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Mixed Risk Aversion*

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We analyze the class of increasing utility functions exhibiting all derivatives of alternating sign. This property, that we call mixed risk aversion, is satisfied by the utility functions most commonly used in financial economics. The utility functions displaying mixed risk aversion can be expressed as mixtures of exponential functions. We characterize stochastic dominance and we find conditions for both mutual aggravation and mutual amelioration of risks when agents are mixed risk averse. Finally, the analysis of the distribution function describing a mixed utility allows one to characterize the behaviour of its indexes of risk aversion and to discuss its implications for portfolio selection. *Journal of Economic Literature* Classification Numbers: D81, G11.

1. INTRODUCTION

Most of the utility functions used to construct examples of choice under uncertainty share the property of having all odd derivatives positive and all even derivatives negative. The aim of this paper is to characterize the class of utility functions exhibiting this property which we call *mixed risk aversion*. Utility functions with this property are called *mixed* and, as follows from Bernstein's theorem, such functions can be expressed as mixtures of exponential utilities.

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Within the paradigm that follows the von Neumann–Morgenstern theory of expected utility, the necessity of imposing additional restrictions on the utility functions representing preferences on the space of random variables was soon recognized. Besides the obvious requirement of risk aversion (or concavity) which allows expected utility maximization, Pratt [12] and Arrow [2] stressed the importance of the property of decreasing risk aversion so as to obtain plausible comparative statics results about the relation between wealth and risk taking by an investor. Thus, an investor with decreasing absolute risk aversion exhibits a demand for risky asset that increases with wealth.

Pratt and Zeckhauser [13] considered afterwards the family of proper risk averse utility functions which constitute a strict subset of the functions with decreasing absolute risk aversion. A proper utility is the one for which an undesirable risk can never be made desirable by the presence of another independent, undesirable risk, that is, these two risks must aggravate each other. In their paper, Pratt and Zeckhauser proved that mixtures of exponential utilities are proper, and that some commonly used utility functions are mixtures of exponential utilities.[1]

As part of the process of refining the set of risk averse expected utility representations, Kimball [10] has introduced the property of standard risk aversion. Standardness means that every undesirable risk is aggravated by every independent, loss-aggravating risk. It should be noticed that, since a loss-aggravating risk is a risk that increases the expected marginal utility, when absolute risk aversion is decreasing, every undesirable risk is loss-aggravating, but the converse is not true. Therefore, standardness implies Pratt and Zeckhauser's properness.

In this paper we take a step further in this refinement strategy and provide a characterization of mixed utility functions in terms of preferences over pairs of sequences of lotteries (or distributions). This characterization might be useful for testing by means of a laboratory experiment whether an individual is mixed risk averse. We also propose two alternative, more technical characterizations for increasing and risk averse preferences. One of these characterizations allows one to show that mixed risk aversion implies standard risk aversion. Such a characterization requires that the measure of absolute prudence be decreasing, which is in turn necessary and sufficient for standardness.

A remarkable feature of the class of mixed utility functions is that it is large enough to allow for several general functional forms. Moreover, these

[1] In a more recent paper, Gollier and Pratt [7] have also introduced the class of risk vulnerable utility functions, which are the ones for which every undesirable risk is aggravated by every independent, unfair risk. Obviously, since an unfair risk is undesirable, a proper utility is risk vulnerable. On the other hand, risk vulnerability implies decreasing absolute risk aversion.

utilities have the interesting property of being completely characterized by the (Lebesgue–Stieltjes) measure describing the mixture of exponential utilities. For instance, this measure contains information about the values of the indexes of absolute and relative risk aversion which are relevant for the comparative statics of simple portfolio selection problems. Therefore, by appropriately choosing a measure over exponential utilities, we can construct examples of expected utility representations with appealing properties and for which we can control the behavior of their indexes of risk aversion.

Finally, it can also be proved that some standard concepts in the theory of decisions under uncertainty, such as stochastic dominance or mutual aggravation of risks, can be easily stated in terms of Laplace transforms when applied to the family of mixed utility functions. With this reformulation those concepts become more operative as some examples suggest.

The paper is organized as follows. In the next section we define mixed risk aversion and we relate this concept to the one of complete monotonicity in order to establish some preliminary results. Section 3 characterizes mixed utilities and discusses their properties. Sections 4 and 5 reformulate the concepts of stochastic dominance and mutual aggravation of risks, respectively, for mixed utility functions. Section 6 studies the link between the distribution function characterizing a mixed utility and its absolute and relative risk aversion indexes. Section 7 analyzes some simple portfolio selection problems for mixed risk averse investors. Section 8 concludes the paper.

2. COMPLETELY MONOTONE FUNCTIONS AND MIXED RISK AVERSION

In this section we present some mathematical results which are useful for the characterization of the class of utility functions we are interested in.

Definition 2.1. A real-valued function $\varphi(w)$ defined on $(0, \infty)$ is completely monotone iff its derivatives $\varphi^n(w)$ of all orders exist and

$$(-1)^n \varphi^n(w) \geqslant 0, \qquad \text{for all} \quad w > 0 \quad \text{and} \quad n = 0, 1, 2, \ldots.$$

Therefore, a function is completely monotone if it is nonnegative and it has odd derivatives that are all nonpositive and even derivatives that are all nonnegative. The following famous theorem due to Bernstein shows that a function is a Laplace transform of a distribution function iff it is completely monotone. Analogously, we can say that the set of negative exponential functions constitutes a basis for the set of completely monotone functions (see Gollier and Kimball [6]).

THEOREM 2.1. *The real-valued function* $\varphi(w)$ *defined on* $(0, \infty)$ *is completely monotone if and only if it has the following functional form*:

$$\varphi(w) = \int_0^\infty e^{-sw}\, dF(s), \qquad \text{for all} \quad w > 0, \tag{1}$$

where F *is a distribution function on* $[0, \infty)$.

Proof. See Feller [5, Section XII.4] or Widder [17, Sections IV.12 and IV.13] for alternative proofs.

Throughout this paper we consider that a distribution function $F(s)$ on $[0, \infty)$ is a nondecreasing and right-continuous map from $[0, \infty)$ into $[0, \infty)$. Obviously, we can rewrite the improper Stieltjes integral in (1) as $\int_{[0, \infty)} e^{-sw}\, dv$, where v is the Lebesgue–Stieltjes measure on $[0, \infty)$ induced by F (see Ash [3, Section 1.4]). Note that $\lim_{s \to \infty} F(s)$ exists because of the monotonicity of F, but this limit is not necessarily finite. This means that $v(A)$ might be equal to infinity when A is an unbounded Borel set.

COROLLARY 2.1. *If* φ *is a completely monotone function and* ψ *is a positive function with a completely monotone first derivative, then the composite function* $\varphi(\psi)$ *is completely monotone. In particular, the function* $\varphi(w) = \beta \exp[-\alpha\psi(w)]$ *is completely monotone for all nonnegative* α *and* β.

Proof. See Section XIII.4 of Feller [5].

COROLLARY 2.2. *If the function* $\varphi(w)$ *defined on* $(0, \infty)$ *is completely monotone and strictly positive, then* $\varphi(w)$ *is log-convex, i.e.*, $\ln(\varphi(w))$ *is convex. The convexity is strict when the support of* F *has at least two points.*

Proof. Just compute

$$\frac{\partial^2[\ln(\varphi(w)]}{\partial w^2} = \frac{[\int_0^\infty s^2 e^{-sw}\, dF(s)][\int_0^\infty e^{-sw}\, dF(s)] - [\int_0^\infty s e^{-sw}\, dF(s)]^2}{[\int_0^\infty e^{-sw}\, dF(s)]^2} \geqslant 0,$$

where the inequality follows from the Cauchy–Schwarz inequality. Q.E.D.

Define, for all $h > 0$, $\Delta_h \varphi(w) = \varphi(w+h) - \varphi(w)$ and, by induction, $\Delta_h^0 \varphi(w) = \varphi(w)$, $\Delta_h^1 \varphi(w) = \Delta_h \varphi(w)$, $\Delta_h^{n+1} \varphi(w) = \Delta_h \Delta_h^n \varphi(w)$.

COROLLARY 2.3. *The function* φ *defined on* $(0, \infty)$ *is completely monotone if and only if, for every nonnegative integer* n, *and for all strictly positive real numbers* w *and* h,

$$(-1)^n \Delta_h^n \varphi(w) \geqslant 0. \tag{2}$$

Proof. The proof is an immediate adaptation of the argument in Akhiezer [1, Section 5.5]. We just have to notice that the function $\varphi(w)$ is completely monotone on $(0, \infty)$ if and only if the function $\hat{\varphi}(z) \equiv \varphi(-z)$ is absolutely monotone for all $z \in (-\infty, 0)$.[2] Q.E.D.

It is important to notice that (2) can be rewritten as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(w+kh) \geqslant 0. \tag{3}$$

Let us assume that an agent has state-independent preferences defined over the space of nonnegative, real-valued random variables and that she has an expected utility representation u of these preferences. Then $\tilde{x}$ is preferred to $\tilde{y}$ if and only if $E[u(\tilde{x})] \geqslant E[u(\tilde{y})]$, where $\tilde{x}$ and $\tilde{y}$ are nonnegative random variables and u is a real-valued Borel measurable function.

Definition 2.2. A real-valued, continuous utility function u defined on $[0, \infty)$ exhibits mixed risk aversion iff it has a completely monotone first derivative on $(0, \infty)$, and $u(0) = 0$.

Utility functions displaying mixed risk aversion are called *mixed*. The requirement of $u(0) = 0$ is just an innocuous normalization for real-valued functions on $[0, \infty)$ since the preference ordering is preserved under affine transformations of the expected utility representation u. The choice of zero as the origin of the domain $[0, \infty)$ is made for convenience. Of course, our analysis can be immediately adapted to different domains and normalizations. In particular, if we consider instead the open domain $(0, \infty)$, all the results of this paper will hold with the obvious exception of the ones referred to the properties of u at zero. Thus, even if many expected utility representations used to model situations of choice under uncertainty exhibit mixed risk aversion, we should point out that Definition 2.2 does not apply to the affine transformations of the logarithmic utility function $u(w) = \ln w$ and of the power function $U(w) = \alpha w^{\alpha}$, with $\alpha < 0$, since they are not finite at zero. Nevertheless, they can be arbitrarily (but not uniformly) approximated for all $w > 0$ by the mixed utilities $\hat{u}(w) = \ln(w+d) - \ln d$ and $\hat{U}(w) = \alpha(w+d)^{\alpha} - \alpha d^{\alpha}$, respectively.

