Probability and Statistics. IDEA. Answers to List 4.

1. For this exercise, see https://brilliant.org/wiki/sum-of-n-n2-or-n3/ if you are not familiar with the required formulae.

(a)

$$\mu_{\tilde{x}} = \sum_{x=1}^{k} x f(x) = \frac{1}{k}\sum_{x=1}^{k} x = \frac{1}{k}\frac{k(k+1)}{2} = \frac{k+1}{2}.$$

(b) Note that $\sum_{x=1}^{k} x^2 = \frac{1}{6} k\,(2k+1)\,(k+1)$

$$\begin{aligned}
\sigma^2_{\tilde{x}} &= \sum_{x=1}^{k}(x-\mu)^2 f(x) = \sum_{x=1}^{k} x^2 f(x) - \mu^2 \\
&= \frac{1}{k}\sum_{x=1}^{k} x^2 - \left(\frac{k+1}{2}\right)^2 = \frac{1}{k}\cdot\frac{k\,(2k+1)\,(k+1)}{6} - \frac{(k+1)^2}{4} \\
&= \frac{k+1}{12}(4k+2-3k-3) = \frac{k+1}{12}(k-1) = \frac{k^2-1}{12}.
\end{aligned}$$

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2. For this exercise, see https://brilliant.org/wiki/geometric-progressions/ if you are not familiar with the required formula.

Note that $\sum_{x=1}^{k} r^x = \dfrac{r\left(1-r^k\right)}{1-r}$ for $r \neq 1$. Then,

$$M_{\tilde{x}}(t) = \mathrm{E}(e^{tx}) = \sum_{x=1}^{k} e^{tx} f(x) = \frac{1}{k}\sum_{x=1}^{k} e^{tx} = \frac{1}{k}\frac{e^t(1-e^{kt})}{1-e^t}, \quad \text{for } t \neq 0,$$

and $M_{\tilde{x}}(0) = 1$. Therefore, $M_{\tilde{x}}(t)$ is finite (and continuous) for all t since the support of $\tilde{x}$ is bounded above and below (please, check that $\lim_{t\to 0} M_{\tilde{x}}(t) = 1$).

Thus, the derivatives of all orders of $M_{\tilde{x}}(t)$ are also continuous.

$$\begin{aligned} M'_{\tilde{x}}(t) &= \frac{1}{k}\frac{\left[e^t-(1+k)e^{(1+k)t}\right](1-e^t)+e^t\left[e^t-e^{(1+k)t}\right]}{(1-e^t)^2} \\ &= \frac{1}{k}\frac{e^t-(1+k)e^{(1+k)t}+ke^{(2+k)t}}{(1-e^t)^2}. \end{aligned}$$

Since $\lim\limits_{t\to 0} M'_{\tilde{x}}(t) = \frac{0}{0}$, by L'Hôpital's Rule,

$$\begin{aligned} \lim_{t\to 0} M'_{\tilde{x}}(t) &= \lim_{t\to 0}\frac{1}{k}\frac{e^t-(1+k)^2e^{(1+k)t}+k(2+k)e^{(2+k)t}}{-2(1-e^t)e^t} \\ &= \lim_{t\to 0}\frac{1}{k}\frac{e^t-(1+k)^3e^{(1+k)t}+k(2+k)^2e^{(2+k)t}}{-2(e^t-2e^{2t})} \\ &= \frac{1}{k}\frac{1-(1+k)^3+k(2+k)^2}{-2(1-2)} \\ &= \frac{1}{k}\frac{k(k+1)}{2}=\frac{k+1}{2}=\mu_{\tilde{x}}. \end{aligned}$$

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3. Let θ = probability that he gets a correct answer in a single question.

$$P\{\text{he gets exactly four correct answers}\} = \binom{8}{4}\theta^4(1-\theta)^4.$$

Note that,

$$\begin{aligned} \theta &= \sum_{i=1}^{3} P\left\{\text{choose } i^{th} \text{ answer}\right\}\cdot P\left\{i^{th} \text{ answer is correct} \,\middle|\, \text{choose } i^{th} \text{ answer}\right\} \\ &= \sum_{i=1}^{3}\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{3} \quad \left(\text{you can skip this step since it is obvious that } \theta=\frac{1}{3}\right). \end{aligned}$$

$$P\{\text{he gets exactly four correct answers}\} = \binom{8}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^4 = 0.1707.$$

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4. Let $\theta =$ probability that each single selected component is without flaw in each lot. This is the case when θ is the proportion of flawless components in a large lot.

(a)

$$\theta = 0.95, \qquad 1 - b(20; 20, 0.95) = 1 - b(0; 20, 0.05) = 1 - 0.3585 = 0.6415$$

(b)

$$\theta = 0.9, \qquad b(20; 20, 0.9) = b(0; 20, 0.1) = 0.1216$$

(c)

$$\theta = 0.8, \qquad b(20; 20, 0.8) = b(0; 20, 0.2) = 0.0115$$

(d)

$$\theta = 0.7, \qquad b(20; 20, 0.7) = b(0; 20, 0.3) = 0.0008.$$

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5. See https://brilliant.org/wiki/geometric-progressions/ . Note thus that $\sum_{x=1}^{\infty} r^x = \dfrac{r}{1-r}$ for $|r| < 1$.

$$\begin{aligned} M_{\tilde{x}}(t) &= \mathrm{E}(e^{tx}) = \sum_{x=1}^{\infty} e^{tx}\theta(1-\theta)^{x-1} = \frac{\theta}{1-\theta}\sum_{x=1}^{\infty} e^{tx}(1-\theta)^x \\ &= \frac{\theta}{1-\theta}\sum_{x=1}^{\infty}\left[e^t(1-\theta)\right]^x = \frac{\theta}{1-\theta}\cdot\frac{e^t(1-\theta)}{1-e^t(1-\theta)} = \frac{\theta e^t}{1-e^t(1-\theta)}, \end{aligned}$$

when $e^t(1-\theta) < 1$. Note that $M_{\tilde{x}}(t)$ is finite in a neihborhood of $t = 0$. More precisely, $M_{\tilde{x}}(t)$ is finite when $1 - e^t(1-\theta) > 0$, i.e., when $t < -\ln(1-\theta)$. Note that $-\ln(1-\theta) > 0$ as $\theta \in (0,1]$.

$$\begin{aligned} M'_{\tilde{x}}(t) &= \frac{\theta e^t\left[1 - e^t(1-\theta)\right] + (1-\theta)e^t\theta e^t}{\left[1 - e^t(1-\theta)\right]^2} \\ &= \frac{\theta e^t}{\left[1 - e^t(1-\theta)\right]^2}. \end{aligned}$$

$$\begin{aligned} M''_{\tilde{x}}(t) &= \frac{\theta e^t\left[1 - e^t(1-\theta)\right]^2 + 2\left[1 - e^t(1-\theta)\right](1-\theta)e^t\theta e^t}{\left[1 - e^t(1-\theta)\right]^4} \\ &= \frac{\theta e^t + \theta(1-\theta)e^{2t}}{\left[1 - e^t(1-\theta)\right]^3}. \end{aligned}$$

$$\mu_{\tilde{x}} = M'_{\tilde{x}}(0) = \frac{\theta}{\left[1 - (1-\theta)\right]^2} = \frac{1}{\theta}.$$

$$\sigma^2_{\tilde{x}} = M''_{\tilde{x}}(0) - \mu^2 = \frac{\theta + \theta(1-\theta)}{(1-1+\theta)^3} - \frac{1}{\theta^2} = \frac{1 + (1-\theta)}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{1-\theta}{\theta^2}.$$

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6. $\theta_1 = \frac{9}{16}; \theta_2 = \frac{3}{16}; \theta_3 = \frac{3}{16}; \theta_4 = \frac{1}{16}$. We see that $\sum_{i=1}^4 \theta_i = 1$. We use the multinomial distribution:

$$\begin{aligned} f\left(4,2,3,0;9,\frac{9}{16},\frac{3}{16},\frac{3}{16},\frac{1}{16}\right) &= \frac{9!}{4!2!3!}\left(\frac{9}{16}\right)^4\left(\frac{3}{16}\right)^2\left(\frac{3}{16}\right)^3\left(\frac{1}{16}\right)^0 \\ &= 1260 \cdot 9^4 \cdot 3^2 \cdot 3^3 \cdot 16^{-9} = 0.02923. \end{aligned}$$

