dfrac{1}{s}e^{-\left(\frac{x-m}{s}\right)}$. Thus,

$$f_{\tilde{x}}(x) = f_{\tilde{w}}\left(e^{-\left(\frac{x-m}{s}\right)}\right)\left|\frac{dw}{dx}\right| = e^{-e^{-\left(\frac{x-m}{s}\right)}}\frac{1}{s}e^{-\left(\frac{x-m}{s}\right)} = \frac{1}{s}e^{-\left(\frac{x-m}{s}\right)}e^{-e^{-\left(\frac{x-m}{s}\right)}} \quad \text{for all } x \in \mathbb{R}.$$

Thus, the random variable $\tilde{x}$ has the generalized extreme value type I (or Gumbel) distribution distribution with location parameter m and scale parameter $s > 0$, $\tilde{x} \sim G(m, s)$. Conversely, if $\tilde{x} \sim G(m, s)$ then $\tilde{w} = e^{-\left(\frac{\tilde{x}-m}{s}\right)}$ is exponential with parameter equal to 1.

(b) Since $\tilde{w}_1$ and $\tilde{w}_2$ are independent, their joint density is

$$f_{\tilde{w}_1,\tilde{w}_2}(w_1, w_2) = f_{\tilde{w}_1}(w_1)f_{\tilde{w}_2}(w_2) = \begin{cases} e^{-w_1}e^{-w_2} & \text{for } w_1 > 0 \text{ and } w_2 > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The distribution function of $\tilde{z} = \tilde{w}_2/\tilde{w}_1$ is

$$F_{\tilde{z}}(z) = P\left\{\tilde{z} \leq z\right\} = P\left\{\frac{\tilde{w}_2}{\tilde{w}_1} \leq z\right\} = \int_A e^{-w_1}e^{-w_2}d(w_1, w_2),$$

where $A = \{(w_1, w_2) \in \mathbb{R}^2 \,|w_2 \leq zw_1 \}$. Hence,

$$F_{\tilde{z}}(z) = \int_0^\infty \int_0^{zw_1} e^{-w_1} e^{-w_2} dw_2 dw_1 = \int_0^\infty e^{-w_1} \left[\int_0^{zw_1} e^{-w_2} dw_2 \right] dw_1$$

$$= \int_0^\infty e^{-w_1} \left[-e^{-w_2} \right]_0^{zw_1} dw_1 = \int_0^\infty e^{-w_1} (1 - e^{-zw_1}) dw_1 = \int_0^\infty \left(e^{-w_1} - e^{-(1+z)w_1} \right) dw_1$$

$$= \left[-e^{-w_1} + \frac{e^{-(1+z)w_1}}{1+z} \right]_0^\infty = 1 - \frac{1}{1+z} = \frac{z}{1+z} \text{ for } z > 0,$$

and $F_{\tilde{z}}(z) = 0$ for $z \leq 0$.

Therefore the density of $\tilde{z}$ is $f_{\tilde{z}}(z) = F'_{\tilde{z}}(z) = \dfrac{1}{(1+z)^2}$ for $z > 0$, and $f_{\tilde{z}}(z) = 0$ otherwise.

(c) Let $\tilde{y} = \tilde{x}_1 - \tilde{x}_2$ and define the random variable $\tilde{q}$ as

$$\tilde{q} = \frac{\tilde{y} - (m_1 - m_2)}{s} = \frac{\tilde{x}_1 - m_1}{s} - \left(\frac{\tilde{x}_2 - m_2}{s} \right)$$

$$= \ln \left(e^{\left(\frac{\tilde{x}_1 - m_1}{s}\right) - \left(\frac{\tilde{x}_2 - m_2}{s}\right)} \right) = \ln \left(\frac{e^{-\left(\frac{\tilde{x}_2 - m_2}{s}\right)}}{e^{-\left(\frac{\tilde{x}_1 - m_1}{s}\right)}} \right).$$

Let $\tilde{w}_i = e^{-\left(\frac{\tilde{x}_i - m_i}{s}\right)}$, $i = 1, 2$. From part (a), since $\tilde{x}_1$ and $\tilde{x}_2$ have the Gumbel distribution and are independent, $\tilde{w}_1$ and $\tilde{w}_2$ are independent and both have the exponential distribution with parameter equal to 1. Therefore, from part (b), the random variable

$$\tilde{z} = \frac{\tilde{w}_2}{\tilde{w}_1} = \frac{e^{-\left(\frac{\tilde{x}_2 - m_2}{s}\right)}}{e^{-\left(\frac{\tilde{x}_1 - m_1}{s}\right)}}$$

has the density function

$$f_{\tilde{z}}(z) = \begin{cases} \dfrac{1}{(1+z)^2} & \text{for } z > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

Note that

$$q = \ln\left(\frac{e^{-\left(\frac{x_2 - m_2}{s}\right)}}{e^{-\left(\frac{x_1 - m_1}{s}\right)}}\right) = \ln z \in (-\infty, \infty) \Longleftrightarrow z = e^q > 0$$

so that $\left|\dfrac{dz}{dq}\right| = e^q$. Thus, the density of $\tilde{q}$ is

$$f_{\tilde{q}}(q) = f_{\tilde{z}}(e^q)e^q = \frac{e^q}{(1+e^q)^2} = \frac{e^{-q}}{(1+e^{-q})^2} = l(q; 0, 1) \text{ for all } q \in \mathbb{R}.$$

where the third equality follows from multiplying the numerator and the denominator by $e^{-2q} > 0$. Thus, we have proved that $\tilde{q}$ has the standard logistic distribution.

Since

$$q = \frac{y - (m_1 - m_2)}{s} \quad \text{so that} \quad \left|\frac{dq}{dy}\right| = \frac{1}{s} > 0,$$

the density of $\tilde{y} = \tilde{x}_1 - \tilde{x}_2 = s\tilde{q} + (m_1 - m_2)$ is

$$f_{\tilde{y}}(y) = f_{\tilde{q}}\left(\frac{\tilde{y} - (m_1 - m_2)}{s}\right)\frac{1}{s} = \frac{e^{-\left(\frac{y-(m_1-m_2)}{s}\right)}}{s\left[1 + e^{-\left(\frac{y-(m_1-m_2)}{s}\right)}\right]^2} = l(y; m_1 - m_2, s),$$

which is the logistic density with location parameter $m_1 - m_2$ and scale parameter s. We could omit this last step by just using the result of Exercise

26.

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