Probability and Statistics. IDEA. Answers to List 3.

1. (a)

$$\mathrm{E}(\tilde{x}) = \int_1^3 x \frac{1}{x \ln 3} dx = \left[\frac{x}{\ln 3}\right]_1^3 = \frac{1}{\ln 3}[3-1] = \frac{2}{\ln 3} = 1.8205.$$

$$\mathrm{E}(\tilde{x}^2) = \int_1^3 x^2 \frac{1}{x \ln 3} dx = \left[\frac{x^2}{2\ln 3}\right]_1^3 = \frac{1}{2\ln 3}[9-1] = \frac{4}{\ln 3} = 3.641.$$

$$\mathrm{E}(\tilde{x}^3) = \int_1^3 x^3 \frac{1}{x \ln 3} dx = \left[\frac{x^3}{3\ln 3}\right]_1^3 = \frac{1}{3\ln 3}[27-1] = \frac{26}{3\ln 3} = 7.8887.$$

(b)

$$\mathrm{E}\left[\tilde{x}^3 + 2\tilde{x}^2 - 3\tilde{x} + 1\right] = \mathrm{E}(\tilde{x}^3) + 2\mathrm{E}(\tilde{x}^2) - 3\mathrm{E}(\tilde{x}) + 1$$

$$= \frac{26}{3\ln 3} + 2\frac{4}{\ln 3} - 3\frac{2}{\ln 3} + 1 = \frac{1}{\ln 3}\left[\frac{26}{3} + 8 - 6\right] + 1 = \frac{32}{3\ln 3} + 1 = 10.709.$$

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2.

$$\mathrm{E}(2^{\tilde{x}}) = \sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^x 2^x = \sum_{x=1}^{\infty} 1 = \infty \text{ !!!}$$

Remark: The paradox is that the price a person would be willing to pay to play this game is limited, but the player's expected gain is infinite. Nicolas Bernoulli posed this paradox and his cousin Daniel Bernoulli resolved it by assuming that individuals are "risk averse". For instance, if the Bernoulli utility function u is logarithmic, the "expected utility" of this game for an individual with an initial wealth of $10,000 will be

$$\mathrm{E}\left[u(10000+2^{\tilde{x}})\right]=\mathrm{E}\left[\ln(10000+2^{\tilde{x}})\right]=\sum_{x=1}^{\infty}\left[\left(\frac{1}{2}\right)^{x}\ln\left(10000+2^{x}\right)\right]=9.2118.$$

In order to find the maximum amount this individual is willing to pay to play this game, we solve for r in the following equation:

$$\mathrm{E}\left[u(10000+2^{\tilde{x}}-r)\right]=\sum_{x=1}^{\infty}\left[\left(\frac{1}{2}\right)^{x}\ln\left(10000+2^{x}-r\right)\right]=\ln\left(10000\right)$$

to get $r=\$14.24$.

If the initial wealth of the individual were $1,000,000, he will be willing to pay at most $20.87 (check it).

If the initial wealth of the individual were $1,000 dollars, he will be willing to pay at most $10.95 (check it).

If the initial wealth of the individual were $100 dollars, he will be willing to pay at most $7.79 (check it).

This numbers agree with the famous Canadian philosopher Ian Hacking when he said "few of us would pay even $25 to enter such a game".

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3. Taking into account that the probability that the game never finishes is zero, define π as the probability that Adams will win all the money:

$\pi=P\{\text{Adams will win all the money}\}$

$1-\pi=P\{\text{Smith will win all the money}\}$

$b=$ Payoff to Adams if he wins the game

$-a=$ Payoff to Adams if he loses the game.

The expected payoff to Adams is

$$\begin{aligned} \mathrm{E}(\tilde{x}) &= 0 = \pi b + (1-\pi)(-a) = \pi(b+a) - a \Longrightarrow \\ \pi(b+a) &= a \Longrightarrow \pi = \frac{a}{b+a} \text{ and } 1-\pi = \frac{b}{b+a}. \end{aligned}$$

What does this suggest about you being able to break the bank at Monte Carlo?

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4.

$$\mu = \int_\Omega \tilde{x} dP = \int_{\{\tilde{x}\geq a\}} \tilde{x} dP + \int_{\{\tilde{x}<a\}} \tilde{x} dP.$$

Since $\tilde{x} \geq 0$, then

$$\int_{\{\tilde{x}<a\}} \tilde{x} dP \geq 0$$

$$\begin{aligned} &\Longrightarrow \quad \mu \geq \int_{\{\tilde{x}\geq a\}} \tilde{x} dP \geq \int_{\{\tilde{x}\geq a\}} a dP = a \int_{\{\tilde{x}\geq a\}} dP \\ &\Longrightarrow \quad \frac{\mu}{a} \geq \underbrace{\int_{\{\tilde{x}\geq a\}} dP}_{=\int_{[a,\infty)} dP_{\tilde{x}}(x)} = \underbrace{P\{\tilde{x} \geq a\}}_{=P_{\tilde{x}}[a,\infty)}. \end{aligned}$$

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5. To get the Moment-Generating Function of $\tilde{x}$:

$$\mathrm{E}\left(e^{t\tilde{x}}\right) = \sum_{x=1}^{\infty} e^{tx} \cdot 2 \cdot \left(\frac{1}{3}\right)^x = 2\sum_{x=1}^{\infty} \left(e^t \cdot \frac{1}{3}\right)^x$$

If $\left(\frac{e^t}{3}\right) < 1$ (or, equivalently, if $t < \ln 3$), then by the property of the geometric series

$$\sum_{x=1}^{\infty}\left(e^t \cdot \frac{1}{3}\right)^x = \left(\frac{\frac{e^t}{3}}{1-\frac{e^t}{3}}\right) = \frac{e^t}{3-e^t}.$$

Hence,

$$M_{\tilde{x}}(t) = \mathrm{E}(e^{t\tilde{x}}) = \frac{2e^t}{3-e^t} \text{ for } t < \ln 3$$

and

$$M'_{\tilde{x}}(t) = 2[e^t(3-e^t)^{-1} + e^t(3-e^t)^{-2}(-1)(-e^t)].$$

Evaluating it at $t = 0$

$$M'_{\tilde{x}}(0) = \frac{3}{2} = \mu'_1.$$

$$\begin{aligned} M''_{\tilde{x}}(t) &= 2e^t[(3-e^t)^{-1} + e^t(3-e^t)^{-2}] \\ &\quad + 2e^t[(-1)(3-e^t)^{-2}(-e^t) + e^t(3-e^t)^{-2} + e^t(-2)(3-e^t)^{-3}(-e^t) \end{aligned}$$

Evaluating it at $t = 0$

$$\begin{aligned} M''_{\tilde{x}}(0) &= 2[(3-1)^{-1} + 1(3-1)^{-2}] + 2\cdot 1[-(3-1)^{-2}(-1) + 1(3-1)^{-2} \\ &\quad + 1(-2)(3-1)^{-3}(-1) = 2\left[\frac{1}{2}+\frac{1}{4}\right] + 2\left[\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right] = \frac{3}{2}+\frac{3}{2} = 3. \end{aligned}$$

Thus, $\mu'_2 = 3$.

Finally, $\sigma^2 = \mu'_2 - (\mu'_1)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$.

Observe how awkward would be to get the moments by direct computation.

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6. The moment-generating function of $\tilde{x}$ is given by:

$$M_{\tilde{x}}(t) = \mathrm{E}(e^{t\tilde{x}}) = \int_0^1 e^{tx} \cdot 1dx = \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{1}{t}\left[e^t - 1\right] = \frac{e^t - 1}{t} \text{ for } t \neq 0.$$

Moreover, $M_{\tilde{x}}(0) = \mathrm{E}(1) = 1$, which holds for all random variables. Therefore, $M_{\tilde{x}}(t)$ is finite in a neighborhood of $t = 0$.

Note that $M_{\tilde{x}}(t)$ is continuous at 0 since $M_{\tilde{x}}(0) = 1 = \lim_{t \to 0} M_{\tilde{x}}(t)$, as follows from L'Hôpital's rule.

Then, all the derivatives of $M_{\tilde{x}}(t)$ are continuous at $t = 0$. To get μ'_1 we need to find $M'_{\tilde{x}}(0)$,

$$M'_{\tilde{x}}(t) = \frac{e^t}{t} + \frac{e^t - 1}{t^2}(-1) = \frac{te^t - e^t + 1}{t^2}.$$

Evaluating it at $t = 0$ we get that

$$M'_{\tilde{x}}(0) = \frac{0}{0}$$

is indeterminate, hence we should look for $\lim_{t \to 0} M'_{\tilde{x}}(t)$. According to L'Hôpital's rule, if $g(x) = \frac{u(x)}{v(x)}$ and both $\lim_{x \to x^*} u(x) = 0$ and $\lim_{x \to x^*} v(x) = 0$ then $\lim_{x \to x^*} g(x) = \lim_{x \to x^*} \frac{u'(x)}{v'(x)}$. Then, since $u'(t) = te^t$ and $v'(t) = 2t$

$$\lim_{t \to 0} M'_{\tilde{x}}(t) = \lim_{t \to 0} \frac{te^t}{2t} = \lim_{t \to 0} \frac{e^t}{2} = \frac{1}{2} = \mu'_1$$

To get μ'_2 we look for $M''_{\tilde{x}}(0)$:

$$M''_{\tilde{x}}(t) = \frac{t^2e^t - 2te^t + 2e^t - 2}{t^3}$$

and

$$M''_{\tilde{x}}(0) = \frac{0}{0},$$

which is indeterminate. To find $\lim\limits_{t\to 0} M''_{\tilde{x}}(t)$ we use L'Hôpital's rule once again.

$$\begin{aligned}
\lim_{t\to 0} M''_{\tilde{x}}(t) &= \lim_{t\to 0} \frac{2te^t + t^2e^t - 2e^t - 2te^t + 2e^t}{3t^2} \\
&= \lim_{t\to 0} \frac{t^2e^t}{3t^2} = \lim_{t\to 0} \frac{e^t}{3} = \frac{1}{3} = \mu'_2
\end{aligned}$$

and

$$\sigma^2 = \mu'_2 - (\mu'_1)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

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7.

$$R_{\tilde{x}}(t) = \ln M_{\tilde{x}}(t)$$

$$R'_{\tilde{x}}(t) = \frac{1}{M_{\tilde{x}}(t)} M'_{\tilde{x}}(t)$$

Since $M_{\tilde{x}}(t)$ evaluated at $t = 0$ is equal to 1

$$R'_{\tilde{x}}(0) = \frac{1}{1} M'_{\tilde{x}}(0) = \mu$$

$$\begin{aligned}
R''_{\tilde{x}}(t) &= \frac{M''_{\tilde{x}}(t)}{M_{\tilde{x}}(t)} + \frac{M'_{\tilde{x}}(t)}{[M_{\tilde{x}}(t)]^2}(-1)M'_{\tilde{x}}(t) \\
&= \frac{1}{[M_{\tilde{x}}(t)]^2}\left\{M_{\tilde{x}}(t)M''_{\tilde{x}}(t) - [M'_{\tilde{x}}(t)]^2\right\}
\end{aligned}$$

Evaluating at $t = 0$

$$\begin{aligned} R''_{\tilde{x}}(0) &= \frac{1}{1}\left\{1 \cdot M''_{\tilde{x}}(0) - [M'_{\tilde{x}}(0)]^2\right\} \\ &= M''_{\tilde{x}}(0) - [M'_{\tilde{x}}(0)]^2 = \mu'_2 - (\mu)^2 = \sigma^2. \end{aligned}$$

To find the mean and the variance of $\tilde{x}$:

$$M_{\tilde{x}}(t) = e^{4(e^t-1)}$$

$$R_{\tilde{x}}(t) = 4(e^t - 1)$$

$$R'_{\tilde{x}}(t) = 4e^t$$

$$R''_{\tilde{x}}(t) = 4e^t$$

$$\mu = R'_{\tilde{x}}(0) = 4$$

$$\sigma^2 = R''_{\tilde{x}}(0) = 4$$

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8.

$$\begin{aligned} M_{\tilde{z}}(t) &= M_{\frac{\tilde{x}-3}{4}}(t) = M_{\tilde{x}-3}\left(\frac{t}{4}\right) = e^{-\frac{3}{4}t} M_{\tilde{x}}\left(\frac{t}{4}\right) \\ &= e^{-\frac{3}{4}t} e^{3\frac{t}{4}+8\left(\frac{t}{4}\right)^2} = e^{t^2/2}. \end{aligned}$$

$$M'_{\tilde{z}}(t) = e^{t^2/2}\frac{2t}{2} = te^{t^2/2}$$

$$M''_{\tilde{z}}(t) = e^{t^2/2} + te^{t^2/2}\left(\frac{2t}{2}\right) = e^{t^2/2} + t^2 e^{t^2/2}$$