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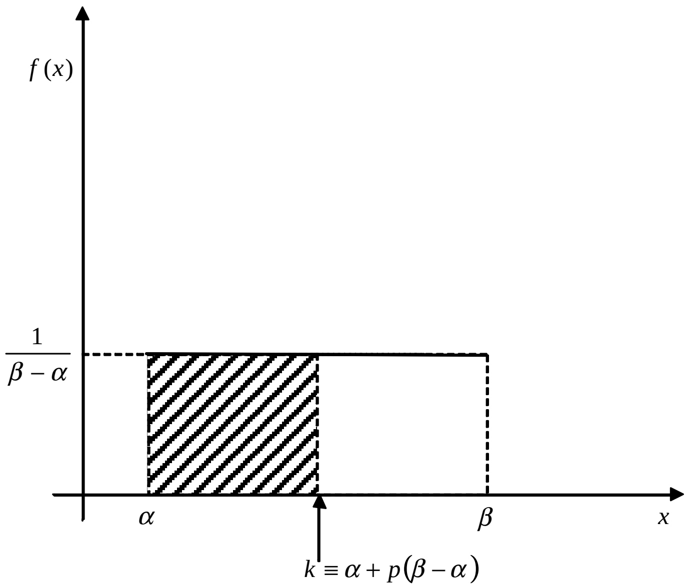
7. To prove that

$$P\left\{\tilde{x} < \alpha + p(\beta - \alpha)\right\} = p.$$

We use the fact that

$$f(x) = \frac{1}{\beta - \alpha}, \text{ for } \alpha < x < \beta.$$



Therefore

$$\begin{aligned} P\left\{\tilde{x} < \alpha + p(\beta - \alpha)\right\} &= \int\limits_{\alpha}^{\alpha+p(\beta-\alpha)} \frac{1}{\beta - \alpha} dx = \frac{1}{\beta - \alpha} \left[x\right]_{\alpha}^{\alpha+p(\beta-\alpha)} \\ &= \frac{1}{\beta - \alpha} \left\{\left[\alpha + p(\beta - \alpha)\right] - \alpha\right\} = \frac{1}{\beta - \alpha} \left\{p(\beta - \alpha)\right\} = p. \end{aligned}$$

Note that the quantile function of the uniform distribution on (α, β) is

$$Q(p) = \alpha + (\beta - \alpha)p \;\text{ for } p \in (0, 1),$$

which directly implies that $P\{\tilde{x} < \alpha + p(\beta - \alpha)\} = p$.

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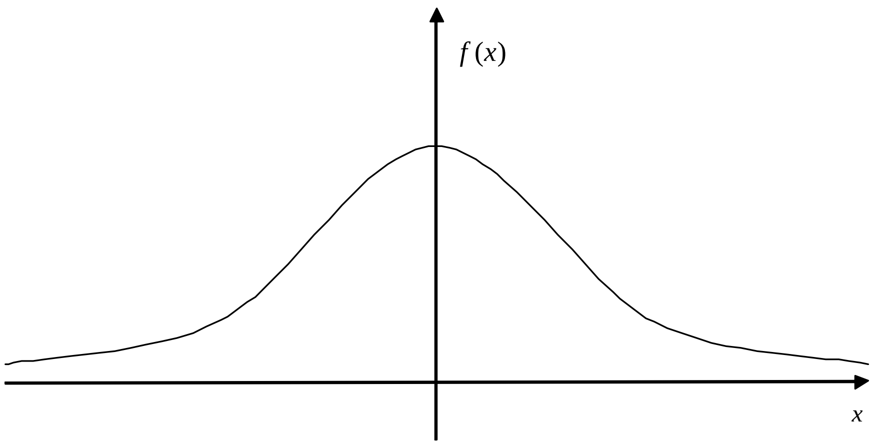
8. Setting $\alpha = 0$ and $\beta = 1$,

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$



$$\mu_1' = \int_{-\infty}^{\infty} x\frac{1}{\pi}\frac{1}{1+x^2}dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{1+x^2}dx = \frac{1}{\pi}\left[\int_{(-\infty,0)}\frac{x}{1+x^2}dx + \int_{[0,\infty)}\frac{x}{1+x^2}dx\right].$$

We make the change of variable $v = 1+x^2 = g^{-1}(x) \geq 1$, so that $x = -\sqrt{v-1}$ for $x \in (-\infty, 0)$ and $x = +\sqrt{v-1}$ for $x \in [0, \infty)$. Then, $|g'(v)| = \frac{1}{2\sqrt{v-1}}$ for all $v = 1 + x^2 \geq 1$.

$$\begin{aligned}
\mu_1' &= \frac{1}{\pi}\left(\int\limits_{(1,\infty)} \frac{-\sqrt{v-1}}{v}\frac{1}{2\sqrt{v-1}}dv + \int\limits_{[1,\infty)} \frac{\sqrt{v-1}}{v}\frac{1}{2\sqrt{v-1}}dv\right) \\
&= \frac{1}{\pi}\left(\int\limits_{(1,\infty)} -\frac{1}{2}\frac{1}{v}dv + \int\limits_{[1,\infty)} \frac{1}{2}\frac{1}{v}dv\right) = \frac{1}{\pi}\left(-\frac{1}{2}\left[\ln v\right]_1^\infty + \frac{1}{2}\left[\ln v\right]_1^\infty\right) \\
&= \frac{1}{2\pi}\left(-\left[\infty - 0\right] + \left[\infty - 0\right]\right) = -\infty + \infty.
\end{aligned}$$

Therefore, μ_1' does not exist.

Note that we can directly see that the primitive (or antiderivative) of $\frac{x}{1+x^2}$ is $\frac{1}{2}\ln\left(1+x^2\right)$. Hence,

$$\begin{aligned}
\mu_1' &= \frac{1}{\pi}\int\limits_{(-\infty,\infty)} \frac{x}{1+x^2}dx = \frac{1}{\pi}\frac{1}{2}\left[\ln\left(1+x^2\right)\right]_{-\infty}^\infty \\
&= \frac{1}{2\pi}\left(\infty - \infty\right) = \infty - \infty.
\end{aligned}$$

$$\begin{aligned}
\mu_2' &= \int\limits_{-\infty}^{\infty} x^2\frac{1}{\pi}\frac{1}{1+x^2}dx = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^2}dx = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right]dx \\
&= \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[1 - \frac{1}{1+x^2}\right]dx = \frac{1}{\pi}\left[x\right]_{-\infty}^\infty - \int\limits_{-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1+x^2}dx.
\end{aligned}$$

Since

$$\begin{aligned}
\int\limits_{-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1+x^2}dx &= \frac{1}{\pi}\left[\arctan x\right]_{-\infty}^\infty = \frac{1}{\pi}\left[\lim_{x\to\infty}\arctan x - \lim_{x\to-\infty}\arctan x\right] \\
&= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{\pi} = 1,
\end{aligned}$$

$$\mu_2' = \frac{1}{\pi}\left[\infty - (-\infty)\right] - 1 = \infty - 1 = \infty.$$

Hence, the second moment is infinite, and the variance does not exist since μ_1' (i.e., the mean of $\tilde{x}$) does not exist. Note that a Cauchy random variable does not belong to the L^1 space and, therefore, cannot belong to the L^2 space either. In fact, in the class notes we have already shown that if $\mathrm{E}(\tilde{x})$ does not exists (i.e., it is of the type $\infty - \infty$), then $\mathrm{E}(\tilde{x}^2) = \infty$.

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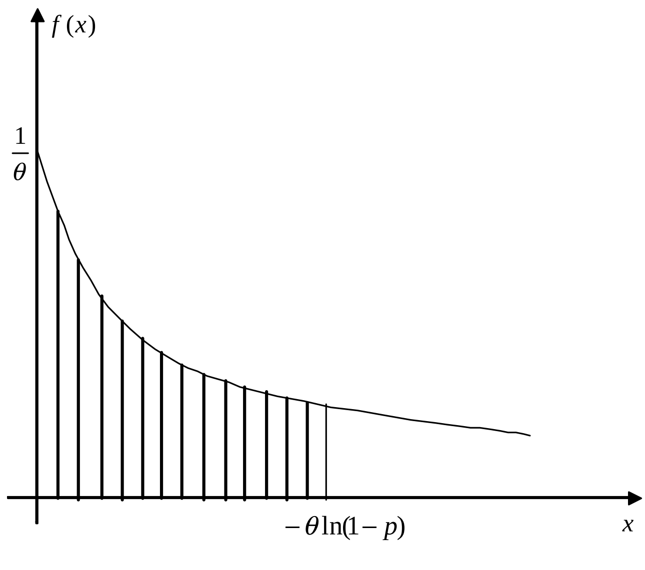
9. $\tilde{x} \sim f(x) = \frac{1}{\theta}e^{-x/\theta}$, for $x > 0$, with $\theta > 0$, and $f(x) = 0$ elsewhere.



$$\begin{aligned}
P\left\{\tilde{x} < -\theta\ln(1-p)\right\} &= \int\limits_0^{-\theta\ln(1-p)} \frac{1}{\theta}e^{-x/\theta}dx = \frac{1}{\theta}\left[\frac{e^{-x/\theta}}{-1/\theta}\right]_0^{-\theta\ln(1-p)} \\
&= -\left[e^{-x/\theta}\right]_0^{-\theta\ln(1-p)} = 1 - e^{-\frac{1}{\theta}[-\theta\ln(1-p)]} = 1 - e^{\ln(1-p)} \\
&= 1 - (1-p) = p.
\end{aligned}$$

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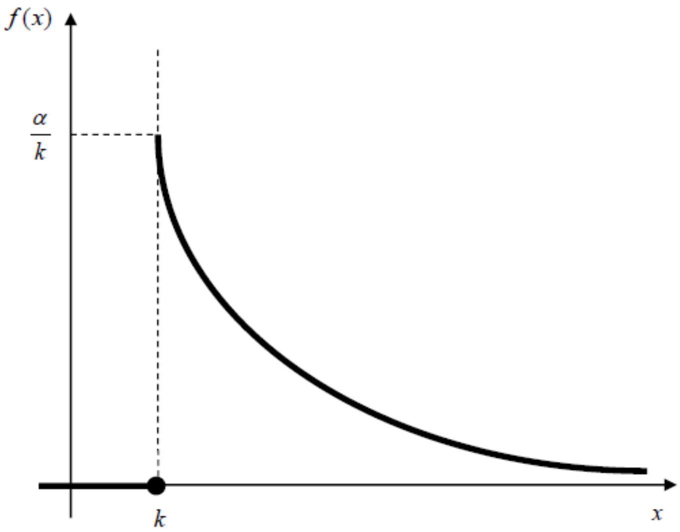
10.

$$f(x) = \begin{cases} \dfrac{\alpha k^\alpha}{x^{\alpha+1}} & \text{for } x > k \\ 0 & \text{elsewhere} \end{cases}$$

with $\alpha > 0$ and $k > 0$.



