Probability and Statistics. IDEA. Answers to List 1.

1. Let us write

$$\begin{aligned}(1+y)^n &= (1+y)^{n-1}(1+y) \\ &= (1+y)^{n-1} + y(1+y)^{n-1}\end{aligned}$$

and equate the coefficient of y^r in $(1+y)^n$ with that in $(1+y)^{n-1} + y(1+y)^{n-1}$. Since the coefficient of y^r in $(1+y)^n$ is $\binom{n}{r}$ and the coefficient of y^r in $(1+y)^{n-1} + y(1+y)^{n-1}$ is the sum of the coefficient of y^r in $(1+y)^{n-1}$, namely, $\binom{n-1}{r}$, and the coefficient of y^{r-1} in $(1+y)^{n-1}$, namely, $\binom{n-1}{r-1}$, we obtain

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1},$$

which completes the proof.

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2. Using the same technique as in the previous problem, let us prove what is asked by equating the coefficients of y^k in the expressions on both sides of the equation

$$(1+y)^{m+n} = (1+y)^m (1+y)^n .$$

The coefficient of y^k in $(1+y)^{m+n}$ is $\binom{m+n}{k}$, and the coefficient of y^k in

$$(1+y)^m (1+y)^n = \left[\binom{m}{0} + \binom{m}{1}y + \ldots\ldots\binom{m}{m}y^m\right] \times \left[\binom{n}{0} + \binom{n}{1}y + \ldots\ldots\binom{n}{n}y^n\right]$$

is the sum of the products that we obtain by multiplying the constant term of the first factor by the coefficient of y^k in the second factor, the coefficient of y in the first factor by the coefficient of y^{k-1} in the second factor ,....., and the coefficient of y^k in the first factor by the constant term of the second factor. Thus, the coefficient of y^k in $(1+y)^m(1+y)^n$ is

$$\binom{m}{0}\binom{n}{k}+\binom{m}{1}\binom{n}{k-1}+\binom{m}{2}\binom{n}{k-2}+....+\binom{m}{k}\binom{n}{0}=\sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r},$$

and this completes the proof.

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3. If A is the event that the job will be completed on time and B is the event that there will be a strike, the given information can be written as $P(B)=0.6$, $P(B^c)=1-0.6=0.4$, $P(A\,|B)=0.35$, and $P(A\,|B^c)=0.85$. Using the theorem of total probability we get:

$$P(A)=P(B)\,P(A\,|B)+P(B^c)\,P(A\,|B^c).$$

Then, substitution of the given numerical values yields

$$P(A)=(0.6)(0.35)+(0.4)(0.85)=0.55.$$

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4. (a) Here we have specified $P(B_n)=\left(\frac{1}{2}\right)^n$, where $B_n=\{I=n\}$, $n=1,2,...$ If A is the event that the coin comes up heads, we have specified $P(A\,|B_n)=e^{-n}$. Hence, from the theorem of total probability,

$$P(A)=\sum_{n=1}^{\infty}P(B_n)\,P(A\,|B_n)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n e^{-n}=\sum_{n=1}^{\infty}\left(\frac{1}{2e}\right)^n$$

$$=\frac{1/2e}{1-(1/2e)}=\frac{1}{2e-1}=0.2254.$$

(b) A^c is the event that the coin comes up tails. Hence,

$$P(A^c)=1-P(A)=1-\frac{1}{2e-1}=\frac{2(e-1)}{2e-1},$$

and $P(A^c\,|B_5) = 1 - P(A\,|B_5) = 1 - e^{-5}$. From Bayes' theorem,

$$P(B_5\,|A^c) = \frac{P(B_5)\,P(A^c\,|B_5)}{P(A^c)} = \frac{\left(\frac{1}{2}\right)^5(1-e^{-5})}{\frac{2(e-1)}{2e-1}}$$

$$= \frac{(1-e^{-5})\,(2e-1)}{64(e-1)} = 0.04.$$

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5. We have $P(B_1) = 0.6$, $P(B_2) = 0.3$, $P(B_3) = 0.1$, $P(A|B_1) = 0.09$, $P(A|B_2) = 0.2$, $P(A|B_3) = 0.06$, where B_i is the event that the car was rented from agency i and A is the event that it needs to be repaired.

