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Stochastic dominance and absolute risk aversion

Abstract In this paper, we propose the infimum of the Arrow–Pratt index of absolute risk aversion as a measure of global risk aversion of a utility function. We show that, for any given arbitrary pair of distributions, there exists a threshold level of global risk aversion such that all increasing concave utility functions with at least as much global risk aversion would rank the two distributions in the same way. Furthermore, this threshold level is sharp in the sense that, for any lower level of global risk aversion, we can find two utility functions in this class yielding opposite preference relations for the two distributions.

1 Introduction

Hadar and Russell (1969) and Rothschild and Stiglitz (1970) proposed the following notion of riskiness: one distribution is riskier than another when the former

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dominates the latter according to the second-order stochastic dominance (SOSD) criterion, i.e. when it is unanimously preferred by all expected utility maximizers who prefer more to less and who are risk averters. Unanimity requires thus that all decision makers agree with the most extreme risk averse preferences, i.e. those giving all the weight to the worst possible outcome. Clearly, for these extreme preferences, which of the two distributions will be preferred depends on the lower tail of the distributions only. However, when we want to verify that the individuals with lower degrees of risk aversion will agree on that ordering as well, we need to compare the two distributions over the entire support. Indeed, the main result in Hadar and Russell (1969) and Rothschild and Stiglitz (1970) is that a pair of distributions can be ranked according to the SOSD criterion if and only if a strong integral condition relating the two distributions is satisfied. This condition is quite stringent so that the ordering on the set of distributions induced by the SOSD criterion is indeed very partial.

A question that naturally arises in the theory of decision under risk is whether the comparison between risky prospects would be facilitated by requiring unanimity only on a subset of the class of increasing and concave utility functions with appealing properties. This task has proven quite unproductive since many additional natural properties imposed on utility functions, like decreasing absolute risk aversion (DARA),[1] do not yield a non-dense basis through which an operative condition relating two distribution functions can be obtained (see Gollier and Kimball 1996). One exception is the class of mixed utility functions, that are those having non-negative odd derivatives and non-positive even derivatives. Caballé and Pomansky (1996) show that the set of negative exponential functions constitutes a basis for that family of utilities. Therefore, a distribution is preferred to another by all individuals with increasing utilities exhibiting sign-alternating derivatives if and only if the Laplace transform of the former is smaller than that of the latter.[2]

In the present paper, we take back the approach of Hadar and Russell (1969) and Rothschild and Stiglitz (1970). However, instead of classifying the pairs of distributions into "uncontroversial" – those that can be ranked according to SOSD – and "controversial" – those that cannot – we wish to associate to every pair of distributions a parameter reflecting how controversial their ranking is. Specifically, for any arbitrary pair of distributions, we wish to characterize the "lowest" degree of risk aversion such that all decision makers with at least this degree of risk aversion would unanimously prefer one distribution over an other. The lower the required degree of risk aversion the less controversial the ranking will be. Notice that the main point of our analysis is not the ordering since this is given by the preferences of the most risk averse individuals. The critical question is how many more individuals will agree with such an ordering.

As shown by Meyer (1977), when two random variables cannot be ranked by SOSD there is always some utility function u (not necessarily concave) such that the resulting distribution of the utility satisfies the integral condition for SOSD.

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[1] A Bernoulli utility belonging to the DARA class exhibits a demand for a risky asset that increases with wealth (Arrow 1970; Pratt (1964)).

[2] The class of mixed utility functions constitutes a subset of the DARA class and includes all the DARA utilities typically found in some economic applications, like the hyperbolic absolute risk aversion, the isoelastic, or the exponential functions. In fact, mixed utilities satisfy other appealing properties found in the literature, like risk vulnerability (Gollier and Pratt 1996), properness (Pratt and Zeckhauser 1987) or standardness (Kimball 1993).

Notice, however, that by not imposing that the critical utility function be concave, its corresponding concave transformations need not be concave either. This is indeed an undesirable feature when individuals face typical optimization problems under risk. Using this fact, our objective is to identify the "least" concave utility function u for which the distributions of the transformed random variables can be ranked according to SOSD. We know then that all the increasing and concave transformations of this utility function u will rank the two original distributions as the function u does and, hence, all the individuals having utilities displaying more absolute risk aversion at each point than that of the threshold utility u will choose unanimously the same random variable.