The following theorem due to Schoenberg [16] provides a functional characterization of mixed functions:

[2] A function $\hat{\varphi}$ defined on $(-\infty, 0)$ is absolutely monotone iff it is nonnegative and its derivatives of all orders are all nonnegative.

THEOREM 2.2. *The utility function $u(w)$ defined on $[0, \infty)$ displays mixed risk aversion if and only if it admits the functional representation*

$$u(w) = \int_0^{\infty} \frac{1 - e^{sw}}{s} \, dF(s), \tag{4}$$

for some distribution function F on $[0, \infty)$ satisfying

$$\int_1^{\infty} \frac{dF(s)}{s} < \infty. \tag{5}$$

Since u is obtained by Riemann integrating a completely monotone function φ which has the functional form given in (1), condition (5) is necessary and sufficient for the convergence of the integral $\int_0^w \varphi(x)\, dx$ defining $u(w)$ for $w \in [0, \infty)$. As in Akhiezer [1], we could also extend the domain of u to $[0, \infty]$, and its range to the extended real numbers, by making $u(\infty) = \lim_{w \to \infty} u(w)$.

As pointed out by Pratt and Zeckhauser [13], most of the utility functions used in applied work have completely monotone first derivatives. For instance, if the utility function u is HARA, i.e., it satisfies $-u''(w)/u'(w) = 1/(a + bw)$ with $a > 0, b > 0$, then (4) holds with $dF(s) = As^{(1/b)-1} e^{-(a/b)s}\, ds$, where A is a scaling factor. Therefore, the HARA utility functions are mixed since they are mixtures of exponential functions characterized by an arbitrarily scaled gamma distribution function.[3] As limit cases of HARA functions, we obtain the isoelastic (or power) function $u(w) = Cw^{\alpha}$ with $0 < \alpha < 1$ when $b = 1/(1-\alpha)$ and $a = 0$. In this case, $F(s)$ is a power distribution with $dF(s) = As^{-\alpha}\, ds$. If $a = 1/\rho$ and $b = 0$, we get the exponential (or CARA) utility function $u(w) = C[1 - e^{-\rho w}]$ with $\rho > 0$ and, then, $F(s)$ is the Dirac distribution, $F(s) = C\rho$ for $s \geqslant \rho$ and $F(s) = 0$ for $0 \leqslant s < \rho$. Finally, if $b = 1$ and $a = d/c$, the utility function is logarithmic, $u(w) = C[\ln(d + cw) - \ln d]$ with $d > 0$ and $c > 0$, and $F(s)$ turns out to be exponential; i.e., $dF(s) = Ae^{-(d/c)\, s}\, ds$.

COROLLARY 2.4. *Let u be a mixed utility function which is analytic at the point $\kappa > 0$ with interval of convergence $(\kappa - \varepsilon, \kappa + \varepsilon)$, where $\kappa \geqslant \varepsilon > 0$. Then $u(w)$ can be expressed as the power series*

$$u(w) = \sum_{n=0}^{\infty} p_n (w - \kappa)\, n, \qquad \text{for} \quad w \in (\kappa - \varepsilon, \kappa + \varepsilon), \tag{6}$$

[3] Cass and Stiglitz [4] proved that two-fund monetary separation holds in an economy populated by investors having those HARA utilities with a common parameter b. Note that we are excluding from our analysis the concave quadratic utility functions, which belong to the HARA class but do not display decreasing absolute risk aversion since $b = -1$.

with

$$p_0 = \int_0^\infty \frac{1 - e^{-s\kappa}}{s} \, dF(s)$$

and

$$p_n = \frac{-(-1)^n}{n!} \int_0^\infty s^{n-1} e^{-s\kappa} \, dF(s), \qquad \text{for} \quad n = 1, 2, ...,$$

where F is a distribution function on $[0, \infty)$.

Proof. It follows directly from applying Taylor's theorem to (4). Q.E.D.

It should be noticed that a mixed utility $u(w)$ is analytic for all $w > 0$, that is, it can be expressed as a power series in $(w - \kappa)$ which converges in some neighborhood of κ, for all $\kappa > 0$ (see Widder [17, Section II.5]). Moreover, it can be proved that if the mixed utility u is characterized by a distribution function having a density f, that is, $F'(s) = f(s)$ for all $s \in (0, \infty)$, and f is bounded above, then the power expansion (6) holds for $w \in (0, 2\kappa)$ and for all $\kappa > 0$. Finally, it is obvious that (6) also holds for $w \in (0, \infty)$ and for all $\kappa > 0$ when $F(s)$ has a discrete support.

3. PROPERTIES AND CHARACTERIZATIONS OF MIXED UTILITY FUNCTIONS

An immediate consequence of Corollary 2.4 is the following property which refers to the response of the expected utility when there is a marginal change in just one of the moments of a small risk (or random variable).

COROLLARY 3.1. *Assume that u is a mixed utility function which is analytic about the point* $\kappa > 0$ *with interval of convergence* $(\kappa - \varepsilon, \kappa + \varepsilon)$, *where* $\kappa > \varepsilon > 0$. *Assume also that* $\tilde{w}$ *is a real-valued random variable whose distribution support is included in the interval* $(\kappa - \varepsilon, \kappa + \varepsilon)$. *Then* $E[u(\tilde{w})]$ *has nonnegative* (*nonpositive*) *partial derivatives with respect to the odd* (*even*) *moments of* $\tilde{w}$, *that is,*

$$-(-1)^r \frac{\partial E[u(\tilde{w})]}{\partial[\mu_r(\tilde{w})]} \geqslant 0, \qquad \text{for} \quad r = 1, 2, ...,$$

where $\mu_r(\tilde{w}) = E[(\tilde{w})^r]$.

Proof. After expanding the binomial expressions $(w - \kappa)^n$ in (6), it can be seen that the coefficients of the terms w^n are positive (negative) when n

is odd (even). The result then follows from evaluating the corresponding expectation. Q.E.D.

Therefore, when a mixed risk averse agent faces a choice between two small risks that *only* differ in the rth moment, she prefers the risk with higher moment if r is odd, and the risk with lower moment if r is even. According to our previous discussion, if the mixed utility u is characterized by a distribution function having a bounded density, then the conclusion of Corollary 3.1 holds for every nonnegative random variable with bounded distribution support. Moreover, the same result also holds for all nonnegative random variables having well-defined moments of all orders if the distribution function describing u has discrete support.

The next propositions provide three characterizations of the utility functions displaying mixed risk aversion. The first one is based on the comparison of two sets of sequences of random variables, whereas the second and the third apply to increasing and concave functions and rely on the behavior of risk aversion indexes and derivatives of all orders.

The following lemmas are crucial for the first characterization we propose in this section:

LEMMA 3.1. *The function $u(w)$ defined on $(0, \infty)$ satisfies $(-1)^n \Delta_h^{n+1} u(w) \geqslant 0$, for every nonnegative integer n and for all real $h > 0$, if and only if $(-1)^n \Delta_h^n \varphi_a(w) \geqslant 0$ for all real $a > 0$, where $\varphi_a(w) \equiv u(w+a) - u(w)$.*

Proof. See the Appendix.

LEMMA 3.2. *Let $u(w)$ be a continuous function on $[0, \infty)$ with $u(0) = 0$. The function $u(w)$ is mixed if and only if $\varphi_h(w) \equiv u(w+h) - u(w)$ is completely monotone with respect to w on $(0, \infty)$, for all $h > 0$.*

Proof. See the Appendix.

Let us now define two sets of sequences of lotteries (or discrete distributions) faced by an agent. The sequences $\{L_n^e(h)\}_{n=1}^{\infty}$ of "even" lotteries are indexed by a positive real number h. For a given $h > 0$, the nth element $L_n^e(h)$ of the even sequence is generated by tossing n times a balanced coin. If the number of heads k is even the individual receives kh dollars, and he receives zero dollars otherwise. Similarly, the sequences $\{L_n^0(h)\}_{n=1}^{\infty}$ of "odd" lotteries are constructed in a manner similar to the even ones except that now the payoff is kh if k is odd and zero otherwise. Table I summarizes the payoffs and probabilities of those two sequences of lotteries.

Note that the lotteries $L_n^e(h)$ and $L_n^0(h)$ have their first $n-1$ moments identical. However, the moments of higher or equal order than n of $L_n^e(h)$

TABLE I

Probabilities and Payoffs of Odd and Even Lotteries

Tosses		Odd lotteries				Even lotteries			
$n=1$	Probabilities:	1/2	1/2			1			
	Payoffs:	0	h			0			
$n=2$	Probabilities:	1/2	1/2			3/4	1/4		
	Payoffs:	0	h			0	$2h$		
$n=3$	Probabilities:	1/2	3/8	1/8		5/8	3/8		
	Payoffs:	0	h	$3h$		0	$2h$		
$n=4$	Probabilities:	1/2	1/4	1/4		9/16	3/8	1/16	
	Payoffs:	0	h	$3h$		0	$2h$	$4h$	
$n=5$	Probabilities:	1/2	5/32	5/16	1/32	17/32	5/16	5/32	
	Payoffs:	0	h	$3h$	$5h$	0	$2h$	$4h$	
$n=6$	Probabilities:	1/2	3/32	5/16	3/32	33/64	15/64	15/64	1/64
	Payoffs:	0	h	$3h$	$5h$	0	$2h$	$4h$	$6h$
...	...	...				...			

are greater than those of $L_n^0(h)$ when n is even, and the converse holds when n is odd, that is,

$$\mu_r[L_n^e(h)] = \mu_r[L_n^0(h)], \qquad \text{for} \quad 0 < r < n,$$

$$\mu_r[L_n^e(h)] \geqslant \mu_r[L_n^0(h)], \qquad \text{for} \quad r \geqslant n, \quad n = 2m, \quad m = 1, 2, ...,$$

$$\mu_r[L_n^e(h)] \leqslant \mu_r[L_n^0(h)], \qquad \text{for} \quad r \geqslant n, \quad n = 2m+1, \quad m = 0, 1, 2, ...$$

PROPOSITION 3.1. *Let $u(w)$ be a continuous utility function on $[0, \infty)$ such that $u(0) = 0$. Then $u(w)$ is mixed if and only if, for all initial wealth $w \geqslant 0$, the odd lottery $L_n^0(h)$ is preferred to the even lottery $L_n^e(h)$, for all $h > 0$ and every positive integer n; i.e.,*

$$E_{L_n^0(h)}[u(w+kh)] \geqslant E_{L_n^e(h)}[u(w+kh)], \qquad n = 1, 2, ..., \quad h > 0, \tag{7}$$

where the subindex in the expectation operator indicates the lottery whose distribution is used to evaluate the expected utility.

