Probability and Statistics. IDEA. Answers to List 2.

1. What one should show is that (i) $f(x)=\dfrac{2x}{k(k+1)}$ is non-negative and (ii) $\sum_{x=1}^{k} f(x)=1$

   (i) Since x and k are positive integers, the numerator and the denominator of $f(x)$ are positive.

   (ii) $\sum_{x=1}^{k}\dfrac{2x}{k(k+1)}=\dfrac{2}{k(k+1)}\sum_{x=1}^{k}x$. Since the last sum is equal to $\dfrac{k(k+1)}{2}$ by the property of the arithmetic progression,

$$\sum_{x=1}^{k}\frac{2x}{k(k+1)}=\frac{2}{k(k+1)}\left[\frac{k(k+1)}{2}\right]=1$$

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2. For $f(x)$ to be a probability function it should satisfy (i) $f(x)\geq 0$ and (ii) $\sum_{x=0}^{\infty} f(x)=1$.

   (i) $0\leq(1-k)k^x$ iff $0\leq k\leq 1$;

   (ii) $\sum_{x=0}^{\infty}(1-k)k^x=(1-k)\sum_{x=0}^{\infty}k^x$. For $k\in(0,1)$ the last sum is equal to $\dfrac{1}{1-k}$ by the property of the geometric progression so that

$$\sum_{x=0}^{\infty}(1-k)k^x=(1-k)\frac{1}{1-k}=1.$$

   Therefore, $f(x)=(1-k)k^x$, for $x=0,1,2,...$, is acceptable as a probability function for all $k\in(0,1)$.

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3. (a) the value of k:

$$\begin{aligned}\int_0^\infty f(z)\,dz &= 1 \Rightarrow \\ \int_0^\infty kze^{-z^2}dz &= -\frac{k}{2}\int_0^\infty \frac{d\left[e^{-z^2}\right]}{dz}dz = -\frac{k}{2}\left[e^{-z^2}\right]_0^\infty = -\frac{k}{2}[0-1] = \frac{k}{2} = 1.\end{aligned}$$

Therefore, $k = 2$.

(b)

$$\begin{aligned}F(z) &= \int_0^z 2te^{-t^2}dt = (-1)\int_0^z \frac{d\left[e^{-t^2}\right]}{dt}dt = -\left[e^{-t^2}\right]_0^z = -\left[e^{-z^2}-1\right] \\ &= 1-e^{-z^2}.\end{aligned}$$

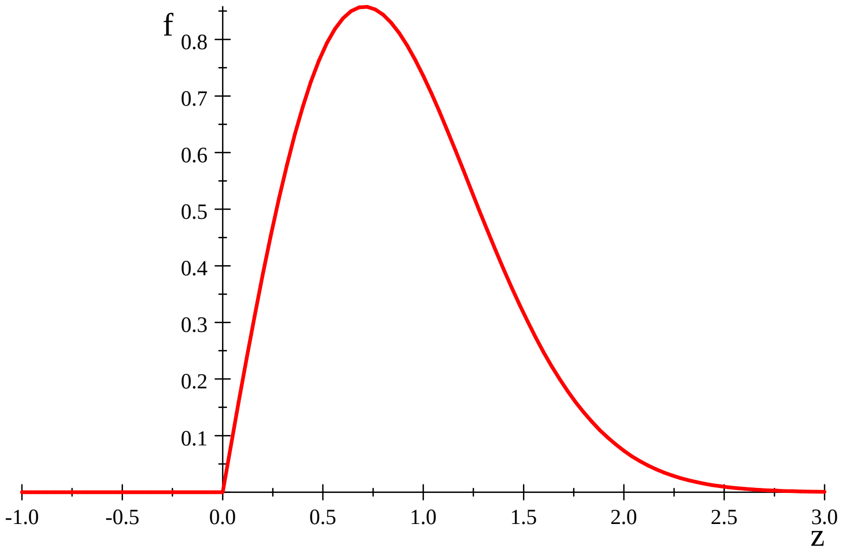
Hence

$$F(z) = \begin{cases} 0 & \text{if} \quad z \le 0 \\ \\ 1-e^{-z^2} & \text{if} \quad z > 0. \end{cases}$$

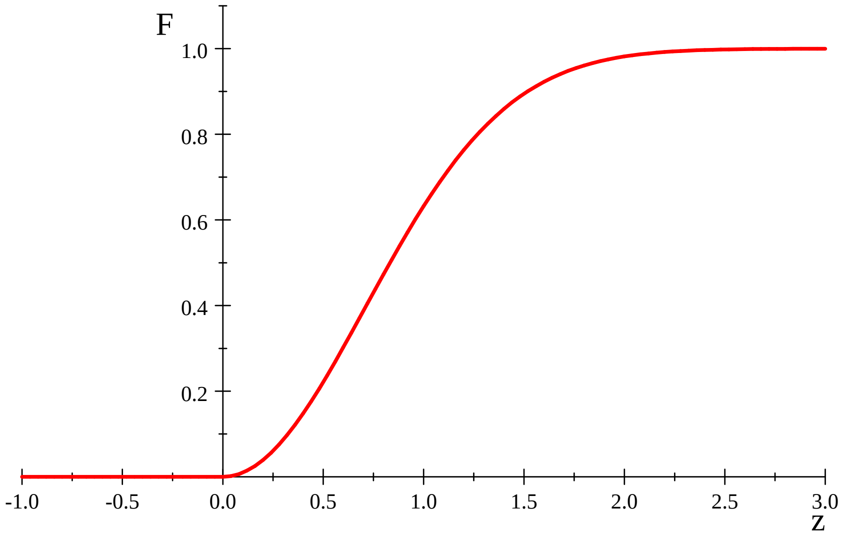
(c) The density function:

$$f(z) = \begin{cases} 2ze^{-z^2} & \text{if} \quad z > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The graph of the density function:



The graph of the distribution function:

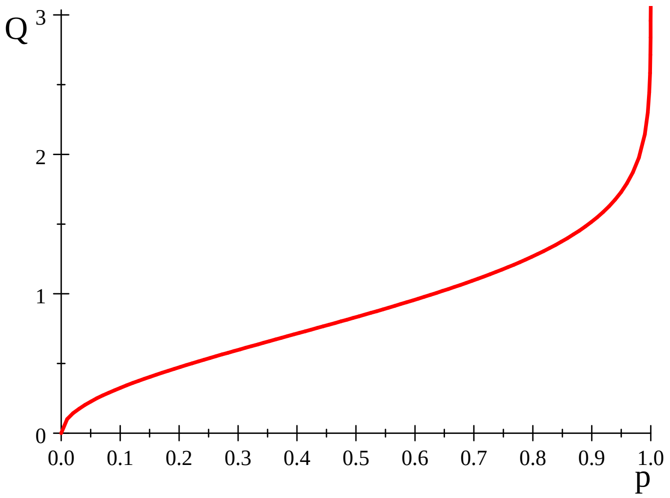


We see that the distribution function is differentiable everywhere since the density is continuous.