Note that the area under the density equals 1:

$$\int_k^\infty \frac{\alpha k^\alpha}{x^{\alpha+1}} dx = k^\alpha \int_k^\infty \alpha x^{-\alpha-1} dx = k^\alpha \left[\alpha \frac{x^{-\alpha}}{(-\alpha)}\right]_k^\infty = -k^\alpha \left[x^{-\alpha}\right]_k^\infty = -k^\alpha \left[0 - k^{-\alpha}\right] = 1$$

$$\mu'_r = \int_k^\infty x^r \frac{\alpha k^\alpha}{x^{\alpha+1}} dx = \alpha k^\alpha \int_k^\infty x^{r-\alpha-1} dx = \alpha k^\alpha \left[\frac{x^{r-\alpha}}{r-\alpha}\right]_k^\infty = \frac{\alpha k^\alpha}{r-\alpha} \left[x^{r-\alpha}\right]_k^\infty,$$

which is finite if and only if $r - \alpha < 0$. Hence μ'_r is finite and equal to $\dfrac{\alpha k^r}{\alpha - r}$ if and only if $r < \alpha$.

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11. (a) $a = c = 0, b > 0, d > -b$.

$$\frac{f'(x)}{f(x)} = \frac{d-x}{bx}$$

$$\ln f(x) = \int \frac{d}{bx}dx - \int \frac{1}{b}dx = \frac{d}{b}\ln x - \frac{x}{b} + K$$

$$f(x) = e^{\frac{d}{b}\ln x - \frac{x}{b} + K} = De^{\frac{d}{b}\ln x}e^{-\frac{x}{b}} = Dx^{\frac{d}{b}}e^{-\frac{x}{b}},$$

which is the density of the gamma distribution for $x > 0$, with $\frac{d}{b} > -1$, $\frac{1}{b} > 0$, and $D = e^K$ is the constant which makes the area under the density equal to one.

(b) $a = c = d = 0, b > 0$.

$$\frac{f'(x)}{f(x)} = \frac{-x}{bx} = -\frac{1}{b}$$

$$\ln f(x) = -\frac{x}{b} + K \rightarrow f(x) = e^{-\frac{x}{b}+K} \rightarrow f(x) = De^{-\frac{x}{b}},$$

which is the density of the exponential distribution for $x > 0$, with $b > 0$.

(c) $a = 0, b = -c, \frac{d-1}{b} < 1, \frac{d}{b} > -1$.

$$\frac{f'(x)}{f(x)} = \frac{d-x}{bx - bx^2} = \frac{d-x}{bx(1-x)}$$

$$\rightarrow \ \ln f(x) = \frac{1}{b}\int \frac{d-x}{x(1-x)}dx$$

$$\begin{aligned}
\ln f(x) &= \frac{1}{b}\int \left[\frac{d}{x} + \frac{d-1}{1-x}\right] dx = \frac{1}{b}\left[d\ln x - (d-1)\ln(1-x)\right] + K \\
f(x) &= \exp\left(\ln x^{\frac{d}{b}} + \ln (1-x)^{\frac{1-d}{b}} + K\right). \\
f(x) &= Dx^{\frac{d}{b}}(1-x)^{\frac{1-d}{b}},
\end{aligned}$$

which is the density of the Beta distribution for $x > 0$ if each exponent is greater than -1.

$\frac{d}{b} > -1 \Longrightarrow$ the exponent of x is greater than -1, and

$\frac{d-1}{b} < 1 \Longrightarrow$the exponent of $(1-x)$ is greater than -1.

(d) $b = c = 0; a > 0$.

$$\frac{f'(x)}{f(x)} = \frac{d-x}{a}$$

$$\begin{aligned}
\ln f(x) &= \frac{1}{a}\int (d-x)\,dx \\
\ln f(x) &= \frac{1}{a}\frac{(d-x)^2}{(-2)} + K \\
f(x) &= e^{-\frac{1}{2a}(x-d)^2+K} \\
f(x) &= D\exp\left(-\frac{1}{2}\left[\frac{x-d}{\sqrt{a}}\right]^2\right),
\end{aligned}$$

which is the normal density when $d = \mu$, $a = \sigma^2 > 0$, and $D = \frac{1}{\sqrt{2\pi}\sqrt{a}} = \frac{1}{\sqrt{2\pi}\sigma}$.

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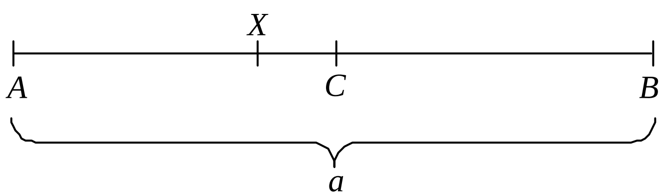
12.



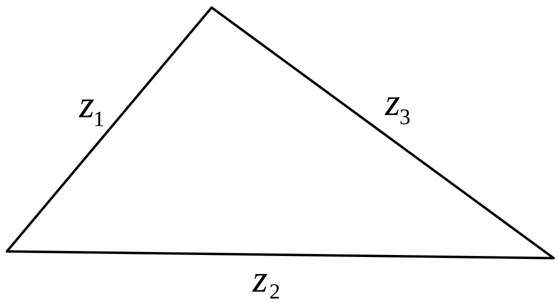
length$(AB) = a$.

length(AC) =length$(CB) = \frac{a}{2}$.

$\tilde{x}$ =length(AX).

$\tilde{x} \sim f(x) = \frac{1}{a}$, for $0 < x < a$

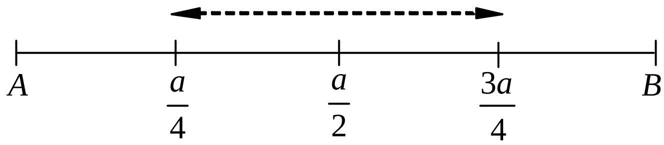
$\pi = P\left\{\left(\tilde{x}, a-\tilde{x}, \frac{a}{2}\right) \text{ will form a triangle}\right\}$



The three segments with lengths $\{z_1, z_2, z_3\}$ will form a triangle if and only if $(z_1+z_2) > z_3,\ (z_1+z_3) > z_2,$ and $(z_2+z_3) > z_1$.

$$\begin{aligned}
\pi &= P\left\{\left\{\tilde{x}+(a-\tilde{x}) > \frac{a}{2}\right\} \cap \left\{\tilde{x}+\frac{a}{2} > (a-\tilde{x})\right\} \cap \left\{(a-\tilde{x})+\frac{a}{2} > \tilde{x}\right\}\right\} \\
&= P\left\{\underbrace{\left\{a > \frac{a}{2}\right\}}_{\text{certain}} \cap \left\{2\tilde{x} > \frac{a}{2}\right\} \cap \left\{\frac{3a}{2} > 2\tilde{x}\right\}\right\} \\
&= P\left\{\left\{\tilde{x} > \frac{a}{4}\right\} \cap \left\{\tilde{x} < \frac{3a}{4}\right\}\right\} = P\left\{\frac{a}{4} < \tilde{x} < \frac{3a}{4}\right\} \\
&= \int_{\frac{a}{4}}^{\frac{3}{4}a} \frac{1}{a} dx = \left[\frac{x}{a}\right]_{\frac{a}{4}}^{\frac{3}{4}a} = \frac{1}{a}\left[\frac{3a}{4} - \frac{a}{4}\right] = \frac{1}{2}
\end{aligned}$$

Therefore, if X lies in the interval $\left(\frac{a}{4}, \frac{3a}{4}\right)$, then a triangle can be formed:



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13. $\tilde{x} \sim f(x) = \frac{1}{40}e^{-x/40}$, for $x > 0$ and $f(x) = 0$, otherwise.

$$F(x) = P\left\{\tilde{x} \leq x\right\} = \int_0^x \frac{1}{40}e^{-z/40}dt = \frac{1}{40}\left[\frac{e^{-z/40}}{-1/40}\right]_0^x = -\left[e^{-z/40}\right]_0^x = 1 - e^{-x/40},$$