It is obvious that finding the "least" concave utility function cannot have an unambiguous answer, even when one chooses to measure the local concavity of a utility function by its absolute risk aversion (ARA) index. To set the ground, we start by analyzing the two extreme types of increasing and concave transformations of the original random variables. First, we consider transformations that are linear (or risk neutral) everywhere except at a single point around which they concentrate all the concavity. Second, we will consider transformations that display an ARA index uniformly distributed over its domain, i.e. these transformations exhibit constant absolute risk aversion (CARA). Clearly, the two types of functions under consideration exhibit a very different behavior of their local ARA indexes. If the function is essentially linear, the infimum (supremum) of the local ARA index over its domain becomes zero (infinite) and, thus, no operative lower bound is obtained in terms of the ARA index. In contrast, the global concavity of a utility belonging to the CARA family is perfectly summarized by the ARA index evaluated at any arbitrary point of its domain. For the first type of functions we obtain the smallest drop of the slope at the kink permitting the ordering of the transformed random variables by SOSD. As for the second type, we prove the existence of a critical minimum value of the ARA index allowing for the SOSD ranking of the two transformed risks.

Our main result follows immediately from the analysis made for the previous two families of functions. If there exists a minimal value of the ARA index for which SOSD holds for the corresponding CARA transformation of the original random variables, then SOSD will hold for all utility functions whose infimum of the ARA index is larger than that threshold value. This is so because the latter functions turn out to be concave transformations of the critical CARA function. However, for all lower values of the infimum of the ARA index, it is possible to find functions for which the SOSD ranking does not apply. Furthermore, if there is a piecewise linear utility function allowing for the SOSD ranking of the given pair of distributions, then we can find functions with an infimum of their ARA index arbitrarily close to zero permitting this ranking.

We then go into examining whether similar results can be obtained with other reasonable measures of global concavity. To this end, we consider two natural alternative measures of global concavity: the supremum of the ARA index and the average ARA index of the utility function over its domain. However, these alternative measures turn out to yield much weaker results concerning our original problem.

Our analysis is based on the use of the key observation, made by Diamond and Stiglitz (1974), Meyer (1977), and Lambert and Hey (1979) that two random

variables can be ranked according to SOSD if and only if any common concave transformation of these random variables can be ranked according to SOSD. Using this fact, we intend to identify the "least" concave utility function u for which the distributions of the transformed random variables can be ranked according to SOSD. Then, all the increasing and concave transformations of this utility function u will rank the two original distributions as the function u does.

In a related paper, Tungodden (2005) uses a similar idea to find a parameter value representing a normative position on how to weigh interests within a population in order to reach a unanimous order between two distributions. Similarly, Le Breton (1994) shows how is it possible to generate less partial preorders than the one implied by SOSD by means of increasing the degree of concavity of different families of utility functions. He illustrates this fact in his Remark 5 by considering the isoelastic (or constant relative risk aversion) family, which is directly related with the Foster–Greer–Thorbecke family of poverty measures considered by Tungodden (2005). Our approach uses instead the class of exponential (or constant absolute risk aversion) family as the building block of our analysis. This is so because every concave transformation of an exponential utility with an ARA index s will display at each point an ARA index larger or equal than s. Therefore, our first target will be to find the threshold ARA index s^* for which all the individual having exponential utilities with a larger ARA index will make a unanimous choice when facing two risky prospects, while such a unanimity will not hold for values below s^*.

The paper is organized as follows. Section 2 reviews some concepts appearing in the literature of decision under risk. Section 3 contains some preliminary results. Section 4 presents the main theorem of the paper. Section 5 considers alternative measures of global risk aversion and their relationship with SOSD. Our concluding remarks in Sect. 4 are followed by an Appendix containing the lengthier proofs.

2 Orderings on distributions

Consider the set of random variables taking values on the interval $[a, b]$. If F_X is the distribution function of the random variable X, then the expectation (or mean) of the distribution of X is $E(F_X) = \int_{[a,b]} z \, dF_X(z)$.[3] Suppose that an agent has a state-independent preference relation defined on the space of random variables and that this preference relation has an expected utility representation (or Bernoulli utility) u. This means that the agent prefers the random variable X with distribution function F_X to the random variable Y with distribution function F_Y whenever

$$\int_{[a,b]} u(z) \, dF_X(z) \geq \int_{[a,b]} u(z) \, dF_Y(z). \tag{1}$$

It is well known that the Bernoulli utility u is unique up to a strictly increasing affine transformation. Note that a state-independent preference relation defined on

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[3] The integral appearing in the expression is the Lebesgue integral with respect to the Lebesgue–Stieltjes measure (or distribution) associated with the distribution function F (see Sect. 1.4 of Ash 1972).

the space of random variables induces a preference relation on the set of distribution functions. Therefore, we will say that F_X is preferred to F_Y by an individual having the Bernoulli utility u, $F_X \underset{u}{\succsim} F_Y$, if (1) holds. Moreover, F_X is strictly preferred to F_Y by an individual with Bernoulli utility u, $F_X \underset{u}{\succ} F_Y$, whenever (1) holds with strict inequality.