Hence,

$$\mu = M'_{\tilde{z}}(0) = 0$$

and

$$\sigma^2 = M''_{\tilde{z}}(0) - \mu^2 = 1 - 0 = 1.$$

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9. (a)

$$\begin{aligned} \mu_{\tilde{y}} &= 2 \cdot 4 - 3 \cdot 9 + 4 \cdot 3 = -7 \\ &\quad \text{or} \\ &\quad \mu_{\tilde{y}} = \alpha^\top \mu = \begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} = -7. \end{aligned}$$

$$\begin{aligned} \sigma^2_{\tilde{y}} &= 4 \cdot \sigma^2_{\tilde{x}_1} + 9 \cdot \sigma^2_{\tilde{x}_2} + 16 \cdot \sigma^2_{\tilde{x}_3} - 2 \cdot 6 \cdot \sigma_{\tilde{x}_1\tilde{x}_2} + 2 \cdot 8 \cdot \sigma_{\tilde{x}_1\tilde{x}_3} - 2 \cdot 12 \cdot \sigma_{\tilde{x}_2\tilde{x}_3} \\ &= 4 \cdot 3 + 9 \cdot 5 + 16 \cdot 7 - 2 \cdot 6 \cdot 1 + 2 \cdot 8 \cdot (-3) - 2 \cdot 12 \cdot (-2) = 157 \\ &\quad \text{or} \\ &\quad \sigma^2_{\tilde{y}} = \alpha^\top \Sigma \alpha, \text{ where } \Sigma = \begin{pmatrix} 3 & 1 & -3 \\ 1 & 5 & -2 \\ -3 & -2 & 7 \end{pmatrix}. \end{aligned}$$

(b)

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\begin{aligned}
\mu_{\tilde{z}} &= 1\cdot 4 + 2\cdot 9 - 1\cdot 3 = 19 \\
&\text{or} \\
&\mu_{\tilde{z}} = \alpha^{\top}\mu = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} = 19.
\end{aligned}
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\begin{aligned}
\sigma^2_{\tilde{z}} &= \sigma^2_{\tilde{x}_1} + 4\cdot\sigma^2_{\tilde{x}_2} + \sigma^2_{\tilde{x}_3} + 2\cdot 2\cdot \sigma_{\tilde{x}_1\tilde{x}_2} - 2\cdot 1\cdot \sigma_{\tilde{x}_1\tilde{x}_3} - 2\cdot 2\cdot \sigma_{\tilde{x}_2\tilde{x}_3} \\
&= 3 + 4\cdot 5 + 7 + 2\cdot 2\cdot 1 - 2\cdot 1\cdot(-3) - 2\cdot 2\cdot(-2) = 48 \\
&\text{or} \\
&\sigma^2_{\tilde{z}} = \alpha^{\top}\Sigma\alpha, \text{ where } \Sigma = \begin{pmatrix} 3 & 1 & -3 \\ 1 & 5 & -2 \\ -3 & -2 & 7 \end{pmatrix}.
\end{aligned}
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10. Note that $\mathrm{Var}(\tilde{w}) = \mathrm{Var}(\tilde{z} - 5) = \mathrm{Var}(\tilde{z})$, where $\tilde{z} \equiv 3\tilde{x} + 4\tilde{y}$. Since

$$\mathrm{Var}(\tilde{z}) = \mathrm{E}(\tilde{z}^2) - (\mathrm{E}(\tilde{z}))^2,$$

we first compute

$$\begin{aligned}
E(\tilde{z}) &= \int_0^2 \int_0^1 (3x+4y)\frac{1}{3}(x+y)dxdy \\
&= \frac{1}{3}\int_0^2 \int_0^1 (3x^2+7xy+4y^2)dxdy \\
&= \frac{1}{3}\int_0^2 \left[3\frac{x^3}{3}+7\frac{x^2}{2}y+4xy^2\right]_0^1 dy \\
&= \frac{1}{3}\int_0^2 \left[1+\frac{7}{2}y+4y^2\right] dy \\
&= \frac{1}{3}\left[y+\frac{7}{2}\frac{y^2}{2}+4\frac{y^3}{3}\right]_0^2 \\
&= \frac{1}{3}\left[2+7+\frac{32}{3}\right] = \frac{59}{9}.
\end{aligned}$$

Second, we compute

$$\begin{aligned}
E(\tilde{z}^2) &= \frac{1}{3}\int_0^2 \int_0^1 (3x+4y)^2(x+y)dxdy \\
&= \frac{1}{3}\int_0^2 \int_0^1 (9x^2+24xy+16y^2)(x+y)dxdy \\
&= \frac{1}{3}\int_0^2 \left[9\frac{x^4}{4}+33\frac{x^3}{3}y+40\frac{x^2}{2}y^2+16y^3x\right]_0^1 dy \\
&= \frac{1}{3}\int_0^2 \left[\frac{9}{4}+11y+20y^2+16y^3\right] dy \\
&= \frac{1}{3}\left[\frac{9}{4}y+11\frac{y^2}{2}+20\frac{y^3}{3}+16\frac{y^4}{4}\right]_0^2 \\
&= \frac{1}{3}\left[\frac{9}{2}+22+\frac{160}{3}+64\right] = \frac{1}{3}\left[\frac{863}{6}\right] = \frac{863}{18}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{Var}(\tilde{w}) &= \mathrm{Var}(\tilde{z}) = \mathrm{E}(\tilde{z}^2) - (\mathrm{E}(\tilde{z}))^2 = \frac{863}{18} - \left(\frac{59}{9}\right)^2 \\
&= \frac{863}{18} - \frac{3481}{81} = \frac{7767 - 6962}{162} = \frac{805}{162}.
\end{aligned}$$

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11. $P(H) = 0.4$

Head in the toss i: $\tilde{x}_i = 1$

Tail in the toss i: $\tilde{x}_i = 0$

Probability function of $\tilde{x}_i$:

x_i	$f(x_i)$
0	0.6
1	0.4

$$\begin{cases} \tilde{z} = \tilde{x}_1 \\ \tilde{w} = \tilde{x}_1 + \tilde{x}_2 \end{cases}$$

$$\mathrm{Cov}(\tilde{z}, \tilde{w}) = \mathrm{E}(\tilde{z} \cdot \tilde{w}) - \mathrm{E}(\tilde{z}) \cdot \mathrm{E}(\tilde{w})$$

where

$$\begin{aligned}
\mathrm{E}(\tilde{z} \cdot \tilde{w}) &= \mathrm{E}\left[\tilde{x}_1 \cdot (\tilde{x}_1 + \tilde{x}_2)\right] \\
&= \mathrm{E}\left(\tilde{x}_1^2 + \tilde{x}_1\tilde{x}_2\right) \\
&= \mathrm{E}\left(\tilde{x}_1^2\right) + \mathrm{E}\left(\tilde{x}_1\tilde{x}_2\right) \\
&= \mathrm{E}\left(\tilde{x}_1^2\right) + \mathrm{E}\left(\tilde{x}_1\right)\mathrm{E}\left(\tilde{x}_2\right),
\end{aligned}$$

where the last inequality follows from the independency between $\tilde{x}_1$ and $\tilde{x}_2$.

Since

$$\begin{aligned} \mathrm{E}\left(\tilde{x}_1^2\right) &= 1^2 \cdot 0.4 + 0^2 \cdot 0.6 = 0.4 \\ \mathrm{E}\left(\tilde{x}_1\right) &= \mathrm{E}\left(\tilde{x}_2\right) = 1 \cdot 0.4 + 0 \cdot 0.6 = 0.4 \end{aligned}$$

$$\mathrm{E}(\tilde{z} \cdot \tilde{w}) = 0.4 + (0.4)^2 = 0.56.$$

Using the fact that

$$\left.\begin{array}{c} \mathrm{E}\left(\tilde{z}\right) = \mathrm{E}\left(\tilde{x}_1\right) = 0.4 \\ \mathrm{E}\left(\tilde{w}\right) = \mathrm{E}\left(\tilde{x}_1\right) + \mathrm{E}\left(\tilde{x}_2\right) = 2 \cdot (0.4) = 0.8 \end{array}\right\} \Longrightarrow \mathrm{E}(\tilde{z})\mathrm{E}(\tilde{w}) = 0.32.$$

Hence,

$$\mathrm{Cov}(\tilde{z}, \tilde{w}) = \mathrm{E}(\tilde{z} \cdot \tilde{w}) - \mathrm{E}(\tilde{z})\mathrm{E}(\tilde{w}) = 0.56 - 0.32 = 0.24.$$

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12.

Outcome	s_1	Prob	Outcome	s_2	Prob	Outcome	s_3	Prob
Tails	0	1/2	Not 6	0	5/6	Not Ace	0	12/13
Heads	1	1/2	6	1	1/6	Ace	1	1/13

(a) $\tilde{z} = \tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3$, where the $\tilde{s}_i$'s are independent.

$$\begin{aligned}
\mathrm{E}(\tilde{z}) &= \mathrm{E}(\tilde{s}_1) + \mathrm{E}(\tilde{s}_2) + \mathrm{E}(\tilde{s}_3) = \mu_{\tilde{s}_1} + \mu_{\tilde{s}_2} + \mu_{\tilde{s}_3} \\
\mathrm{Var}(\tilde{z}) &= \mathrm{Var}(\tilde{s}_1) + \mathrm{Var}(\tilde{s}_2) + \mathrm{Var}(\tilde{s}_3) = \sigma^2_{\tilde{s}_1} + \sigma^2_{\tilde{s}_2} + \sigma^2_{\tilde{s}_3}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(\tilde{s}_1) = 0 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{2} = \tfrac{1}{2}, &\quad \mathrm{E}(\tilde{s}_1^2) = 0^2 \cdot \tfrac{1}{2} + 1^2 \cdot \tfrac{1}{2} = \tfrac{1}{2}, \\
\mathrm{E}(\tilde{s}_2) = 0 \cdot \tfrac{5}{6} + 1 \cdot \tfrac{1}{6} = \tfrac{1}{6}, &\quad \mathrm{E}(\tilde{s}_2^2) = 0^2 \cdot \tfrac{5}{6} + 1^2 \cdot \tfrac{1}{6} = \tfrac{1}{6}, \\
\mathrm{E}(\tilde{s}_3) = 0 \cdot \tfrac{12}{13} + 1 \cdot \tfrac{1}{13} = \tfrac{1}{13}, &\quad \mathrm{E}(\tilde{s}_3^2) = 0^2 \cdot \tfrac{12}{13} + 1^2 \cdot \tfrac{1}{13} = \tfrac{1}{13}.
\end{aligned}$$

Then

$$\begin{aligned}
\mu_{\tilde{s}_1} = \tfrac{1}{2}, &\quad \sigma^2_{\tilde{s}_1} = \tfrac{1}{2} - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}, \\
\mu_{\tilde{s}_2} = \tfrac{1}{6}, &\quad \sigma^2_{\tilde{s}_2} = \tfrac{1}{6} - \left(\tfrac{1}{6}\right)^2 = \tfrac{5}{36}, \\
\mu_{\tilde{s}_3} = \tfrac{1}{13}, &\quad \sigma^2_{\tilde{s}_3} = \tfrac{1}{13} - \left(\tfrac{1}{13}\right)^2 = \tfrac{12}{169}.
\end{aligned}$$

Hence

$$\mu_{\tilde{z}} = \frac{1}{2} + \frac{1}{6} + \frac{1}{13} = \frac{58}{78} = 0.743$$

and

$$\mathrm{Var}(\tilde{z}) = \frac{1}{4} + \frac{5}{36} + \frac{12}{169} = 0.46$$

Then

$$\mu_{\tilde{z}} = 0.743$$

and

$$\sigma_{\tilde{z}} = (0.46)^{1/2} = 0.678.$$

(b) $\tilde{w} = \tilde{s}_{11} + \tilde{s}_{12} + \tilde{s}_{13} + \tilde{s}_{21} + \tilde{s}_{22} + \tilde{s}_3$

$$\mu_{\tilde{w}} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{13} = 1.91$$

and

$$\mathrm{Var}(\tilde{w}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{5}{36} + \frac{5}{36} + \frac{12}{169} = 1.0988$$

Then,

$$\mu_{\tilde{w}} = 1.91$$

and

$$\sigma_{\tilde{w}} = (1.0988)^{1/2} = 1.0482.$$

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13. (a)

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_{\tilde{y}}(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{1} \frac{2}{3}(x+2y)dx = \frac{1}{3}(1+4y)$$

for $0 < y < 1$, and $f_{\tilde{y}}(y) = 0$ elsewhere.

Therefore, if $0 < y < 1$,

$$f_{\tilde{x}|\tilde{y}}(x|y) = \frac{f(x,y)}{f_{\tilde{y}}(y)} = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(1+4y)} = \frac{2x+4y}{1+4y}$$

for $0 < x < 1$, and $f_{\tilde{x}|\tilde{y}}(x|y) = 0$ elsewhere.