Proof. For $n = 1$, (7) becomes $(\frac{1}{2})\, u(w) + (\frac{1}{2})\, u(w+h) \geqslant u(w)$. For $n = 2$, we have $(\frac{1}{2})\, u(w) + (\frac{1}{2})\, u(w+h) \geqslant (\frac{3}{4})\, u(w) + (\frac{1}{4})\, u(w+2h)$. For $n = 3$, $(\frac{1}{2})\, u(w) + (\frac{3}{8})\, u(w+h) + (\frac{1}{8})\, u(w+3h) \geqslant (\frac{5}{8})\, u(w) + (\frac{3}{8})\, u(w+2h)$, and so on. Therefore, by induction we derive the following inequalities:

$$u(w) \sum_{k=0}^{m} \binom{2m}{2k} + \sum_{k=1}^{m} \binom{2m}{2k-1} u(w+[2k-1]\,h)$$
$$\geqslant u(w) \sum_{k=1}^{m} \binom{2m}{2k-1} + \sum_{k=0}^{m} \binom{2m}{2k} u(w+2kh),$$
$$\text{for} \quad n=2m, \quad m=1, 2, ...,$$

and

$$u(w) \sum_{k=0}^{m} \binom{2m+1}{2k} + \sum_{k=1}^{m} \binom{2m+1}{2k-1} u(w+[2k-1]\,h)$$
$$\geqslant u(w) \sum_{k=1}^{m} \binom{2m+1}{2k-1} + \sum_{k=0}^{m} \binom{2m+1}{2k} u(w+2kh),$$
$$\text{for} \quad n=2m+1, \quad m=0, 1, 2, ...$$

These two last inequalities can be compactly rewritten as

$$\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} u(w+kh) \geqslant 0, \qquad \text{for} \quad n=1, 2, ...,$$

which, from the equivalence between (2) and (3), becomes

$$(-1)^{n-1} \Delta_h^n u(w) \geqslant 0, \qquad \text{for} \quad n=1, 2, ... \tag{8}$$

From Lemma 3.1, condition (8) holds iff $(-1)^{n-1} \Delta_h^{n-1} \varphi_a(w) \geqslant 0$ for all $a>0$ and $n=1, 2, ...,$ where $\varphi_a(w) \equiv u(w+a)-u(w)$. Therefore, $\varphi_a(w)$ is completely monotone as dictated by Corollary 2. 3. Finally, since Lemma 3.2 tells us that $\varphi_a(w)$ is completely monotone iff $u(w)$ is mixed, we conclude that (8) holds if and only if $u(w)$ is mixed. Q.E.D.

The previous proposition might be useful for testing by means of a laboratory experiment whether an individual is mixed risk averse. Of course, such a test would suffer obvious (and typical) limitations since it would be necessary to choose an appropriate grid of values for the elementary payoff h and wealth w, and a finite number of tosses n.

Kimball [9] introduced the index of absolute prudence as a measure of the strength of the precautionary saving motive in an intertemporal context when the future endowments are uncertain. This measure of prudence shifts up a derivative the measure of absolute risk aversion. In general, we can define the nth order index of absolute risk aversion as $A_n(w) = -u^{n+1}(w)/u^n(w)$, for $n=1, 2, ...$ The function $A_1(w)$ corresponds to the Arrow–Pratt index of absolute risk aversion, whereas $A_2(w)$ is the aforementioned Kimball index of prudence.

PROPOSITION 3.2. *Let the continuous utility function* $u(w)$ *defined on* $[0, \infty)$ *be increasing, concave and smooth on* $(0, \infty)$ *with* $u(0)=0$ *and* $u^n(w) \neq 0$ *for all* $w>0$ *and* $n=1, 2, \ldots$. *Then* $u(w)$ *is mixed if and only if* $A_n(w)$ *is nonincreasing for all* $n=1, 2, \ldots$.

Proof. (Sufficiency) It can be immediately shown that the requirement of nonincreasing $A_n(w)$ is equivalent to $u^{n+2}(w) \cdot u^n(w) \geqslant [u^{n+1}(w)]^2$, which implies that $u^{n+2}(w)$ and $u^n(w)$ have the same sign for $n=1, 2, \ldots$, and $w>0$. The assumption of monotonicity and concavity allows us to conclude inductively that u has positive odd derivatives and negative even derivatives.

(Necessity) The odd derivatives $u^{2n-1}(w)$, for $n=1, 2, \ldots$, are completely monotone by assumption. On the other hand, the negative of the even derivatives $-u^{2n}(w)$, for $n=1, 2, \ldots$, are also completely monotone functions. Complete monotonicity implies log-convexity (see Corollary 2.2), and log-convexity is in turn equivalent to having $A_n(w)$ nonincreasing for $n=1, 2, \ldots$. Q.E.D.

The previous proposition shows that mixed utilities constitute a strict subset of the class of utility functions displaying standard risk aversion, i.e., utility functions for which every loss-aggravating risk aggravates S every independent, undesirable risk (see Kimball [10]).[4] Kimball proves that an utility function is standard risk averse iff $A_2(w)$ is nonincreasing on $(0, \infty)$ so that the characterization of Proposition 3.2 is clearly more y stringent.[5] Needless to say, this characterization is quite technical since we lack an economic interpretation of $A_n(w)$ for values of n greater than 2.

The following proposition is also technical and it can be viewed as an alternative definition of mixed risk aversion for increasing and concave utility functions:

PROPOSITION 3.3. *Let the continuous utility function* $u(w)$ *defined on* $[0, \infty)$ *be increasing, concave, and smooth on* $(0, \infty)$ *with* $u(0)=0$. *Then* $u(w)$ *is mixed if and only if the derivatives of all orders of* $u(w)$ *are either uniformly nonpositive or uniformly nonnegative.*

Proof. It follows immediately from adapting the argument in Ingersoll [8, p. 41].

[4] A risk $\tilde{x}$ is undesirable iff $E[u(\tilde{w}+\tilde{x})] \leqslant E[u(\tilde{w})]$ for all random background wealth $\tilde{w}$. A risk $\tilde{x}$ is loss-aggravating iff $E[u'(\tilde{w}+\tilde{x})] \geqslant E[u'(\tilde{w})]$ for all $\tilde{w}$. The definitions of desirable and loss-ameliorating risks are obtained by just reversing the weak inequalities in the previous definitions.

[5] Pratt and Zeckhauser [13] have also shown that a utility function with a completely monotone first derivative is proper. Proper utilities are those for which every two independent, undesirable risks are mutually aggravated. Kimball [10] shows in turn that standardness implies properness.

4. STOCHASTIC DOMINANCE AND MIXED RISK AVERSION

If we consider a subset $\mathscr{C}$ of the family of real-valued, Borel measurable utility functions defined on $[0, \infty)$, we say that the random variable $\tilde{x}_1$ $\mathscr{C}$-stochastically dominates the random variable $\tilde{x}_2$ iff $E[u(\tilde{x}_1)] \geqslant E[u(\tilde{x}_2)]$ for all $u \in \mathscr{C}$. Thus, when $\mathscr{C}$ is the set of continuous increasing (concave) utility functions, $\mathscr{C}$-stochastic dominance coincides with the concept of first- (second-) degree stochastic dominance.

Let $\mathscr{M}$ be the set of mixed utility functions. Then the following proposition characterizes $\mathscr{M}$-stochastic dominance by requiring that all Laplace transforms of the dominated distribution be greater than those of the dominating distribution.

Proposition 4.1. *Let G_1 and G_2 be the distribution functions of the nonnegative random variables $\tilde{x}_l$ and $\tilde{x}_2$, respectively. Then $\tilde{x}_1$ $\mathscr{M}$-stochastically dominates $\tilde{x}_2$ if and only if*

$$\int_0^\infty e^{-sz}\, dG_1(z) \leqslant \int_0^\infty e^{-sz}\, dG_2(z), \qquad \textit{for all} \quad s \geqslant 0. \tag{9}$$

Proof. (Sufficiency) Every u belonging to $\mathscr{M}$ can be written as in (4). The result follows from exchanging the order of integration of the corresponding expected utility and simplifying. Note that the exchange of the order of integration is justified by Fubini's theorem since the distribution functions $G_1(z)$, $G_2(z)$, and $F(s)$ define σ-finite measures on the Borel sets of $[0, \infty)$ (see Ash [3, pp. 9 and 103]).

(Necessity) By contradiction. Suppose that (9) does not hold for some nonempty set Ξ of values of s. Construct then a utility function $u \in \mathscr{M}$ by using a distribution function F whose support is Ξ. Then (4) and (9) imply that $E[u(\tilde{x}_1)] < E[u(\tilde{x}_2)]$, which contradicts the assumption that $\tilde{x}_1$ $\mathscr{M}$-stochastically dominates $\tilde{x}_2$. Q.E.D.