Definition 2.1 (a) *The distribution function F_X dominates the distribution function F_Y according to the SOSD criterion, $F_X \underset{D}{\succsim} F_Y$, if $F_X \underset{u}{\succsim} F_Y$ for all the Bernoulli utility functions u that are increasing and concave.*
(b) *The distribution function F_X strictly dominates the distribution function F_Y according to the SOSD criterion, $F_X \underset{D}{\succ} F_Y$, if $F_X \underset{u}{\succ} F_Y$ for all the Bernoulli utility functions u that are increasing and strictly concave.*

Therefore, if $F_X \underset{D}{\succsim} F_Y$, then all the individuals who prefer more to less and are risk averse will prefer the random variable X to the random variable Y. Moreover, a related result appearing in the theory of income inequality establishes that SOSD between two distribution functions is equivalent to dominance of the corresponding generalized Lorenz curves (Shorrocks 1983). According to the well-known analysis of Hadar and Russell (1969) and Rothschild and Stiglitz (1970), we can state the following famous result:

Proposition 2.2 $F_X \underset{D}{\succsim} F_Y$ *if and only if*

$$\int_a^x [F_X(z) - F_Y(z)]\, dz \le 0 \ \ for\ all\ x \in [a,b]\,. \tag{2}$$

Moreover, $F_X \underset{D}{\succ} F_Y$ if the previous inequality is strict for all $x \in (a,b)$.

Consider now an increasing and concave function u and two random variables X and Y. Let $F_{u(X)}$ and $F_{u(Y)}$ be the distribution functions associated with the transformed random variables $u(X)$ and $u(Y)$, respectively. The following corollary, arising from the papers of Diamond and Stiglitz (1974), Meyer (1977), and Lambert and Hey (1979), will play a crucial role in our analysis:

Corollary 2.3 $F_{u(X)} \underset{D}{\succsim} (\underset{D}{\succ}) F_{u(Y)}$ *if and only if* $F_{v(X)} \underset{D}{\succsim} (\underset{D}{\succ}) F_{v(Y)}$ *for all the Bernoulli utility functions v that are increasing and (strictly) concave transformations of u.*

Proof Obvious from Definition 2.1 since $F_{u(X)} \underset{D}{\succsim} (\underset{D}{\succ}) F_{u(Y)}$ if and only if $F_{u(X)} \underset{v}{\succsim} (\underset{v}{\succ}) F_{u(Y)}$ for all the Bernoulli utility functions v that are increasing and (strictly) concave transformations of u. □

The order induced on the set of distribution functions by the SOSD criterion is very partial as the distributions that can be ranked according to that criterion

constitute indeed a very small subset of distribution functions. This can be easily deduced from just looking at the stringent integral condition (2). In contrast, the leximin (or lexicografic maximin) criterion discussed in Rawls (1974), which makes preferable the distribution with the better worst possible outcome, induces a quite complete ordering on the set of distributions. Before defining more precisely this lexicographic criterion, we need the following definition that will be extensively used in the rest of the paper:

Definition 2.4 *The right-continuous function g defined on* $[a, b]$ *changes sign at* x *if there exist two real numbers* $\varepsilon > 0$ *and* $\eta \geq 0$ *such that the following two conditions hold:*

(i) $g(z) \cdot g(y) \leq 0$ *for all* $(z, y) \in (x - \varepsilon, x) \times [x, x + \eta]$ *and*
(ii) $g(z) \cdot g(y) < 0$ *for some* $(z, y) \in (x - \varepsilon, x) \times [x, x + \eta]$.

Definition 2.5 (a) *The distribution function* F_X *strictly dominates the distribution function* F_Y *according to the leximin criterion,* $F_X \underset{M}{\succ} F_Y$, *if there exists a* $\hat{z} \in (a, b)$ *such that* $F_X(z) \leq F_Y(z)$ *for all* $z \in [a, \hat{z})$, *and* $F_X(z) < F_Y(z)$ *for some* $z \in [a, \hat{z})$.
   (b) *The distribution function* F_X *dominates the distribution function* F_Y *according to the leximin criterion,* $F_X \underset{M}{\succsim} F_Y$, *if either* $F_X \underset{M}{\succ} F_Y$ *or* $F_X(z) = F_Y(z)$ *for all* $z \in [a, b]$.