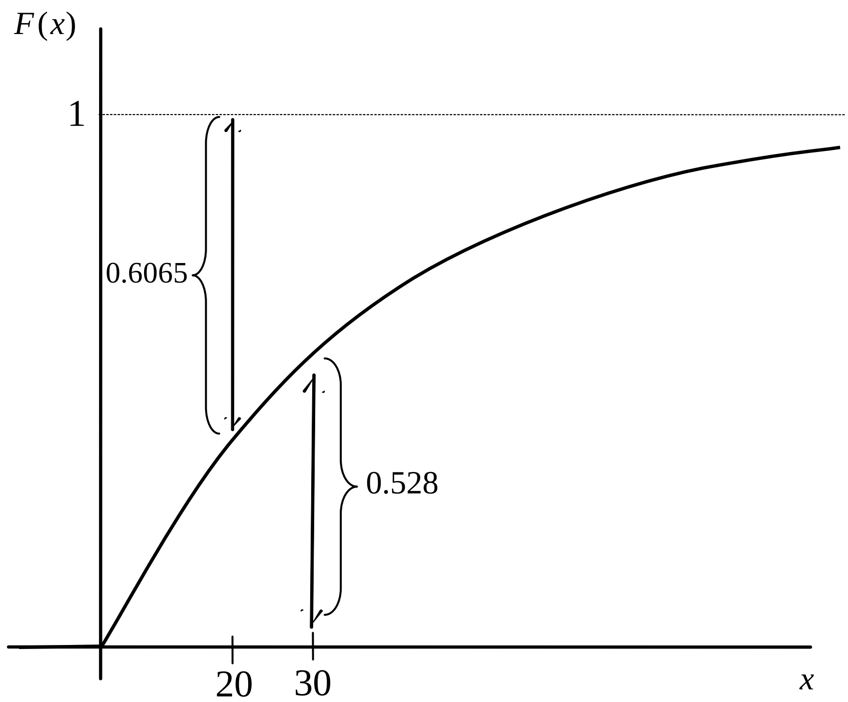
for $x > 0$ and $F(x) = 0$, otherwise.

(a)

$$P\left\{\tilde{x} \geq 20\right\} = 1 - F(20) = 1 - \left[1 - e^{-20/40}\right] = e^{-1/2} = 0.6065.$$

(b)

$$P\left\{\tilde{x} \leq 30\right\} = F(30) = 1 - e^{-30/40} = 1 - 0.472 = 0.528.$$



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14. $\tilde{x} \sim \mathrm{N}(\mu, \sigma^2)$ so that $M_{\tilde{x}}(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. Let $w \equiv \mu t + \frac{1}{2}\sigma^2 t^2$.

(1)

$$M_{\tilde{x}}(t) = e^w$$

(2)

$$M'_{\tilde{x}}(t) = e^w w'$$

(3)

$$M''_{\tilde{x}}(t) = e^w w' w' + e^w w'' = e^w \left\{ (w')^2 + w'' \right\}$$

(4)

$$
\begin{aligned}
M_{\tilde{x}}'''(t) &= e^w w' \left[(w')^2 + w'' \right] + e^w \left[2\,(w'w'') + w''' \right] \\
&= e^w \left\{ (w')^3 + w'w'' + 2w'w'' + w''' \right\} = e^w \left\{ (w')^3 + 3w'w'' + w''' \right\}
\end{aligned}
$$

(5)

$$
\begin{aligned}
M_{\tilde{x}}^{IV}(t) &= e^w w' \left\{ (w')^3 + 3w'w'' + w''' \right\} + e^w \left\{ 3\,(w')^2\, w'' + 3\,(w'')^2 + 3w'w''' + w^{IV} \right\} \\
&= e^w \left\{ (w')^4 + 6\,(w')^2\, w'' + 4w'w''' + 3\,(w'')^2 + w^{IV} \right\}
\end{aligned}
$$

(6) $w(t) = \mu t + \frac{1}{2}\sigma^2 t^2 \rightarrow w(0) = 0$

(7) $w'(t) = \mu + \sigma^2 t \rightarrow w'(0) = \mu$

(8) $w''(t) = \sigma^2 \rightarrow w''(0) = \sigma^2$

(9) $w'''(t) = w^{IV}(t) = 0 \rightarrow w'''(0) = w^{IV}(0) = 0$

$$
\begin{aligned}
\mu_1' &= M_{\tilde{x}}'(0) = e^0 \mu = \mu \\
\mu_2' &= M_{\tilde{x}}''(0) = e^0 \left\{ \mu^2 + \sigma^2 \right\} = \mu^2 + \sigma^2 \\
\mu_3' &= M_{\tilde{x}}'''(0) = e^0 \left\{ \mu^3 + 3\mu\sigma^2 + 0 \right\} = \mu^3 + 3\mu\sigma^2 \\
\mu_4' &= M_{\tilde{x}}^{IV}(0) = e^0 \left\{ \mu^4 + 6\mu^2\sigma^2 + 4\,(\mu \cdot 0) + 3\left(\sigma^2\right)^2 + 0 \right\} = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.
\end{aligned}
$$

Using the formula

$$
\mu_k = \sum_{n=0}^{k} \binom{k}{n} \mu_n' \left[-\mu \right]^{k-n},
$$

we get (check it!)

$$\begin{aligned}
\mu_3 &= \mu_3' - 3\mu\mu_2' + 2\mu^3 = \left(\mu^3 + 3\mu\sigma^2\right) - 3\mu\left(\sigma^2 + \mu^2\right) + 2\mu^3 \\
&= \mu^3\left(1 - 3 + 2\right) + \mu\sigma^2\left(3 - 3\right) = 0 \\
\mu_4 &= \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4 \\
&= \left(\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4\right) - 4\mu\left(\mu^3 + 3\mu\sigma^2\right) + 6\mu^2\left(\mu^2 + \sigma^2\right) - 3\mu^4 \\
&= \mu^4\left[1 - 4 + 6 - 3\right] + \mu^2\sigma^2\left[6 - 12 + 6\right] + 3\sigma^4 = 3\sigma^4.
\end{aligned}$$

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15. $\tilde{x} \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$$P\left\{\mu - k\sigma < \tilde{x} < \mu + k\sigma\right\} = P\left\{-k < \tilde{z} < k\right\}, \text{where } \tilde{z} = \frac{\tilde{x} - \mu}{\sigma} \sim \mathrm{N}\left(0, 1\right).$$

(a)

$$P\left\{\mu - \sigma < \tilde{x} < \mu + \sigma\right\} = P\left\{-1 < \tilde{z} < 1\right\} = 2 \cdot 0.3413 = 0.6826$$

(b)

$$P\left\{\mu - 2\sigma < \tilde{x} < \mu + 2\sigma\right\} = P\left\{-2 < \tilde{z} < 2\right\} = 2 \cdot 0.4772 = 0.9544$$

(c)

$$P\left\{\mu - 3\sigma < \tilde{x} < \mu + 3\sigma\right\} = P\left\{-3 < \tilde{z} < 3\right\} = 3 \cdot 0.4987 = 0.9974$$

(d)

$$P\left\{\mu - 4\sigma < \tilde{x} < \mu + 4\sigma\right\} = P\left\{-4 < \tilde{z} < 4\right\} = 2 \cdot 0.49997 = 0.99994$$

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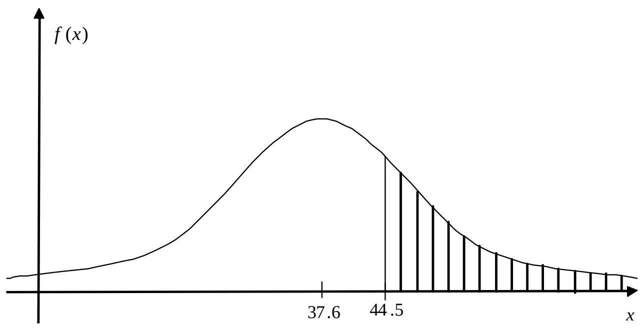
16. $\tilde{x} \sim \mathrm{N}\left(37.6, (4.6)^2\right)$

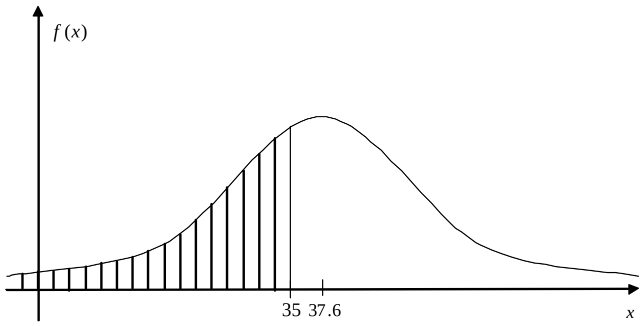
(a)

$$\begin{aligned} P\left\{\tilde{x} > 44.5\right\} &= 0.5 - P\left\{0 < \tilde{z} < \frac{44.5 - 37.6}{4.6}\right\} \\ &= 0.5 - P\left\{0 < \tilde{z} < 1.5\right\} = 0.5 - 0.4332 = 0.0668. \end{aligned}$$