Then,

$$f_{\tilde{x}|\tilde{y}}\left(x \,\middle|\, \frac{1}{2}\right) = \begin{cases} \frac{2}{3}(x+1) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus

$$\begin{aligned}
\mathrm{E}\left(\tilde{x}\,\middle|\,\tilde{y}=\frac{1}{2}\right) &= \int\limits_0^1 \frac{2}{3}x(x+1)dx = \frac{5}{9} \\
\mathrm{E}\left(\tilde{x}^2\,\middle|\,\tilde{y}=\frac{1}{2}\right) &= \int\limits_0^1 \frac{2}{3}x^2(x+1)dx = \frac{7}{18}
\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}
\mathrm{Var}\left(\tilde{x}\,\middle|\,\tilde{y}=\frac{1}{2}\right) &= \mathrm{E}\left(\tilde{x}^2\,\middle|\,\tilde{y}=\frac{1}{2}\right) - \left[\mathrm{E}\left(\tilde{x}\,\middle|\,\tilde{y}=\frac{1}{2}\right)\right]^2 \\
&= \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}
\end{aligned}$$

(b)

$$\mathrm{E}(\tilde{x}) = \int\limits_0^1 \int\limits_0^1 x\frac{2}{3}(x+2y)dxdy = \frac{5}{9},$$

$$\mathrm{E}(\tilde{y}) = \int\limits_0^1 \int\limits_0^1 y\frac{2}{3}(x+2y)dxdy = \frac{11}{18},$$

$$\mathrm{E}(\tilde{x}\cdot\tilde{y}) = \int\limits_0^1 \int\limits_0^1 xy\frac{2}{3}(x+2y)dxdy = \frac{1}{3}.$$

Therefore,

$$\mathrm{Cov}(\tilde{x},\tilde{y}) = \mathrm{E}(\tilde{x}\cdot\tilde{y}) - \mathrm{E}(\tilde{x})\,\mathrm{E}(\tilde{y}) = \frac{1}{3} - \left(\frac{5}{9}\right)\cdot\left(\frac{11}{18}\right) = -\frac{1}{162}$$

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14. (a)

$$
\begin{aligned}
f_{\tilde{x}}(x) &= \int_{0^+}^{1} \left(-\frac{1}{2}\right) \ln(xy) dy = -\frac{1}{2}\left[\int_{0^+}^{1} (\ln x)\, dy + \int_{0^+}^{1} (\ln y)\, dy\right] \\
&= -\frac{1}{2}\left\{\underbrace{[(\ln x) y]_0^1}_{A} + \underbrace{[y \ln y - y]_{0^+}^1}_{B}\right\} \\
&= -\frac{1}{2}[\ln x - 1] = \frac{1}{2}[1 - \ln x], \quad \text{for } 0 < x < 1.
\end{aligned}
$$

Note:

$$A = (\ln x) \cdot 1 - (\ln x) \cdot 0 = \ln x.$$

$$
\begin{aligned}
B &= [1 \cdot (\ln 1) - 1] - \lim_{y \to 0^+} [y \ln y - y] \\
&= (0 - 1) - \left[\lim_{y \to 0^+} (y \ln y) - 0\right] \\
&= -1 - \lim_{y \to 0^+} [y \ln y] = -1 - 0 = -1,
\end{aligned}
$$

since

$$\lim_{y \to 0^+} [y \ln y] = \lim_{y \to 0^+} \frac{\ln y}{\frac{1}{y}} = \frac{-\infty}{\infty},$$

by L'Hôpital's rule

$$\lim_{y \to 0^+} \frac{\ln y}{\frac{1}{y}} = \lim_{y \to 0^+} \frac{\frac{1}{y}}{\frac{-1}{y^2}} = \lim_{y \to 0^+} (-y) = 0$$

$\Longrightarrow$

$$
f_{\tilde{x}}(x) = \begin{cases} \frac{1}{2}(1 - \ln x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}
$$

(b) Symmetrically,

$$f_{\tilde{y}}(y) = \begin{cases} \frac{1}{2}(1 - \ln y) & \text{for } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, if $0 < y < 1$,

$$f_{\tilde{x}|\tilde{y}}(x|y) = \frac{f(x,y)}{f_{\tilde{y}}(y)} = \frac{-\frac{1}{2}\ln(xy)}{\frac{1}{2}(1-\ln y)} = -\frac{\ln x + \ln y}{1 - \ln y}, \quad \text{for } 0 < x < 1.$$

Therefore, if $0 < y < 1$,

$$f_{\tilde{x}|\tilde{y}}(x|y) = \begin{cases} -\dfrac{\ln x + \ln y}{1 - \ln y} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(c)

$$\begin{aligned} f(x,y) &\stackrel{?}{=} f_{\tilde{x}}(x) \cdot f_{\tilde{y}}(y), \text{ for } \ 0 < x < 1\ , 0 < y < 1 \\ -\frac{1}{2}\ln(xy) &\stackrel{?}{=} \frac{1}{2}(1 - \ln x) \cdot \frac{1}{2}(1 - \ln y) \\ -\frac{1}{2}\ln(xy) &\stackrel{?}{=} \frac{1}{4}(1 - \ln x - \ln y + (\ln x)(\ln y)) \\ -\frac{1}{2}\ln x - \frac{1}{2}\ln y &\stackrel{?}{=} \frac{1}{4} - \frac{1}{4}\ln x - \frac{1}{4}\ln y + \frac{1}{4}(\ln x)(\ln y) \\ 0 \quad &\neq \quad \frac{1}{4} + \frac{1}{4}\ln x + \frac{1}{4}\ln y + \frac{1}{4}(\ln x)(\ln y) \end{aligned}$$

$\Longleftrightarrow$ $\tilde{x}$ and $\tilde{y}$ are not independent

(d)

$$
\begin{aligned}
\mathrm{E}(\tilde{x}) &= \int_{0+}^{1} x\left[\frac{1}{2}(1-\ln x)\right]dx = \int_{0+}^{1}\frac{x}{2}dx - \frac{1}{2}\int_{0+}^{1} x\ln x dx \\
\text{by parts} \quad \rightarrow &= \left[\frac{x^2}{4}\right]_{0+}^{1} - \frac{1}{2}\left\{\left[\frac{x^2}{2}\ln x\right]_{0+}^{1} - \int_{0+}^{1}\frac{x^2}{2}\frac{1}{x}dx\right\}
\end{aligned}
$$

$u = \ln x, \quad v = \frac{x^2}{2}$

$u' = \frac{1}{x}, \quad v' = x$

$$
\begin{aligned}
&= \frac{1}{4} - \frac{1}{2}\left((0-0) - \frac{1}{2}\left[\frac{x^2}{2}\right]_{0+}^{1}\right) \\
&= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}
\end{aligned}
$$

(e) If $0 < y < 1$,

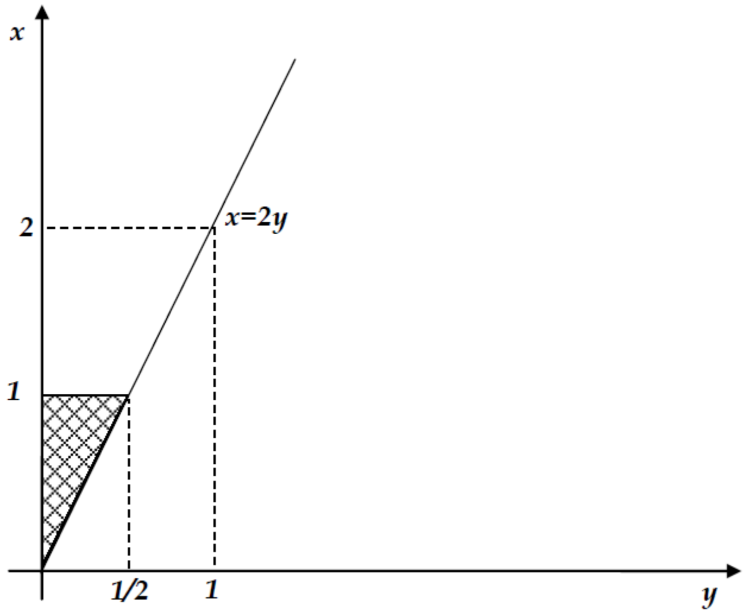
$$
\begin{aligned}
\mathrm{E}(\tilde{x}|\ \tilde{y} = y) &= \int_{0+}^{1} x f_{\tilde{x}|\tilde{y}}(x|y)dx = \int_{0+}^{1} x\left(-\frac{\ln x + \ln y}{1-\ln y}\right)dx \\
&= -\frac{1}{1-\ln y}\left[\int_{0+}^{1} x\ln x dx + \ln y\int_{0+}^{1} x dx\right] \\
&= -\frac{1}{1-\ln y}\left[\underbrace{\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_{0+}^{1}}_{=-\frac{1}{4}} + \ln y\underbrace{\left[\frac{x^2}{2}\right]_{0+}^{1}}_{=\frac{1}{2}}\right] \\
&= -\frac{1}{1-\ln y}\left[-\frac{1}{4} + \frac{1}{2}\ln y\right] = \frac{1}{4}\left[\frac{1-2\ln y}{1-\ln y}\right].
\end{aligned}
$$

$\Longrightarrow$

$$
\mathrm{E}(\tilde{x}|\tilde{y}) = \frac{1}{4}\left[\frac{1-2\ln\tilde{y}}{1-\ln\tilde{y}}\right].
$$

Obviously, $\mathrm{E}(\tilde{x}|\tilde{y})$ is measurable w.r.t the σ-algebra generated by $\tilde{y}$ since it is a Borel measurable transformation of $\tilde{y}$.

(f)



$$P\left\{\tilde{x} > 2\tilde{y}\right\} = \int_{0}^{\frac{1}{2}} \int_{2y}^{1} f(x,y)dxdy = -\frac{1}{2}\int_{0}^{\frac{1}{2}} \int_{2y}^{1} \ln(xy)dxdy = \frac{1}{4}+\frac{1}{8}\ln 2 = 0.33664.$$

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15. In this exercise, we are basically applying Bayes' theorem to find the conditional pdf/pmf of $\tilde{y}$ given $\tilde{x}_1 = x_1, ..., \tilde{x}_n = x_n$.

(a) The conditional density of $\tilde{y}$, given $\tilde{x}_1 = x_1, ..., \tilde{x}_n = x_n$, is

$$f_{\tilde{y}|\tilde{x}_1,...,\tilde{x}_n}(y|x_1, ..., x_n) = \frac{f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y, x_1, ..., x_n)}{f_{\tilde{x}_1,...,\tilde{x}_n}(x_1, ..., x_n)}.$$

Since $\tilde{x}_1, ..., \tilde{x}_n$ are independent random variables with the same density $h(\cdot; y)$,

$$\begin{aligned} f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y, x_1, ..., x_n) &= g(y) \cdot f_{\tilde{x}_1,...,\tilde{x}_n|\tilde{y}}(x_1, ..., x_n|y) \\ &= g(y) \cdot \prod_{i=1}^{n} h(x_i; y). \end{aligned}$$

Therefore,

$$f_{\tilde{x}}(x_1,...,x_n) = \int\limits_{\mathbb{R}} f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y, x_1, ..., x_n)dy = \int\limits_{\mathbb{R}} \left[g(y) \cdot \prod_{i=1}^{n} h(x_i; y) \right] dy > 0.$$

Then,

$$\mathrm{E}\left(\tilde{y}\left|\tilde{x}_1 = x_1, ..., \tilde{x}_n = x_n\right.\right) = \int\limits_{\mathbb{R}} y f_{\tilde{y}|\tilde{x}_1,...,\tilde{x}_n}(y|x_1, ..., x_n)dy =$$

$$\int\limits_{\mathbb{R}} y \cdot \left(\frac{g(y) \cdot \prod\limits_{i=1}^{n} h(x_i; y)}{\int\limits_{\mathbb{R}} \left[g(y) \cdot \prod\limits_{i=1}^{n} h(x_i; y) \right]} \right) dy = \frac{\int\limits_{\mathbb{R}} \left[y \cdot g(y) \cdot \prod\limits_{i=1}^{n} h(x_i; y) \right] dy}{\int\limits_{\mathbb{R}} \left[g(y) \cdot \prod\limits_{i=1}^{n} h(x_i; y) \right] dy}.$$

(b) The conditional probability function of $\tilde{y}$, given $\tilde{x}_1 = x_1, ..., \tilde{x}_n = x_n$, is

$$f_{\tilde{y}|\tilde{x}_1,...,\tilde{x}_n}(y|x_1, ..., x_n) = \frac{f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y, x_1, ..., x_n)}{f_{\tilde{x}_1,...,\tilde{x}_n}(x_1, ..., x_n)}.$$

Since $\tilde{x}_1, ..., \tilde{x}_n$ are independent random variables with the same probability function $h(\cdot; y)$,

$$\begin{aligned} f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y, x_1, ..., x_n) &= g(y) \cdot f_{\tilde{x}_1,...,\tilde{x}_n|\tilde{y}}(x_1, ..., x_n|y) \\ &= g(y) \cdot \prod_{i=1}^{n} h(x_i; y). \end{aligned}$$

Therefore,

$$f_{\tilde{x}_1,...,\tilde{x}_n}(x_1, ..., x_n) = \sum_{y \in \tilde{y}(\Omega)} f_{\tilde{y},\tilde{x}_1,...,\tilde{x}_n}(y, x_1, ..., x_n) = \sum_{y \in \tilde{y}(\Omega)} \left[g(y) \cdot \prod_{i=1}^{n} h(x_i; y) \right] > 0.$$

Then,

$$\mathrm{E}\left(\tilde{y}\,|\tilde{x}_1 = x_1, ..., \tilde{x}_n = x_n\right) = \sum_{y\in\tilde{y}(\Omega)} y f_{\tilde{y}|\tilde{x}_1,...,\tilde{x}_n}(y|x_1, ..., x_n) =$$

$$\sum_{y\in\tilde{y}(\Omega)} y \cdot \left(\frac{g(y)\cdot\prod\limits_{i=1}^{n} h(x_i;y)}{\sum\limits_{y\in\tilde{y}(\Omega)}\left[g(y)\cdot\prod\limits_{i=1}^{n} h(x_i;y)\right]} \right) = \frac{\sum\limits_{y\in\tilde{y}(\Omega)} y\cdot g(y)\cdot\prod\limits_{i=1}^{n} h(x_i;y)}{\sum\limits_{y\in\tilde{y}(\Omega)}\left[g(y)\cdot\prod\limits_{i=1}^{n} h(x_i;y)\right]}.$$

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16. Since

$$P(A|\mathbb{I}_{B_n} = 1) = \frac{P(A\cap B_n)}{P(B_n)}, \forall A\in\mathcal{F}, \forall B_n$$

and, thus,

$$P(A|\mathbb{I}_{B_n} = 1) = \frac{P(A)}{P(B_n)}, \quad \forall A\in\mathcal{F} \ \text{ with } A\subset B_n,$$

$$\mathrm{E}(\tilde{x}|\mathbb{I}_{B_n} = 1) = \int_{B_n} \frac{1}{P(B_n)}\tilde{x}dP$$