Clearly, the ordering induced by the leximin criterion is much more complete than that induced by the SOSD criterion. For instance, all pairs $\{F_X, F_Y\}$ of distribution functions for which the function $F_X - F_Y$ changes sign a finite number (including zero) of times can be ranked according to the former criterion.
   We will restrict our attention throughout the paper to pairs of distributions functions $\{F_X, F_Y\}$ of random variables taking values on the interval $[a, b]$ that satisfy the following assumption:

Assumption M The function $F_X - F_Y$ changes sign a finite number of times and $F_X \underset{M}{\succ} F_Y$, whereas neither $F_X \underset{D}{\succsim} F_Y$ nor $F_Y \underset{D}{\succsim} F_X$.

On the one hand, the assumption of a finite number of changes of sign is sufficient to ensure that there is a point on $[a, b]$ of first sign change according to Definition 2.4.[4] On the other hand, the assumption of $F_X \underset{M}{\succ} F_Y$ and neither $F_X \underset{D}{\succsim} F_Y$ nor $F_Y \underset{D}{\succsim} F_X$ is made without loss of generality whenever the two random variables under consideration cannot be ranked according to the SOSD criterion and the function $F_X - F_Y$ changes sign a finite number of times.
   Consider the case where the distribution functions F_X and F_Y cannot be ranked by SOSD. Suppose that we could find a utility function u such that the transformed

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[4] Consider in this respect the case where $F_{\bar{x}}$ and $F_{\bar{y}}$ satisfy $F_{\bar{x}} < F_{\bar{y}}$ on $[a, c)$, $F_{\bar{x}} = F_{\bar{y}}$ at c and they intersect at all points of the form $c + (1/n)$ for every positive integer n. Obviously, the function $F_{\bar{x}} - F_{\bar{y}}$ changes sign infinitely many times and there is no point of first change of sign according to Definition 2.4.

random variables satisfy $F_{u(X)} \underset{D}{\succsim} F_{u(Y)}$. Then, by Corollary 2.3, the random variable X will be preferred to Y by all agents having Bernoulli utility functions that are increasing and concave transformations of u. The rest of the paper is devoted to find the "least concave" utility function permitting the SOSD ranking of the distributions associated with the transformed random variables when the original pair of distributions satisfies Assumption M.

3 Local risk neutrality almost everywhere

We start by considering essentially linear transformations of two given random variables. This means that, if we view these transformations as utility functions, they display risk neutrality everywhere except at a point where they exhibit a kink. The next proposition shows explicitly how we can construct a common increasing and concave transformation of two random variables having distributions that cannot be ranked according to the SOSD criterion so as to obtain SOSD for the corresponding transformed random variables. If one of the two random variables is strictly preferred to an other according to the leximin criterion, then the integral condition (2) will be satisfied for an interval of low realizations of these variables. Our strategy consists on scaling down the larger values of both random variables so that the previous integral condition will hold for the whole range of values of the transformed random variables.

Proposition 3.1 *Consider the class of continuous functions with the following functional form:*

$$k(z; \alpha, z_1) = \begin{cases} z & for\ z \in [a, z_1) \\ \alpha z + (1 - \alpha)z_1 & for\ z \in [z_1, b]. \end{cases} \tag{3}$$

Assume that the pair of distribution functions $\{F_X, F_Y\}$ satisfies Assumption M. Then, there exist two real numbers $\alpha^* \in (0, 1)$ and $z_1 \in (a, b)$ such that $F_{k(X;\alpha^*,z_1)} \underset{D}{\succsim} F_{k(Y;\alpha^*,z_1)}$ and $F_{k(X;\alpha,z_1)} \underset{D}{\succ} F_{k(Y;\alpha,z_1)}$ for all $\alpha < \alpha^*$, whereas neither $F_{k(X;\alpha,z_1)} \underset{D}{\succsim} F_{k(Y;\alpha,z_1)}$ nor $F_{k(Y;\alpha,z_1)} \underset{D}{\succsim} F_{k(X;\alpha,z_1)}$ for all $\alpha > \alpha^*$.

Proof See the Appendix □

The next corollary shows that the pair (α^*, z_1) is in some sense unique. In particular, if we had chosen a point different from z_1, as defined in Eq. (11) of the Appendix, in the functional form of the function $k(\cdot; \alpha, z_1)$, the value of the maximal slope α^*, as defined in Eq. (18) of the Appendix, should be smaller in order to preserve stochastic dominance. It follows then that our characterization of α^* is sharp.