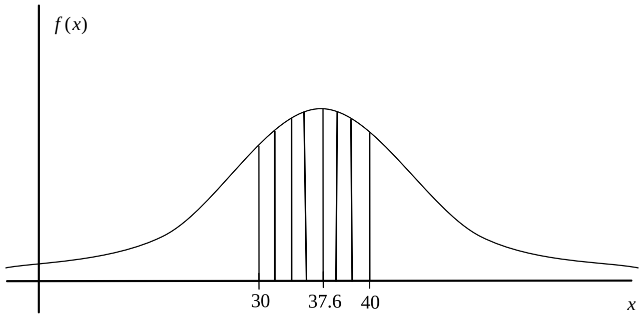
(b)

$$\begin{aligned} P\left\{\tilde{x} < 35\right\} &= 0.5 - P\left\{\frac{35 - 37.6}{4.6} < \tilde{z} < 0\right\} = 0.5 - P\left\{0 < \tilde{z} < \frac{37.6 - 35}{4.6}\right\} \\ &= 0.5 - P\left\{0 < \tilde{z} < 0.565\right\} = 0.5 - 0.214 = 0.286. \end{aligned}$$



(c)

$$
\begin{aligned}
P\{30 < \tilde{x} < 40\} &= P\left\{\frac{30-37.6}{4.6} < \tilde{z} < 0\right\} + P\left\{0 < \tilde{z} < \frac{40-37.6}{4.6}\right\} \\
&= P\left\{0 < \tilde{z} < \frac{37.6-30}{4.6}\right\} + P\left\{0 < \tilde{z} < \frac{40-37.6}{4.6}\right\} \\
&= P\{0 < \tilde{z} < 1.625\} + P\{0 < \tilde{z} < 0.522\} = 0.4507 + 0.1992 \\
&= 0.6499 \approx 0.65.
\end{aligned}
$$



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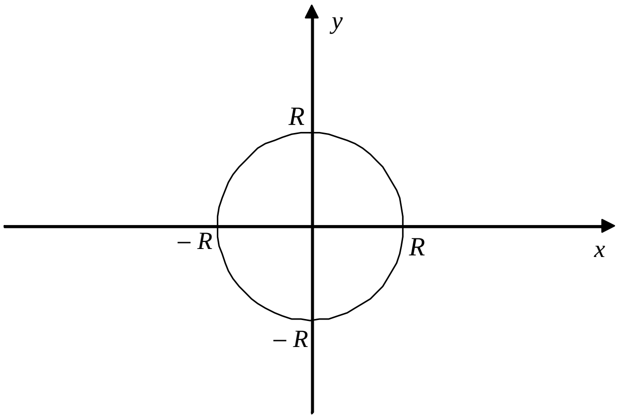
17. $(\tilde{x}, \tilde{y}) \sim \mathrm{MN}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right\}$, with $\sigma_1^2 = \sigma_2^2 = \sigma$ and $\sigma_{12} = 0$ for the circular case.

$$f(x,y) = \frac{1}{(2\pi)\,\sigma\sigma} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma^2}+\frac{y^2}{\sigma^2}\right]}, \quad \text{for } -\infty < x < \infty, \quad -\infty < y < \infty$$

$$P\left\{\tilde{x}^2 + \tilde{y}^2 < R^2\right\} = \int\limits_C \frac{1}{(2\pi)\,\sigma^2} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma^2}+\frac{y^2}{\sigma^2}\right]} d\,(x,y)\,,$$

where $C = \{(x, y \in \mathbb{R}^2 \,| x^2 + y^2 < R^2\}\,,$



This integration is best carried out by transforming variables to work in polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Longrightarrow r^2 = x^2 + y^2$$

$$P\left\{\tilde{x}^2 + \tilde{y}^2 < R^2\right\} = P\left\{\widetilde{r} < R\right\} = 4 \int\limits_0^{\pi/2} \int\limits_0^{R} \frac{1}{(2\pi)\,\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r dr d\theta$$

<u>Remark:</u> r is the absolute value of the determinant of the Jacobian of the transformation.

Let us now make the change of variable $u = r^2 > 0 \Longleftrightarrow r = u^{1/2} > 0.$

Thus, $\dfrac{dr}{du} = \dfrac{1}{2}u^{-1/2} > 0$

$$P\left\{\tilde{x}^2 + \tilde{y}^2 < R^2\right\} = \frac{2}{\pi\sigma^2}\int_0^{\pi/2}\int_0^{R^2} e^{-\frac{u}{2\sigma^2}} u^{1/2}\frac{1}{2}u^{-1/2} du d\theta = \frac{2}{\pi\sigma^2}\int_0^{\pi/2}\left[\int_0^{R^2} e^{-\frac{u}{2\sigma^2}}\frac{1}{2}du\right] d\theta$$

$$= \frac{1}{\pi\sigma^2}\left[-\frac{e^{-\frac{u}{2\sigma^2}}}{\frac{1}{2\sigma^2}}\right]_0^{R^2}\int_0^{\pi/2} d\theta = \frac{2}{\pi}\left(1 - e^{-\frac{R^2}{2\sigma^2}}\right)[\theta]_0^{\pi/2} = \frac{2}{\pi}\left(1 - e^{-\frac{R^2}{2\sigma^2}}\right)\frac{\pi}{2} = 1 - e^{-\frac{R^2}{2\sigma^2}}$$

(a) For $\sigma = 12$ and $R = 6$,

$$P\left\{\tilde{x}^2 + \tilde{y}^2 < 36\right\} = 1 - e^{-\frac{36}{288}} = 1 - e^{-0.125} = 1 - 0.8825 = 0.1175$$

(b)

$$\begin{aligned} 0.8 &= 1 - e^{-\frac{c^2}{2\sigma^2}} \rightarrow e^{-\frac{c^2}{288}} = 0.2 \rightarrow -\frac{c^2}{288} = \ln(0.2), \\ c^2 &= -288\ln(0.2) = -288(-1.609) = 463.52, \end{aligned}$$

$$c = 21.53$$

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18. (a) $\tilde{y} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$, $\qquad y \sim \mathrm{N}(\mathrm{E}(\tilde{y}), \mathrm{Var}(\tilde{y}))$

$$\begin{aligned} \mathrm{E}(\tilde{y}) &= \mu_1 + \mu_2 + \mu_3 = 2 \\ \mathrm{Var}(\tilde{y}) &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23} = 20 \\ P\{\tilde{y} \le 3\} &= P\left\{\tilde{z} \le \frac{3-2}{\sqrt{20}}\right\} = P\{\tilde{z} \le 0.223\} = \mathrm{N}(0.223) \\ &= 0.5 + P\{0 \le \tilde{z} \le 0.223\} = 0.5 + 0.088 = 0.588. \end{aligned}$$

(b) Recall that, if we partition the mean vector and the variance covariance matrix, we get

$$\mu = \begin{pmatrix} 1 \\ 0 \\ - \\ 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 9 & 1 & | & 1 \\ 1 & 3 & | & 1 \\ - & - & - & - \\ 1 & 1 & | & 2 \end{pmatrix}.$$

$$\begin{aligned} \mathrm{E}\left((\tilde{x}_1, \tilde{x}_2)^\top | \tilde{x}_3 = 0 \right) &= \binom{1}{0} + \binom{1}{1} 2^{-1} (0-1) = \binom{\frac{1}{2}}{-\frac{1}{2}} \equiv \hat{\mu} \\ \mathrm{Var}\left((\tilde{x}_1, \tilde{x}_2)^\top | \tilde{x}_3 = 0 \right) &= \begin{pmatrix} 9 & 1 \\ 1 & 3 \end{pmatrix} - \binom{1}{1} 2^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{17}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix} \equiv \hat{\Sigma} \end{aligned}$$

$$f_{\tilde{x}_1, \tilde{x}_2 | \tilde{x}_3} (x_1, x_2 | 0) = \frac{1}{(2\pi) \left| \hat{\Sigma} \right|^{\frac{1}{2}}} \exp \left(-\frac{1}{2} (x - \hat{\mu})^\top \hat{\Sigma}^{-1} (x - \hat{\mu}) \right)$$

where $x^\top = (x_1, x_2)^\top \in \mathbb{R}^2$, $\left| \hat{\Sigma} \right|^{\frac{1}{2}} = (21)^{1/2} = 4.5826$, and $\hat{\Sigma}^{-1} = \begin{pmatrix} \frac{5}{42} & -\frac{1}{42} \\ -\frac{1}{42} & \frac{17}{42} \end{pmatrix}$.