(d) Solving for p in $F(z) = p$, for $p \in (0,1)$ we get the quantile function

$$Q(p) = \left[\ln\left(\frac{1}{1-p}\right)\right]^{1/2} \quad \text{for} \quad p \in (0,1)\,,$$

whose plot is



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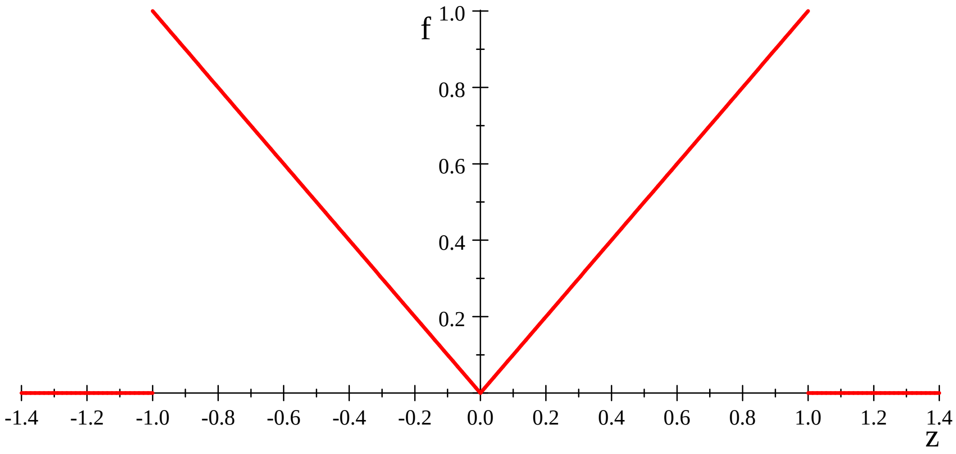
4. (a) $\int_{-\infty}^{\infty} f(z)dz = \int_{-1}^{1} f(z)dz = k \Rightarrow k = 1$

(b) The graph of the density function:

$$f(z) = \begin{cases} 0 & \text{if} \quad z \leq -1 \\ -z & \text{if} \quad -1 < z < 0 \\ z & \text{if} \quad 0 \leq z < 1 \\ 0 & \text{if} \quad 1 \leq z \end{cases}$$



The distribution function:

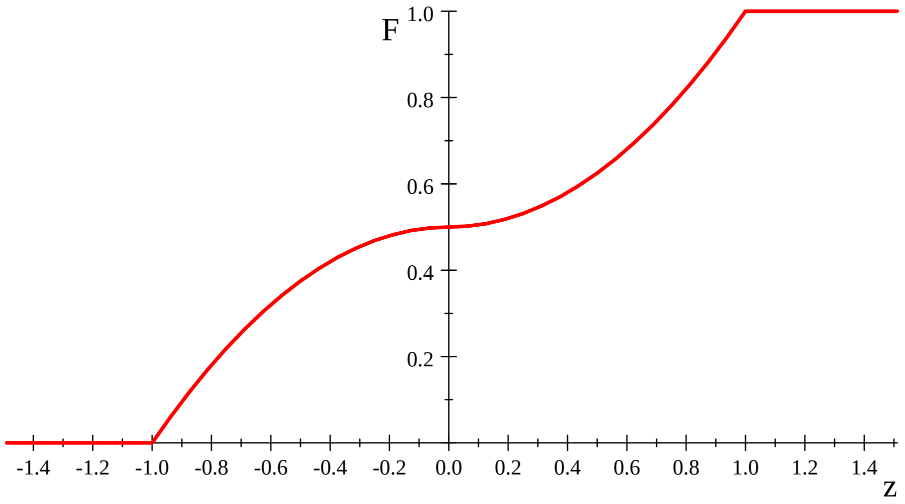
$$F\left(z\right)=\int_{-1}^{z}-tdt=\left[-\frac{t}{2}\right]_{-1}^{z}=\frac{1}{2}\left(1-z^{2}\right)\ \text{for}\ -1<z<0$$

$$F\left(z\right)=\frac{1}{2}+\int_{0}^{z}tdt=\frac{1}{2}+\left[\frac{t^{2}}{2}\right]_{0}^{z}=\frac{1}{2}\left(1+z^{2}\right)\ \text{for}\ 0\leq z<1$$

$$F\left(z\right)=0\ \text{for}\ z<-1\ \text{and}\ F\left(z\right)=1\ \text{for}\ z>1.$$

The graph of the distribution function:

$$F(z)=\begin{cases}0 & \text{if}\quad z\leq -1\\ \frac{1}{2}\left(1-z^{2}\right) & \text{if}\quad -1<z<0\\ \frac{1}{2}\left(1+z^{2}\right) & \text{if}\quad 0\leq z<1\\ 1 & \text{if}\quad 1\leq z\end{cases}$$



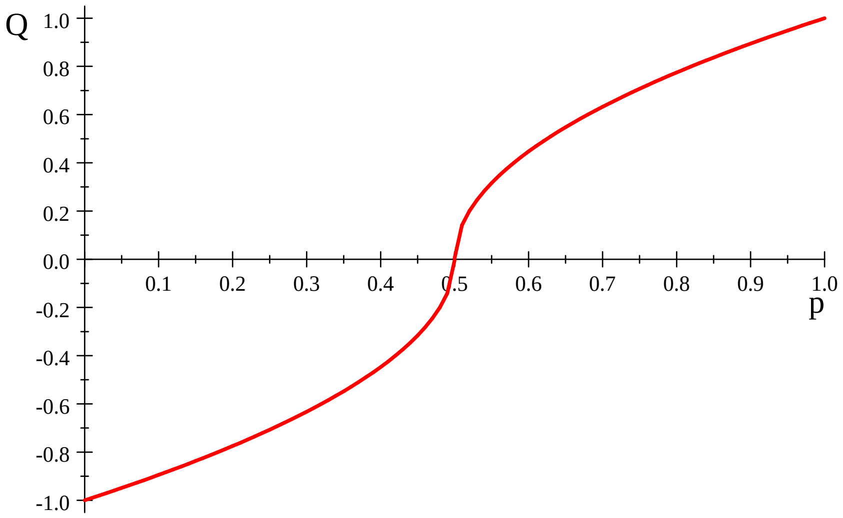
We see that the distribution is not differentiable (i.e., exhibits kinks) at the points $z = 1$ and $z = -1$ where the density is discontinuous.