Corollary 3.2 *Assume that the pair of distribution functions $\{F_X, F_Y\}$ satisfies Assumption M. Consider the set of pairs of numbers $\{\hat{\alpha}, \hat{z}\} \in (0, 1) \times (a, b)$ for which $F_{k(X;\hat{\alpha},\hat{z})} \underset{D}{\succsim} F_{k(Y;\hat{\alpha},\hat{z})}$ and $F_{k(X;\alpha,\hat{z})} \underset{D}{\succ} F_{k(Y;\alpha,\hat{z})}$ for all $\alpha < \hat{\alpha}$, whereas neither $F_{k(X;\alpha,\hat{z})} \underset{D}{\succsim} F_{k(Y;\alpha,\hat{z})}$ nor $F_{k(Y;\alpha,\hat{z})} \underset{D}{\succsim} F_{k(X;\alpha,\hat{z})}$ for all $\alpha > \hat{\alpha}$. Then, $\hat{\alpha} \leq \alpha^*$.*

Proof See the Appendix □

We next provide a technical remark concerning the location of the value z_M defined in Eq. (22) of the Appendix when the two original distributions have the same mean.[5] Notice that the value z_M could be located at the upper limit of the interval $[a, b]$. However, if we assume that the distributions of the random variables X and Y satisfy $E(F_X) = E(F_Y)$, then $z_M < b$. The value z_M plays a crucial role in the definition of the the critical slope α^* (see Eq. (18) of the Appendix).

Corollary 3.3 *Assume that the pair of distribution functions $\{F_X, F_Y\}$ satisfies Assumption M and $E(F_X) = E(F_Y)$. Then, $z_M \in (a, b)$ and the function $F_X - F_Y$ changes sign at z_M.*

Proof See the Appendix   □

Note then that, when the pair $\{F_X, F_Y\}$ satisfies Assumption M with $E(F_X) = E(F_Y)$, the function $F_X - F_Y$ must change sign both at z_1 and at z_M, which agrees with the first statement in the proof of the previous corollary.

The transformation $k(\cdot; \alpha^*, z_1)$ of the original random variables proposed in Proposition 3.1 in order to obtain SOSD has the undesirable property of being non-differentiable. Obviously, all the increasing and concave transformations of the function $k(\cdot; \alpha^*, z_1)$ are also non-differentiable at z_1. However, these functions can be arbitrarily approximated by a differentiable function, as the next proposition shows:

Proposition 3.4 *Consider the class of functions defined on $[a, b]$ with the following functional form:*

$$q(z; \varepsilon, \beta, z_1) = \begin{cases} (1 + \varepsilon)z - \varepsilon z_1 + \varepsilon^2 & for \ z \in [a, z_1 - \varepsilon] \\ g(z) & for \ z \in (z_1 - \varepsilon, z_1 + \varepsilon) \ (4) \\ (\beta - \varepsilon)z + (1 - \beta + \varepsilon)z_1 + \varepsilon^2 & for \ z \in [z_1 + \varepsilon, b]. \end{cases}$$

Assume that the pair of distribution functions $\{F_X, F_Y\}$ satisfies Assumption M. Then, for all $\beta \in (0, \alpha^*)$, there exists a real number $\varepsilon > 0$ and a function $g(\cdot)$ such that the function $q(\cdot; \varepsilon, \beta, z_1)$ is smooth, increasing, concave, and satisfies $F_{q(X;\eta,\beta,z_1)} \underset{D}{\succ} F_{q(Y;\eta,\beta,z_1)}$ for all $\eta \in (0, \varepsilon]$.

The intuition behind the previous proposition is straightforward as, according to Proposition 3.1, we can choose a real number $\beta \in (0, \alpha^*)$ for which $F_{k(\tilde{x};\beta,z_1)} \underset{D}{\succ} F_{k(\tilde{y};\beta,z_1)}$, where z_1 and α^* are defined in Eqs. (11) and (18) of the Appendix respectively. Since we have strict SOSD, we can slightly perturb the continuous function $k(\cdot, \beta, z_1)$, whose functional form is given in (3), while preserving strict SOSD. Hence, there exists a sufficiently small real number $\varepsilon > 0$ for which we can find a smooth, increasing and concave function g inducing the desired properties on the function $q(z; \varepsilon, \beta, z_1)$ with the functional form given in (4) and, in particular, $F_{q(\tilde{x};\varepsilon,\beta,z_1)} \underset{D}{\succ} F_{q(\tilde{y};\varepsilon,\beta,z_1)}$. The corresponding technical proof can be found in Caballé and Esteban (2003).

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[5] This is the scenario considered by Atkinson (1970).