This proposition also tells us that in order to verify if $\tilde{x}_1$ is preferred to $\tilde{x}_2$ by all mixed risk averse individuals, it is necessary and sufficient to check that $\tilde{x}_1$ is preferred to $\tilde{x}_2$ by all individuals having CARA utilities; i.e., $u(w) = 1 - e^{-sw}$, for all $s \geqslant 0$. Obviously, these individuals constitute a strict subset of the mixed risk averse individuals. The following example shows an application of Proposition 4.1:

Example 4.1. Consider the random variables $\tilde{x}_1$ taking the values 2 and 4 with probabilities 3/4 and 1/4, respectively, and $\tilde{x}_2$ taking the values 1 and 3 with probabilities 1/4 and 3/4, respectively. Note that these two random variables have the same mean. In this example, the inequality (9) should become

$$3z^2 + z^4 \leqslant z + 3z^3, \qquad \text{where} \quad z = e^{-s}, \quad \text{for all} \quad s \geqslant 0,$$

which in turn becomes $(1-z)^3 \geqslant 0$, and this inequality holds since $z \in (0, 1]$. Therefore, $\tilde{x}_1$ $\mathscr{M}$-stochastically dominates $\tilde{x}_2$. However, note that $\tilde{x}_1$ does not dominate $\tilde{x}_2$ in the sense of second-degree stochastic dominance. To see the latter, consider the following concave utility:

$$v(w) = \begin{cases} w & \text{for} \quad 0 \leqslant w \leqslant 3.2 \\ 3.2 & \text{for} \quad w > 3.2. \end{cases}$$

Then $E[v(\tilde{x}_1)] = 2.3$ and $E[v(\tilde{x}_2)] = 2.5$, which proves that $\tilde{x}_1$ cannot dominate $\tilde{x}_2$ in the sense of second-degree stochastic dominance.

The next proposition provides an alternative characterization of $\mathscr{M}$-stochastic dominance:

PROPOSITION 4.2. *Let G_1 and G_2 be the distribution functions of the non-negative random variables $\tilde{x}_1$ and $\tilde{x}_2$, respectively. Then $\tilde{x}_1$ $\mathscr{M}$-stochastically dominates $\tilde{x}_2$ iff*

$$\int_0^\infty \frac{dG_1(z)}{(z+\tau)^n} \leqslant \int_0^\infty \frac{dG_2(z)}{(z+\tau)^n}, \qquad \textit{for all real} \quad \tau > 0, \quad \textit{and} \quad n = 1, 2, ... \tag{10}$$

Proof. Let us define $\Psi(s) = P_2(s) - P_1(s)$, where $P_i(s)$ is the Laplace transform of the distribution function G_i; i.e., $P_i(s) = \int_0^\infty e^{-sz}\, dG_i(z)$.

Therefore, (9) can be written as $\Psi(s) \geqslant 0$ for all $s \geqslant 0$, so that the function $\Lambda = \int_0^\infty e^{-s\tau} \Psi(s)\, ds$, for $\tau > 0$, is a Laplace transform of a distribution function if and only if (9) holds. Theorem 2.1 tells us that $\Lambda(\tau)$ is a Laplace transform of a distribution function if and only if it is completely monotone. Observe also that

$$\begin{aligned} \Lambda(\tau) &= \int_0^\infty e^{-s\tau} \Psi(s)\, ds = \int_0^\infty e^{-s\tau} P_2(s)\, ds - \int_0^\infty e^{-s\tau} P_1(s)\, ds \\ &= \int_0^\infty \int_0^\infty e^{-s(z+\tau)}\, ds\, dG_2(z) - \int_0^\infty \int_0^\infty e^{-s(z+\tau)}\, ds\, dG_1(z) \\ &= \int_0^\infty \frac{dG_2(z)}{(z+\tau)} - \int_0^\infty \frac{dG_1(z)}{(z+\tau)}, \end{aligned} \tag{11}$$

where the third equality comes from substituting $P_i(s)$, $i = 1, 2$, and exchanging the order of integration, and the fourth equality is obtained from computing the inner Riemann integral. By successively differentiating

(11), it follows that the condition of complete monotonicity of $\Lambda(\tau)$ is equivalent to (10). Q.E.D.

This proposition also allows us to look at a subset of $\mathscr{M}$ so as to verify the ordering implied by $\mathscr{M}$-stochastic dominance. In this case, it is enough to verify the order relation for preferences represented by utility functions of the form $u(w) = 1/\tau^n - 1/(w+\tau)^n$, for all $\tau > 0$ and $n = 1, 2, \ldots$. It can be checked that this family of utility functions is equivalent to the class of HARA utilities satisfying $-u''(w)/u'(w) = 1/(a+bw)$ for all $a > 0$ and $b = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.

5. AGGRAVATION AND AMELIORATION OF RISKS FOR MIXED UTILITIES

As we have already mentioned, Pratt and Zeckhauser [13] and Kimball [10] have characterized two nested subsets of utility functions exhibiting decreasing absolute risk aversion. A key element of those characterizations are the concepts of aggravation and amelioration of random variables (or risks). In this section we are interested in conditions for both mutual aggravation and amelioration of risks when agents are mixed risk averse.

Assume that the random variables $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$ are mutually independent, and consider the lottery consisting of receiving the random payoff $\tilde{w}+\tilde{x}$ with probability $\frac{1}{2}$ and the payoff $\tilde{w}+\tilde{y}$ also with probability $\frac{1}{2}$. Consider now a second lottery consisting of getting the random payoff $\tilde{w}$ with probability $\frac{1}{2}$ and $\tilde{w}+\tilde{x}+\tilde{y}$ with probability $\frac{1}{2}$. We say that $\tilde{x}$ and $\tilde{y}$ aggravate each other iff the former lottery is preferred to the latter; i.e.,

$$\tfrac{1}{2}E[u(\tilde{w}+\tilde{x})] + \tfrac{1}{2}E[u(\tilde{w}+\tilde{y})] \geqslant \tfrac{1}{2}E[u(\tilde{w})] + \tfrac{1}{2}E[u(\tilde{w}+\tilde{x}+\tilde{y})], \quad \text{for all} \quad \tilde{w}. \tag{12}$$

Similarly, we say that the random variables $\tilde{x}$ and $\tilde{y}$ ameliorate each other iff the weak inequality in (12) is reversed. Therefore, the concept of aggravation (amelioration) of risks provides a particular, precise meaning to the notion of substitutability (complementarity) between risks.

Assume that individuals have utility functions belonging to the class $\mathscr{M}$, and that the risks to be evaluated are nonnegative. Let $P_w(s)$, $P_x(s)$ and $P_y(s)$ denote the Laplace transforms of the distribution functions of $\tilde{w}$, $\tilde{x}$ and $\tilde{y}$, respectively. Since Proposition 4.1 identifies the $\mathscr{M}$-stochastic dominance ordering with the ordering of Laplace transforms, it is easy to see that condition (12) holds for all $u \in \mathscr{M}$ if and only if

$$P_w(s) + P_w(s)\cdot P_x(s)\cdot P_y(s) \geqslant P_w(s)\cdot P_x(s) + P_w(s)\cdot P_y(s), \quad \text{for all} \quad s \geqslant 0. \tag{13}$$

Therefore, we have the following proposition obtained from just simplifying (13):

PROPOSITION 5.1. *Assume that the risks $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$ are independent, and that the risks $\tilde{w}$, $\tilde{w}+\tilde{x}$, $\tilde{w}+\tilde{y}$, and $\tilde{w}+\tilde{x}+\tilde{y}$ are nonnegative. Then the risks $\tilde{x}$ and $\tilde{y}$ aggravate each other for all mixed risk averse individuals if and only if*

$$[1-P_x(s)][1-P_y(s)] \geqslant 0, \qquad \textit{for all} \quad s \geqslant 0. \tag{14}$$

Therefore, the property of aggravation between $\tilde{x}$ and $\tilde{y}$ is independent of the characteristics of the (possibly random) initial wealth $\tilde{w}$. The next two examples illustrate the previous result:

EXAMPLE 5.1. Let $\tilde{w}$ be nonrandom and equal to $w>0$, $\tilde{x}$ takes the values 0 and $h \in (0, w)$ with the two values being equiprobable, and $\tilde{y}$ takes the values $-h$, 0, and h with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. It can be shown that $P_x(s)=(1+z)/2$ and $P_y(s)=((1/z)+2+z)/4$, where $z=e^{-sh} \in (0, 1]$. Therefore, $1-P_x(s)=(1-z)/2$ and $1-P_y(s)=-((1-z)^2/4z)$ so that $[1-P_x(s)][1-P_y(s)]=-((1-z)^3/8z) \geqslant 0$, which proves that $\tilde{x}$ and $\tilde{y}$ ameliorate each other. The amelioration is strict if the utility functions are strictly concave since then $\zeta=\sup\{s \mid F(s)>0\}>0$ and $e^{-\zeta h}<1$.

EXAMPLE 5.2. Let $\tilde{w}=w>0$, and both $\tilde{x}$ and $\tilde{y}$ take the values $-h$, 0, and h with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively, where $h \in (0, w/2)$. Then some computations yield $[1-P_x(s)][1-P_y(s)]=(1-z)^4/16z^2 \geqslant 0$, since $z=e^{-sh} \in (0, 1]$. Therefore, $\tilde{x}$ and $\tilde{y}$ aggravate each other. Again, strict concavity implies strict aggravation.

The following corollary states sufficient conditions for either mutual aggravation or amelioration of risks:

COROLLARY 5.1. *Assume that the risks $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$ are independent and that the risks $\tilde{w}$, $\tilde{w}+\tilde{x}$, $\tilde{w}+\tilde{y}$, and $\tilde{w}+\tilde{x}+\tilde{y}$ are nonnegative. Assume also that individuals have mixed risk averse preferences. Then,*

(a) *The risks $\tilde{x}$ and $\tilde{y}$ aggravate each other if they are nonnegative.*

(b) *The risks $\tilde{x}$ and $\tilde{y}$ aggravate each other if $E(\tilde{x}) \leqslant 0$ and $E(\tilde{y}) \leqslant 0$.*

(c) *The risks $\tilde{x}$ and $\tilde{y}$ ameliorate each other if $E(\tilde{x}) \leqslant 0$ and $\tilde{y}$ is nonnegative.*

Proof. (a) If the random variables $\tilde{x}$ and $\tilde{y}$ take only nonnegative values, then $P_x(s) \leqslant 1$ and $P_y(s) \leqslant 1$ for all $s \geqslant 0$, which is sufficient for (14).