$\Longrightarrow$

$$\begin{aligned}\sum_n P(B_n)\mathrm{E}(\tilde{x}\,|\mathbb{I}_{B_n} = 1) &= \sum_n P(B_n)\frac{1}{P(B_n)}\int_{B_n}\tilde{x}dP \\ &= \sum_n\int_{B_n}\tilde{x}dP = \int_{\Omega}\tilde{x}dP = \mathrm{E}(\tilde{x}).\end{aligned}$$

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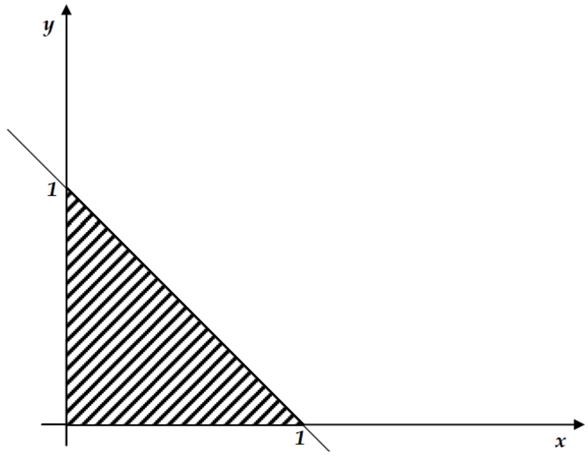
17.



$$\begin{aligned}
\mu_{\tilde{x}} &= \mathrm{E}(\tilde{x}) = \int_0^1 \int_0^{1-x} 2x dy dx = \frac{1}{3} \\
\mu_{\tilde{y}} &= \mathrm{E}(\tilde{y}) = \int_0^1 \int_0^{1-x} 2y dy dx = \frac{1}{3} \\
\mathrm{E}(\tilde{x} \cdot \tilde{y}) &= \int_0^1 \int_0^{1-x} 2xy dy dx = \frac{1}{12} \\
\sigma^2_{\tilde{x},\tilde{y}} &= \mathrm{Cov}(\tilde{x}, \tilde{y}) = \mathrm{E}(\tilde{x} \cdot \tilde{y}) - \mathrm{E}(\tilde{x}) \cdot \mathrm{E}(\tilde{y}) = \frac{1}{12} - \frac{1}{9} = -\frac{1}{36}.
\end{aligned}$$

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18.

$$M_{\tilde{x}}(t) = \mathrm{E}\left(e^{t\tilde{x}}\right) = \sum_{x \in \tilde{x}(\Omega)} e^{tx} f_{\tilde{x}}(x),$$

where $\tilde{x} =$ the number of black balls extracted in the second round. Let $\tilde{y}$ the number of black balls extracted in the first round. From the theorem of total probability, we have

$$P\{\tilde{x} = 0\} = P\{\tilde{y} = 0\} \cdot P\{\tilde{x} = 0 \,|\tilde{y} = 0\} + P\{\tilde{y} = 1\} \cdot P\{\tilde{x} = 0 \,|\tilde{y} = 1\}$$

$$= \frac{2}{5}\left(\frac{2}{5} \cdot \frac{1}{4}\right) + \frac{3}{5} \cdot \frac{2}{5} = \frac{1}{25} + \frac{6}{25} = \frac{7}{25}.$$

Similary,

$$P\{\tilde{x}=1\} = P\{\tilde{y}=0\}\cdot P\{\tilde{x}=1\,|\tilde{y}=0\} + P\{\tilde{y}=1\}\cdot P\{\tilde{x}=1\,|\tilde{y}=1\}$$

$$= \frac{2}{5}\left[\left(\frac{3}{5}\cdot\frac{2}{4}\right)+\left(\frac{2}{5}\cdot\frac{3}{4}\right)\right]+\frac{3}{5}\cdot\frac{3}{5}=\frac{15}{25},$$

and

$$P\{\tilde{x}=2\} = P\{\tilde{y}=0\}\cdot P\{\tilde{x}=2\,|\tilde{y}=0\} + P\{\tilde{y}=1\}\cdot \underbrace{P\{\tilde{x}=2\,|\tilde{y}=1\}}_{=0}$$

$$P\{\tilde{x}=2\} = \frac{2}{5}\left(\frac{3}{5}\cdot\frac{2}{4}\right)+0=\frac{3}{25}.$$

Thus, the moment-generating function is

$$M_{\tilde{x}}(t) = \mathrm{E}\left(e^{t\tilde{x}}\right) = \frac{7}{25}e^{t\cdot 0}+\frac{15}{25}e^{t\cdot 1}+\frac{3}{25}e^{t\cdot 2}=\frac{1}{25}(7+15e^{t}+3e^{2t}).$$

Note that $M_{\tilde{x}}(0)=1$

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19. Obviously,

$$\begin{aligned} M_{\tilde{x}}(t) &= \mathrm{E}\left(e^{t\tilde{x}}\right)=\sum_{k=0}^{\infty}e^{tk}f(k)=p+e^{t}pq+e^{2t}q^{2}p+e^{3t}q^{3}p+\dots \\ &= p\left[\left(e^{t}q\right)^{0}+\left(e^{t}q\right)^{1}+\left(e^{t}q\right)^{2}+\dots\right]=\frac{p}{1-e^{t}q}, \end{aligned}$$

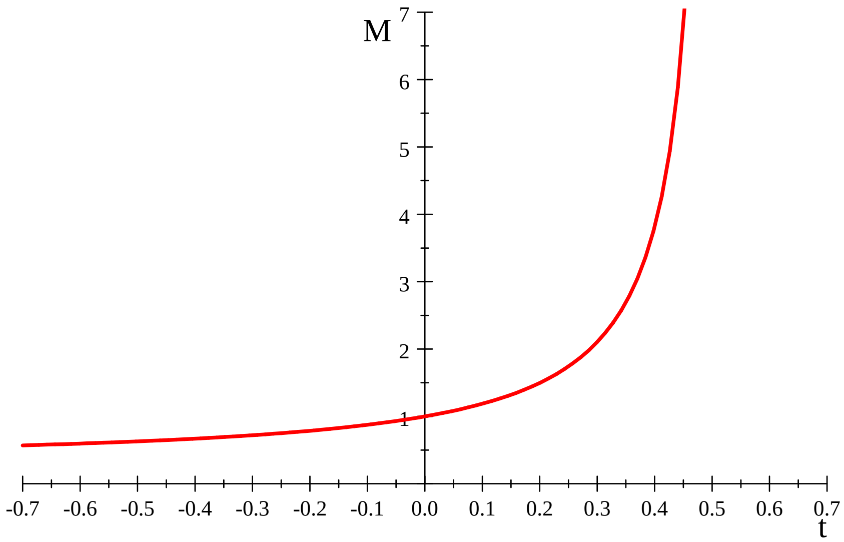
when $e^{t}q<1$, that is, when $t<-\ln q$. Note that $-\ln q>0$ since $q\in(0,1)$. Therefore, $M_{\tilde{x}}(t)$ is finite in a neighborhood of $t=0$.

Note that

$$M_{\tilde{x}}(0) = \frac{p}{1-q} = 1$$

since $p + q = 1$.

The following graph shows the moment generation function for $p = 0.4$, $q = 1 - p = 0.6$. We see that it is finite for $t < -\ln 0.6 = 0.5108$. The function has an asymptote at $t = -\ln 0.6 = 0.5108$.



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20. We know that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots.$$

Therefore,

$$M_{\tilde{x}}(t) = \mathrm{E}(e^{t\tilde{x}}) = \mathrm{E}(1) + \mathrm{E}\left(\tilde{x}\right)t + \mathrm{E}\left(\tilde{x}^2\right)\frac{t^2}{2!} + \mathrm{E}\left(\tilde{x}^3\right)\frac{t^3}{3!} + \dots$$

Since $\mathrm{E}\left(\tilde{x}^k\right) = k$, we have

$$\mathrm{E}\left(e^{t\tilde{x}}\right) = 1 + t + 2\frac{t^2}{2!} + 3\frac{t^3}{3!} + \ldots$$

$$1 + t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + ... = 1 + te^t.$$

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21. We know that $M_{\tilde{x}}(0) = \mathrm{E}\left(e^0\right) = 1$ for all $\tilde{x}$. However, $g(0) = 4$, thus $g(t)$ can not be a moment-generating function.

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22. (a)

$$M_{\tilde{x}}(t) = p_0 e^{0t} + p_1 e^{1t} + p_2 e^{2t} = p_0 + p_1 e^t + p_2 e^{2t},$$

with $p_2 = 1 - p_0 - p_1$.

(b)

$$M'_{\tilde{x}}(t) = p_1 e^t + 2p_2 e^{2t},$$

and, thus,

$$\mathrm{E}(\tilde{x}) = M'_{\tilde{x}}(0) = p_1 + 2p_2 = p_1 + 2\left(1 - p_0 - p_1\right) = 2 - 2p_0 - p_1.$$

(c)

$$M''_{\tilde{x}}(t) = p_1 e^t + 4p_2 e^{2t}$$

and, thus,

$$\mathrm{E}(\tilde{x}^2) = M''_{\tilde{x}}(0) = p_1 + 4p_2 = p_1 + 4\,(1 - p_0 - p_1) = 4 - 4p_0 - 3p_1$$

(d)

$$\begin{aligned}
\mathrm{Var}(\tilde{x}) &= \mathrm{E}(\tilde{x}^2) - [\mathrm{E}(\tilde{x})]^2 = 4 - 4p_0 - 3p_1 - (2 - 2p_0 - p_1)^2 \\
&= 4p_0 - 4p_0^2 - 4p_0p_1 + p_1 - p_1^2.
\end{aligned}$$

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23. (a) $k + \frac{k}{2} + \frac{k}{3} = \frac{11}{6}k$. Thus, $k = \frac{6}{11}$.

(b) $M_{\tilde{x}}(t) = \mathrm{E}\left(e^{t\tilde{x}}\right) = \frac{6}{11}e^t + \frac{3}{11}e^{2t} + \frac{2}{11}e^{3t}$.

(c) $M'_{\tilde{x}}(t) = \frac{6}{11}e^t + \frac{6}{11}e^{2t} + \frac{6}{11}e^{3t}$, $\mathrm{E}\left(\tilde{x}\right) = M'_{\tilde{x}}(0) = \frac{18}{11}$.

$M''_{\tilde{x}}(t) = \frac{6}{11}e^t + \frac{12}{11}e^{2t} + \frac{18}{11}e^{3t}$, $\mathrm{E}\left(\tilde{x}^2\right) = M''_{\tilde{x}}(0) = \frac{36}{11}$.

$M^{IV}_{\tilde{x}}(t) = \frac{6}{11}e^t + \frac{48}{11}e^{2t} + \frac{162}{11}e^{3t}$, $\mathrm{E}\left(\tilde{x}^4\right) = M^{IV}_{\tilde{x}}(0) = \frac{216}{11} = 19.636$.

(d) $\mathrm{Var}\left(\tilde{x}\right) = \mathrm{E}\left(\tilde{x}^2\right) - \left[\mathrm{E}\left(\tilde{x}\right)\right]^2 = \frac{36}{11} - \left(\frac{18}{11}\right)^2 = \frac{72}{121} = 0.595$.

(e) $\mathrm{Var}\left(\tilde{y}\right) = \mathrm{Var}\left(3\tilde{x}^2 + 4\right) = 9\mathrm{Var}\left(\tilde{x}^2\right) = 9\left[\mathrm{E}\left(\tilde{x}^4\right) - \left[\mathrm{E}\left(\tilde{x}^2\right)\right]^2\right]$

$= 9\left[\frac{216}{11} - \left(\frac{36}{11}\right)^2\right] = \frac{9720}{121} = 80.331$.

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24. (a)

$$f \in \mathrm{L}^p \iff \int_\Omega |f|^p \, d\mu < \infty.$$

Let us consider just the non-trivial case where $s \geq r > 0$. Thus,

$$\int_\Omega |f|^r \, d\mu = \int_{\{|f|^r<1\}} |f|^r \, d\mu + \int_{\{|f|^r\geq 1\}} |f|^r \, d\mu \leq \int_{\{|f|^r<1\}} d\mu + \int_{\{|f|^r\geq 1\}} \left(|f|^r\right)^{s/r} d\mu$$

$$\leq \mu\{\omega \in \Omega \,||f(\omega)|^r < 1\} + \int_{\{|f|^r \geq 1\}} |f|^s \, d\mu + \int_{\{|f|^r < 1\}} |f|^s \, d\mu$$

$$\leq \mu(\Omega) + \int_{\Omega} |f|^s \, d\mu < \infty$$

since the first term is finite because μ is finite and the second term is finite because $f \in \mathrm{L}^s$.

(b)

$$\int_{\mathbb{R}} |f(x)| \, dx = \int_{[1,\infty)} \frac{1}{x} dx = \int_1^{\infty} x^{-1} dx =$$

$$\lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} (\ln b) - 0 = \infty \Rightarrow f \notin \mathrm{L}^1.$$

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \int_{[1,\infty)} \frac{1}{x^2} dx = \int_1^{\infty} x^{-2} dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^b = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = 0 - (-1) = 1 \Rightarrow f \in \mathrm{L}^2.$$

This example does not contradict the result in (a) since the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ is not finite (it is just σ-finite).