(a) Using the theorem of total probability,

$$P(A) = \sum_{i=1}^{3} P(B_i)P(A|B_i) = 0.12$$

(b)

$$P(B_2|A) = \frac{P(B_2)P(A|B_2)}{P(A)} = 0.5$$

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6. (a) $\displaystyle\binom{n}{x} = \frac{\prod_{i=0}^{x-1}(n-i)}{x!} = \frac{n(n-1)(n-2)\cdot\ldots\cdot(n-x+1)}{x(x-1)(x-2)\cdot\ldots\cdot 2\cdot 1} = \frac{n!}{x!(n-x)!}.$

(b) $\displaystyle\binom{n}{x} = \frac{n(n-1)(n-2)\ldots\cdot 2\cdot 1\cdot \overset{\downarrow}{0}\cdot\ldots\cdot(n-x+1)}{x(x-1)(x-2)\cdot\ldots\cdot 2\cdot 1} = 0.$

(c) $\displaystyle\binom{-r}{x} = \frac{\overbrace{-r(-r-1)(-r-2)\cdot\ldots\cdot(-r-x+2)\cdot(-r-x+1)}^{x\text{ terms}}}{x(x-1)(x-2)\cdot\ldots\cdot 2\cdot 1}$

$\displaystyle = (-1)^x\cdot\frac{r(r+1)(r+2)\cdot\ldots\cdot(r+x-2)\cdot(r+x-1)}{x(x-1)(x-2)\cdot\ldots\cdot 2\cdot 1}$

$\displaystyle = (-1)^x\cdot\frac{(r+x-1)\cdot(r+x-2)\cdot\ldots\cdot(r+1)\cdot r}{x(x-1)(x-2)\cdot\ldots\cdot 2\cdot 1}$

$\displaystyle = (-1)^x\cdot\frac{(r+x-1)!}{x!(r-1)!} = (-1)^x\binom{r+x-1}{x}.$

(d) Obviously, if n and x are two natural numbers and n is strictly positive, then $\displaystyle\binom{n+x-1}{x} = \binom{n+x-1}{n-1}$ since in this case $n+x-1$, x, and $n-1$

are all natural with $n+x-1 \geq x$ and $n+x-1 \geq n-1$. Therefore, using part (c), we get

$$\binom{-n}{x} = (-1)^x \binom{n+x-1}{x} = (-1)^x \binom{n+x-1}{n-1}.$$

(e) Use part (c) to obtain

$$\binom{-1}{x} = (-1)^x \binom{1+x-1}{x} = (-1)^x \binom{x}{x} = (-1)^x.$$

$$\binom{-2}{x} = (-1)^x \binom{2+x-1}{x} = (-1)^x \binom{x+1}{x} = (-1)^x (x+1) = (-1)^x (1+x).$$

(f) Making Taylor's expansion around $z=0$, we get $(1+z)^r = \sum\limits_{x=0}^{\infty} \binom{r}{x} z^x$ (Please, do it!).

(g) Just apply Newton's binomial expansion.

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7. We use the formula $P(A \cap B) = P(B) \cdot P(A|B)$ when $P(B) > 0$.

$E_1 =$ The first individual speaks English.

$F_1 =$ The first individual speaks French.

$E_2 =$ The second individual speaks English.

$F_2 =$ The second individual speaks French.

$$P\{\text{they understand each other}\} = P((E_1 \cap F_1^c) \cap E_2) + P((F_1 \cap E_1^c) \cap F_2) +$$

$$P(E_1 \cap F_1) = P(E_1 \cap F_1^c) \cdot P(E_2 | E_1 \cap F_1^c) + P(F_1 \cap E_1^c) \cdot P(F_2 | F_1 \cap E_1^c) +$$

$$+ P(E_1 \cap F_1) = \left(\frac{60}{100} \cdot \frac{69}{99} \right) + \left(\frac{30}{100} \cdot \frac{39}{99} \right) + \left(\frac{10}{100} \right) = \frac{7}{11}.$$