(c) $P\left\{-\frac{1}{2} < \tilde{z} < \frac{1}{2}\right\} = P_{\tilde{z}}\left(-\frac{1}{2}, \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{5}{8} - \frac{3}{8} = \frac{1}{4}.$

(d) Solving for p in $F(z) = p$, for $p \in (0,1)$ we get the quantile function

$$Q(p) = \begin{cases} -(1-2p)^{1/2} & \text{if} \quad 0 < p < 1/2 \\ \\ (2p-1)^{1/2} & \text{if} \quad 1/2 \leq p < 1 \end{cases}$$

whose plot is



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5. The distribution function is:

$$F\left(x\right)=\int_0^x \frac{1}{9}te^{-t/3}dt \text{ for } x>0$$

Integrating by parts: $\int u(t)v'(t)dt = u(t)v(t) - \int v(t)u'(t)dt$ and letting $u(t)=t \Longrightarrow u'(t)=1$ and $v'(t)=e^{-t/3} \Longrightarrow v(t)=-3e^{-t/3}$

$$\int te^{-t/3}dt = t\left(-3e^{-t/3}\right) - \int\left(-3e^{-t/3}\right)dt = -3te^{-t/3} - 9e^{-t/3} + C$$

Hence,

$$\begin{aligned} F\left(x\right) &= \frac{1}{9}\left[-3te^{-t/3}-9e^{-t/3}\right]_0^x = \frac{1}{9}\left[\left(-3xe^{-x/3}-9e^{-x/3}\right)-(0-9)\right] \\ &= 1-e^{-x/3}-\frac{1}{3}xe^{-x/3} \end{aligned}$$

Then,

$$F\left(x\right)=\begin{cases} 1-e^{-x/3}-\frac{1}{3}xe^{-x/3} & \text{for } x>0 \\ \\ 0 & \text{for } x\leq 0 \end{cases}$$

$(a)\ P\{\tilde{x} \leq 6\} = P_{\tilde{x}}(-\infty, 6] = F(6) = 0.594$

$(b)\ P\{\tilde{x} > 9\} = P_{\tilde{x}}(9, \infty) = 1 - P\{\tilde{x} \leq 9\} = 1 - F(9) = 1 - 0.801 = 0.199.$

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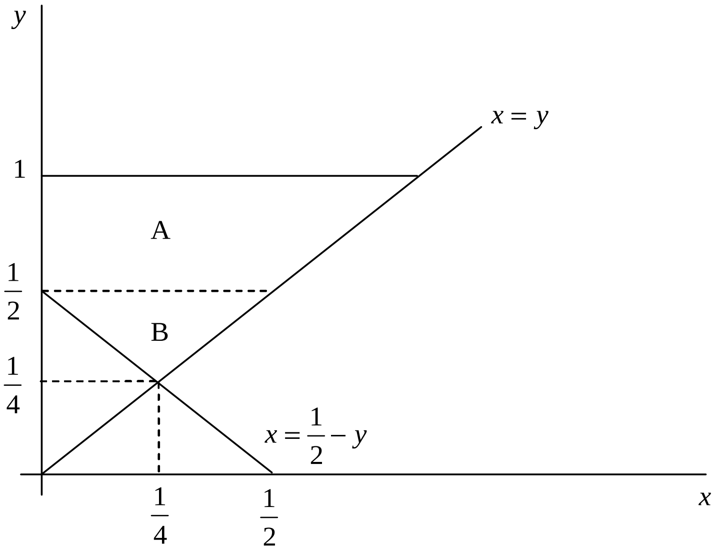

$$\begin{aligned} P\{\tilde{x} + \tilde{y} > 1/2\} &= P_{\tilde{x},\tilde{y}}(A \cup B) = \int\limits_A f(x,y)d(x,y) + \int\limits_B f(x,y)d(x,y) \\ &= \int\limits_{1/2}^{1} \int\limits_{0}^{y} \frac{1}{y} dxdy + \int\limits_{1/4}^{1/2} \int\limits_{\frac{1}{2}-y}^{y} \frac{1}{y} dxdy = 1 - \frac{\ln 2}{2} = 0.65343. \end{aligned}$$

Note that

$$\int_{\frac{1}{2}}^{1} \int_{0}^{y} \frac{1}{y} dxdy = \int_{\frac{1}{2}}^{1} \left[\frac{1}{y} x\right]_0^y dy = \int_{\frac{1}{2}}^{1} 1dy = [y]_{\frac{1}{2}}^{1} = 1 - \frac{1}{2} = \frac{1}{2}$$

and

$$\begin{aligned}
\int_{\frac{1}{4}}^{\frac{1}{2}}\int_{\frac{1}{2}-y}^{y}\frac{1}{y}dxdy &= \int_{\frac{1}{4}}^{\frac{1}{2}}\left[\frac{1}{y}x\right]_{\frac{1}{2}-y}^{y}dy = \int_{\frac{1}{4}}^{\frac{1}{2}}\left[\frac{y}{y}-\frac{\left(\frac{1}{2}-y\right)}{y}\right]dy \\
&= \int_{\frac{1}{4}}^{\frac{1}{2}}\left[1-\frac{1}{2y}+1\right]dy = \left[2y-\frac{1}{2}\ln y\right]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{2}-\frac{1}{2}\ln 2.
\end{aligned}$$

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7. (a) $P\{\widetilde{p} < 0.3, \widetilde{s} > 2\} =$

$$\begin{aligned}
\int_{0.2}^{0.3}\int_{2}^{\infty}5pe^{-ps}dsdp &= 5\int_{0.2}^{0.3}\left[\frac{5pe^{-ps}}{(-p)}\right]_{2}^{\infty}dp = (-5)\int_{0.2}^{0.3}\left[0-e^{-2p}\right]dp = \\
5\left[\frac{e^{-2p}}{(-2)}\right]_{0.2}^{0.3} &= \frac{5}{2}\left[e^{-0.4}-e^{-0.6}\right] = 0.3038
\end{aligned}$$