(b) Since $P_x(s) = \int_{-\infty}^{\infty} e^{-sz}\, dG_x(z)$, where $G_x(z)$ is the distribution function of $\tilde{x}$, we have $P_x(0) = 1$, $P'_x(0) = -\int_{-\infty}^{\infty} z\, dG_x(z) = -E(\tilde{x}) \geqslant 0$, and $P''_x(s) = \int_{-\infty}^{\infty} z^2 e^{-sz}\, dG_x(z) \geqslant 0$ for all $s \geqslant 0$. Therefore, $P_x(s)$ is a function defined on $[0, \infty)$ that is convex and reaches its minimum at zero so that $P_x(s) \geqslant 1$ for all $s \geqslant 0$. Since the same holds for $P_y(s)$, the condition (14) is always fulfilled.

(c) By assumption, and as it follows from parts (a) and (b), $P_x(s) \leqslant 1$ and $P_y(s) \geqslant 1$. Therefore, $[1 - P_x(s)][1 - P_y(s)] \leqslant 0$, for all $s \geqslant 0$. Q.E.D.

Kimball [10] has in fact proved that part (b) holds for the larger class of proper utility functions. Under the assumptions in (b), $\tilde{x}$ and $\tilde{y}$ are clearly undesirable and hence the risks $\tilde{x}$ and $\tilde{y}$ must aggravate each other for all u proper. We just repeat the result for mixed risk averse preferences because of the simplicity of its proof.

Finally, it should be noticed that a random variable is undesirable for all mixed risk averse individuals if and only if it is loss-aggravating for all such individuals. It is straightforward to see that the necessary and sufficient condition for a risk $\tilde{x}$ to be loss-aggravating (and undesirable) for all mixed utilities is that $P_x(s) \geqslant 1$ for all $s \geqslant 0$. This implies, according to (14), that two independent risks aggravate each other for all mixed utilities if and only if both of them are either loss-aggravating or loss-ameliorating for all such utilities.

6. ABSOLUTE AND RELATIVE RISK AVERSION OF MIXED UTILITIES

Mixed utility functions have some features that facilitate the characterization of their indexes of absolute and relative risk aversion, $A(w) = -u''(w)/u'(w)$ and $R(w) = -wu''(w)/u'(w)$, respectively. These indexes are crucial for the comparative statics of the simplest portfolio selection problem, in which investors must allocate their wealth between a riskless asset and a risky asset (or portfolio) having a positive risk premium. As Proposition 3.2 shows, $A(w)$ is nonincreasing for all mixed utilities and it is strictly decreasing when the support of F has at least two points (see Corollary 2.2). Therefore, a mixed risk averse investor increases the optimal amount invested in the risky asset as her wealth increases (see Arrow [2]).

It should also be noticed that, when the absolute risk aversion approaches infinity (zero), the optimal amount invested in the risky asset goes to zero (infinity). On the other hand, the proportion of wealth invested in the risky asset tends to zero (infinity) as the relative risk aversion approaches infinity (zero).

Clearly, the nth order derivative at the origin, $u^n(0) \equiv \lim_{w \to 0} u^n(w)$, is finite iff $\int_0^\infty s^{n-1}\, dF(s) < \infty$. Moreover, $u'(w) > 0$ for all $w \in [0, \infty)$ if and only if $\int_0^\infty dF(s) > 0$. Finally, it is also straightforward to see that $\lim_{w \to \infty} u'(w) > 0$ iff $F(0) > 0$.

The next proposition provides results concerning the behaviour of the two indexes of risk aversion at the origin.

PROPOSITION 6.1. *Assume that u is a mixed utility characterized by a distribution function F with $\int_0^\infty dF(s) > 0$. Then,*

(a) *if $\int_0^\infty s\, dF(s) < \infty$, then $\lim_{w \to 0} A(w) < \infty$ and $\lim_{w \to 0} R(w) = 0$.*

(b) *if $\int_0^\infty s\, dF(s)$ is not finite, then $\lim_{w \to 0} A(w) = \infty$.*

Proof. Part (a) is obvious.

For part (b), if $\int_0^\infty dF(s) < \infty$, then $\lim_{w \to 0} u'(w)$ is finite and $\lim_{w \to 0} u''(w)$ is unbounded so that $\lim_{w \to 0} A(w) = \infty$. If $\int_0^\infty dF(s)$ is not finite, then $\lim_{w \to 0} u'(w) = \infty$ and $\lim_{w \to 0} u''(w) = -\infty$. Moreover, $1/A(w)$ is nonnegative and increasing in w so that $\lim_{w \to 0} 1/A(w)$ exists. Therefore, Hôpital's theorem implies that $\lim_{w \to 0} 1/A(w) = \lim_{w \to 0} (-u'(w)/u''(w)) = \lim_{w \to 0} (-u(w)/u'(w)) = 0$; that is, $\lim_{w \to 0} A(w) = \infty$. Q.E.D.

The next two propositions characterize the behaviour of $A(w)$ and $R(w)$ for high values of w. We first state the following technical lemma:

LEMMA 6.4. *Assume that*

(a) *$\int_0^\infty |\varphi(s)|\, e^{-sw}\, dF(s)$ exists for all $w \geqslant 0$, and*

(b) *there exists a strictly positive real number c such that $\int_0^a dF(s) > 0$ and $\varphi(a) > 0$, for all $a \in (0, c)$.*

Then there exists a positive real number w_0 such that $\int_0^\infty \varphi(s)\, e^{-sw}\, dF(s) > 0$ for all $w > w_0$.

Proof. See the Appendix.

PROPOSITION 6.2. *Let u be a mixed utility characterized by a distribution function F such that $\int_0^\infty dF(s) > 0$. Then $\lim_{w \to \infty} A(w) = s_0$, where $s_0 \equiv \inf\{s \mid F(s) > 0\}$.*

Proof. Since $A(w)$ is nonnegative and decreasing, $\lim_{w \to \infty} A(w)$ exists. Assume first that $s_0 = 0$ so that $\int_0^a dF(s) > 0$ for all $a > 0$. We proceed by contradiction and assume that $c \equiv \lim_{w \to \infty} A(w) > 0$. Note that $-u''(w)/u'(w) \geqslant c$ or, equivalently, $cu'(w) + u''(w) \geqslant 0$, for all $w > 0$.

However, if $\int_0^\infty s\,dF(s) < \infty$ for all $w \geqslant 0$, we can apply Lemma 6.1 to obtain that $cu'(w) + u''(w) = \int_0^\infty (c-s)\,e^{-sw}\,dF(s) > 0$ for sufficiently high values of w, which constitutes a contradiction.

It should be noticed that we can always assume that $\int_0^\infty s\,dF(s) < \infty$ since, if not, we can consider instead the mixed utility function $\hat{u}(w) = u(w+b) - u(b)$, with $b > 0$. The limit at infinity of the absolute risk aversion of $\hat{u}(w)$ is the same as that of $u(w)$. Note that $\hat{u}''(w) = \int_0^\infty s e^{-sw}\,d\hat{F}(s)$, where $d\hat{F}(s) = e^{-sb}\,dF(s)$. Hence, $\hat{u}''(0) = \int_0^\infty s\,d\hat{F}(s) = \int_0^\infty s e^{-sb}\,dF(s) = u''(b) < \infty$.

Assume now that $s_0 > 0$. Then $u'(w) = \int_{s_0}^\infty e^{-sw}\,dF(s) = \int_0^\infty e^{-(t+s_0)w}\,dF(t+s_0) = \exp(-s_0 w)\,\bar{u}'(w)$, where $\bar{u}'(w) = \int_0^\infty e^{-tw}\,dF(t+s_0)$. Therefore, $t_0 \equiv \inf\{t \mid F(t+s_0) > 0\} = 0$, and then $\lim_{w\to\infty} (-\bar{u}''(w))/\bar{u}'(w) = 0$. Finally, compute the index of absolute risk aversion of $u(w)$,

$$A(w) = \frac{s_0 \exp(-s_0 w)\,\bar{u}'(w) - \exp(-s_0 w)\,\bar{u}''(w)}{\exp(-s_0 w)\,\bar{u}'(w)} = s_0 - \frac{\bar{u}''(w)}{\bar{u}'(w)},$$

which implies that $\lim_{w\to\infty} A(w) = s_0$. Q.E.D.

In particular, $\lim_{w\to\infty} A(w) = 0$ when $s_0 = 0$, that is, the individual tends to be absolutely risk neutral for high levels of wealth in this case. Moreover, if $s_0 > 0$ then $\lim_{w\to\infty} R(w) = \infty$ so that the fraction of wealth invested in the risky asset goes to zero as wealth tends to infinity. As we are going to show, the properties at the origin of the distribution function F are also crucial to give a richer description of the behaviour of the relative risk aversion index for high levels of wealth.

Definition 6.1. Let F be a distribution function satisfying $F(s) > 0$ for all $s > 0$. The distribution function F is said to be of regular variation at the origin with exponent ρ iff

$$\lim_{s\to 0} \frac{F(ts)}{F(s)} = t^\rho, \qquad \text{with} \quad 0 \leqslant \rho < \infty.$$

Lemma 6.2. *Let $\varphi(w)$ be a Laplace transform of the distribution function $F(s)$. If F varies regularly at the origin with exponent $\rho \in [0, \infty)$ and $ws = 1$, then the ratio $\varphi(w)/F(s)$ converges to $\Gamma(\rho+1)$ as $w \to \infty$ (or, equivalently, as $s \to 0$), where $\Gamma(\cdot)$ denotes the gamma function; i.e., $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y}\,dy$ for $x > 0$.*

Proof. It follows immediately from adapting the Tauberian Theorems 1 and 3 in Section XIII.5 of Feller [5]. Q.E.D.