(c) Note that the vector $(\tilde{z}, \tilde{x}_3)^\top$ is multivariate normal since it is an affine transformation of the multivariate normal vector $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^\top$:

$$\begin{pmatrix} \tilde{z} \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}.$$

Then, we use the fact that, under multivariate normality and $\mathrm{Cov}(\tilde{z}, \tilde{x}_3) = 0 \Leftrightarrow$

$\tilde{z}$ and $\tilde{x}_3$ are independent normal random variables.

$$\operatorname{Cov}(\tilde{z}, \tilde{x}_3) = \operatorname{E}(\tilde{z} \cdot \tilde{x}_3) - \operatorname{E}(\tilde{z}) \cdot \operatorname{E}(\tilde{x}_3)$$

$$\begin{aligned}
\operatorname{E}(\tilde{z} \cdot \tilde{x}_3) &= \operatorname{E}((a\tilde{x}_1 + \tilde{x}_2) \cdot \tilde{x}_3) = a\operatorname{E}(\tilde{x}_1 \cdot \tilde{x}_3) + \operatorname{E}(\tilde{x}_2 \cdot \tilde{x}_3) \\
&= a(\sigma_{13} + \mu_1\mu_3) + (\sigma_{23} + \mu_2\mu_3) = a(1+1) + (1+0) \\
&= 2a + 1
\end{aligned}$$

$$\operatorname{E}(\tilde{z}) = \operatorname{E}(a\tilde{x}_1 + \tilde{x}_2) = a\mu_1 + \mu_2 = a$$

$$\operatorname{E}(\tilde{x}_3) = \mu_3 = 1$$

$\Longrightarrow$

$$\operatorname{Cov}(\tilde{z}, \tilde{x}_3) = 2a + 1 - a = a + 1 = 0 \Leftrightarrow a = -1$$

Alternatively,

$$\begin{aligned}
\operatorname{Cov}(\tilde{z}, \tilde{x}_3) &= \operatorname{Cov}(a\tilde{x}_1 + \tilde{x}_2, \tilde{x}_3) = a\operatorname{Cov}(\tilde{x}_1, \tilde{x}_3) + \operatorname{Cov}(\tilde{x}_2, \tilde{x}_3) \\
&= a + 1 = 0 \Leftrightarrow a = -1
\end{aligned}$$

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19. Moment-generating function of $\tilde{x}$ (note that k is a strictly positive integer):

$$M_{\tilde{x}}(t) = \sum_{x=k}^{\infty} e^{tx} b^*(x; k, \theta) = \sum_{x=k}^{\infty} e^{tx} \binom{x-1}{k-1} \theta^k (1-\theta)^{x-k}$$

Make the change of variable $y = x - k$ so that

$$M_{\tilde{x}}(t) = \sum_{y=0}^{\infty} e^{t(y+k)} \binom{y+k-1}{k-1} \theta^k (1-\theta)^y = \left(\theta e^t\right)^k \sum_{y=0}^{\infty} \binom{k+y-1}{k-1} \left[(1-\theta)e^t\right]^y$$

From part (d) of Exercise 6 of List 1:

$$\binom{-k}{y} = (-1)^y \binom{k+y-1}{k-1} \Leftrightarrow \binom{k+y-1}{k-1} = (-1)^y \binom{-k}{y}$$

$\Longrightarrow$

$$\begin{aligned} M_{\tilde{x}}(t) &= \left(\theta e^t\right)^k \sum_{y=0}^{\infty} (-1)^y \binom{-k}{y} \left[(1-\theta)e^t\right]^y \\ &= \left(\theta e^t\right)^k \sum_{y=0}^{\infty} \binom{-k}{y} \left[-(1-\theta)e^t\right]^y. \end{aligned}$$

From Part (f) of Exercise 6 of List 1:

$$(1+z)^\alpha = \sum_{y=0}^{\infty} \binom{\alpha}{y} z^y, \text{ for all } \alpha \text{ and } z \text{ real with } |z| < 1.$$

Note that $(1-\theta)e^t < 1$ when $t < -\ln(1-\theta)$. Therefore, making $\alpha = -k$ and $z = -(1-\theta)e^t$, we get

$$M_{\tilde{x}}(t) = \left(\theta e^t\right)^k \left(1 - (1-\theta)e^t\right)^{-k} = \left(\frac{\theta e^t}{1-(1-\theta)e^t}\right)^k, \quad \text{for } t < -\ln(1-\theta).$$

Alternatively, we can obtain the moment-generating function of the Pascal (or negative binomial) distribution by just observing that a Pascal random variable $\tilde{x}$ with the parameters k and θ is the sum of k geometric random variables $\tilde{x}_i$ with the parameter θ, $\tilde{x} = \sum_{i=1}^{k} \tilde{x}_i$. This is so because $\tilde{x}_i$ is the number of independent trials needed to get one success and, thus, $\sum_{i=1}^{k} \tilde{x}_i$ is the number of trials needed to get k successes. Therefore, since the random variables $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_k$ are independent, $M_{\tilde{x}}(t) = \prod_{i=1}^{k} M_{\tilde{x}_i}(t)$. Moreover, from Exercise 5 of this list we know that the moment-generating function of a geometric random variable $\tilde{x}_i$ is

$$M_{\tilde{x}_i}(t) = \frac{\theta e^t}{1-(1-\theta)e^t}, \quad \text{for } t < -\ln(1-\theta), \quad i = 1, 2, ..., k.$$

Therefore,

$$M_{\tilde{x}}(t) = \prod_{i=1}^{k} M_{\tilde{x}_i}(t) = \left(\frac{\theta e^t}{1-(1-\theta)e^t}\right)^k, \quad \text{for } t < -\ln(1-\theta).$$

We can now compute the mean of the Pascal distribution:

$$\begin{aligned} M'_{\tilde{x}}(t) &= k\left(\theta e^t\right)^{k-1}\theta e^t\left(1-(1-\theta)\,e^t\right)^{-k} \\ &\quad + \left(\theta e^t\right)^k(-k)\left(1-(1-\theta)\,e^t\right)^{-k-1}\left(-(1-\theta)\,e^t\right). \end{aligned}$$

$$\begin{aligned} \mu &= M'_{\tilde{x}}(0) = k\theta^{k-1}\theta\theta^{-k} + \theta^k(-k)\,\theta^{-k-1}\left(-(1-\theta)\right) \\ &= k + k\theta^{-1}(1-\theta) = k\left(1+\frac{1-\theta}{\theta}\right) = \frac{k}{\theta}. \end{aligned}$$

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20. (a) Geometric distribution

$\theta = 0.51$, $x = 4$.

$$\begin{aligned} g(x,\theta) &= \theta(1-\theta)^{x-1} \\ &= (0.51)(0.49)^3 = 0.06 \end{aligned}$$

(b) Pascal distribution

$\theta = 0.51,\ k = 2,\ x = 4.$

$$b^*(x, k, \theta) = \binom{x-1}{k-1}\theta^k(1-\theta)^{x-k} = \binom{3}{1}(0.51)^2(0.49)^2 = 0.187$$

(c)

$$\mathrm{E}(\tilde{x}) = \frac{2}{\theta} = \frac{2}{0.51} = 3.92157$$

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21. (a)

$$f_{\tilde{x}}(x) = \begin{cases} \dfrac{1}{\pi}e^{-\frac{1}{2}x^2}\int\limits_0^\infty e^{-\frac{1}{2}y^2}dy & \text{if } x \le 0 \\ \\ \dfrac{1}{\pi}e^{-\frac{1}{2}x^2}\int\limits_{-\infty}^0 e^{-\frac{1}{2}y^2}dy & \text{if } x > 0 \end{cases}$$

Since

$$\int\limits_0^\infty e^{-\frac{1}{2}t^2}dt = \int\limits_{-\infty}^0 e^{-\frac{1}{2}t^2}dt = \sqrt{\frac{\pi}{2}},$$

we have

$$f_{\tilde{x}}(x) = \frac{1}{\pi}e^{-\frac{1}{2}x^2}\sqrt{\frac{\pi}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \quad \text{for } -\infty < x < \infty$$

Equivalently,

$$f_{\tilde{y}}(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \quad \text{for } -\infty < y < \infty$$

$\Longrightarrow \tilde{x} \sim \mathrm{N}(0,1)$ and $\tilde{y} \sim \mathrm{N}(0,1)$.

(b) Let $x > 0$ and $y > 0 \Longrightarrow f(x,y) = 0$ and $f_{\tilde{x}}(x) > 0,\ f_{\tilde{y}}(y) > 0.$

Therefore,

$$\left.\begin{array}{c} f(x,y) \neq f_{\tilde{x}}(x) \cdot f_{\tilde{y}}(y) \\ \text{for a set with positive Lebesgue measure} \end{array}\right\} \Longleftrightarrow \tilde{x} \text{ and } \tilde{y} \text{ are not independent.}$$

(c) $(\tilde{x}, \tilde{y})$ is not multivariate normally distributed, since $f(x,y) = 0$ in some set with positive Lebesgue measure, while a multivariate normal density is positive for all $(x,y) \in \mathbb{R}^2$.

While multivariate normality of $(\tilde{x}, \tilde{y})$ implies normality of both $\tilde{x}$ and $\tilde{y}$, normality of both $\tilde{x}$ and $\tilde{y}$ does not imply multivariate normality of the vector $(\tilde{x}, \tilde{y})$, as can be seen in this example.