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8. (a) The number of possible orders is 10!. There are 9 possible pairs of positions next to each other. They can stay there in 2 different orders. On the other 8 positions the others can stay in 8! different orders. Thus, the probability that two given persons stay next to each other is

$$\frac{9 \cdot 2 \cdot 8!}{10!} = \frac{1}{5}.$$

(b) Now the only difference is that, there are 10 possible pair of positions next to each other. Thus, the probability is

$$\frac{10 \cdot 2 \cdot 8!}{10!} = \frac{2}{9}.$$

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9. Note that the probability that we can make at least 3 successive extractions is equal to the probability of extracting a white ball in both the first and the second round. Moreover, since after a round we replace the ball, we need to compute first the probability $P(W)$ that we choose a white ball in a given round, and then $[P(W)]^2$ will be the probability that we will extract a white ball in both the first and the second round.

Note that, from the theorem of total probability,

$$P(W) = \sum_{k=2}^{12} P(k)P(W|k),$$

where k is the sum of the points of the dices. We can calculate that

$$P(k) = \frac{k-1}{36},$$

for $k = 2, \ldots, 7$, and

$$P(k) = \frac{12-k+1}{36},$$

for $k = 8, \ldots, 12$. Since $P(W|k) = \dfrac{k-2}{k-1}$ for $k = 2, \ldots, 7$ and $P(W|k) = \dfrac{12-k}{12-k+1}$ for $k = 8, \ldots, 12$, we have

$$P(W) = \left[\sum_{k=2}^{7}\left(\frac{k-1}{36} \cdot \frac{k-2}{k-1}\right)\right] + \left[\sum_{k=8}^{12}\left(\frac{12-k+1}{36} \cdot \frac{12-k}{12-k+1}\right)\right]$$

$$= \left(\sum_{k=2}^{7} \frac{k-2}{36}\right) + \left(\sum_{k=8}^{12} \frac{12-k}{36}\right) = \frac{25}{36}.$$

Therefore, the solution is $[P(W)]^2 = \left(\frac{25}{36}\right)^2 = 0.48225.$

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10. (a) The probability $P(i)$ that there are exactly i red balls in the box is $\frac{1}{n+1}$, with $i = 0, 1, 2, ...n$. Let R be the event that we picked a red ball. Let $P(R|i)$ be the probability of picking a red ball given that there are i red balls in the box. Obviously, $P(R|i) = \frac{i}{n}$. We are asked to to find $P(k|R)$. We apply Bayes' theorem,

$$P(k|R) = \frac{P(k)P(R|k)}{\sum_{i=0}^{n} P(i)P(R|i)} = \frac{\frac{1}{n+1}\frac{k}{n}}{\sum_{i=0}^{n} \frac{1}{n+1}\frac{i}{n}} = \frac{k}{\sum_{i=0}^{n} i} = \frac{k}{\frac{n(n+1)}{2}} = \frac{2k}{n(n+1)}.$$

(b) The probability $P(i)$ that there are exactly i red balls in the box is $\frac{1}{7}$ for $i = 0, 1, ..., 6$. Let $\{R_1 \cap R_2\}$ be the event where we pick two red balls in two extractions with replacement. Let $P(R_1 \cap R_2|i)$ be the probability of extracting two red balls with replacement given that there are i white balls in the box. Obviously, $P(R_1 \cap R_2|i) = \left(\frac{i}{6}\right)^2 = \frac{i^2}{36}$. We are asked to to find $P(5|R_1 \cap R_2)$. We apply Bayes' theorem,

$$P(5|R_1 \cap R_2) = \frac{P(5)P(R_1 \cap R_2|5)}{\sum\limits_{i=0}^{6} P(i)P(R_1 \cap R_2|i)} = \frac{\frac{1}{7} \cdot \frac{25}{36}}{\sum\limits_{i=0}^{6} \frac{1}{7} \cdot \frac{i^2}{36}} = \frac{25}{\sum\limits_{i=0}^{6} i^2} = \frac{25}{91} = 0.2747.$$

(c)

$\{B1\}$ = the first extracted ball is black

$\{B2\}$ = the second extracted ball is black

$\{j\}$ = there are j black balls in the box, i.e, there are $7 - j$ red balls in the box, $j = 0, 1, ..., 7$.