(b) $P\{0.25 < \widetilde{p} < 0.3, \widetilde{s} < 1\} =$

$$\int_{0.25}^{0.3}\int_{0}^{1}5pe^{-ps}dsdp = 5\int_{0.25}^{0.3}\left[\frac{pe^{-ps}}{(-p)}\right]_{0}^{1}dp = 5\int_{0.25}^{0.3}\left(-e^{-p}+1\right)dp =$$

$$5\left[e^{-p}+p\right]_{0.25}^{0.3} = 5\left[e^{-0.3}+0.3-e^{-0.25}-0.25\right] = 0.06$$

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8. (a)

$$f_{\tilde{x},\tilde{y}}(x,y) = \sum_{z=1}^{2}\frac{xyz}{108} = \frac{xy}{108}+\frac{2xy}{108} = \frac{xy}{36},\ x=1,2,3;\ y=1,2,3$$

(b)

$$f_{\tilde{x},\tilde{z}}(x,z) = \sum_{y=1}^{3}\frac{xyz}{108} = \frac{xz}{108}+\frac{2xz}{108}+\frac{3xz}{108} = \frac{xz}{18},\ x=1,2,3;\ z=1,2,$$

(c)

$$f_{\tilde{x}}(x) = \sum_{z=1}^{2}f_{\tilde{x},\tilde{z}}(x,z) = \sum_{z=1}^{2}\frac{xz}{18} = \frac{x}{18}+\frac{2x}{18} = \frac{x}{6},\ x=1,2,3$$

(d)

$$f_{\tilde{z}|\tilde{x},\tilde{y}}\left(z\,|1,2\right)=\frac{f\left(1,2,z\right)}{f_{\tilde{x},\tilde{y}}\left(1,2\right)}=\frac{z}{3},\ \ z=1,2$$

(e)

$$f_{\tilde{y},\tilde{z}|\tilde{x}}\left(y,z\,|3\right)=\left.\frac{f\left(3,y,z\right)}{f_{\tilde{x}}\left(3\right)}\right|_{x=3}=\frac{yz}{18},\ \ y=1,2,3;\ \ z=1,2$$

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9. (a) The joint probability function and the marginal probability functions of $\tilde{x}$ and $\tilde{y}$ are given by the following table:

$x\backslash y$	-1	1	$f_{\tilde{x}}(x)$
-1	$1/4$	$1/4$	$1/2$
1	$1/4$	$1/4$	$1/2$
$f_{\tilde{y}}(y)$	$1/2$	$1/2$	

Then, $f_{\tilde{x}}\left(x\right)\cdot f_{\tilde{y}}\left(y\right)=f\left(x,y\right)=\frac{1}{4}$ for all $\left(x,y\right)\in\tilde{x}\left(\Omega\right)\times\tilde{y}\left(\Omega\right)\Rightarrow x$ and y are independent.

(b) The joint probability function and the marginal probability functions of $\tilde{x}$ and $\tilde{y}$ are given by the following table:

$x\backslash y$	0	1	$f_{\tilde{x}}(x)$
0	$1/3$	$1/3$	$2/3$
1	0	$1/3$	$1/3$
$f_{\tilde{y}}(y)$	$1/3$	$2/3$	

We see that $f_{\tilde{x}}\left(0\right)\cdot f_{\tilde{y}}\left(0\right)=f_{\tilde{x}}\left(1\right)\cdot f_{\tilde{y}}\left(1\right)=\frac{2}{9}\neq\frac{1}{3}=f\left(0,0\right)=f\left(1,1\right),$ $f_{\tilde{x}}\left(0\right)\cdot f_{\tilde{y}}\left(1\right)=\frac{4}{9}\neq\frac{1}{3}=f\left(0,1\right)$ and $f_{\tilde{x}}\left(1\right)\cdot f_{\tilde{y}}\left(0\right)=\frac{1}{9}\neq 0=f\left(1,0\right)\Rightarrow$ x and y are not independent.

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10. $(a)\ f_{\tilde{x}}(x) = \int_0^2 \frac{1}{4}(2x+y)\,dy = \frac{1}{4}\left[2xy + \frac{y^2}{2}\right]_0^2 = \frac{1}{4}[(4x+2)-0] = x + \frac{1}{2}$ for $0 < x < 1$

Hence,

$$f_{\tilde{x}}(x) = \begin{cases} x + \frac{1}{2} & \text{for } 0 < x < 1 \\ \\ 0 & \text{elsewhere} \end{cases}$$

$(b)\ f_{\tilde{y}}(y) = \int_0^1 \frac{1}{4}(2x+y)\,dx = \frac{1}{4}\left[\frac{2x^2}{2} + xy\right]_0^1 = \frac{1}{4}[1+y]$ for $0 < y < 2$

Hence,

$$f_{\tilde{y}}(y) = \begin{cases} \frac{1}{4}(1+y) & \text{for } 0 < y < 2 \\ \\ 0 & \text{elsewhere} \end{cases}$$

$(c)\ f_{\tilde{x}|\tilde{y}}(x\,|1) = \dfrac{f(x,1)}{f_{\tilde{y}}(1)} = \dfrac{\frac{1}{4}(2x+1)}{\frac{1}{4}(1+1)} = x + \dfrac{1}{2}$ for $0 < x < 1$

Hence,

$$f_{\tilde{x}|\tilde{y}}(x\,|1) = \begin{cases} x + \frac{1}{2} & \text{for } 0 < x < 1 \\ \\ 0 & \text{elsewhere} \end{cases}$$

$(d)\ f_{\tilde{y}|\tilde{x}}(y\,|1/4) = \dfrac{f(1/4,y)}{f_{\tilde{x}}(1/4)} = \dfrac{\frac{1}{4}\left(2\cdot\frac{1}{4}+y\right)}{\frac{1}{4}+\frac{1}{2}} = \dfrac{2y+1}{6}$ for $0 < y < 2$

Hence

$$f_{\tilde{y}|\tilde{x}}(y\,|1/4) = \begin{cases} \frac{2y+1}{6} & \text{for } 0 < y < 2 \\ \\ 0 & \text{elsewhere} \end{cases}$$