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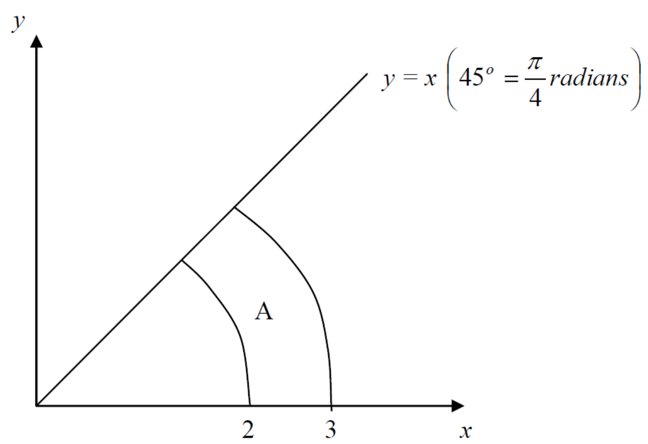
22. (a) The region A is



By changing to polar coordinates,

$$\int_A k(x^2+y^2)^{1/2} d(x,y) = k\int_0^{\frac{\pi}{4}}\int_2^3 r \cdot r dr d\theta = k\left(\int_2^3 r^2 dr\right)\left(\int_0^{\frac{\pi}{4}} d\theta\right) = k \cdot \frac{19\pi}{12}.$$

We have to choose k such that the above integral equals 1. Thus, $k = \dfrac{12}{19\pi}$.

(b) Similarly,

$$\int_A k(x^2+y^2)^{1/2} d(x,y) = k \int_0^{\pi}\int_3^4 r \cdot r dr d\theta = k\left(\int_3^4 r^2 dr\right)\left(\int_0^{\pi} d\theta\right) = k \cdot \frac{37\pi}{3} = 1.$$

Thus, $k = \dfrac{3}{37\pi}$.

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23. (a) Using the change to polar coordinates,

$$\begin{cases} x = s\cos\theta \\ y = s\sin\theta \end{cases} \Longrightarrow s^2 = x^2 + y^2,$$

and noticing that s is the absolute value of the determinant of the Jacobian of the transformation, we get

$$\int_C d(x,y) = \int_C 1 d(x,y) = \left[\int_0^r s ds\right] \cdot \left[\int_0^{2\pi} d\theta\right] = \frac{r^2}{2} \cdot 2\pi = \pi r^2.$$

This double integral gives us the area of a circle since we are integrating the constant 1 over the circle so that the volume is equal to the area.

(b) Using the equation of a circumference, $x^2 + y^2 = r^2$, we define $y = f(x) = (r^2 - x^2)^{1/2}$, and we find the area of a quarter of circle multiplied by 4 as folllows:

$$4\int_0^r \left(r^2 - x^2\right)^{1/2} dx = 4\int_{(0,r)} \left(r^2 - x^2\right)^{1/2} dx = ...$$

We now make the change of variable $x = g(\alpha) = r \sin\alpha$ for $x \in (0, r) \iff \alpha \in (0, \pi/2) \iff \sin\alpha \in (0, 1)$. Then

$$\frac{dx}{d\alpha} = g'(\alpha) = r\cos\alpha > 0.$$

The inequality comes from the fact that $\alpha \in (0, \pi/2)$, which implies that $\cos\alpha > 0$.

$$\ldots = 4 \int\limits_{(0,\pi/2)} \Big(r^2 - \underbrace{r^2 \sin^2\alpha}_{x^2}\Big)^{1/2} |g'(\alpha)|\, d\alpha = 4 \int\limits_{(0,\pi/2)} \left(r^2 - r^2 \sin^2\alpha\right)^{1/2} r\,(\cos\alpha)\, d\alpha$$

$$= 4r^2 \int\limits_{(0,\pi/2)} \underbrace{\left(1 - \sin^2\alpha\right)^{1/2}}_{\cos^2\alpha} (\cos\alpha)\, d\alpha = 4r^2 \int_0^{\pi/2} \left(\cos^2\alpha\right) d\alpha$$

$$= 4r^2 \int_0^{\pi/2} \frac{1 + \cos(2\alpha)}{2} d\alpha = 4r^2 \left(\left[\frac{\alpha}{2}\right]_0^{\pi/2} + \underbrace{\left[\frac{\sin(2\alpha)}{4}\right]_0^{\pi/2}}_{0} \right) = 4r^2 \frac{\pi}{4} = \pi r^2.$$

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24. Changing to polar coordinates:

$$\frac{1}{\pi} \left[\int_0^6 e^{-r^2} r dr\right] \left[\int_0^{2\pi} d\theta\right] = \frac{1}{\pi} \cdot \frac{1}{2} \left[-e^{-r^2}\right]_0^6 \cdot (2\pi) = 1 - e^{-36}$$

Thus, the volume equals to $1 - e^{-36}$.

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25. (a) Multinomial distribution, with parameters $n = 10$, $\theta_1 = 0.45$,

$\theta_2 = 0.43$, $\theta_3 = 0.08$, $\theta_4 = 0.04$, so

$$f(3,4,2,1;n,\theta_1,\theta_2,\theta_3,\theta_4) = \frac{10!}{3!4!2!1!}(0.45)^3(0.43)^4(0.08)^2(0.04)^1 = 0.01052.$$

(b) We have 7 possible disjoint events satisfying the desired property:

A = At least one of them is 0,

B = All of them are AB,

C = Two of them are AB and the other is neither 0 nor AB (that is, the other is either A or B),

D = Two of them are A and the other is AB,

E = Two of them are B and the other is AB,

F = All of them are A,

G = All of them are B.

$$P(A) = 1 - \frac{3!}{0!3!}(0.55)^3,$$

$$P(B) = \frac{3!}{0!3!}(0.04)^3,$$

$$P(C) = \frac{3!}{1!2!}(0.04)^2\,0.51,$$

$$P(D) = \frac{3!}{1!2!}(0.43)^2\,0.04,$$

$$P(E) = \frac{3!}{1!2!}(0.08)^2\,0.04,$$

$$P(F) = \frac{3!}{0!3!}(0.43)^3,$$

$$P(G) = \frac{3!}{0!3!}(0.08)^3.$$

Therefore,

$$P\{\text{success}\} = P(A)+P(B)+P(C)+P(D)+P(E)+P(F)+P(G) = 0.93911.$$

(c) Geometric distribution with parameter $\theta = 0.04$,

$$P\{\tilde{x} = 5\} = (0.96)^4 \cdot 0.04 = 0.033974.$$

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26. (a) Using the multivariate hypergeometric distribution:

$$\frac{\binom{7}{3}\binom{3}{2}\binom{2}{0}}{\binom{12}{5}} = 0.1326.$$

(b) Using the multinomial distribution:

$$\frac{5!}{3!2!0!}\left(\frac{7}{12}\right)^3\left(\frac{3}{12}\right)^2\left(\frac{2}{12}\right)^0 = 0.1241.$$

(c) Obvious. Since the extractions are with replacement in part (b), the outcome of every extraction does not affect the probabilities of the outcomes of the other extractions. Therefore, we are facing 5 independent and identical trials. This is not so in part (a).

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27. (a) Write $M = \left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}dy\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy.$

Then, changing to polar coordinates,

$$M=\int_0^{2\pi}\int_0^{\infty}e^{-r^2}rdrd\theta.$$

Now, make a new change of variable, $s=r^2$,

$$M=\int_0^{2\pi}\int_0^{\infty}\frac{e^{-s}}{2}dsd\theta=\frac{1}{2}\int_0^{2\pi}\int_0^{\infty}\left[-e^{-s}\right]_0^{\infty}d\theta=\frac{1}{2}\int_0^{2\pi}1d\theta=\pi.$$

Hence, $\int_{-\infty}^{\infty}e^{-x^2}dx=\int_{-\infty}^{\infty}e^{-y^2}dy=\sqrt{M}=\sqrt{\pi}$.

(b) Since e^{-x^2} is a symmetric function with respect to 0, we get $\int_0^{\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

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28. (a) Note that, if the vector $(\tilde{z},\tilde{y})$ were multivariate normal, then

$$P\{\tilde{z}+\tilde{y}=0\}=\int_A n(z,y)d(z,y)=0, \qquad (*)$$

where $n(z,y)$ is the density of the multivariate normal distribution of the random vector $(\tilde{z},\tilde{y})$ and $A=\{(z,y)\in\mathbb{R}^2\,|z+y=0\}$. The second equality in $(*)$ follows from the fact that A is a set with zero Lebesgue measure in $\mathbb{R}^2$.

However, $\tilde{z}+\tilde{y}=\tilde{z}+\tilde{z}\tilde{x}=\tilde{z}(1+\tilde{x})$. Therefore, $\tilde{z}+\tilde{y}=0$ if and only if either the event $\{\tilde{z}=0\}$ or the event $\{\tilde{x}=-1\}$ or both occur. Therefore,

$$P\{\tilde{z}+\tilde{y}=0\}=P(\{\tilde{z}=0\}\cup\{\tilde{x}=-1\})$$

$$=P\{\tilde{z}=0\}+P\{\tilde{x}=-1\}-P(\{\tilde{z}=0\}\cap\{\tilde{x}=-1\})$$

$$=P\{\tilde{z}=0\}+P\{\tilde{x}=-1\}-P\{\tilde{z}=0\}\cdot P\{\tilde{x}=-1\}=0+\frac{1}{2}-0\cdot\frac{1}{2}=\frac{1}{2},$$

which is a contradiction with $(*)$. Note that the third equality in the previous expression follows from the independence between $\tilde{z}$ and $\tilde{x}$, which implies that $P(\{\tilde{z}=0\}\cap\{\tilde{x}=-1\}) = P\{\tilde{z}=0\}\cdot P\{\tilde{x}=-1\}$. Therefore, the vector $(\tilde{z},\tilde{y})$ cannot be multivariate normal.