The following Proposition shows that, if the non-differentiable function $k(z; \alpha, z_1)$ is picked so that neither $F_{k(X;\alpha,z_1)} \succsim_D F_{k(Y;\alpha,z_1)}$ nor $F_{k(Y;\alpha,z_1)} \succsim_D F_{k(X;\alpha,z_1)}$, then this function can also be approximated by a smooth function:

Proposition 3.5 *Consider the class of functions with the functional form given in (4). Assume that the pair of distribution functions $\{F_X, F_Y\}$ satisfies Assumption M. Then, for all $\beta \in (\alpha^*, 1)$, there exists a real number $\varepsilon > 0$ and a function $g(\cdot)$ such that the function $q(\cdot; \varepsilon, \beta, z_1)$ is smooth, increasing, concave, and neither $F_{q(X;\eta,\beta,z_1)} \succsim_D F_{q(Y;\eta,\beta,z_1)}$ nor $F_{q(Y;\eta,\beta,z_1)} \succsim_D F_{q(X;\eta,\beta,z_1)}$ for all $\eta \in (0, \varepsilon]$.*

Proof See the Appendix □

Let u be a twice continuously differentiable function on (a, b). The Arrow–Pratt index of absolute risk aversion (ARA) of the function u at $z \in (a, b)$ is $A_u(z) = -u''(z) / u'(z)$ (see Arrow 1970; and Pratt (1964)). The following corollary characterizes the limiting behavior of the infimum and the supremum of the ARA index of the function $q(\cdot; \varepsilon, \beta, z_1)$ defined in (4) as ε becomes arbitrarily small:

Corollary 3.6 (a) $\displaystyle \inf_{z \in (a,b)} \left(\lim_{\varepsilon \to 0} A_{q(\cdot;\varepsilon,\beta,z_1)}(z) \right) = 0$;

(b) $\displaystyle \sup_{z \in (a,b)} \left(\lim_{\varepsilon \to 0} A_{q(\cdot;\varepsilon,\beta,z_1)}(z) \right) = \infty$.

Part (a) is obvious. For part (b) Notice that, since all the concavity is concentrated on the interval $(z_1 - \varepsilon, z_1 + \varepsilon)$, the supremum of the ARA index on that interval tends to infinity when ε tends to zero. The precise proof of the previous Corollary can also be found in Caballé and Esteban (2003).

4 The infimum of absolute risk aversion

We have considered so far utility functions that exhibit local risk neutrality almost everywhere except in a small neighborhood of a point where all the risk aversion is concentrated. In the first part of this section, we will use a completely different approach since instead of concentrating all the concavity in a small interval, we are going to consider transformations of the original random variables through functions that have all the risk aversion uniformly distributed over its domain. Consider the class of increasing, concave and twice continuously differentiable utility functions $r(\cdot; s)$ exhibiting an ARA index, $A_{r(\cdot;s)}(z)$, equal to the constant $s > 0$ for all $z \in (a, b)$. These functions exhibiting constant absolute risk aversion (CARA) have a functional form that is an increasing affine transformation of the function $-e^{-sz}$. We will next show that, given two random variables X and Y such that $F_X \succ_M F_Y$, then there exists a CARA utility function $r(\cdot; s)$ with an ARA index s for which $F_{r(X;s)} \succsim_D F_{r(Y;s)}$. Recall that, if one of the two random variables is strictly preferred to the other according to the leximin criterion, then the integral condition (2) will hold for an interval of low realizations of these variables. A CARA transformation of these variables attaches a relative lower weight to high

realizations, and this relative weight decreases with the ARA index s. Therefore, for a sufficiently large value of s the integral condition (2) will be satisfied over the whole range of values of the transformed random variables. The following proposition establishes the basic existence result:

Proposition 4.1 *Assume that the pair of distribution functions* $\{F_X, F_Y\}$ *satisfies Assumption M. Then, there exists a real number* s^* *such that* $F_{r(X;s^*)} \underset{D}{\succsim} F_{r(Y;s^*)}$ *and* $F_{r(X;s)} \underset{D}{\succ} F_{r(Y;s)}$ *for all* $s > s^*$, *whereas neither* $F_{r(X;s)} \underset{D}{\succsim} F_{r(Y;s)}$ *nor* $F_{r(Y;s)} \underset{D}{\succsim} F_{r(X;s)}$ *for all* $s < s^*$.

Proof See the Appendix   □

The following corollary extends the previous proposition to functions that are not necessarily CARA. In order to obtain SOSD between two random variables we only require a sufficiently large value of the ARA index on some interval (a, z_0) with $z_0 < b$.

Corollary 4.2 *Assume that the pair of distribution functions* $\{F_X, F_Y\}$ *satisfies Assumption M. Then, there exists a pair of real numbers* $\{\hat{s}, z_0\} \in (0, \infty) \times (a, b)$ *such that* $F_X \underset{u}{\succsim} (\underset{u}{\succ}) F_Y$ *for every twice differentiable, increasing and concave Bernoulli utility function* u *satisfying* $A_u(z) \geq (>)\hat{s}$ *for* $z \in (a, z_0)$.