If the utility function u is mixed, then the marginal utility $u'(w)$ is completely monotone and it is thus the Laplace transform of some distribution function $F(s)$. Let us define the distribution function $F_1(s) = \int_0^s \tau\, dF(\tau)$ whose Laplace transform is equal to $-u''(w)$ since $dF_1(s) = s\, dF(s)$ and $-u''(w) = \int_0^\infty e^{-sw} s\, dF(s)$.

Lemma 6.3. *Let F be a distribution function of regular variation at the origin with exponent $\rho \in [0, \infty)$. Then,*

(a) *$F_1(s)$ also varies regularly at the origin with exponent $\rho + 1$, and*

(b) $\lim_{s \to 0} F_1(s)/sF(s) = \rho/(\rho + 1)$.

Proof. See the Appendix.

Proposition 6.3. *Let $u(w)$ be a mixed utility function characterized by a distribution function F of regular variation at the origin with exponent $\rho \in [0, \infty)$. Then the relative risk aversion of $u(w)$ converges to ρ as w tends to infinity.*

Proof. The regular variation of F implies that $\inf\{s \mid F(s) > 0\} = 0$. As follows from Proposition 6.2, the absolute risk aversion of u tends to zero as $w \to \infty$. Moreover, Proposition 6.2 also tells us that, in order to find the limit of the relative risk aversion, we can first compute the limit of the absolute risk aversion as $w \to \infty$ with $ws = 1$ (which means that $s \to 0$), and then multiply this limit by w. From Lemmas 6.2 and 6.3(a), and since $u'(w)$ and $-u''(w)$ are Laplace transforms of $F(s)$ and $F_1(s)$, respectively, it follows that $u'(w)/F(s)$ converges to $\Gamma(\rho + 1)$, whereas $-u''(w)/F_1(s)$ converges to $\Gamma(\rho + 2)$ as w tends to infinity with $ws = 1$. Therefore, $A(w) = -u''(w)/u'(w)$ converges to $F_1(s)\,\Gamma(\rho + 2)/F(s)\,\Gamma(\rho + 1)$, which in turn converges to $\rho s = \rho/w$ because $F_1(s)/F(s)$ tends to $\rho s/(\rho + 1)$ as s goes to zero (see part (b) of Lemma 6.3) and $\Gamma(\rho + 2)/\Gamma(\rho + 1) = \rho + 1$. Hence, the result follows since $R(w) = wA(w)$. Q.E.D.

This last proposition allows us to conclude that, for high levels of wealth and small risks, mixed risk averse investors would behave as if they had constant relative risk aversion, provided the regular variation hypothesis holds. Therefore, if an investor must allocate her wealth between a risky and a riskless asset, the wealth elasticity of her demand for the risky asset approaches unity as her wealth tends to infinity.

Example 6.1. A simple illustration of the last proposition is the power utility function $u(w) = Cw^\alpha$ with $0 < \alpha < 1$. This function has a constant relative risk aversion index equal to $1 - \alpha$ which clearly coincides with the exponent of regular variation of its associated distribution function $F(s) = As^{1-\alpha}$. More generally, if u is HARA, i.e., $-u''(w)/u'(w) = 1/(a + bw)$ with $b > 0$, then the limit of the relative risk aversion as $w \to \infty$ is $1/b$, which is in turn equal to the exponent of regular variation of its corresponding gamma distribution. Obviously, the absolute risk aversion approaches zero as $w \to \infty$.

EXAMPLE 6.2. As an example of a strictly concave, mixed utility function with both absolute and relative risk aversion vanishing at infinity, we can consider the utility function having the derivative $u'(w) = \exp(-(w/1+w))$, which is completely monotone according to Corollary 2.1 and, therefore, after imposing the appropriate boundary condition, $u(w)$ is mixed. Clearly, both the absolute risk aversion $A(w) = 1/(1+w)^2$ and the relative risk aversion $R(w) = w/(1+w)^2$ tend to zero as $w \to \infty$. Moreover, $\lim_{w \to \infty} u'(w) = 1/e > 0$, which means that the associated distribution function F satisfies $F(0) > 0$. In fact, the density function associated with the utility of this example is $f(s) = e^{-s} \sum_{k=1}^{\infty} (s^{k-1}/k!\, \Gamma(k)) + \delta(s)$, where $\delta(s)$ is the Dirac delta function. Hence, its distribution function F satisfies $\lim_{s \to 0} F(ts)/F(s) = 1$ so that it varies regularly at the origin with exponent $\rho = 0$. Here, both the amount and the proportion of the optimal investment in risky assets increase as wealth tends to infinity.[6]

EXAMPLE 6.3. Finally, as an example of a mixed utility function with vanishing absolute risk aversion but an unbounded limit of relative risk aversion, we can consider the utility function having the derivative $u'(w) = \exp(-w^{\alpha})$ with $\alpha \in (0, 1)$. This derivative is clearly completely monotone (see Corollary 2.1). The utility u is characterized by a stable distribution F with parameter α (see Feller [5, Section XIII.6]) for which the assumption of Proposition 6.3 does not hold. Then $A(w) = \alpha w^{\alpha - 1}$ and $R(w) = \alpha w^{\alpha}$ so that $\lim_{w \to \infty} A(w) = 0$ and $\lim_{w \to \infty} R(w) = \infty$. Therefore, for the utility considered in this example, the amount invested in the risky asset tends to infinity as wealth increases without bound, whereas the fraction of wealth invested in the risky asset goes to zero.

7. PORTFOLIO SELECTION AND MIXED RISK AVERSION

In addition to the limit of relative risk aversion for low and high levels of wealth, it is also possible to provide some results about the global behaviour of this index which are relevant for the theory of investment. The following two examples illustrate the importance of relative risk aversion:

[6] A trivial example satisfying $A(w) = R(w) = 0$ for all $w \geqslant 0$ is given by the linear, mixed utility $u(w) = Cw$ whose associated density function is the Dirac delta function $\delta(s)$; i.e., $F(s) = C$ for all $s \geqslant 0$.

EXAMPLE 7.1. Consider the typical saving problem faced by an individual who has a given wealth w_0 today which he has to distribute between consumption today $c_0 \geqslant 0$ and consumption tomorrow $c_1 \geqslant 0$. He saves what is not consumed today, and the investment yields a nonrandom return $\theta > 0$ per dollar saved. The individual maximizes the following additive separable utility:

$$v(c_0) + u(c_1),$$

with $v' \geqslant 0, u' \geqslant 0, v'' \leqslant 0$, and $u'' \leqslant 0$. It is well known that the optimal saving is locally increasing (decreasing) in the return θ iff the relative risk aversion of u, evaluated at the optimal consumption, is less (greater) than unity.

EXAMPLE 7.2. Another context in which the relative risk aversion index plays a key role is in the problem of portfolio selection with pure securities. Assume a two-period economy in which there are S states of nature in the second period. The investor has to distribute her first-period wealth w_0 between first-period consumption $c_0 \geqslant 0$ and investment in S pure securities, indexed by i, which will finance second-period consumption $c_i \geqslant 0$ in each state. Security i has a gross return $\theta_i > 0$ if state i occurs and zero otherwise. The probability of state i is $\pi_i > 0$ with $\sum_{i=1}^{s} \pi_i = 1$. Let z_i be the wealth invested in security i. Therefore, the problem faced by the investor is to choose the vector $(z_1, ..., z_s)$ in order to maximize the expected utility

$$v(c_0) + \sum_{i=1}^{s} \pi_i u(c_i), \tag{15}$$

subject to

$$c_0 = w_0 - \sum_{i=1}^{s} z_i \geqslant 0 \tag{16}$$

and

$$c_i = \theta_i z_i \geqslant 0, \tag{17}$$

with $v' \geqslant 0, u' \geqslant 0, v'' \leqslant 0$, and $u'' \leqslant 0$. The following proposition provides the comparative statics of the optimal portfolio:[7]

[7] Obviously, when financial markets are complete, the returns θ_i and the amounts z_i^* invested in each pure security $(i = 1, ..., S)$ can be derived from both the returns and the optimal portfolio of the existing (not necessarily pure) securities.

PROPOSITION 7.1. *Let* $(z_1^*, ..., z_s^*)$ *be a solution to the above portfolio selection problem, then* $\partial c_0^*/\partial\theta_k \leqslant 0$, $\partial z_k^*/\partial\theta_k \geqslant 0$, *and* $\partial z_i^*/\partial\theta_k \leqslant 0$ *for* $i \neq k$ *if and only if* $R(c_k^*) = -c_k^* u''(c_k^*)/u'(c_k^*) \leqslant 1$, *where* $c_0^* = w_0 - \sum_{i=1}^{s} z_i^*$ *and* $c_k^* = \theta_k z_k^*$.

Proof. Substituting (16) and (17) in the objective function (15), we obtain the following first-order condition characterizing an interior solution:

$$v'(c_0^*) = \pi_i \theta_i u'(c_i^*), \qquad i = 1, ..., S. \tag{18}$$

Differentiating (18) with respect to θ_k we obtain

$$v''(c_0^*) \frac{\partial c_0^*}{\partial \theta_k} = \pi_k u'(c_k^*) + \pi_k \theta_k^2 u''(c_k^*) \frac{\partial z_k^*}{\partial \theta_k} + \pi_k \theta_k z_k^* u''(c_k^*), \qquad \text{for} \quad i = k, \tag{19}$$

and

$$v''(c_0^*) \frac{\partial c_0^*}{\partial \theta_k} = \pi_i \theta_i^2 u''(c_i^*) \frac{\partial z_i^*}{\partial \theta_k}, \qquad \text{for} \quad i \neq k. \tag{20}$$

Divide both sides of (19) by $\pi_k \theta_k^2 u''(c_k^*)$, and divide also both sides of (20) by $\pi_i \theta_i^2 u''(c_i^*)$, for $i \neq k$. Adding the resulting S equations yields

$$v''(c_0^*) \left(\sum_{i=1}^{S} \frac{1}{\pi_i \theta_i^2 u''(c_i^*)} \right) \frac{\partial c_0^*}{\partial \theta_k} = \frac{u'(c_k^*) + \theta_k z_k^* u''(c_k^*)}{\theta^2 u''(c_k^*)} + \sum_{i=1}^{S} \frac{\partial z_i^*}{\partial \theta_k}. \tag{21}$$

Furthermore, (16) implies that $\sum_{i=1}^{S} \partial z_i^*/\partial\theta_k = -(\partial c_0^*/\partial\theta_k)$ so that (21) becomes

$$\left[1 + v''(c_0^*) \left(\sum_{i=1}^{S} \frac{1}{\pi_i \theta_i^2 u''(c_i^*)} \right) \right] \frac{\partial c_0^*}{\partial \theta_k} = \frac{u'(c_k^*) + c_k^* u''(c_k^*)}{\theta^2 u''(c_k^*)}. \tag{22}$$

The term within the square brackets of the LHS of (22) is positive, whereas the denominator of the RHS is negative. Hence, $\partial c_0^*/\partial\theta_k \leqslant 0$ iff the numerator of the RHS is negative, which in turn is clearly equivalent to have $R(c_k^*) \leqslant 1$.