(b) Use the theorem of total probability to get

$$P\{\tilde{y}\le y\} = P\{\tilde{x}=1\}\cdot P\{\tilde{y}\le y|\,\tilde{x}=1\} + P\{\tilde{x}=-1\}\cdot P\{\tilde{y}\le y|\,\tilde{x}=-1\}$$

$$=\frac{1}{2}\cdot P\{\tilde{z}\le y|\,\tilde{x}=1\}+\frac{1}{2}\cdot P\{-\tilde{z}\le y|\,\tilde{x}=-1\}$$

$$=\frac{1}{2}\cdot P\{\tilde{z}\le y\}+\frac{1}{2}\cdot P\{-\tilde{z}\le y\}=\frac{1}{2}\cdot P\{\tilde{z}\le y\}+\frac{1}{2}\cdot P\{\tilde{z}\le y\}$$

$$=P\{\tilde{z}\le y\}=\mathrm{N}(y), \tag{$**$}$$

where $\mathrm{N}(\cdot)$ is the standard normal distribution function. Note that the second equality in the previous expression follows since $\tilde{y}=\tilde{z}$ when $\tilde{x}=1$ and $\tilde{y}=-\tilde{z}$ when $\tilde{x}=-1$. The third equality follows from the independence between $\tilde{z}$ and $\tilde{x}$. The fourth equality follows since $P\{-\tilde{z}\le y\}=P\{\tilde{z}\ge -y\}=P\{\tilde{z}\le y\}$, that is, $\tilde{z}$ and $-\tilde{z}$ have the same standard normal distribution. The last equality in $(**)$ follows from the standard normality of $\tilde{z}$.

(c)

$$\mathrm{Cov}(\tilde{z},\tilde{y})=\mathrm{E}(\tilde{z}\cdot\tilde{y})-\mathrm{E}(\tilde{z})\cdot\mathrm{E}(\tilde{y})=\mathrm{E}(\tilde{z}\cdot\tilde{y})-0\cdot 0=\mathrm{E}(\mathrm{E}(\tilde{z}\cdot\tilde{y}\,|\tilde{x}))$$

$$=P\{\tilde{x}=1\}\,\mathrm{E}\left(\tilde{z}^2\right)+P\{\tilde{x}=-1\}\,\mathrm{E}\left(-\tilde{z}^2\right)=\frac{1}{2}\cdot\mathrm{E}\left(\tilde{z}^2\right)-\frac{1}{2}\mathrm{E}\left(\tilde{z}^2\right)$$

$$=\frac{1}{2}\cdot\mathrm{Var}(\tilde{z})-\frac{1}{2}\mathrm{Var}(\tilde{z})=\frac{1}{2}\cdot 1-\frac{1}{2}\cdot 1=0.$$

(d) Since $\tilde{z}\in[0,1/2)$ implies that $\tilde{y}=\tilde{z}\tilde{x}\in(-1/2,1/2)$, then $\{\tilde{y}>1\}\cap\{\tilde{z}\in[0,1/2)\}=\emptyset$.

Therefore, $P(\{\tilde{y} > 1\} \cap \{\tilde{z} \in [0, 1/2)\}) = P(\emptyset) = 0$. However,

$$0 = P(\{\tilde{y} > 1\} \cap \{\tilde{z} \in [0, 1/2)\}) \neq P\{\tilde{y} > 1\} \cdot P\{\tilde{z} \in [0, 1/2)\} > 0$$

since $P\{\tilde{y} > 1\} > 0$ and $P\{\tilde{z} \in [0, 1/2)\} > 0$ due to the normality of $\tilde{y}$ and $\tilde{z}$.

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29.

$$\mathrm{E}(\tilde{x}) = \int_0^1 x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \int_0^1 \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\alpha+1+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+1)\Gamma(\alpha+1+\beta)} x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(\alpha+1+\beta)} \cdot \underbrace{\int_0^1 \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1}\,dx}_{=1}.$$

since it is the area under the beta density

with parameters $\alpha+1$ and β

Then, using the properties of the gamma function Γ, we get

$$\frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(\alpha+1+\beta)} = \frac{\Gamma(\alpha+\beta)\cdot\overbrace{\alpha\cdot\Gamma(\alpha)}^{\Gamma(\alpha+1)}}{\Gamma(\alpha)\cdot\underbrace{(\alpha+\beta)\cdot\Gamma(\alpha+\beta)}_{\Gamma(\alpha+1+\beta)}} = \frac{\alpha}{\alpha+\beta}.$$

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30. Making the change to polar coordinates,

$$\int_A \frac{k}{(x^2+y^2)^3} d(x,y) = k \cdot \left[\int_1^{\sqrt{3}} \frac{1}{(r^2)^3} r dr\right] \cdot \left[\int_\pi^{3\pi/2} d\theta\right] = k \cdot \left[\int_1^{\sqrt{3}} \frac{1}{r^5} dr\right] \cdot [\theta]_\pi^{3\pi/2}$$

$$k \cdot \left[-\frac{r^{-4}}{4}\right]_1^{\sqrt{3}} \cdot \frac{\pi}{2} = k \cdot \frac{2}{9} \cdot \frac{\pi}{2} = k \cdot \frac{\pi}{9} = 1 \Longrightarrow k = \frac{9}{\pi}$$

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31. Let $\tilde{\theta}$ be the smaller positive angle (with vertex at the center of the circumference) that separates the two points on the circumference. Then, the distance between the two points is $2r \sin\left(\frac{\tilde{\theta}}{2}\right)$ (think about it!). The expected distance is $\mathrm{E}\left[2r \sin\left(\frac{\tilde{\theta}}{2}\right)\right] = 2r\mathrm{E}\left[\sin\left(\frac{\tilde{\theta}}{2}\right)\right]$. Clearly,

$$f_{\tilde{\theta}}(\theta) = \begin{cases} \dfrac{1}{\pi} & \text{for } 0 < \theta < \pi \\ \\ 0 & \text{otherwise.} \end{cases}$$

That is, the distribution of the angle $\tilde{\theta}$ is obviously absolutely continuous and uniform on $(0, \pi)$.

Then,

$$2r\mathrm{E}\left[\sin\left(\frac{\tilde{\theta}}{2}\right)\right] = 2r \int_0^\pi \frac{1}{\pi} \sin\left(\frac{\theta}{2}\right) d\theta = \frac{2r}{\pi} \left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^\pi = \frac{4r}{\pi}.$$

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32. Since the standard normal density is symmetric around zero, we have $P\{\tilde{z} \geq b\} = P\{\tilde{z} \leq -b\}$. Moreover, from the result in part (a) of Exercise 33 of List 3, we have

$$P\{\tilde{z} \geq b\} \leq \frac{M_{\tilde{z}}(t)}{e^{bt}} = \frac{e^{t^2/2}}{e^{bt}} \quad \text{for all } b \in \mathbb{R} \text{ and all } t \geq 0.$$

Consider a b non-negative and make $t = b \geq 0$ in the previous expression so that

$$P\{\tilde{z} \geq b\} \leq \frac{e^{b^2/2}}{e^{b^2}} = e^{-b^2/2} \text{ for all } b \geq 0.$$

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33. (a) We use the multivariate hypergeometric distribution:

$$h(2,2,0;4,6,3,2,1) = \frac{\binom{3}{2}\binom{2}{2}\binom{1}{0}}{\binom{6}{4}} = \frac{1}{5}.$$

(b) We use the multinomial distribution:

$$m(2,2,0;4,1/2,1/3,1/6) = \frac{4!}{2!2!0!}\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right)^0 = \frac{1}{6}.$$

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34. Since

$$l(x;m,s) = \frac{e^{-\left(\frac{x-m}{s}\right)}}{s\left[1+e^{-\left(\frac{x-m}{s}\right)}\right]^2},$$

then

$$l(m-x;m,s) = \frac{e^{-\left(\frac{-x}{s}\right)}}{s\left[1+e^{-\left(\frac{-x}{s}\right)}\right]^2} = \frac{e^{x/s}}{s\left(1+e^{x/s}\right)^2},$$

and

$$l(m+x;m,s) = \frac{e^{-\left(\frac{x}{s}\right)}}{s\left[1+e^{-\left(\frac{x}{s}\right)}\right]^2} = \frac{e^{-x/s}}{s\left(1+e^{-x/s}\right)^2}.$$

Let us check that $l(m-x,m,s) = l(m+x,m,s)$, i.e.,

$$\frac{e^{x/s}}{s\left(1+e^{x/s}\right)^2} = \frac{e^{-x/s}}{s\left(1+e^{-x/s}\right)^2}$$

$$\iff e^{x/s}\left(1+e^{-x/s}\right)^2 = e^{-x/s}\left(1+e^{x/s}\right)^2$$

$$\iff e^{x/s}\left(1+e^{-2x/s}+2e^{-x/s}\right) = e^{-x/s}\left(1+e^{2x/s}+2e^{x/s}\right)$$

$$\iff e^{x/s}+e^{-x/s}+2 = e^{-x/s}+e^{x/s}+2,$$

which is what we wanted to check.

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