$$P(B1 \cap B2) = \sum_{j=0}^{7} P(j) \cdot P(B1 \cap B_2 | j)$$

$$= \sum_{j=0}^{7} \frac{1}{8} \cdot \left(\frac{j}{7} \cdot \frac{j-1}{6} \right) = \frac{1}{336} \sum_{j=0}^{7} j \cdot (j-1) = \frac{1}{336} \cdot 112 = \frac{1}{3}.$$

Note that

$$P(B1 \cap B2) = \sum_{j=0}^{7} \overbrace{P(j)}^{\frac{1}{8}} \cdot \underbrace{\underbrace{\overbrace{P(B1 \,|j)}^{\frac{j}{7}}}_{\frac{P(B1 \cap j)}{P(j)}} \cdot \underbrace{\overbrace{P(B2 \,|B1 \cap j)}^{\frac{j-1}{6}}}_{\frac{P(B2 \cap (B1 \cap j))}{P(B1 \cap j)}}}_{\underbrace{\frac{P((B1 \cap B2) \cap j)}{P(j)}}_{P(B1 \cap B2|j)}}$$

$\{R2\}$ = the second extracted ball is red.

Then,

$$P(B1 \cap R2) = \sum_{j=0}^{7} P(j) \cdot P(B1 \cap R2 \,|j)$$

$$= \sum_{j=0}^{7} \frac{1}{8} \cdot \frac{j}{7} \cdot \frac{7-j}{6} = \frac{1}{336} \sum_{j=0}^{7} j \cdot (7-j) = \frac{1}{336} \cdot 56 = \frac{1}{6}.$$

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11. Let

F_1 = to extract a figure from the first deck in the first extraction

F_2 = to extract a figure in the second extraction.

From the theorem of total probability,

$$P(F_2) = P(F_1 \cap \{1\})P(F_2|F_1 \cap \{1\}) + P(F_1^c \cap \{1\})P(F_2|F_1^c \cap \{1\}) +$$

$$P(F_1 \cap \{2\})P(F_2|F_1 \cap \{2\}) + P(F_1^c \cap \{2\})P(F_2|F_1^c \cap \{2\})$$

$$+ P(\{3,4,5,6\}) \, P(F_2| \{3,4,5,6\}). \tag{1}$$

Note that

$$P(F_1 \cap \{1\}) = P(F_1) \cdot P(\{1\}) = \frac{12}{48} \cdot \frac{1}{6} = \frac{1}{24},$$

$$P(F_1^c \cap \{1\}) = P(F_1^c) \cdot P(\{1\}) = \frac{36}{48} \cdot \frac{1}{6} \cdot = \frac{1}{8},$$
$$P(F_1 \cap \{2\}) = P(F_1) \cdot P(\{2\}) = \frac{12}{48} \cdot \frac{1}{6} = \frac{1}{24},$$
$$P(F_1^c \cap \{2\}) = P(F_1^c) \cdot P(\{2\}) = \frac{36}{48} \cdot \frac{1}{6} = \frac{1}{8},$$

and

$$P(\{3, 4, 5, 6\}) = \frac{4}{6} = \frac{2}{3}.$$

Moreover,

$$P(F_2 | F_1 \cap \{1\}) = \frac{11}{47},$$
$$P(F_2 | F_1^c \cap \{1\}) = \frac{12}{47},$$
$$P(F_2 | F_1 \cap \{2\}) = \frac{13}{49},$$
$$P(F_2 | F_1^c \cap \{2\}) = \frac{12}{49},$$

and

$$P(F_2 | \{3, 4, 5, 6\}) = \frac{24}{96} = \frac{1}{4}.$$

Thus, from (1) we have

$$P(F_2) = \left(\frac{1}{24} \cdot \frac{11}{47}\right) + \left(\frac{1}{8} \cdot \frac{12}{47}\right) + \left(\frac{1}{24} \cdot \frac{13}{49}\right) + \left(\frac{1}{8} \cdot \frac{12}{49}\right) + \left(\frac{2}{3} \cdot \frac{1}{4}\right) = \frac{1}{4}.$$

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12. (a)

$B_t = \{\text{Bulls win the game } t\}$

$W_t = \{\text{Warriors win the game } t\}$

Assumptions:

$P(B_t) = 0.6$

$P(B_t | B_{t-1}) = 0.7$

$P(B_t | W_{t-1}) = 0.5$

(i) $P(W_t | B_{t-1}) = 1 - P(B_t | B_{t-1}) = 1 - 0.7 = 0.3$

(ii) $P(B_1 \cap B_2 \cap B_3 \cap B_4)$

$= P(B_1) \cdot P(B_2 | B_1) \cdot P(B_3 | B_1 \cap B_2) \cdot P(B_4 | B_1 \cap B_2 \cap B_3)$

$= P(B_1) \cdot P(B_2 | B_1) \cdot P(B_3 | B_2) \cdot P(B_4 | B_3)$

$= 0.6 \cdot 0.7 \cdot 0.7 \cdot 0.7 = 0.2058$

(iii) $P(W_1 \cap W_2 \cap W_3 \cap W_4)$

$= P(W_1) \cdot P(W_2 | W_1) \cdot P(W_3 | W_1 \cap W_2) \cdot P(W_4 | W_1 \cap W_2 \cap W_3)$

$= P(W_1) \cdot P(W_2 | W_1) \cdot P(W_3 | W_2) \cdot P(W_4 | W_3)$

$= 0.4 \cdot 0.5 \cdot 0.5 \cdot 0.5 = 0.05$

(iv) By Bayes' Theorem:

$$P(W_1 | B_2) = \frac{P(W_1) \cdot P(B_2 | W_1)}{P(W_1) \cdot P(B_2 | W_1) + P(B_1) \cdot P(B_2 | B_1)}$$

$$= \frac{0.4 \cdot 0.5}{(0.4 \cdot 0.5) + (0.6 \cdot 0.7)} = \frac{0.2}{0.62} = 0.3226$$

(b)

$P(B_t) = 0.5.$

$P(B_t | B_{t-1}) = P(B_t | W_{t-1}) = 0.5.$

$P\{4 \text{ game series under conditions } (a)\} =$

$$P\left(\left(\underbrace{B_1 \cap B_2 \cap B_3 \cap B_4}_{B}\right) \cup \left(\underbrace{W_1 \cap W_2 \cap W_3 \cap W_4}_{W}\right)\right)$$

$= P(B) + P(W) = 0.2058 + 0.05 = 0.2558$, since B and W are disjoint events.

$P\{4 \text{ game series under conditions } (b)\} =$

$$P\left(\left(\underbrace{B_1 \cap B_2 \cap B_3 \cap B_4}_{B}\right) \cup \left(\underbrace{W_1 \cap W_2 \cap W_3 \cap W_4}_{W}\right)\right)$$

$= P(B) + P(W) = (0.5)^4 + (0.5)^4 = 0.125.$

$0.125 < 0.2558 \leftarrow$ More than twice more likely under conditions of (a).

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13. (a)

$$P\{\text{the three have passed the Micro exam}\} = \frac{14}{20}\frac{13}{19}\frac{12}{18} = \frac{91}{285}.$$

$$P\{\text{the three have passed the Macro exam}\} = \frac{8}{20}\frac{7}{19}\frac{6}{18} = \frac{14}{285}.$$

Since only two students have passed both exams, the events

$$\{\text{the three have passed the Micro exam}\}$$

and

$$\{\text{the three have passed the Macro exam}\}$$

are disjoint so that their intersection has zero probability. Therefore,

$$P\{\text{the three have passed the same exam}\} = \frac{91}{285} + \frac{14}{285} = \frac{7}{19} = 0.36842.$$

(b)

$$P\{\text{the three have passed the Micro exam}\}\,|\{\text{the three have passed the same exam}\} =$$

$$= \frac{P\{\text{the three have passed the Micro exam}\} \cap \{\text{the three have passed the same exam}\}}{P\{\text{the three have passed the same exam}\}}$$

$$= \frac{P\{\text{the three have passed the Micro exam}\}}{P\{\text{the three have passed the same exam}\}} = \frac{\frac{91}{285}}{\frac{7}{19}} = \frac{13}{15} = 0.86667.$$

Note that

$$\{\text{the three have passed the Micro exam}\} \subset \{\text{the three have passed the same exam}\}$$

so that

$$(\{\text{the three have passed the Micro exam}\} \cap \{\text{the three have passed the same exam}\})$$

$$= \{\text{the three have passed the Micro exam}\}.$$

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14. (a) $\frac{1}{4} = P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. A and B are independent.