Proof See the Appendix   □

As follows from the proofs of the previous corollary and of Proposition 4.1, the upper limit z_0 of the interval where strict concavity is required turns out to be the smallest value at which the function $F_X(z) - F_Y(z)$ changes sign. Moreover, the critical value $\hat{s}$ of the ARA index on the interval $(0, z_0)$ is given by the value of s solving Eq. (27) of the Appendix.

Until now we have analyzed two basic families of functions for which the transformations of the random variables X and Y through these functions can be ranked according to the SOSD criterion. One family is that of the CARA functions $r(\cdot; s)$, which we have just analyzed. Another family is formed by functions that have all the risk aversion concentrated on a small interval of its domain. The functional form of a function belonging to the latter class is given in (4). Note that the function $q(\cdot; \varepsilon, \beta, z_1)$ is an increasing, concave and smooth function that is linear for all values that do not belong to the interval $(z_1 - \varepsilon, z_1 + \varepsilon)$. Moreover, the derivative of $q(\cdot; \varepsilon, \beta, z_1)$ is equal to $1 + \varepsilon$ for all values of the interval $(a, z_1 - \varepsilon)$, while its derivative is equal to $\beta - \varepsilon$ for all values belonging to $(z_1 + \varepsilon, b)$.

We propose now a global measure of risk aversion (the infimum of the ARA index) inducing a partition over the set of increasing and concave functions so that all the utility functions displaying more global risk aversion than a threshold level rank one distribution over the other. Consider thus the following partition of the set of increasing and concave utility functions on $[a, b]$ that are twice continuously differentiable on (a, b). A function u belongs to the class $I(s)$ if the infimum of the ARA index over its domain is s,

$$u \in I(s), \quad \text{whenever} \quad \inf_{z \in (a,b)} A_u(z) = s.$$

Notice that all $u \in I(s)$ with $s \geq s^*$ are increasing and concave transformations of the CARA utility with an ARA index equal to s^*. Therefore, an implication of Corollary 2.3 is that, if $F_{r(X;s^*)} \underset{D}{\succ} F_{r(Y;s^*)}$ then $F_{u(X)} \underset{D}{\succ} F_{u(Y)}$ for all $u \in I(s)$ with $s \geq s^*$.

From this observation, we derive the main result of our paper, which is stated in the following theorem:

Theorem 4.3 *Assume that the pair of distribution functions $\{F_X, F_Y\}$ satisfies Assumption M. Then, there exists a real number $s^* > 0$ such that*

(a) $F_{u(X)} \underset{D}{\succ} F_{u(Y)}$ *for all $u \in I(s)$ with $s > s^*$*

(b) *For all $s \in (0, s^*)$ there exists a $u \in I(s)$ such that neither $F_{\hat{u}(X)} \underset{D}{\succsim} F_{\hat{u}(Y)}$ nor*

$F_{\hat{u}(Y)} \underset{D}{\succsim} F_{\hat{u}(X)}$.

Proof (a) Let s^* be the real number defined in Proposition 4.1. Notice that, if $u \in I(s)$ with $s > s^*$, then u is an increasing and concave transformation of the CARA utility $r(\cdot; s)$ since $A_u(z) \geq s$ for all $z \in (a, b)$ (see Pratt 1964. Then, as $F_{r(X;s)} \underset{D}{\succ} F_{r(Y;s)}$, we must have $F_{u(X)} \underset{D}{\succ} F_{u(Y)}$ as follows from Corollary 2.3.

(b) Obvious from Proposition 4.1 since the CARA utility $\hat{u}(\cdot) \equiv r(\cdot; s)$ belongs to $I(s)$. □

The previous theorem implies that we can always find a function u, with an arbitrarily given value of the infimum of its ARA index, for which the random variables $u(X)$ and $u(Y)$ can be compared according to the SOSD criterion or, equivalently, according to the generalized Lorenz curve criterion. In fact, part (a) says that, for sufficiently large values of the infimum of the ARA index, SOSD between two random variables always holds. On the contrary, part (b) tells us that, if a concave transformation of two random variables does not generate SOSD, then that transformation must exhibit a small value of the infimum of its ARA index.

Note that, if we have an arbitrary set V of random variables such that each pair of random variables from this set satisfies assumption M, then we can always find a value s^* so that for all $u \in I(s)$ with $s > s^*$ the set of random variables $\{u(X) \mid X \in V\}$ can be ordered according to the SOSD criterion. Obviously, a large value of s^* will indicate a substantial degree of disagreement in this economy since a large fraction of the less risk averse individuals will not agree with the aforementioned ordering. A small value of s^* will be a signal of a larger social consensus.