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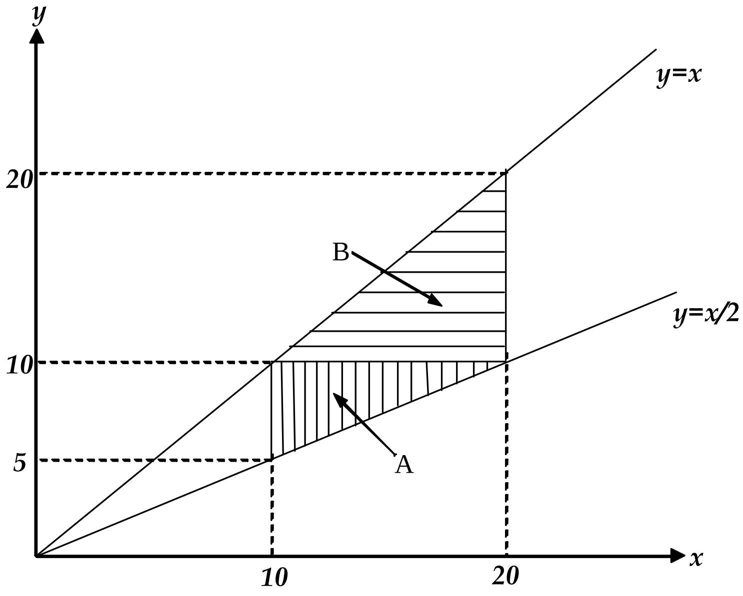
11. (a) $f_{\tilde{x}}(x) = \int_{x/2}^{x} f(x,y)\,dy = \int_{x/2}^{x} \frac{1}{25}\left(\frac{20}{x} - 1\right) dy = \frac{1}{25}\left[\left(\frac{20}{x} - 1\right) y\right]_{x/2}^{x} =$ $\frac{1}{25}\left(10 - \frac{x}{2}\right)$, for $10 < x < 20$.

Hence,

$$f_{\tilde{x}}(x) = \begin{cases} \frac{1}{25}\left(10 - \frac{x}{2}\right) & \text{for } 10 < x < 20 \\ \\ 0 & \text{elsewhere.} \end{cases}$$



Region A: $5 < y \leq 10$

$$\int_{10}^{2y} f(x,y)\,dx = \int_{10}^{2y} \frac{1}{25}\left(\frac{20}{x} - 1\right) dx = \frac{1}{25}\left[20\ln x - x\right]_{10}^{2y}$$

$$= \frac{1}{25}\left(20\ln\left(\frac{2y}{10}\right) - 2y + 10\right) \text{ for } 5 < y \leq 10$$

Region B: $10 < y < 20$

$$\int_y^{20} f(x,y)\,dx = \int_y^{20} \frac{1}{25}\left(\frac{20}{x}-1\right)dx = \frac{1}{25}\left[20\ln x - x\right]_y^{20}$$

$$= \tfrac{1}{25}\left[20\ln\left(\frac{20}{y}\right) + y - 20\right] \text{ for } 10 < y < 20$$

Hence,

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{1}{25}\left(20\ln\left(\dfrac{2y}{10}\right) - 2y + 10\right) & \text{for } 5 < y \leq 10 \\ \\ \dfrac{1}{25}\left[20\ln\left(\dfrac{20}{y}\right) + y - 20\right] & \text{for } 10 < y < 20 \\ \\ 0 & \text{elsewhere} \end{cases}$$

(b) $f_{\tilde{y}|\tilde{x}}(y\,|12) = \dfrac{f(12,y)}{f_{\tilde{x}}(12)} = \dfrac{\frac{1}{25}\left(\frac{20}{12}-1\right)}{4/25} = \dfrac{1}{6}$ for $6 < y < 12$.

Hence,

$$f_{\tilde{y}|\tilde{x}}(y\,|12) = \begin{cases} \dfrac{1}{6} & \text{for } 6 < y < 12 \\ \\ 0 & \text{elsewhere} \end{cases}$$

(c)

$$P\{\tilde{y} > 8\,|\tilde{x} = 12\} = \int_8^{12} \frac{1}{6}dy = \left[\frac{y}{6}\right]_8^{12} = \frac{2}{3}$$

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12. Let $\tilde{x}$ be an arbitrary random variable with a density, and take $\tilde{y} \equiv \tilde{x}$. Then necessarily $(\tilde{x},\tilde{y}) \in L$, where $L = \{(x,y) \in \mathbb{R}^2\,|x = y\}$. Note that the Lebesgue measure of the set L in $\mathbb{R}^2$ is zero. Therefore, if $(\tilde{x},\tilde{y})$ has

density f, then

$$1 = P\{(\tilde{x},\tilde{y}) \in L\} = \int\limits_{L} f(x,y)d(x,y) = 0.$$

A contradiction.

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13. (a) On the one hand,

$$P\{\tilde{x} \in A, \tilde{y} \in B\} = \int\limits_{A} P(\tilde{y} \in B \mid \tilde{x} = x) f_{\tilde{x}}(x) dx$$

$$= \int\limits_{A} \left[\left(\sum_{y \in B} f_{\tilde{y}|\tilde{x}}(y \,|x) \right) f_{\tilde{x}}(x) \right] dx = \sum_{y \in B} \left[\int\limits_{A} \frac{f_{\tilde{x}}(x) f_{\tilde{y}|\tilde{x}}(y \,|x)}{f_{\tilde{y}}(y)} dx \right] \cdot f_{\tilde{y}}(y),$$

where the last equality arises from exchanging the order of integration and summation and from dividing and multiplying by $f_{\tilde{y}}(y)$.

In particular,

$$P\{\tilde{x} \in A, \tilde{y} = y\} = \left[\int\limits_{A} \frac{f_{\tilde{x}}(x) f_{\tilde{y}|\tilde{x}}(y \,|x)}{f_{\tilde{y}}(y)} dx \right] \cdot f_{\tilde{y}}(y). \qquad (1)$$

On the other hand,

$$\begin{aligned} P\{\tilde{x} \in A, \tilde{y} = y\} &= P\{\tilde{x} \in A \,|\tilde{y} = y\} f_{\tilde{y}}(y) \\ &= \left[\int\limits_{A} f_{\tilde{x}|\tilde{y}}(x \,|y) dx \right] \cdot f_{\tilde{y}}(y). \end{aligned} \qquad (2)$$

Combining (1) and (2) we get that

$$f_{\tilde{x}|\tilde{y}}(x \,|y) = \frac{f_{\tilde{x}}(x) f_{\tilde{y}|\tilde{x}}(y \,|x)}{f_{\tilde{y}}(y)}, \text{ as desired.}$$

(b) On the one hand,

$$P\{\tilde{x} \in A, \tilde{y} \in B\} = \sum_{x \in A} P(\tilde{y} \in B \mid \tilde{x} = x) f_{\tilde{x}}(x)$$

$$= \sum_{x \in A} \left[\left(\int\limits_B f_{\tilde{y}|\tilde{x}}(y\,|x\,)dy \right) f_{\tilde{x}}(x) \right]$$

$$= \int\limits_B \left[\sum_{x \in A} \frac{f_{\tilde{x}}(x) f_{\tilde{y}|\tilde{x}}(y\,|x\,)}{f_{\tilde{y}}(y)} \right] f_{\tilde{y}}(y)dy, \tag{3}$$

where the last equality arises from exchanging the order of summation and integration and from dividing and multiplying by $f_{\tilde{y}}(y)$.