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25. (a) If $x \in (0,1)$ and $y \in (0,1)$, then

$$F(x,y) = \int_0^x \int_0^y \frac{6}{5} \left(x + y^2\right) dy dx = \frac{6}{5} \left(\frac{x^2 y}{2} + \frac{x y^3}{3}\right)$$

Moreover, $F(x,y) = F(x,1)$ if $y > 1$, $F(x,y) = F(1,y)$ if $x > 1$. Therefore,

$$F(x,y) = \begin{cases} 0, & \text{if } x \leq 0 \text{ or } y \leq 0 \\ \frac{6}{5}\left(\frac{x^2 y}{2} + \frac{xy^3}{3}\right), & \text{if } x \in (0,1) \text{ and } y \in (0,1) \\ \frac{6}{5}\left(\frac{x^2}{2} + \frac{x}{3}\right), & \text{if } x \in (0,1) \text{ and } y \geq 1 \\ \frac{6}{5}\left(\frac{y}{2} + \frac{y^3}{3}\right), & \text{if } x \geq 1 \text{ and } y \in (0,1) \\ 1, & \text{if } x \geq 1 \text{ and } y \geq 1, \end{cases}$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{6}{5}\left(x + y^2\right) \text{ if } x \in (0,1) \text{ and } y \in (0,1),$$

and $\frac{\partial^2 F(x,y)}{\partial x \partial y} = 0$ if (x,y) lies in the interior of the complement of $(0,1) \times (0,1)$, which is the complement of $[0,1] \times [0,1]$. Note that the boundary of $(0,1) \times (0,1)$ has zero Lebesgue measure in $\mathbb{R}^2$

(b)

$$f_{\tilde{y}}(y) = \int_0^1 \frac{6}{5}\left(x + y^2\right) dx = \frac{3}{5}\left(2y^2 + 1\right) \text{ if } y \in (0,1),$$

and $f_{\tilde{y}}(y) = 0$ otherwise.

$\Rightarrow$

$$f_{\tilde{y}}\left(\frac{1}{3}\right) = \frac{11}{15}$$

and

$$f\left(x, \frac{1}{3}\right) = \begin{cases} \frac{6}{5}\left(x + \frac{1}{9}\right) & \text{for } x \in (0,1) \\ \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$

$$f_{\tilde{x}|\tilde{y}}\left(x \,\middle|\, \frac{1}{3}\right) = \begin{cases} \frac{\frac{6}{5}\left(x + \frac{1}{9}\right)}{\frac{11}{15}} = \frac{18x+2}{11} & \text{for } x \in (0,1) \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\mathrm{Var}\left(\tilde{x} \,\middle|\, \tilde{y} = \frac{1}{3}\right) = \mathrm{E}\left(\tilde{x}^2 \,\middle|\, \tilde{y} = \frac{1}{3}\right) - \left[\mathrm{E}\left(\tilde{x} \,\middle|\, \tilde{y} = \frac{1}{3}\right)\right]^2$$

$$= \int_0^1 x^2 \left(\frac{18x+2}{11}\right) dx - \left[\int_0^1 x \left(\frac{18x+2}{11}\right)\right]^2 = \underbrace{\frac{31}{66}}_{0.4697} - \underbrace{\left(\frac{7}{11}\right)^2}_{(0.6364)^2} = \frac{47}{726} = 0.06474.$$

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26. (a) $f_{\tilde{x},\tilde{y},\tilde{z}} : \tilde{x}\left(\Omega\right) \times \tilde{y}\left(\Omega\right) \times \tilde{z}\left(\Omega\right) \longrightarrow \left[0,1\right],$

$$f_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z) = \begin{cases} \dfrac{1}{4} & \text{if } x=1, y=0, z=1 \\ \\ \dfrac{1}{4} & \text{if } x=1, y=0, z=0 \\ \\ \dfrac{1}{4} & \text{if } x=0, y=1, z=1 \\ \\ \dfrac{1}{4} & \text{if } x=0, y=1, z=0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

(b) $f_{\tilde{x},\tilde{y}} : \tilde{x}\left(\Omega\right) \times \tilde{y}\left(\Omega\right) \longrightarrow \left[0,1\right]$, $f_{\tilde{x},\tilde{z}} : \tilde{x}\left(\Omega\right) \times \tilde{z}\left(\Omega\right) \longrightarrow \left[0,1\right]$, and $f_{\tilde{y},\tilde{z}} : \tilde{y}\left(\Omega\right) \times \tilde{z}\left(\Omega\right) \longrightarrow \left[0,1\right]$.

$$f_{\tilde{x},\tilde{y}}(x,y) = \begin{cases} \dfrac{1}{2} & \text{if } x=1, y=0 \\ \\ \dfrac{1}{2} & \text{if } x=0, y=1 \\ \\ 0 & \text{otherwise,} \end{cases}$$

$$f_{\tilde{x},\tilde{z}}(x,z) = \begin{cases} \dfrac{1}{4} & \text{if } x=1, z=1 \\[2em] \dfrac{1}{4} & \text{if } x=1, z=0 \\[2em] \dfrac{1}{4} & \text{if } x=0, z=1 \\[2em] \dfrac{1}{4} & \text{if } x=0, z=0, \end{cases}$$

$$f_{\tilde{y},\tilde{z}}(y,z) = \begin{cases} \dfrac{1}{4} & \text{if } y=0, z=1 \\[2em] \dfrac{1}{4} & \text{if } y=0, z=0 \\[2em] \dfrac{1}{4} & \text{if } y=1, z=1 \\[2em] \dfrac{1}{4} & \text{if } y=1, z=0. \end{cases}$$

Finally,

$$f_{\tilde{x}}(x) = \begin{cases} \dfrac{1}{2} & \text{if } x=1 \\[2em] \dfrac{1}{2} & \text{if } x=0, \end{cases}$$

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{1}{2} & \text{if } y=1 \\[2em] \dfrac{1}{2} & \text{if } y=0, \end{cases}$$

$$f_{\tilde{z}}(z) = \begin{cases} \dfrac{1}{2} & \text{if } z = 1 \\ \\ \dfrac{1}{2} & \text{if } z = 0. \end{cases}$$

The following tables summarize the joint probability functions and the corresponding marginal probability functions:

$y \backslash x$	0	1	
0	0	1/2	1/2
1	1/2	0	1/2
	1/2	1/2	

$z \backslash x$	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	

$z \backslash y$	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	

We see that $\tilde{x}$ and $\tilde{y}$ are not independent since, for instance,

$$f_{\tilde{x},\tilde{y}}(1,0) = \frac{1}{2} \neq f_{\tilde{x}}(1) \cdot f_{\tilde{y}}(0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Moreover, $\tilde{x}$ and $\tilde{z}$ are independent since $f_{\tilde{x},\tilde{z}}(x,z) = f_{\tilde{x}}(x) \cdot f_{\tilde{z}}(z)$ for all $(x,z) \in \tilde{x}(\Omega) \times \tilde{z}(\Omega) = \{0,1\} \times \{0,1\}$.

Finally, $\tilde{y}$ and $\tilde{z}$ are also independent since $f_{\tilde{y},\tilde{z}}(y,z) = f_{\tilde{y}}(y) \cdot f_{\tilde{z}}(z)$ for all $(y,z) \in \tilde{y}(\Omega) \times \tilde{z}(\Omega) = \{0,1\} \times \{0,1\}$.

(c)

$$\mathrm{Cov}(\tilde{x},\tilde{y}) = \mathrm{E}(\tilde{x}\cdot\tilde{y}) - \mathrm{E}(\tilde{x})\,\mathrm{E}(\tilde{y}) = 0 - \frac{1}{2}\cdot\frac{1}{2} = -\frac{1}{4},$$

$\mathrm{Cov}(\tilde{x},\tilde{z}) = 0$ (from independency) and $\mathrm{Cov}(\tilde{y},\tilde{z}) = 0$ (from independency).

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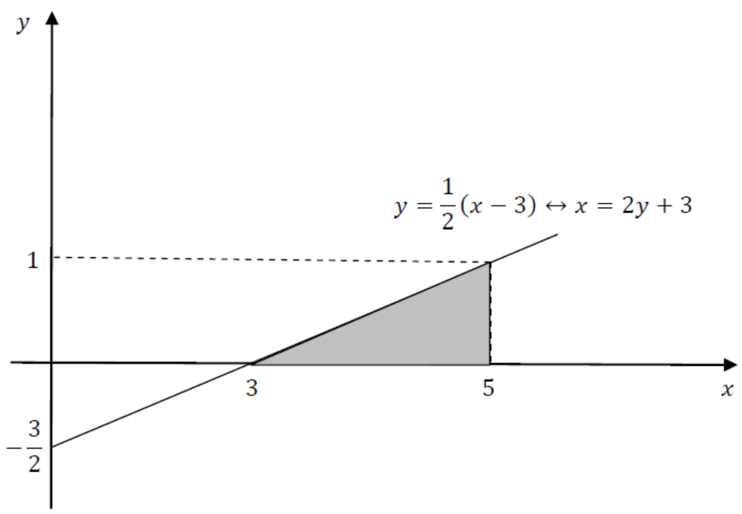
27.



(a) $\int_3^5 \int_0^{\frac{1}{2}(x-3)} ky\,dy\,dx = \frac{k}{3} = 1 \Longrightarrow k = 3.$

(b) $\mathrm{E}(\tilde{x}) = \int_3^5 \int_0^{\frac{1}{2}(x-3)} x \cdot 3y\,dy\,dx = \frac{9}{2}$, $\mathrm{E}(\tilde{y}) = \int_3^5 \int_0^{\frac{1}{2}(x-3)} y \cdot 3y\,dy\,dx = \frac{1}{2}.$

(c) $\mathrm{E}(\tilde{x}^2) = \int_3^5 \int_0^{\frac{1}{2}(x-3)} x^2 \cdot 3y\,dy\,dx = \frac{102}{5}$ so that $\mathrm{Var}(\tilde{x}) = \mathrm{E}(\tilde{x}^2) - [\mathrm{E}(\tilde{x})]^2 = \frac{102}{5} - \left(\frac{9}{2}\right)^2 = \frac{3}{20}$;

$\mathrm{E}(\tilde{y}^2) = \int_3^5 \int_0^{\frac{1}{2}(x-3)} y^2 \cdot 3y\,dy\,dx = \frac{3}{10}$ so that $\mathrm{Var}(\tilde{y}) = \mathrm{E}(\tilde{y}^2) - [\mathrm{E}(\tilde{y})]^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}.$

(d) $\mathrm{E}(\tilde{x}\cdot\tilde{y}) = \int_3^5 \int_0^{\frac{1}{2}(x-3)} xy\cdot 3ydydx = \dfrac{23}{10}$ so that $\mathrm{Cov}(\tilde{x},\tilde{y}) = \dfrac{23}{10} - \left(\dfrac{9}{2}\cdot\dfrac{1}{2}\right) = \dfrac{1}{20}$.

$$\rho = \frac{\mathrm{Cov}(\tilde{x},\tilde{y})}{\sqrt{\mathrm{Var}(\tilde{x})}\cdot\sqrt{\mathrm{Var}(\tilde{y})}} = \frac{\frac{1}{20}}{\sqrt{\frac{3}{20}}\cdot\sqrt{\frac{1}{20}}} = \frac{1}{\sqrt{3}} = 0.57735.$$

Since $\mathrm{Cov}(\tilde{x},\tilde{y}) \neq 0$, then the random variables $\tilde{x}$ and $\tilde{y}$ are not independent.

(e)

$$f_{\tilde{y}}(y) = \begin{cases} \int_{2y+3}^5 3ydx = 6y(1-y) & \text{if } y\in(0,1) \\ \\ 0 & \text{otherwise.}\end{cases}$$

(f)

$$f_{\tilde{x},\tilde{y}}(x,3/4) = \begin{cases} 3\cdot\frac{3}{4} = \frac{9}{4} & \text{if } x\in\left(\frac{9}{2},5\right) \\ \\ 0 & \text{otherwise.}\end{cases}$$

$$f_{\tilde{y}}\left(\frac{3}{4}\right) = 6\cdot\frac{3}{4}\cdot\left(1-\frac{3}{4}\right) = \frac{9}{8}.$$

$$f_{\tilde{x}|\tilde{y}}(x\,|3/4) = \begin{cases} \dfrac{f_{\tilde{x},\tilde{y}}(x,3/4)}{f_{\tilde{y}}(3/4)} = \dfrac{9/4}{9/8} = 2 & \text{if } x\in\left(\frac{9}{2},5\right) \\ \\ 0 & \text{otherwise.}\end{cases}$$

(g) $\mathrm{E}(\tilde{x}\,|\tilde{y}=3/4) = \int_{\frac{9}{2}}^5 x\cdot 2dx = \dfrac{19}{4}$.

(h) $\mathrm{Var}(\tilde{x}\,|\tilde{y}=3/4) = \mathrm{E}(\tilde{x}^2\,|\tilde{y}=3/4) - [\mathrm{E}(\tilde{x}\,|\tilde{y}=3/4)]^2 = \dfrac{271}{12} - \left(\dfrac{19}{4}\right)^2 = \dfrac{1}{48}$.

since $\mathrm{E}(\tilde{x}^2\,|\tilde{y}=3/4) = \int_{\frac{9}{2}}^{5} x^2 \cdot 2dx = \dfrac{271}{12}$.