$\frac{1}{4} = P(A \cap C) = P(A) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. A and C are independent.

$\frac{1}{4} = P(B \cap C) = P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. B and C are independent.

$\frac{1}{4} = P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. A, B, and C are not independent.

(b) $P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = 1.$

(c) Note that, in general, $G = (G \cap H) \cup (G \cap H^c)$ and, obviously $(G \cap H) \cap (G \cap H^c) = \emptyset$.

Therefore, $P(G) = P(G \cap H) + P(G \cap H^c)$ so that $P(G \cap H^c) = P(G) - P(G \cap H)$.

Therefore, $P((A \cap B) \cap C^c) = P(A \cap B) - P(A \cap B \cap C) = \frac{1}{4} - \frac{1}{4} = 0.$

(d) $0.48 = P(D \cap E) = P(D) \cdot P(E) = 0.6 \cdot 0.8 = 0.48$. D and E are independent.

$0.3 = P(D \cap F) = P(D) \cdot P(F) = 0.6 \cdot 0.5 = 0.3$. D and F are independent.

$0.38 = P(E \cap F) \neq P(E) \cdot P(F) = 0.8 \cdot 0.5 = 0.4$. E and F are not independent.

$0.24 = P(D \cap E \cap F) = P(D) \cdot P(E) \cdot P(F) = 0.6 \cdot 0.8 \cdot 0.5 = 0.24$. However, D, E, and F are not independent since $P(E \cap F) \neq P(E) \cdot P(F)$.

(e) $P((D \cap E) \cap F^c) = P(D \cap E) - P(D \cap E \cap F) = 0.48 - 0.24 = 0.24$.

(f) $P(D \cup E \cup F)$

$= P(D) + P(E) + P(F) - P(D \cap E) - P(D \cap F) - P(E \cap F) + P(D \cap E \cap F)$

$= 0.6 + 0.8 + 0.5 - 0.48 - 0.3 - 0.38 + 0.24 = 0.98$.

$P((D \cup E \cup F)^c) = 1 - P(D \cup E \cup F) = 1 - 0.98 = 0.02$.

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15. Note that $P(B_i \cap C) > 0$ for $i = 1, 2, ...$ implies that $P(C) > 0$. Let us check that the expression in the statement holds,

$$P(A | C) = \sum_i P(B_i | C) \cdot P(A | B_i \cap C), \tag{1}$$

where

$$P(A | C) = \frac{P(A \cap C)}{P(C)},$$

$$P(B_i | C) = \frac{P(B_i \cap C)}{P(C)},$$

$$P(A | B_i \cap C) = \frac{P(A \cap B_i \cap C)}{P(B_i \cap C)}.$$

Thus, (1) becomes

$$\frac{P(A \cap C)}{P(C)} = \sum_i \frac{P(B_i \cap C)}{P(C)} \cdot \frac{P(A \cap B_i \cap C)}{P(B_i \cap C)}$$

$$= \frac{1}{P(C)} \sum_i P((A \cap C) \cap B_i). \tag{2}$$

Note that $\{(A \cap C) \cap B_1, (A \cap C) \cap B_2, ...\}$ is a countable partition of the event $A \cap C$. Thus, using the countable additivity property of probability, we get

$$\sum_i P((A \cap B_i) \cap C) = P(A \cap C).$$

Therefore, (2) becomes

$$\frac{P(A \cap C)}{P(C)} = \frac{P(A \cap C)}{P(C)}. \qquad Q.E.D.$$

Note that, if $C = \Omega$, then we recover from (1) the original theorem of total probability,

$$P(A) = \sum_i P(B_i) \cdot P(A | B_i), \quad \text{for all } A \in \mathcal{F},$$

since $B_i \cap \Omega = B_i$, $P(A | \Omega) = \dfrac{P(A \cap \Omega)}{P(\Omega)} = \dfrac{P(A)}{1} = P(A)$, and similarly $P(B_i | \Omega) = P(B_i)$.

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16. Let us consider an arbitrary individual from the set of n individuals and consider the following events:

$S = \{$the individual picks the shortest straw$\}$,

$j = \{$the individual is placed in the jth position$\}$,

$p = \{$the individual picks a straw$\}$,

$p \cap j = \{$the individual picks a straw and is placed in the jth position$\}$.