On the other hand,

$$P\{\tilde{x} \in A, \tilde{y} \in B\} = \int\limits_B P(\tilde{x} \in A \mid \tilde{y} \in y) f_{\tilde{y}}(y)dy$$

$$= \int\limits_B \left[\sum_{x \in A} f_{\tilde{x}|\tilde{y}}(x\,|y\,) \right] f_{\tilde{y}}(y)dy. \tag{4}$$

Comparing (3) and (4), we can make

$$f_{\tilde{x}|\tilde{y}}(x\,|y\,) = \frac{f_{\tilde{x}}(x) f_{\tilde{y}|\tilde{x}}(y\,|x\,)}{f_{\tilde{y}}(y)}, \text{ as desired.}$$

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14.

$$P\{\tilde{x} \in B\,|A\} = P\{\tilde{x} \in B\,|\tilde{x} \in B_0\} = \frac{P\{\tilde{x} \in \{B \cap B_0\}\}}{P(A)}$$

$$= \frac{1}{P(A)} \int\limits_{B \cap B_0} f(x)dx = \int\limits_B \frac{f(x)}{P(A)} \mathbb{I}_{B_0}(x)dx, \text{ as desired,}$$

since

$$\frac{f(x)}{P(A)} \mathbb{I}_{B_0}(x) = \frac{f(x)}{P(A)} \text{ if } x \in B_0$$

and

$$\frac{f(x)}{P(A)} \mathbb{I}_{B_0}(x) = 0 \text{ if } x \notin B_0.$$

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15. (a)

$$P\{\tilde{x}<1/2\}=\int_{-\infty}^{1/2}f(x)dx=\int_{0}^{1/2}1dx=[x]_0^{1/2}=\frac{1}{2},$$

$$P(\{\tilde{x}<1/3\}\cap\{\tilde{x}<1/2\})=P\{\tilde{x}<1/3\}=\int_{0}^{1/3}1dx=[x]_0^{1/3}=\frac{1}{3},$$

$$P(\{\tilde{x}<3/4\}\cap\{\tilde{x}<1/2\})=P\{\tilde{x}<1/2\}=\int_{0}^{1/2}1dx=[x]_0^{1/2}=\frac{1}{2},$$

$$\begin{aligned}P(\{1/3<\tilde{x}<3/4\}\cap\{\tilde{x}<1/2\})&=P\{1/3<\tilde{x}<1/2\}\\&=\int_{1/3}^{1/2}1dx=[x]_{1/3}^{1/2}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}.\end{aligned}$$

Thus,

$$P\{\tilde{x}<1/3\mid\tilde{x}<1/2\}=\frac{P(\{\tilde{x}<1/3\}\cap\{\tilde{x}<1/2\})}{P\{\tilde{x}<1/2\}}=\frac{1/3}{1/2}=\frac{2}{3},$$

$$P\{\tilde{x}<3/4\mid\tilde{x}<1/2\}=\frac{P(\{\tilde{x}<3/4\}\cap\{\tilde{x}<1/2\})}{P\{\tilde{x}<1/2\}}=\frac{1/2}{1/2}=1,$$

$$P\{1/3<\tilde{x}<3/4\mid\tilde{x}<1/2\}=\frac{P(\{1/3<\tilde{x}<3/4\}\cap\{\tilde{x}<1/2\})}{P\{\tilde{x}<1/2\}}=\frac{1/6}{1/2}=\frac{1}{3}.$$

(b) Let us compute the previous conditional probabilities using the conditional density of Exercise 14:

$$f_{\tilde{x}|A}(x\,|A)=\begin{cases}\dfrac{f(x)}{P(A)}=\dfrac{f(x)}{1/2}=2f(x) & \text{if } x<1/2\\[2ex] 0 & \text{if } x\geq 1/2,\end{cases}$$

where $A=\{\tilde{x}<1/2\}$.

$$f_{\tilde{x}|A}(x\,|A)=\begin{cases}2f(x)=2\cdot 1=2 & \text{if } x\in(0,1) \text{ and } x<1/2\\[2ex] 2f(x)=2\cdot 0=0 & \text{if } x\notin(0,1) \text{ and } x<1/2\\[2ex] 0 & \text{if } x\geq 1/2,\end{cases}$$

The previous conditional density simplifies to

$$f_{\tilde{x}|A}\left(x\,|A\right) = \begin{cases} 2 & \text{if } x \in (0,1/2) \\ \\ 0 & \text{otherwise.} \end{cases}$$

$$P\left\{\tilde{x} < 1/3 \mid \tilde{x} < 1/2\right\} = \int_{-\infty}^{1/3} f_{\tilde{x}|A}\left(x\,|A\right)dx = \int_{0}^{1/3} 2dx = \left[2x\right]_0^{1/3} = \frac{2}{3},$$

$$P\left\{\tilde{x} < 3/4 \mid \tilde{x} < 1/2\right\} = \int_{-\infty}^{3/4} f_{\tilde{x}|A}\left(x\,|A\right)dx = \int_{0}^{1/2} 2dx = \left[2x\right]_0^{1/2} = 1,$$

$$P\left\{1/3 < \tilde{x} < 3/4 \mid \tilde{x} < 1/2\right\} = \int_{1/3}^{3/4} f_{\tilde{x}|A}\left(x\,|A\right)dx = \int_{1/3}^{1/2} 2dx = \left[2x\right]_{1/3}^{1/2} = 1-\frac{2}{3} = \frac{1}{3}.$$

(c)

$$P\left\{\tilde{x} \in B \mid \tilde{x} = x\right\} = \begin{cases} 1 & \text{if } x \in B \\ \\ 0 & \text{if } x \notin B. \end{cases}$$