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28. (a) The partition induced on Ω by $\tilde{x}$ is $F(\tilde{x}) = \{\{1,4\},\{3\},\{2,5\},\{6\}\}$. The σ-algebra generated by the partition $F(\tilde{x})$ is the σ-algebra induced on Ω by $\tilde{x}$,

$$\mathcal{F}(\tilde{x}) = \{\emptyset, \Omega, \{1,4\},\{3\},\{2,5\},\{6\},\{1,3,4\},\{1,2,4,5\},\{1,4,6\},\{2,3,5\},$$
$$\{3,6\},\{2,5,6\},\{2,3,5,6\},\{1,2,4,5,6\},\{1,3,4,6\},\{1,2,3,4,5\}\}.$$

(b)

$$\mathrm{E}\left(\tilde{x}\,|\mathcal{F}(\tilde{x})\right) = \tilde{x} = \begin{cases} 0 & \text{if } \omega \in \{1,4\} \\ 1 & \text{if } \omega = 3 \\ 2 & \text{if } \omega \in \{2,5\} \\ 3 & \text{if } \omega = 6; \end{cases}$$

$$\mathcal{G} = \{\emptyset, \Omega, \{1\},\{2,3,4\},\{5,6\},\{1,2,3,4\},\{1,5,6\},\{2,3,4,5,6\}\};$$

Since

$$P\{\tilde{x}=0\,|\{1\}\} = \frac{P(\{1,4\}\cap\{1\})}{P\{1\}} = \frac{P\{1\}}{P\{1\}} = 1,$$

$$P\{\tilde{x}=0\,|\{2,3,4\}\} = \frac{P(\{1,4\}\cap\{2,3,4\})}{P\{2,3,4\}} = \frac{P\{4\}}{P\{2,3,4\}} = \frac{1}{3},$$

$$P\{\tilde{x}=0\,|\{5,6\}\} = \frac{P(\{1,4\}\cap\{5,6\})}{P\{5,6\}} = \frac{P(\emptyset)}{P\{5,6\}} = 0,$$

$$P\{\tilde{x}=1\,|\{1\}\} = \frac{P(\{3\}\cap\{1\})}{P\{1\}} = \frac{P(\emptyset)}{P\{1\}} = 0,$$

$$P\{\tilde{x}=1\,|\{2,3,4\}\} = \frac{P(\{2,3,4\}\cap\{3\})}{P\{2,3,4\}} = \frac{P\{3\}}{P\{2,3,4\}} = \frac{1}{3},$$

$$P\{\tilde{x}=1\,|\{5,6\}\}=\frac{P(\{3\}\cap\{5,6\})}{P\{5,6\}}=\frac{P(\emptyset)}{P\{5,6\}}=0,$$

$$P\{\tilde{x}=2\,|\{1\}\}=\frac{P(\{2,5\}\cap\{1\})}{P\{1\}}=\frac{P(\emptyset)}{P\{1\}}=0,$$

$$P\{\tilde{x}=2\,|\{2,3,4\}\}=\frac{P(\{2,5\}\cap\{2,3,4\})}{P\{2,3,4\}}=\frac{P\{2\}}{P\{2,3,4\}}=\frac{1}{3},$$

$$P\{\tilde{x}=2\,|\{5,6\}\}=\frac{P(\{2,5\}\cap\{5,6\})}{P\{5,6\}}=\frac{P\{5\}}{P\{5,6\}}=\frac{1}{2},$$

$$P\{\tilde{x}=3\,|\{1\}\}=\frac{P(\{6\}\cap\{1\})}{P\{1\}}=\frac{P(\emptyset)}{P\{1\}}=0,$$

$$P\{\tilde{x}=3\,|\{2,3,4\}\}=\frac{P(\{6\}\cap\{2,3,4\})}{P\{2,3,4\}}=\frac{P(\emptyset)}{P\{2,3,4\}}=0,$$

$$P\{\tilde{x}=3\,|\{5,6\}\}=\frac{P(\{6\}\cap\{5,6\})}{P\{5,6\}}=\frac{P\{6\}}{P\{5,6\}}=\frac{1}{2},$$

then

$$\mathrm{E}(\tilde{x}\,|\mathcal{G})=\begin{cases}0\cdot 1+1\cdot 0+2\cdot 0+3\cdot 0=0 & \text{if } \omega=1\\[2ex] 0\cdot\frac{1}{3}+1\cdot\frac{1}{3}+2\cdot\frac{1}{3}+3\cdot 0=1 & \text{if } \omega\in\{2,3,4\}\\[2ex] 0\cdot 0+1\cdot 0+2\cdot\frac{1}{2}+3\cdot\frac{1}{2}=\frac{5}{2} & \text{if } \omega\in\{5,6\}\,.\end{cases}$$

Since

$$P\{\tilde{x}=0\}=P\{1,4\}=\frac{1}{3},$$

$$P\{\tilde{x}=1\}=P\{3\}=\frac{1}{6},$$

$$P\{\tilde{x}=2\}=P\{2,5\}=\frac{1}{3},$$

$$P\{\tilde{x}=3\}=P\{6\}=\frac{1}{6},$$

then

$$\mathrm{E}\left(\tilde{x}\,|\mathcal{H}\right) = \mathrm{E}\left(\tilde{x}\right) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{6} = \frac{4}{3}$$

for $\omega \in \{1, 2, 3, 4, 5, 6\} = \Omega$.

Note that, since,

$$P\{1\} = \frac{1}{6},$$

$$P\{2, 3, 4\} = \frac{1}{2},$$

$$P\{5, 6\} = \frac{1}{3},$$

then

$$\mathrm{E}\left(\mathrm{E}\left(\tilde{x}\,|\mathcal{G}\right)\right) = \mathrm{E}\left(\mathrm{E}\left(\tilde{x}\,|\mathcal{G}\right)|\mathcal{H}\right) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{2} + \frac{5}{2} \cdot \frac{1}{3} = \frac{4}{3} = \mathrm{E}\left(\tilde{x}\,|\mathcal{H}\right) = \mathrm{E}\left(\tilde{x}\right).$$

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29. (a)

$$\mathrm{Cov}\left(\tilde{y}, \sum_{i=1}^{n} c_i \tilde{x}_i\right) = \mathrm{E}\left([\tilde{y} - \mathrm{E}(\tilde{y})] \cdot \left[\sum_{i=1}^{n} c_i \tilde{x}_i - \mathrm{E}\left(\sum_{i=1}^{n} c_i \tilde{x}_i\right)\right]\right) =$$

$$\mathrm{E}\left([\tilde{y} - \mathrm{E}(\tilde{y})] \cdot \left[\sum_{i=1}^{n} c_i \tilde{x}_i - \left(\sum_{i=1}^{n} c_i \mathrm{E}\left(\tilde{x}_i\right)\right)\right]\right) = \mathrm{E}\left([\tilde{y} - \mathrm{E}(\tilde{y})] \cdot \left[\sum_{i=1}^{n} \left(c_i \tilde{x}_i - c_i \mathrm{E}\left(\tilde{x}_i\right)\right)\right]\right)$$

$$= \mathrm{E}\left[\sum_{i=1}^{n} c_i \left(\tilde{y} - \mathrm{E}(\tilde{y})\right) \cdot \left(\tilde{x}_i - \mathrm{E}\left(\tilde{x}_i\right)\right)\right] = \sum_{i=1}^{n} c_i \cdot \mathrm{E}\left[\left(\tilde{y} - \mathrm{E}(\tilde{y})\right) \cdot \left(\tilde{x}_i - \mathrm{E}\left(\tilde{x}_i\right)\right)\right]$$

$$= \sum_{i=1}^{n} c_i \mathrm{Cov}\left(\widetilde{y}, \tilde{x}_i\right).$$

(b) Make $\widetilde{y}=\sum\limits_{j=1}^{m} b_j\tilde{y}_j$ in (a) so that

$$\operatorname{Cov}\left(\sum_{j=1}^{m} b_j\tilde{y}_j, \sum_{i=1}^{n} c_i\tilde{x}_i\right)=\sum_{i=1}^{n} c_i \operatorname{Cov}\left(\sum_{j=1}^{m} b_j\tilde{y}_j, \tilde{x}_i\right).$$

Then, use (a) again for $\operatorname{Cov}\left(\sum\limits_{j=1}^{m} b_j\tilde{y}_j, \tilde{x}_i\right)$ so that

$$\sum_{i=1}^{n} c_i \operatorname{Cov}\left(\sum_{j=1}^{m} b_j\tilde{y}_j, \tilde{x}_i\right)=\sum_{i=1}^{n} c_i \sum_{j=1}^{m} b_j \operatorname{Cov}\left(\tilde{y}_j, \tilde{x}_i\right)=\sum_{j=1}^{m}\sum_{i=1}^{n} b_j c_i \operatorname{Cov}\left(\tilde{y}_j, \tilde{x}_i\right).$$

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30. (a) Equality (iii) is always true. We know that

$$\operatorname{Var}(\tilde{x}\,|\tilde{y}\,)=\operatorname{E}\left(\tilde{x}^2\,|\tilde{y}\,\right)-\left[\operatorname{E}(\tilde{x}\,|\tilde{y}\,)\right]^2$$

so that, using the theorem of total expectation, $\operatorname{E}(\operatorname{E}(\tilde{x}\,|\tilde{y}\,))=\operatorname{E}(\tilde{x})$, we have

$$\operatorname{E}(\operatorname{Var}(\tilde{x}\,|\tilde{y}\,))=\operatorname{E}\left(\operatorname{E}\left(\tilde{x}^2\,|\tilde{y}\,\right)-\left[\operatorname{E}(\tilde{x}\,|\tilde{y}\,)\right]^2\right)$$

$$=\operatorname{E}\left(\operatorname{E}\left(\tilde{x}^2\,|\tilde{y}\,\right)\right)-\operatorname{E}\left(\left[\operatorname{E}(\tilde{x}\,|\tilde{y}\,)\right]^2\right)=\operatorname{E}\left(\tilde{x}^2\right)-\operatorname{E}\left(\left[\operatorname{E}(\tilde{x}\,|\tilde{y}\,)\right]^2\right)$$

and

$$\operatorname{Var}(\operatorname{E}(\tilde{x}\,|\tilde{y}\,))=\operatorname{E}\left(\left[\operatorname{E}(\tilde{x}\,|\tilde{y}\,)\right]^2\right)-\left[\operatorname{E}(\operatorname{E}(\tilde{x}\,|\tilde{y}\,))\right]^2=\operatorname{E}\left(\left[\operatorname{E}(\tilde{x}\,|\tilde{y}\,)\right]^2\right)-\left[\operatorname{E}(\tilde{x})\right]^2.$$

Therefore,

$$\operatorname{E}(\operatorname{Var}(\tilde{x}\,|\tilde{y}\,))+\operatorname{Var}(\operatorname{E}(\tilde{x}\,|\tilde{y}\,))=\operatorname{E}\left(\tilde{x}^2\right)-\left[\operatorname{E}(\tilde{x})\right]^2=\operatorname{Var}(\tilde{x}).$$

This relationship is called Eve's law (EVE: Expectation-Variance-Expectation.) In fact, it should be called Evve's law since EVVE stands for Expectation-Variance-Variance-Expectation.

(b) If $\tilde{x}$ and $\tilde{y}$ are independent then $\mathrm{E}(\tilde{x}\,|\tilde{y}\,) = \mathrm{E}(\tilde{x})$ and $\mathrm{Var}(\tilde{x}\,|\tilde{y}\,) = \mathrm{Var}(\tilde{x})$.

In this case, equality (ii) is also true,

$$\mathrm{E}(\mathrm{Var}(\tilde{x}\,|\tilde{y}\,)) = \mathrm{E}(\mathrm{Var}(\tilde{x})) = \mathrm{Var}(\tilde{x})$$

since $\mathrm{Var}(\tilde{x})$ is a real number.

Equality (iv) is also true when $\tilde{x}$ and $\tilde{y}$ are independent. In this case,

$$\mathrm{Var}(\mathrm{E}(\tilde{x}\,|\tilde{y}\,)) = \mathrm{Var}(\mathrm{E}(\tilde{x})) = 0,$$

since $\mathrm{E}(\tilde{x})$ is a real number. Moreover, we know that $\mathrm{E}(\mathrm{Var}(\tilde{x})) = \mathrm{Var}(\tilde{x})$ always holds. Therefore,

$$\mathrm{E}(\mathrm{Var}(\tilde{x})) + \mathrm{Var}(\mathrm{E}(\tilde{x}\,|\tilde{y}\,)) = \mathrm{Var}(\tilde{x}) + 0 = \mathrm{Var}(\tilde{x})\,.$$

Note that the equality (i) cannot be true in general since, if $\tilde{x}$ and $\tilde{y}$ are independent, then $\mathrm{E}(\tilde{x}\,|\tilde{y}\,) = \mathrm{E}(\tilde{x})$. In this case, $\mathrm{Var}(\mathrm{E}(\tilde{x}\,|\tilde{y}\,)) = \mathrm{Var}(\mathrm{E}(\tilde{x})) = 0$ since $\mathrm{E}(\tilde{x})$ is a real number and, hence, $\mathrm{Var}(\mathrm{E}(\tilde{x}\,|\tilde{y}\,)) = 0 \neq \mathrm{Var}(\tilde{x}) > 0$.