Moreover, (20) implies that $\text{sign}(\partial c_0^*/\partial\theta_k) = \text{sign}(\partial z_i^*/\partial\theta_k)$ for $i \neq k$ and, on the other hand, (16) implies that $\partial z_k^*/\partial\theta_k = -(\partial c_0^*/\partial\theta_k) - \sum_{i \neq k} \partial z_i^*/\partial\theta_k$ so that $\partial z_i^*/\partial\theta_k \leqslant 0$ and $\partial z_k^*/\partial\theta_k \geqslant 0$ iff $R(c_k^*) \leqslant 1$. Q.E.D.

Obviously, if $S = 1$, Proposition 7.1 implies the standard result discussed in Example 7.1. On the other hand, the previous proposition generalizes the theorem of Mitchell [11] by allowing the investor to consume also in the first period of her life.

The next two propositions characterize the behaviour of the relative risk aversion for a mixed utility function depending on the properties of its associated distribution function $F(s)$. To this end, first define the function $\Psi(w) \equiv wu'(w)$ and observe that, when $u'(w) > 0$, $R(w) \leqslant 1$ if and only if $\Psi'(w) \geqslant 0$.

Proposition 7.2. *Let $u(w)$ be a mixed utility function characterized by the distribution function $F(s)$ with $\int_0^\infty dF(s) > 0$. Then,*

(a) $\max_{w \in (0, \infty)} R(w) > 1$ *if* $\inf\{s \mid F(s) > 0\} > 0$.

(b) $\lim_{w \to 0} R(w) < 1$.

Proof. (a) Let $s_0 \equiv \inf\{s \mid F(s) > 0\}$ so that

$$\Psi(w) = w \int_{s_0}^\infty e^{-sw} \, dF(s) \leqslant w \exp(-s_0 w) \int_{s_0}^\infty dF(s),$$

and, if $\int_{s_0}^\infty dF(s) < \infty$, we get $\lim_{w \to \infty} \Psi(w) = 0$ by taking the limit of both sides. Since $\Psi(w) > 0$ for $w \in (0, \infty)$, the latter limit implies that $\Psi'(w)$ has to become negative for sufficiently high values of w.

As in the proof of Proposition 6.2, we can safely assume that $\int_{s_0}^\infty dF(s) < \infty$ since, if not, we can consider instead the mixed utility $\hat{u}(w) = u(w+b) - u(b)$, with $b > 0$. The limit at infinity of the function $\hat{\Psi}(w) \equiv w\hat{u}'(w)$ is the same as that of $\Psi(w)$, and $\hat{u}'(w) = \int_0^\infty e^{-sw} \, d\hat{F}(s)$, where $d\hat{F}(s) = e^{-sb} \, dF(s)$. Therefore, $\hat{u}'(0) = \int_0^\infty d\hat{F}(s) = \int_0^\infty e^{-sb} \, dF(s) = u'(b) < \infty$.

(b) Note that

$$\Psi'(w) = u'(w) + wu''(w) = \int_0^\infty e^{-sw} \, dF(s) - \int_0^\infty wse^{-sw} \, dF(s), \qquad (23)$$

so that $\lim_{w \to 0} \Psi'(w) = \int_0^\infty dF(s) > 0$. Q.E.D.

Proposition 7.3. *Let $u(w)$ be a mixed utility function characterized by a distribution function $F(s)$ having a continuously differentiable density $f(s)$ on $(0, \infty)$ and such that $\int_0^\infty dF(s) > 0$. Then,*

(a) $R(w) \leqslant 1$ *for all* $w > 0$ *if* $f(s)$ *is monotonically decreasing.*

(b) $\max_{w \in (0, \infty)} R(w) > 1$ *if* $\lim_{s \to 0} f'(s)$ *exists and is strictly positive.*

Proof. (a) Observe that (23) becomes

$$\Psi'(w) = \int_0^\infty e^{-sw} f(s)\, ds - \int_0^\infty w e^{-sw} s f(s)\, ds. \tag{24}$$

The integral $-\int_0^\infty we^{-sw}sf(s)\,ds$ can be written as $\int_0^\infty sf(s)\,d(e^{-sw})$ which, after integrating by parts, becomes equal to $-\int_0^\infty e^{-sw}[f(s)+sf'(s)]\,ds$. Substituting in (24), we get $\Psi'(w) = -\int_0^\infty e^{-sw}sf'(s)\,ds$. Thus, part (a) follows since $\Psi'(w) \geqslant 0$ for all $w > 0$ when $f'(s) \leqslant 0$ for all $s > 0$.

(b) If $\lim_{s\to 0} f'(s) > 0$ then the function $w\Psi'(w) = -\int_0^\infty swe^{-sw} f'(s)\,ds$ becomes strictly negative for sufficiently high values of w. Hence, taking into account that $\lim_{w\to 0} \Psi'(w) > 0$ (see part (b) of Proposition 7.2), we conclude that $\Psi'(w)$ is not monotonic and thus $\max_{w\in(0,\infty)} R(w) > 1$. Q.E.D.

Part (a) of Proposition 7.3 implies that the comparative statics exercises in Examples 7.1 and 7.2 can be unambiguously signed when the density f is decreasing. According to part (b) of Proposition 7.2, the same result also holds for low levels of initial wealth and small returns. However, under the assumptions considered in the other parts of these propositions, the signs of the comparative statics exercises remain ambiguous depending on both the level of initial wealth and the returns structure.

The following famous example will illustrate the relationship between the properties of the density function f and the effects of mean-preserving spreads on portfolio choice:

EXAMPLE 7.3. Consider a risk averse individual with initial wealth w_0 to be invested in a risky asset A with random gross return $\tilde{R}_A$ and a riskless asset with return R_f. There is another asset B with gross return $\tilde{R}_B$ which is riskier than $\tilde{R}_A$ according to the definition given by Rothschild and Stiglitz [14]; that is, $\tilde{R}_A$ dominates $\tilde{R}_B$ in the sense of second-degree stochastic dominance. If now the investor has to invest in risky asset B and the riskless asset, we want to know under which conditions the amount invested in risky asset B is less than the amount invested in risky asset A. This change in the portfolio composition would seem more natural than the opposite since $\tilde{R}_B$ is obtained from subjecting $\tilde{R}_A$ to a mean-preserving spread. Rothschild and Stiglitz [15] gave the following set of sufficient conditions for the natural result: the relative risk aversion is less than unity and increasing, and the absolute risk aversion is decreasing.

The next proposition provides a different sufficient condition for mixed risk averse investors.

PROPOSITION 7.4. *Assume that an investor has a mixed utility function u characterized by a distribution function having a decreasing and continuously* *differentiable density on* $(0, \infty)$. *Assume also that her initial wealth is* $w_0 > 0$ *and that the random gross returns* $\tilde{R}_A$ *and* $\tilde{R}_B$ *are both nonnegative with* $E(\tilde{R}_A) = E(\tilde{R}_B) > R_f > 0$. *If* $\tilde{R}_B$ *is riskier than* $\tilde{R}_A$, *then the amount invested in asset* B *is less than in asset* A.

Proof. Let z be the optimal amount invested in risky asset A. This amount z is positive because asset A has a positive risk premium. The first-order condition of the portfolio selection problem is

$$E[u'(R_f w_0 + (\tilde{R}_A - R_f) z)(\tilde{R}_A - R_f)] = 0. \tag{25}$$

Since the LHS of (25) is decreasing in z, the optimal amount invested in asset B will be less than z if

$$E[u'(R_f w_0 + (\tilde{R}_B - R_f) z)(\tilde{R}_B - R_f)] \geqslant 0. \tag{26}$$

Since $\tilde{R}_B$ is riskier than $\tilde{R}_A$, a sufficient condition for (26) is that the function $v(x) = u'(R_f w_0 + (x - R_f) z)(x - R_f)$ is concave. Define $w = R_f w_0 + (x - R_f) z$, so that

$$u'(R_f w_0 + (x - R_f) z)(x - R_f) = u'(w) \left[\frac{w - R_f w_0}{z} \right] = \frac{wu'(z)}{z} - \left(\frac{R_f w_0}{z} \right) u'(w).$$

Recall that if $\Psi(w) = wu'(w)$, then $\Psi'(w) = -\int_0^\infty e^{-sw} sf'(s)\, ds$. Moreover, $\Psi''(w) = \int_0^\infty e^{-sw} s^2 f'(s)\, ds \leqslant 0$ because $f'(s) \leqslant 0$ for all $s > 0$. Hence, $wu'(w)/z$ is concave in w. Furthermore, $(R_f w_0/z)\, u'(w)$ is convex in w since $u'(w)$ is convex. This proves in turn the concavity of $v(x)$. Q.E.D.

Therefore, the assumptions of mixed risk aversion and decreasing, differentiable density, which imply decreasing absolute risk aversion and $R(w) \leqslant 1$, allow one to dispense with the condition of increasing relative risk aversion in order to obtain the same natural conclusion in Example 7.3. Note in this respect that the condition $\Psi''(w) = 2u''(w) + wu'''(w) \leqslant 0$, which appears in the proof of Proposition 7.4, is neither necessary nor sufficient for increasing relative risk aversion.