Theorem of total probability:

$$\int_{\mathbb{R}} P\left\{\tilde{x} \in B \mid \tilde{x} = x\right\} dP_{\tilde{x}}(x)$$

$$= \int_{B} P\left\{\tilde{x} \in B \mid \tilde{x} = x\right\} dP_{\tilde{x}}(x) + \int_{B^c} P\left\{\tilde{x} \in B \mid \tilde{x} = x\right\} dP_{\tilde{x}}(x)$$

$$= \int_{B} 1 \cdot dP_{\tilde{x}}(x) + \int_{B^c} 0 \cdot dP_{\tilde{x}}(x) = \int_{B} dP_{\tilde{x}}(x) = P_{\tilde{x}}(B) = P\left\{\tilde{x} \in B\right\}.$$

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16. If either $x < 0$ or $y < 0$, it follows immediately that $F(x,y) = 0$. For $0 < x < 1$ and $0 < y < 1$ (Region I of the figure) we get

$$F(x,y) = \int_0^y\int_0^x (s+t)dsdt = \frac{1}{2}xy(x+y),$$

for $x > 1$ and $0 < y < 1$ (Region II of the figure) we get

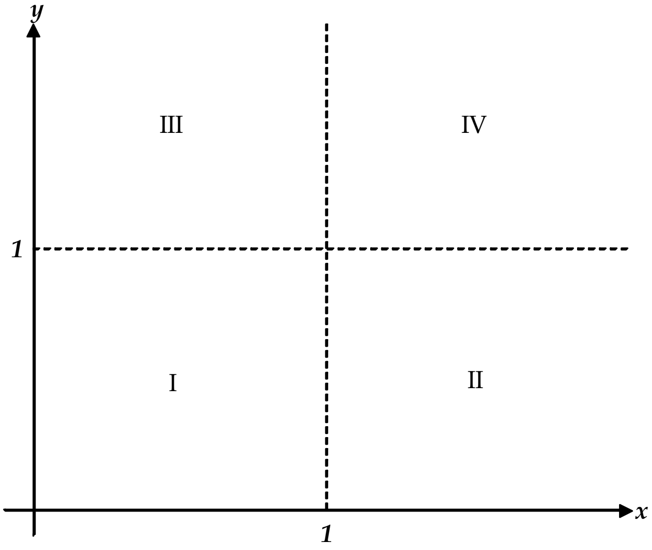
$$F(x,y) = \int_0^y\int_0^1 (s+t)dsdt = \frac{1}{2}y(y+1),$$

for $0 < x < 1$ and $y > 1$ (Region III of the figure) we get

$$F(x,y) = \int_0^1 \int_0^x (s+t)\,ds\,dt = \frac{1}{2}x(x+1),$$

and for $x > 1$ and $y > 1$ (Region IV of the figure) we get

$$F(x,y) = \int_0^1 \int_0^1 (s+t)\,ds\,dt = 1.$$



Since the joint distribution function is everywhere continuous, the boundaries between any two of these regions can be included in either one, and we can write

$$F(x,y)=\begin{cases} 0 & \text{for } x \le 0 \text{ or } y \le 0 \\ \\ \frac{1}{2}xy(x+y) & \text{for } 0<x<1,\, 0<y<1 \\ \\ \frac{1}{2}y(y+1) & \text{for } x \ge 1,\, 0<y<1 \\ \\ \frac{1}{2}x(x+1) & \text{for } 0<x<1,\, y \ge 1 \\ \\ 1 & \text{for } x \ge 1,\, y \ge 1 \end{cases}$$

It is immediate to see that

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = x+y, \text{ for } x \in (0,1)\,,\ y \in (0,1)$$

and

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = 0, \text{ for } (x,y) \in C,$$

where C is the interior of the complement of Region I. Note that the distribution function $F(x,y)$ is not differentiable at the boundary of Region I (except at the point $(0,0)$). This boundary has zero Lebesgue measure. Therefore, $\dfrac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$ a.e. with respect to Lebesgue measure in $\mathbb{R}^2$. Note also that the density function $f(x,y)$ is not continuous at the boundary of Region I (except at the point $(0,0)$).

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17. (a) $\int_0^1 \int_1^2 c(x^2-y)dxdy = \frac{11}{6}c$. Thus, $c = 6/11$.

(b) If $x \in (1,2)$, then $f_{\tilde{x}}(x) = \int_0^1 \frac{6}{11}(x^2 - y)dy = \frac{3}{11}\left(2x^2 - 1\right)$, so that

$$f_{\tilde{x}}(x) = \begin{cases} \dfrac{3}{11}\left(2x^2 - 1\right) & \text{if } x \in (1,2) \\ \\ 0 & \text{otherwise.} \end{cases}$$

If $y \in (0,1)$, then $f_{\tilde{y}}(y) = \int_1^2 \frac{6}{11}(x^2 - y)dx = \frac{2}{11}(7 - 3y)$, so that

$$f_{\tilde{y}}(y) = \begin{cases} \dfrac{2}{11}(7 - 3y) & \text{if } y \in (0,1) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $y \in (0,1)$, then $F_{\tilde{y}}(y) = P\left\{\tilde{y} \leq y\right\} = \int_0^y \frac{2}{11}(7 - 3y)dy = \dfrac{y\left(14 - 3y\right)}{11}$, so that

$$F_{\tilde{y}}(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \\ \dfrac{y\left(14 - 3y\right)}{11} & \text{if } y \in (0,1) \\ \\ 1 & \text{if } y \geq 1. \end{cases}$$

(c) For $(x,y) \in (1,2) \times (0,1)$, we have $f_{\tilde{x}}(x) \cdot f_{\tilde{y}}(y) = \frac{3}{11}\left(2x^2 - 1\right)\frac{2}{11}\left(7 - 3y\right) = \frac{6}{121}\left(14x^2 - 6x^2 y + 3y + 7\right)$, which is not equal to $f(x,y) = \frac{6}{11}(x^2 - y)$. Therefore, $\tilde{x}$ and $\tilde{y}$ are not independent.