(c)

$$\mathrm{E}\left[(\tilde{x} - \mathrm{E}(\tilde{x}\,|\tilde{y}\,)) \cdot h(\tilde{y})\right] = \mathrm{E}\left[\tilde{x}h(\tilde{y}) - h(\tilde{y})\,\mathrm{E}(\tilde{x}\,|\tilde{y}\,)\right]$$

$$= \mathrm{E}\left[\tilde{x}h(\tilde{y})\right] - \mathrm{E}\left[h(\tilde{y})\,\mathrm{E}(\tilde{x}\,|\tilde{y}\,)\right]$$

$$= \mathrm{E}\left[\tilde{x}h(\tilde{y})\right] - \mathrm{E}\left[\mathrm{E}(\tilde{x}h(\tilde{y})\,|\tilde{y}\,)\right] = \mathrm{E}\left[\tilde{x}h(\tilde{y})\right] - \mathrm{E}\left[\tilde{x}h(\tilde{y})\right] = 0. \qquad (*)$$

Note that

$$\operatorname{Cov}\left[\left(\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right), h\left(\tilde{y}\right)\right]=\mathrm{E}\left[\left(\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right)\cdot h\left(\tilde{y}\right)\right]-\left(\mathrm{E}\left[\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right]\right)\cdot\mathrm{E}\left[h\left(\tilde{y}\right)\right]$$

We already know that $\mathrm{E}\left[\left(\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right)\cdot h\left(\tilde{y}\right)\right]=0$ from $(*)$ so that

$$\operatorname{Cov}\left[\left(\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right), h\left(\tilde{y}\right)\right]=0-\underbrace{\left(\mathrm{E}\left[\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right]\right)}_{=0}\cdot\mathrm{E}\left[h\left(\tilde{y}\right)\right]=0$$

since $\mathrm{E}\left[\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right]=\mathrm{E}\left(\tilde{x}\right)-\mathrm{E}\left[\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)\right]=\mathrm{E}\left(\tilde{x}\right)-\mathrm{E}\left(\tilde{x}\right)=0.$

This result tells us that the prediction error for $\tilde{x}$ using the information provided by $\tilde{y}$ should be uncorrelated with any function of $\tilde{y}$. Obviously, if there is some correlation between the error $\tilde{x}-\mathrm{E}\left(\tilde{x}\left|\tilde{y}\right.\right)$ and $h\left(\tilde{y}\right)$ then we should use this correlation to improve our prediction about $\tilde{x}$.

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31. (a) The partition induced on Ω by $\tilde{x}$ is $F(\tilde{x})=\left\{\{1,2,3\},\{4,5\},\{6\}\right\}$. The σ-algebra generated by the partition $F(\tilde{x})$ is the σ-algebra induced on Ω by $\tilde{x}$,

$$\mathcal{F}(\tilde{x})=\left\{\emptyset,\Omega,\{1,2,3\},\{4,5\},\{6\},\{1,2,3,4,5\},\{1,2,3,6\},\{4,5,6\}\right\}$$

(b)

$$\mathrm{E}\left(\tilde{x}\left|\mathcal{F}(\tilde{x})\right.\right)=\tilde{x}=\begin{cases}1 & \text{if } \omega\in\{1,2,3\}\\ 2 & \text{if } \omega\in\{4,5\}\\ 3 & \text{if } \omega=6;\end{cases}$$

$$\mathcal{G}=\left\{\emptyset,\Omega,\{1,2\},\{3,4\},\{5,6\},\{1,2,3,4\},\{1,2,5,6\},\{3,4,5,6\}\right\}.$$

Since

$$P\{\tilde{x}=1\,|\{1,2\}\}=\frac{P(\{1,2,3\}\cap\{1,2\})}{P\{1,2\}}=\frac{P\{1,2\}}{P\{1,2\}}=1,$$

$$P\{\tilde{x}=1\,|\{3,4\}\}=\frac{P(\{1,2,3\}\cap\{3,4\})}{P\{3,4\}}=\frac{P\{3\}}{P\{3,4\}}=\frac{1}{2},$$

$$P\{\tilde{x}=1\,|\{5,6\}\}=\frac{P(\{1,2,3\}\cap\{5,6\})}{P\{5,6\}}=\frac{P(\emptyset)}{P\{5,6\}}=0,$$

$$P\{\tilde{x}=2\,|\{1,2\}\}=\frac{P(\{4,5\}\cap\{1,2\})}{P\{1,2\}}=\frac{P(\emptyset)}{P\{1,2\}}=0,$$

$$P\{\tilde{x}=2\,|\{3,4\}\}=\frac{P(\{4,5\}\cap\{3,4\})}{P\{3,4\}}=\frac{P\{4\}}{P\{3,4\}}=\frac{1}{2},$$

$$P\{\tilde{x}=2\,|\{5,6\}\}=\frac{P(\{4,5\}\cap\{5,6\})}{P\{5,6\}}=\frac{P\{5\}}{P\{5,6\}}=\frac{1}{2},$$

$$P\{\tilde{x}=3\,|\{1,2\}\}=\frac{P(\{6\}\cap\{1,2\})}{P\{1,2\}}=\frac{P(\emptyset)}{P\{1,2\}}=0,$$

$$P\{\tilde{x}=3\,|\{3,4\}\}=\frac{P(\{6\}\cap\{3,4\})}{P\{3,4\}}=\frac{P(\emptyset)}{P\{3,4\}}=0,$$

$$P\{\tilde{x}=3\,|\{5,6\}\}=\frac{P(\{6\}\cap\{5,6\})}{P\{5,6\}}=\frac{P\{6\}}{P\{5,6\}}=\frac{1}{2},$$

we get

$$\mathrm{E}(\tilde{x}\,|\mathcal{G})=\begin{cases}1\cdot 1+2\cdot 0+3\cdot 0=1 & \text{if } \omega=\{1,2\}\\[2ex] 1\cdot\frac{1}{2}+2\cdot\frac{1}{2}+3\cdot 0=\frac{3}{2} & \text{if } \omega\in\{3,4\}\\[2ex] 1\cdot 0+2\cdot\frac{1}{2}+3\cdot\frac{1}{2}=\frac{5}{2} & \text{if } \omega\in\{5,6\}\,.\end{cases}$$

Since

$$P\{\tilde{x}=1\}=P\{1,2,3\}=\frac{1}{2},$$

$$P\{\tilde{x}=2\}=P\{4,5\}=\frac{1}{3},$$

$$P\{\tilde{x}=3\}=P\{6\}=\frac{1}{6},$$

we get

$$\mathrm{E}(\tilde{x}\,|\mathcal{H})=\mathrm{E}(\tilde{x})=1\cdot\frac{1}{2}+2\cdot\frac{1}{3}+3\cdot\frac{1}{6}=\frac{5}{3}$$

$$\text{for } \omega\in\{1,2,3,4,5,6\}=\Omega.$$

Note also that, since

$$P\{1,2\}=\frac{1}{2},$$

$$P\{3,4\}=\frac{1}{2},$$

$$P\{5,6\}=\frac{1}{2},$$

we get

$$\mathrm{E}(\mathrm{E}(\tilde{x}\,|\mathcal{G}))=\mathrm{E}(\mathrm{E}(\tilde{x}\,|\mathcal{G})\,|\mathcal{H})=1\cdot\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{2}+\frac{5}{2}\cdot\frac{1}{2}=\frac{5}{3}=\mathrm{E}(\tilde{x}\,|\mathcal{H})=\mathrm{E}(\tilde{x}).$$

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32. By making the corresponding Taylor expansion around zero, we get that

$$e^y=\sum_{r=0}^{\infty}\frac{y^r}{r!},$$

$$\cos y=1+0-\frac{y^2}{2!}+0+\frac{y^4}{4!}+...,$$

and

$$\sin y=0+y+0-\frac{y^3}{3!}+0+\frac{y^5}{5!}+...$$

Recall that $i = (0, 1)$ and $i^2 = -1 = (-1, 0)$. Therefore, for all $z \in \mathbb{R}$, we have

$$e^{iz} = \sum_{r=0}^{\infty} \frac{(iz)^r}{r!} = 1 + iz + \frac{i^2 z^2}{2!} + \frac{i^3 z^3}{3!} + \frac{i^4 z^4}{4!} + \frac{i^5 z^5}{5!} + ...$$

$$= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - ...,$$

$$\cos z = 1 + 0 - \frac{z^2}{2!} + 0 + \frac{z^4}{4!} + ...,$$

and

$$i \sin z = 0 + iz + 0 - \frac{iz^3}{3!} + 0 + \frac{iz^5}{5!} + ...$$

Hence, we obtain the famous Euler's formula,

$$e^{iz} = \cos z + i \sin z = 1 \cdot (\cos z) + i \cdot \sin z$$

$$= \cos z \cdot (1, 0) + \sin z \cdot (0, 1) = (\cos z, \sin z)\,, \text{ for all } z \in \mathbb{R},$$

and thus

$$e^{itx} = \cos(tx) + i \sin(tx) = (\cos(tx), \sin(tx))\,, \text{ for all } t \in \mathbb{R}, x \in \mathbb{R}.$$

Then,

$$\varphi_{\tilde{x}}(t) = \mathrm{E}\left(e^{it\tilde{x}}\right) = \mathrm{E}\left[\cos(t\tilde{x}) + i \sin(t\tilde{x})\right] = \int_{\mathbb{R}} \left[\cos(tx) + i \sin(tx)\right] dP_{\tilde{x}}(x)$$

$$= \int_{\mathbb{R}} \cos(tx)\, dP_{\tilde{x}}(x) + i \int_{\mathbb{R}} \sin(tx)\, dP_{\tilde{x}}(x)$$

$$= \mathrm{E}\left[\cos(t\tilde{x})\right] + i\mathrm{E}\left[\sin(t\tilde{x})\right] = \left(\mathrm{E}\left[\cos(t\tilde{x})\right], \mathrm{E}\left[\sin(t\tilde{x})\right]\right).$$

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33. (a) Let $t \geq 0$. The function $g(y) = e^{ty}$ is non-decreasing in y (its derivative is $g'(y) = te^{ty} \geq 0$ for $t \geq 0$). Thus,

$$P\{\tilde{y} \geq c\} = P\left\{e^{t\tilde{y}} \geq e^{ct}\right\}.$$

Define $\tilde{x} = e^{t\tilde{y}}$ and $a = e^{ct}$ and note that $\tilde{x}$ is non-negative and a is positive. Therefore, applying Markov's inequality,

$$P\{\tilde{y} \geq c\} = P\{\tilde{x} \geq a\} \leq \frac{\mathrm{E}(\tilde{x})}{a} = \frac{\mathrm{E}\left(e^{t\tilde{y}}\right)}{e^{ct}} = \frac{M_{\tilde{y}}(t)}{e^{ct}}.$$

Thus,

$$M_{\tilde{y}}(t) \geq e^{ct} P\{\tilde{y} \geq c\} \qquad \text{for all } c \in \mathbb{R} \text{ and all } t \geq 0.$$

Let $t \leq 0$. The function $g(y) = e^{ty}$ is non-increasing in y (its derivative is $g'(y) = te^{ty} \leq 0$ for $t \leq 0$). As follows from the previous argument, we thus have

$$P\{\tilde{y} \leq c\} = P\left\{e^{t\tilde{y}} \geq e^{ct}\right\} \leq \frac{M_{\tilde{y}}(t)}{e^{ct}} \quad \text{for all } c \in \mathbb{R} \text{ and all } t \leq 0,$$

so that

$$M_{\tilde{y}}(t) \geq e^{ct} P\{\tilde{y} \leq c\} \quad \text{for all } c \in \mathbb{R} \text{ and all } t \leq 0.$$

(b) Since $g(x) = e^{tx}$ is a convex function of x (its second derivative is $g''(x) = t^2 e^{tx} \geq 0$ for all t), Jensen's inequality tells us that, for g convex,

$$M_{\tilde{x}}(t) = \underbrace{\mathrm{E}\left(e^{t\tilde{x}}\right)}_{\mathrm{E}[g(\tilde{x})]} \geq \underbrace{e^{t\mathrm{E}(\tilde{x})}}_{g(\mathrm{E}(\tilde{x}))} = e^{\mu t}.$$

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34. The partition induced on Ω by $\tilde{x}$ is $F(\tilde{x}) = \{\{1,2\},\{3,4,5,6\}\}$. The σ-algebra generated by the partition $F(\tilde{x})$ is the σ-algebra induced on Ω by $\tilde{x}$,

$$\mathcal{F}(\tilde{x}) = \{\emptyset, \Omega, \{1,2\}, \{3,4,5,6\}\}$$

$$\mathrm{E}(\tilde{x}\,|\mathcal{F}(\tilde{x})) = \tilde{x} = \begin{cases} 2 & \text{if } \omega \in \{1,2\} \\ \\ 3 & \text{if } \omega \in \{3,4,5,6\}. \end{cases}$$

$$\mathcal{G} = \{\emptyset, \Omega, \{1,2,3\}, \{4,5,6\}\}.$$

Since

$$P\{\tilde{x} = 2\,|\{1,2,3\}\} = \frac{P(\{1,2\} \cap \{1,2,3\})}{P\{1,2,3\}} = \frac{P\{1,2\}}{P\{1,2,3\}} = \frac{1/3}{1/2} = \frac{2}{3},$$

$$P\{\tilde{x} = 3\,|\{1,2,3\}\} = \frac{P(\{3,4,5,6\} \cap \{1,2,3\})}{P\{1,2,3\}} = \frac{P\{3\}}{P\{1,2,3\}} = \frac{1/6}{1/2} = \frac{1}{3},$$

$$P\{\tilde{x} = 2\,|\{4,5,6\}\} = \frac{P(\{1,2\} \cap \{4,5,6\})}{P\{4,5,6\}} = \frac{P(\emptyset)}{P\{4,5,6\}} = 0,$$

$$P\{\tilde{x} = 3\,|\{4,5,6\}\} = \frac{P(\{3,4,5,6\} \cap \{4,5,6\})}{P\{4,5,6\}} = \frac{P\{4,5,6\}}{P\{4,5,6\}} = 1,$$

$$\mathrm{E}(\tilde{x}\,|\mathcal{G}) = \begin{cases} \left(\frac{2}{3} \cdot 2\right) + \left(\frac{1}{3} \cdot 3\right) = \frac{7}{3} & \text{if } \omega = \{1,2,3\} \\ \\ (2 \cdot 0) + (3 \cdot 1) = 3 & \text{if } \omega \in \{4,5,6\}. \end{cases}$$

Since

$$P\{\tilde{x} = 2\} = P\{1,2\} = \frac{1}{3},$$

$$P\{\tilde{x} = 3\} = P\{3,4,5,6\} = \frac{2}{3},$$

$$\mathrm{E}(\tilde{x}\,|\mathcal{H}) = \mathrm{E}(\tilde{x}) = \left(\frac{1}{3}\cdot 2\right) + \left(\frac{2}{3}\cdot 3\right) = \frac{8}{3} \text{ for } \omega \in \{1,2,3,4,5,6\} = \Omega.$$

Note also that, since

$$P\{1,2,3\} = \frac{1}{2},$$

$$P\{3,4,5\} = \frac{1}{2},$$

we get

$$\mathrm{E}(\mathrm{E}(\tilde{x}\,|\mathcal{G})) = \mathrm{E}(\mathrm{E}(\tilde{x}\,|\mathcal{G})\,|\mathcal{H}) = \left(\frac{1}{2}\cdot\frac{7}{3}\right) + \left(\frac{1}{2}\cdot 3\right) = \frac{8}{3} = \mathrm{E}(\tilde{x}\,|\mathcal{H}) = \mathrm{E}(\tilde{x}).$$

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35. (a) $\mathrm{E}(\tilde{x}) \geq \mathrm{GE}(\tilde{x})$.