8. CONCLUSION AND EXTENSIONS

In this paper we have analyzed the class of mixed utility functions, that is, utility functions whose first derivatives are Laplace transforms. One of the most interesting properties of such utilities is that the characteristics of the associated distribution functions allow one to extract information about their measures of risk aversion. Moreover, we have shown that the concepts of stochastic dominance and aggravation of risks become more operative when they are applied to this set of utilities.

The concept of mixed risk aversion also has interesting computational implications. Since a distribution function can be approximated by a step function, the construction of algorithms to solve portfolio problems for mixed utilities is enormously simplified. Those algorithms should deal with functions of the type $u(w) = b_0 - \sum_i a_i \exp(-s_i w)$, which can be easily handled.

As Cass and Stiglitz [4] have shown, the HARA utilities are the ones for which two-fund monetary separation holds for all distributions of the vector of risky returns. Since the utilities belonging to the class HARA are mixed, an interesting subject of further research would be to consider the class of mixed utilities and restrict appropriately the set of return distributions so as to obtain separation theorems for this larger family of utilities. We believe that such theorems should exploit the relationship between the distribution of returns and the distribution characterizing a mixed utility.

Another possible extension of our work would be to refine even more the set of utility functions. An immediate restriction would be to consider the family of utilities whose first derivatives are Laplace transforms of infinitely divisible distribution functions.[8] A utility function $u(w)$ belonging to this family has a first derivative which can be written as $u'(w) = \exp(-\Psi(w))$, where Ψ has a completely monotone first derivative. Hence, an immediate consequence is that the absolute risk aversion of u is completely monotone. We leave the analysis of such a property for future research.

APPENDIX

Proof of Lemma 3.1. (Necessity) For $n = 1$ the assumption implies that $u(w) + u(w + 2h) \leqslant 2u(w + h)$ so that u is concave and thus continuous on $(0, \infty)$. Therefore, the function $\varphi_a(w)$ is continuous with respect to a and w, and $\Delta_h^n \varphi_a(w)$ is continuous with respect a, w, and h. Consider the set $D \subset \mathbb{R}^2$ such that $D = \{(a, h) \in \mathbb{R}^2 \mid a = kh$, where k is a positive integer$\}$. Then $\varphi_a(w) = \varphi_{kh}(w) = \sum_{i=0}^{k-1} \Delta_h^1 u(w + ih)$ and $\Delta_h^n \varphi_a(w) = \sum_{i=0}^{k-1} \Delta_h^{n+1} u(w + ih)$. Since $(-1)^n \Delta_h^{n+1} u(w + ih) \geqslant 0$, the result then follows from the denseness of the set D on $\mathbb{R}^2$ and the continuity of both $u(w)$ and $\Delta_h^n \varphi_a(w)$.

(Sufficiency) Make $a = h$ and obtain $\Delta_h^n \varphi_h(w) = \Delta_h^{n+1} u(w)$. Q.E.D.

[8] A distribution function is infinitely divisible iff, for every natural number n, it can be represented as the distribution of the sum of n independent random variables having a common distribution. For instance, the gamma distribution is infinitely divisible.

Proof of Lemma 3.2. (Necessity) Since $u(w)$ is mixed, there exists a distribution function $F(s)$ for which (4) and (5) hold. Define the distribution function $F_h(s)$ such that $dF_h(s) = (1 - e^{-sh}/s)\, dF(s)$. Using the fact that $\varphi_h(w) = u(w+h) - u(w)$, it is easy to check that $\varphi_h(w) = \int_0^\infty e^{-sw}\, dF_h(s)$ for $w \in (0, \infty)$. Therefore, we conclude that $\varphi_h(w)$ is completely monotone from Theorem 2.1.

(Sufficiency) If $\varphi_h(w)$ is complete monotone, then there exists a distribution function $F_h(s)$ such that $\varphi_h(w) = \int_0^\infty e^{-sw}\, dF_h(s)$. Define the distribution function $F(s)$ satisfying $dF(s) = (s/(1 - e^{-sh}))\, dF_h(s)$. Our next goal is to prove that $F(s)$ is independent of h. To this end, first note that $\varphi_{h+a}(w) = u(w+h+a) - u(w) = [u(w+h+a) - u(w+h)] + [u(w+h) - u(w)] = \varphi_a(w+h) + \varphi_h(w)$. Therefore, since a Laplace transform is uniquely determined by its associated distribution function, we have

$$dF_{h+a}(s) = e^{-sh}\, dF_a(s) + dF_h(s), \qquad \text{for all} \quad h > 0 \qquad \text{and} \qquad a > 0. \tag{27}$$

The solution $dF_h(s)$ of the measure equation (27) is clearly increasing in h. For $a = h$ (27) becomes $dF_{2h}(s) = (1 + e^{-sh})\, dF_h(s)$, whereas for $a = 2h$ it becomes $dF_{3h}(s) = e^{-sh}\, dF_{2h}(s) + dF_h(s) = (1 + e^{-sh} + e^{-2sh})\, dF_h(s)$. Hence, by induction we get $dF_{nh}(s) = (\sum_{k=0}^{n-1} e^{-ksh})\, dF_h(s)$. Taking the limit as n tends to infinitely yields $\lim_{a \to \infty} dF_a(s) = (1/(1 - e^{-sh}))\, dF_h(s)$. Therefore, $\lim_{a \to \infty} dF_a(s)$ is finite for all $s > 0$. This proves that both $(1/(1 - e^{-sh}))\, dF_h(s)$ and $dF(s) = (s/(1 - e^{-sh}))\, dF_h(s)$ are independent of h.

Finally, notice that $u(h) = \varphi_h(0) = \int_0^\infty dF_h(s) = \int_0^\infty ((1 - e^{-sh})/s)\, dF(s)$. Moreover, $\int_1^\infty (dF(s)/s) = \int_1^\infty (1/(1 - e^{-sh}))\, dF_h(s) \leqslant (1/(1 - e^{-h})) \int_1^\infty dF_h(s) < \infty$, where the last inequality follows because $\int_1^\infty dF_h(s) \leqslant \int_0^\infty dF_h(s) = \varphi_h(0) = u(h) < \infty$. Therefore, Theorem 2.2 tells us that $u(w)$ displays mixed risk aversion. Q.E.D.

Proof of Lemma 6.1. We have that $\int_0^\infty \varphi(s)\, e^{-sw}\, dF(s) \leqslant \int_c^\infty |\varphi(s)|\, e^{-sw}\, dF(s) \leqslant Me^{-cw}$ for all $w > 0$, where $M \equiv \int_c^\infty |\varphi(s)|\, dF(s)$ is finite by assumption (a). Moreover, there exists a positive real number $w_0 > 0$ such that $\int_0^c \varphi(s)\, e^{(c-s)w}\, dF(s) > M$ for all $w > w_0$, as follows from assumption (b). Therefore, $\int_0^c \varphi(s)\, e^{-sw}\, dF(s) > Me^{-cw}$, and hence $\int_0^\infty \varphi(s)\, e^{-sw}\, dF(s) = \int_0^c \varphi(s)\, e^{-sw}\, dF(s) + \int_c^\infty \varphi(s)\, e^{-sw}\, dF(s) > 0$ for all $w > w_0$. Q.E.D.

Proof of Lemma 6.3. (a) Note that

$$\lim_{s \to 0} \frac{F_1(ts)}{F_1(s)} = \lim_{s \to 0} \frac{\int_0^{ts} \tau\, dF(\tau)}{\int_0^s \tau\, dF(\tau)} = \lim_{s \to 0} \frac{tsF(ts) - \int_0^{ts} F(\tau)\, d\tau}{sF(s) - \int_0^s F(\tau)\, d\tau}$$

$$= \lim_{s \to 0} \frac{(tF(ts)/F(s)) - \int_0^{ts} (F(\tau/sF(s))\, d\tau}{1 - \int_0^s (F(\tau)/sF(s))\, d\tau}, \tag{28}$$

where the first equality in (28) follows from integrating by parts, and the second from dividing both numerator and denominator by $sF(s)$. The limit of the first term in the last numerator of (28) is equal to $t^{\rho+1}$ as dictated by the regular variation of F. Moreover,

$$\lim_{s\to 0}\int_0^{ts}\frac{F(\tau)}{sF(s)}\,d\tau=\lim_{s\to 0}\int_0^{t}\frac{F(\eta s)}{F(s)}\,d\eta=\int_0^{t}\eta^{\rho}\,d\eta=\frac{t^{\rho+1}}{\rho+1}, \tag{29}$$

where the first equality is obtained by making the change of variable $\tau=\eta s$. For the second equality, it should be noticed that the regular variation of F allows us to apply the Lebesgue convergence theorem. The last equality follows from just performing the Riemann integral.

Concerning the denominator of (28), we obtain in a similar fashion

$$\lim_{s\to 0}\int_0^{s}\frac{F(\tau)}{sF(s)}\,d\tau=\lim_{s\to 0}\int_0^{1}\frac{F(\eta s)}{F(s)}\,d\eta=\int_0^{1}\eta^{\rho}\,d\eta=\frac{1}{\rho+1}. \tag{30}$$

After substituting (29) and (30) into (28), we get

$$\lim_{s\to 0}\frac{F_1(ts)}{F_1(s)}=\frac{t^{\rho+1}-(t^{\rho+1}/(\rho+1))}{1-(1/(\rho+1))}=t^{\rho+1},$$

which proves the regular variation at the origin with exponent $\rho+1$ of $F_1(s)$.

(b) Note that

$$\lim_{s\to 0}\frac{F_1(s)}{sF(s)}=\lim_{s\to 0}\frac{\int_0^s\tau\,dF(\tau)}{sF(s)}=1-\lim_{s\to 0}\int_0^{s}\frac{F(\tau)}{sF(s)}\,d\tau, \tag{31}$$

where the last equality comes from integrating by parts. Furthermore, $\lim_{s\to 0}\int_0^s(F(\tau)/sF(s))\,d\tau=1/(\rho+1)$, as shown in (30). Substituting in (31), we get the desired conclusion. Q.E.D.

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