(d) $f_{\tilde{x}|\tilde{y}}\left(x \,\middle|\, \frac{1}{4}\right) = \dfrac{f\left(x, \frac{1}{4}\right)}{f_{\tilde{y}}\left(\frac{1}{4}\right)}$, where $f_{\tilde{y}}\left(\frac{1}{4}\right) = \frac{2}{11}\left(7 - 3 \cdot \frac{1}{4}\right) = \dfrac{25}{22}$ and, if $x \in (1,2)$, then $f\left(x, \frac{1}{4}\right) = \frac{6}{11}\left(x^2 - \frac{1}{4}\right)$, so that

$$f\left(x, \frac{1}{4}\right) = \begin{cases} \dfrac{6}{11}\left(x^2 - \dfrac{1}{4}\right) & \text{if } x \in (1,2) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$f_{\tilde{x}|\tilde{y}}\left(x \,\middle|\, \frac{1}{4}\right) = \begin{cases} \dfrac{\frac{6}{11}\left(x^2 - \frac{1}{4}\right)}{25/22} = \dfrac{3}{25}\left(4x^2 - 1\right) & \text{if } x \in (1,2) \\[2em] \dfrac{0}{25/22} = 0 & \text{otherwise.} \end{cases}$$

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18. (a) - Properties of F^+ :

(i) $\lim\limits_{(x,y)\to(\infty,\infty)} F^+(x,y) = \min\{1,1\} = 1.$

(ii) $\lim\limits_{x\to-\infty} F^+(x,y) = \min\{0, F_{\tilde{y}}(y)\} = 0.$

(iii) $\lim\limits_{y\to-\infty} F^+(x,y) = \min\{F_{\tilde{x}}(x), 0, \} = 0.$

(iv) F^+ is non-decreasing:

$$(x_1,y_1) \le (x_2,y_2) \Rightarrow F^+(x_1,y_1) = \min\{F_{\tilde{x}}(x_1), F_{\tilde{y}}(y_1)\}$$

$$\le \min\{F_{\tilde{x}}(x_2), F_{\tilde{y}}(y_2)\} = F^+(x_2,y_2).$$

(v) F^+ is right-continuous:

$$\lim_{(s,t)\to(x,y)^+} F^+(s,t) = \lim_{(s,t)\to(x,y)^+} \left[\min\{F_{\tilde{x}}(s), F_{\tilde{y}}(t)\}\right]$$

$$= \min\{F_{\tilde{x}}(x), F_{\tilde{y}}(y)\} = F^+(x,y).$$

Therefore, F^+ is a distribution function associated with a probability measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

- Properties of F^-:

(i) $\lim\limits_{(x,y)\to(\infty,\infty)} F^-(x,y) = \max\{1+1-1, 0\} = 1.$

(ii) $\lim\limits_{x\to-\infty} F^-(x,y) = \max\{0 + F_{\tilde{y}}(y) - 1, 0\} = 0.$

(iii) $\lim_{y\to-\infty} F^{-}(x,y)=\max\{F_{\tilde{x}}(x)+0-1,0\}=0.$

(iv) F^{-} is non-decreasing:

$$(x_1,y_1)\le(x_2,y_2)\Rightarrow F^{-}(x_1,y_1)=\max\{F_{\tilde{x}}(x_1)+F_{\tilde{y}}(y_1)-1,0\}$$

$$\le\max\{F_{\tilde{x}}(x_2)+F_{\tilde{y}}(y_2)-1,0\}=F^{-}(x_2,y_2).$$

(v) F^{-} is right-continuous:

$$\lim_{(s,t)\to(x,y)^{+}} F^{-}(s,t)=\lim_{(s,t)\to(x,y)^{+}}[\max\{F_{\tilde{x}}(s)+F_{\tilde{y}}(t)-1,0\}]$$

$$=\max\{F_{\tilde{x}}(x)+F_{\tilde{y}}(y)-1,0\}=F^{-}(x,y).$$

Therefore, F^{-} is a distribution function associated with a probability measure on $(\mathbb{R}^2,\mathcal{B}(\mathbb{R}^2))$.

(b) Marginals of F^{+} :

$F^{+}_{\tilde{x}}(x)=\lim_{y\to\infty}F^{+}(x,y)=\lim_{y\to\infty}[\min\{F_{\tilde{x}}(x),F_{\tilde{y}}(y)\}]=\min\{F_{\tilde{x}}(x),1\}=F_{\tilde{x}}(x).$

Similarly, $F^{+}_{\tilde{y}}(y)=F_{\tilde{y}}(y)$.

Marginals of F^{-} :

$$F^{-}_{\tilde{x}}(x)=\lim_{y\to\infty}F^{-}(x,y)=\lim_{y\to\infty}[\max\{F_{\tilde{x}}(x)+F_{\tilde{y}}(y)-1,0\}]$$

$$=\max\{F_{\tilde{x}}(x)+1-1,0\}=F_{\tilde{x}}(x).$$

Similarly, $F^{-}_{\tilde{y}}(y)=F_{\tilde{y}}(y)$

(c) We know that, for any two events A and B,

(i) $P(A\cap B)\le\min\{P(A),P(B)\}$ since $(A\cap B)\subset A$ and $(A\cap B)\subset B$.

(ii) $P(A\cup B)\le P(A)+P(B)$.

Let $A = \{\tilde{x} \le x\}$, $B = \{\tilde{y} \le y\}$. Then, (i) implies that

$$F(x,y) = P\left(\underbrace{\{\tilde{x} \le x\}}_{A} \cap \underbrace{\{\tilde{y} \le y\}}_{B}\right) \le \min\{P(A), P(B)\}$$

$$= \min\{F_{\tilde{x}}(x), F_{\tilde{y}}(y)\} = F^{+}(x,y).$$

Moreover,

$$P(A^c \cup B^c) = P\left((A \cap B)^c\right) = 1 - P(A \cap B).$$

Then,

$$P\left(\underbrace{\{\tilde{x} > x\}}_{A^c} \cup \underbrace{\{\tilde{y} > y\}}_{B^c}\right) = 1 - P\left(\underbrace{\{\tilde{x} \le x\}}_{A} \cap \underbrace{\{\tilde{y} \le y\}}_{B}\right)$$

$$= 1 - F(x,y) \underset{\substack{\uparrow \\ \text{from (ii)}}}{\le} P\underbrace{\{\tilde{x} > x\}}_{A^c} + P\underbrace{\{\tilde{y} > y\}}_{B^c} = [1 - F_{\tilde{x}}(x)] + [1 - F_{\tilde{y}}(y)]$$

$$\iff F(x,y) \ge F_{\tilde{x}}(x) + F_{\tilde{y}}(y) - 1$$

and, moreover, $F(x,y) \ge 0$. Therefore,

$$F(x,y) \ge \max\{F_{\tilde{x}}(x) + F_{\tilde{y}}(y) - 1, 0\} = F^{-}(x,y).$$

Summing up,

$$F^{-}(x,y) \le F(x,y) \le F^{+}(x,y), \quad \forall (x,y) \in \mathbb{R}^2.$$