Note that p and p^c constitute a partition of the sample space. Therefore, applying the theorem of Exercise 15, we have

$$P(S | j) = P(p | j) \cdot P(S | p \cap j) + P(p^c | j) \cdot P(S | p^c \cap j)$$

$P(S | p \cap j)$ is the probability that an individual placed in the position j and who actually picks a straw picks the shortest straw. Then, $P(S | p \cap j) = \dfrac{1}{n - (j-1)}$ since there are $n - (j-1)$ straws left in the fist when the individual placed in the jth position has to pick a straw.

Obviously, $P(S\,|p^c \cap j) = 0$ since this individual placed in the jth position does not pick a straw and, thus, he cannot pick the shortest.

$$P(p\,|j) = P\{\text{None of the previous } j-1 \text{ individuals has picked the shortest straw}\}$$

$$= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \ldots \cdot \frac{n-(j-2)}{n-(j-3)} \cdot \frac{n-(j-1)}{n-(j-2)} = \frac{n-(j-1)}{n}.$$

Therefore,

$$P(S\,|j) = P(p\,|j) \cdot P(S\,|p \cap j) + P(p^c\,|j) \cdot P(S\,|p^c \cap j)$$

$$= \frac{n-(j-1)}{n} \cdot \frac{1}{n-(j-1)} + \underbrace{P(p^c\,|j)}_{\frac{j-1}{n}} \cdot 0 = \frac{1}{n}, \quad \text{for all } j = 1, 2, ..., n.$$

Therefore, the order in which individuals pick the straw does not affect their probability of picking the shortest straw.

One implication of the previous result is that, if you were one of the six individuals playing Russian roulette with a six-chamber revolver with one bullet in the cylinder and the gun is spun just once at the beginning of the game, no matter if you go first, second, ... or sixth, your odds of seeing what a bullet feels like would be the same. This is so because the more rounds that are played, the more likely the bullet is to appear, but this is offset by the fact that the game is less likely to reach that round (so for instance, in the first round, there is a 100% chance that the trigger of the gun will be pulled, and a 1/6 probability that the chamber will contain the bullet. In the sixth round, there is a 100% chance that the chamber contains the bullet, but only a 1/6 probability that the game actually gets there.) Thus, in this case, according to our previous formula with $n = 6$,

$$P(\{\text{getting the bullet}\}\,|j) = \frac{1}{6}, \quad \text{for all } j = 1, 2, ..., 6.$$

However, if the rules were different, there may be an advantage to playing later. Imagine that the rule is the following: the first player spins and shoots, then the second one spins and shoots, and so on. In this scenario, the game is effectively "reset" each time. That is, the likelihood of the bullet being in any of the chambers is totally unaffected by the chamber it was in the previous turn. So this time the bullet is no more likely to appear as the game continues. However, the chance of the game continuing still goes down over time (as it is more likely that the bullet would have appeared at some point). Let us formalize that: the first player has a 1/6 probability of getting the bullet when he pulls the trigger. If he survives, the second player also has a 1/6 probability of getting the bullet when he pulls the trigger. However, the second player only has a 5/6 probability of ever pulling the trigger at all, so he has a slightly better chance of surviving. In this case, let us define the following events:

$B = \{\text{the individual gets the bullet}\}$,

$j = \{\text{the individual is placed in the } j\text{th position}\}$,

$p = \{\text{the individual pulls the trigger}\}$,

$p \cap j = \{\text{the individual pulls the trigger and is placed in the } j\text{th position}\}$.

Applying the theorem of total conditional probability,

$$P(B|j) = P(p|j) \cdot P(B|p \cap j) + P(p^c|j) \cdot P(B|p^c \cap j)$$

$$= \frac{6-(j-1)}{6} . \frac{1}{6} + \underbrace{P(p^c|j)}_{\frac{j-1}{6}} \cdot 0 = \frac{6-(j-1)}{36} = \frac{7-j}{36}, \quad \text{for all } j = 1, 2, ..., 6,$$

which is decreasing in j. For example, $P(B|1) = 1/6$ whereas $P(B|6) = 1/36$.

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