Since $g(x) = \ln(x)$ is an increasing and concave function for $x \geq 0$, Jensen's inequality implies that

$$\ln \mathrm{E}(\tilde{x}) \geq \mathrm{E}[\ln \tilde{x}] = \ln(\exp(\mathrm{E}[\ln \tilde{x}])),$$

which implies that

$$\mathrm{E}(\tilde{x}) \geq \exp(\mathrm{E}[\ln \tilde{x}]) \equiv \mathrm{GE}(\tilde{x}).$$

(b) $\mathrm{GE}(\tilde{x}) \geq \mathrm{HE}(\tilde{x})$.

From part (a) and since $\tilde{x}^{-1} \geq 0$, we have

$$[\mathrm{HE}(\tilde{x})]^{-1} = \mathrm{E}\left[\tilde{x}^{-1}\right] \geq \mathrm{GE}\left[\tilde{x}^{-1}\right] = \exp(\mathrm{E}\left[\ln\left(\tilde{x}^{-1}\right)\right]) = \exp(\mathrm{E}[-\ln \tilde{x}])$$

$$= \exp(-\mathrm{E}[\ln \tilde{x}]) = [\exp(\mathrm{E}[\ln \tilde{x}])]^{-1} = [\mathrm{GE}(\tilde{x})]^{-1},$$

which implies that

$$\mathrm{GE}(\tilde{x}) \geq \mathrm{HE}(\tilde{x}).$$

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36. (a) $\left(\mathrm{E}\left[\tilde{x}^2\right]\right)^{1/2} \geq \mathrm{E}(\tilde{x})$.

From Cauchy-Schwarz inequality,

$$\mathrm{E}(\tilde{x}) = \int_{\Omega} \tilde{x}\,dP = \int_{\Omega} (\tilde{x}\cdot 1)\,dP \leq \left[\int_{\Omega} \tilde{x}^2 dP\right]^{1/2} \underbrace{\left[\int_{\Omega} 1^2 dP\right]^{1/2}}_{=1} = \left(\mathrm{E}\left[\tilde{x}^2\right]\right)^{1/2}.$$

(b) (i)

$$\mathrm{E}_0(\tilde{x}) \equiv \lim_{p\to 0} \mathrm{E}_p(\tilde{x}) = \lim_{p\to 0} \left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{1/p} = \lim_{p\to 0} \exp\left[\ln\left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{1/p}\right] = \lim_{p\to 0} \exp\left[\frac{1}{p}\ln\left(\mathrm{E}\left[\tilde{x}^p\right]\right)\right] =$$

$$\exp\left[\lim_{p\to 0}\frac{\ln\left(\mathrm{E}\left[\tilde{x}^p\right]\right)}{p}\right] = \exp\left(\frac{\lim\limits_{p\to 0}\dfrac{\mathrm{E}\left[\tilde{x}^p \ln \tilde{x}\right]}{\mathrm{E}\left[\tilde{x}^p\right]}}{1}\right) = \exp\left(\frac{\mathrm{E}\left[1\cdot \ln\tilde{x}\right]}{\mathrm{E}(1)}\right) = \exp(\mathrm{E}\left[\ln\tilde{x}\right]) = \mathrm{GE}(\tilde{x}),$$

where the fifth equality follows from L'Hôpital's rule.

(ii)

$$\mathrm{E}_{-1}(\tilde{x}) = \left(\mathrm{E}\left[\tilde{x}^{-1}\right]\right)^{-1} = \mathrm{HE}(\tilde{x}).$$

(iii)

$$\mathrm{E}_{\infty}(\tilde{x}) \equiv \lim_{p\to\infty} \mathrm{E}_p(\tilde{x}) = \lim_{p\to\infty} \left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{1/p} = x_{\max}\cdot \lim_{p\to\infty} \left(\mathrm{E}\left[\left(\frac{\tilde{x}}{x_{\max}}\right)^p\right]\right)^{1/p} = x_{\max}$$

since

$$\lim_{p\to\infty} \left(\mathrm{E}\left[\left(\frac{\tilde{x}}{x_{\max}}\right)^p\right]\right)^{1/p} = 1$$

as $\dfrac{\tilde{x}}{x_{\max}} \leq 1.$

(iv) Note that $\mathrm{E}_{-p}(\tilde{x}) = \left(\mathrm{E}_p\left[\tilde{x}^{-1}\right]\right)^{-1}$. Thus,

$$\mathrm{E}_{-\infty}(\tilde{x}) \equiv \lim_{p\to-\infty} \mathrm{E}_p(\tilde{x}) = \lim_{p\to\infty} \mathrm{E}_{-p}(\tilde{x}) = \lim_{p\to\infty} \left(\mathrm{E}_p\left[\tilde{x}^{-1}\right]\right)^{-1} = \left(\lim_{p\to\infty} \mathrm{E}_p\left[\tilde{x}^{-1}\right]\right)^{-1} =$$

$$\left(\lim_{p\to\infty} \mathrm{E}_p\left[\frac{1}{\tilde{x}}\right]\right)^{-1} = \left[\lim_{p\to\infty} \left(\mathrm{E}\left[\left(\frac{1}{\tilde{x}}\right)^p\right]\right)^{1/p}\right]^{-1} = x_{\min}\cdot\left[\lim_{p\to\infty} \left(\mathrm{E}\left[\left(\frac{x_{\min}}{\tilde{x}}\right)^p\right]\right)^{1/p}\right]^{-1} = x_{\min}$$

since

$$\lim_{p\to\infty} \left(\mathrm{E}\left[\left(\frac{x_{\min}}{\tilde{x}}\right)^p\right]\right)^{1/p} = 1$$

as $\dfrac{x_{\min}}{\tilde{x}} \leq 1.$

(v) We have to consider the following three cases:

Case I: $q > p > 0$.

For this case, we have $q/p > 1$.

$$\mathrm{E}_p(\tilde{x}) = \left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{1/p} = \left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{(1/p)(q/p)(p/q)}$$

$$= \left[\left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{(q/p)}\right]^{(1/p)(p/q)} = \left[\left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{(q/p)}\right]^{1/q} \tag{1}$$

Note that the function $g(t) = t^{q/p}$ is convex for $t \geq 0$ when $q/p > 1$ since

$$g'(t) = \frac{q}{p} t^{(q/p)-1}$$

and

$$g''(t) = \underbrace{\overbrace{\frac{q}{p}}^{+}\overbrace{\left(\frac{q}{p} - 1\right)}^{+}}_{+} t^{(q/p)-2} \geq 0.$$

Using Jensen's inequality, we get

$$g\left(\mathrm{E}\left[\tilde{x}^{p}\right]\right)=\left(\mathrm{E}\left[\tilde{x}^{p}\right]\right)^{q/p}\leq\mathrm{E}\left[g\left(\tilde{x}^{p}\right)\right]=\mathrm{E}\left[\tilde{x}^{p(q/p)}\right]=\mathrm{E}\left[\tilde{x}^{q}\right]. \tag{2}$$

Combining (1) and (2), we get

$$\mathrm{E}_{p}\left(\tilde{x}\right)=\left[\left(\mathrm{E}\left[\tilde{x}^{p}\right]\right)^{(q/p)}\right]^{1/q}\leq\left(\mathrm{E}\left[\tilde{x}^{q}\right]\right)^{1/q}=\mathrm{E}_{q}\left(\tilde{x}\right). \tag{3}$$

Note that the inequality in the previous expression holds because the exponent $1/q$ is positive.

Case II: $q>0>p$.

For this case, we have $q/p<0$.

Note that the function $g(t)=t^{q/p}$ is also convex for $t\geq 0$ when $q/p<0$ since

$$g''(t)=\underbrace{\overbrace{\frac{q}{p}}^{-}\overbrace{\left(\frac{q}{p}-1\right)}^{-}}_{+}t^{(q/p)-2}\geq 0.$$

We thus follow the same steps as in Case I to obtain the inequality in (3). Note that in this case, the exponent $1/q$ in (3) is also positive, which allows us to preserve the inequality.

Case III: $0>q>p$.

For this case, we have $0<q/p<1$.

Note that now the function $g(t)=t^{q/p}$ is concave for $t\geq 0$ when $0<q/p<1$ since

$$g''(t)=\underbrace{\overbrace{\frac{q}{p}}^{+}\overbrace{\left(\frac{q}{p}-1\right)}^{-}}_{-}t^{(q/p)-2}\leq 0.$$

Thus, applying Jensen's inequality, the inequality in (2) is reversed,

$$g\left(\mathrm{E}\left[\tilde{x}^p\right]\right)=\left(\mathrm{E}\left[\tilde{x}^p\right]\right)^{q/p}\geq\mathrm{E}\left[g\left(\tilde{x}^p\right)\right]=\mathrm{E}\left[\tilde{x}^{p(q/p)}\right]=\mathrm{E}\left[\tilde{x}^q\right],$$

which combined with (1) delivers again the inequality in (3) since the exponent $1/q$ in (3) is now negative, which allows us to reverse the inequality.

Note that from the previous results, we immediately obtain the inequalities in Exercise 35 since $1>0>-1$ and, thus,

$$\mathrm{E}\left(\tilde{x}\right)=\mathrm{E}_1\left(\tilde{x}\right)\geq\mathrm{GE}\left(\tilde{x}\right)=\mathrm{E}_0\left(\tilde{x}\right)\geq\mathrm{HE}\left(\tilde{x}\right)=\mathrm{E}_{-1}\left(\tilde{x}\right).$$

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37. Making the Taylor's expansion around zero (or McLaurin expansion) of $f(y)=e^y$, we get

$$e^y=\sum_{r=0}^{\infty}\frac{y^r}{r!}$$

so that, for $y=1$, we obtain

$$e=\sum_{r=0}^{\infty}\frac{1}{r!}.$$

Using the Newton's binomial theorem, we get

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^x=\lim_{x\to\infty}\sum_{r=0}^{x}\binom{x}{r}\left(\frac{1}{x}\right)^r1^{x-r}=\lim_{x\to\infty}\sum_{r=0}^{x}\binom{x}{r}\frac{1}{x^r}$$

$$=\lim_{x\to\infty}\sum_{r=0}^{x}\frac{x!}{r!(x-r)!}\frac{1}{x^r}=\lim_{x\to\infty}\sum_{r=0}^{x}\frac{x(x-1)(x-2)\cdots(x-r+1)}{r!}\frac{1}{x^r}$$

$$=\lim_{x\to\infty}\sum_{r=0}^{x}\frac{1}{r!}\frac{x(x-1)(x-2)\cdots(x-r+1)}{x^r}$$

$$= \lim_{x\to\infty} \sum_{r=0}^{x} \frac{1}{r!} \frac{\overbrace{x^r + a_1 x^{r-1} + a_2 x^{r-2} + ... + a_{r-2}x^2 + a_{r-1}x + a_r}^{\text{Polynomial of order } r}}{x^r}$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \underbrace{\lim_{x\to\infty} \frac{x^r + a_1 x^{r-1} + a_2 x^{r-2} + ... + a_{r-2}x^2 + a_{r-1}x + a_r}{x^r}}_{=1} = \sum_{r=0}^{\infty} \frac{1}{r!} = e.$$

(b) $e > 2$ follows from

$$e = \sum_{r=0}^{\infty} \frac{1}{r!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... > 2.$$

We can find the following indefinite integrals by integrating by parts:

$$\int \ln x dx = \int \underbrace{1}_{f} \cdot \underbrace{\ln x}_{G} dx = x(\ln x - 1),$$

$$\int (\ln x)^2 \, dx = \int \underbrace{1}_{f} \cdot \underbrace{(\ln x)^2}_{G} dx = x\left[(\ln x)^2 - 2\ln x + 2\right],$$

$$\int (\ln x)^3 \, dx = \int \underbrace{1}_{f} \cdot \underbrace{(\ln x)^3}_{G} dx = x\left[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6\right].$$

Thus,

$$\int_{(1,e]} (\ln x)^3 dx = 6 - 2e > 0,$$

where the inequality follows because $e > 1$ and $\ln x$ is strictly positive for $x > 1$. Thus, we are integrating a strictly positive function on $(1, e]$. Note that $6 - 2e > 0$ holds if and only if $e < 3$.

Note that

$$\int_{(1,e]} (\ln x)^2 dx = e - 2 > 0$$

since we are integrating a square and, thus, a strictly positive function on $(1, e]$.

This is an alternative proof of $e > 2$. In fact, $e = 2.7183...$

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