In order to provide a complete picture, we provide the following corollary accruing from Proposition 3.4

Corollary 4.4 *For all $s \in (0, s^*)$ there exists a $u \in I(s)$ such that $F_{u(X)} \underset{D}{\succ} F_{u(Y)}$.*

Proof Consider the utility function $q(z; \varepsilon, \beta, z_1)$ characterized in Proposition 3.4 so that $F_{q(X;\varepsilon,\beta,z_1)} \underset{D}{\succ} F_{q(Y;\varepsilon,\beta,z_1)}$. Clearly, the infimum of the ARA index of $q(z; \varepsilon, \beta, z_1)$ is zero. Obviously, any concave function w will satisfy $F_{w(q(X;\varepsilon,\beta,z_1))} \underset{D}{\succ} F_{w(q(Y;\varepsilon,\beta,z_1))}$ as follows from Corollary 2.3. Therefore, the infimum of the ARA index of $u(\cdot) \equiv w(q(\cdot; \varepsilon, \beta, z_1))$ can take any positive value. □

Therefore, the previous corollary and part (a) of Theorem 4.3 imply that we can always find a function u, with an arbitrarily given value of the infimum of the ARA index, for which the random variables $u(X)$ and $u(Y)$ can be compared according to the SOSD criterion

The following corollary characterizes explicitly the critical value s^* of the infimum of the ARA index above which unanimity is reached:

Corollary 4.5 *The critical value s^* defined in Theorem 4.3 is the smallest positive real number s satisfying*

$$\int_a^x [F_X(z) - F_Y(z)] e^{-sz} dz \le 0 \quad for\ all\ x \in [a, b]. \tag{5}$$

Proof Obviously, the critical value s^* of the ARA index yielding stochastic dominance for CARA utilities is the smallest positive real value of s satisfying

$$\int_{r(a;s)}^y \left[F_{r(X;s)}(r) - F_{r(Y;s)}(r) \right] dr \le 0 \quad for\ all\ y \in [r(a;s), r(b;s)]. \tag{6}$$

From Proposition 4.1, we know that such a critical value s^* exists. Therefore, by performing the corresponding change of variable in (6), s^* turns out to be the smallest positive real value of s satisfying (5).   □

We should point out that the search for the critical value s^* can be a difficult task since we should evaluate the integral appearing in (5) for all $x \in [a, b]$. However, if we assume that the random variables X and Y are absolutely continuous and, hence, have density functions f_X and f_Y, respectively, we can find the critical value s^* characterized in the previous corollary by means of the following simple algorithm:

1. For each value of s find the values of x in the interval $[a, b]$ that are candidates for maximizing the integral

$$\int_a^x [F_X(z) - F_Y(z)] e^{-sz} \, dz. \tag{7}$$

In order to do so, pick the values of x satisfying the first-order condition $F_X(x) = F_Y(x)$ and the second-order condition $f_X(x) \le f_Y(x)$. Note that these values are independent of s.

2. Pick the smallest positive real number s satisfying

$$\int_a^x [F_X(z) - F_Y(z)] e^{-sz} \, dz \le 0,$$

for all the values of x obtained in the previous step.

Note that, if X and Y were discrete, the values of x obtained in step 1 correspond to those for which the function $F_X(z) - F_Y(z)$ changes from positive to negative sign.

5 Alternative notions of global risk aversion

There are many alternative ways by which one can define a global measure based on the ARA index. We will next briefly discuss two of them, namely, the supremum of the ARA index and the average of the ARA index over the utility domain. The kind of results that can be obtained with these two measures are much less appealing than those obtained with the measure based on the infimum of the ARA index. The reader is referred to Caballé and Esteban (2003) for further details.

Consider the following partition of the set of increasing and concave utility functions on $[a, b]$ that are twice continuously differentiable on (a, b). A function u belongs to the class $P(s)$ if the supremum of the ARA index over its domain is s,

$$u \in P(s), \quad \text{whenever} \sup_{z \in (a.b)} A_u(z) = s.$$

Concerning the supremum of the ARA index as a measure of global concavity, we can always find a function u with an arbitrarily given value of the supremum of its ARA index, for which the random variables $u(X)$ and $u(Y)$ cannot be ranked according to the SOSD criterion. Moreover, for a sufficiently high value s of the supremum of the ARA index, it is possible to order two given random variables for some utility function belonging to $P(s)$. However, SOSD turns out to be unfeasible for sufficiently small values of the supremum of the ARA index.

Obviously, the ARA index cannot be properly applied to non-differentiable utility functions. Consider then the index of thriftiness that has been proposed as a global measure of concavity for general strictly increasing functions (see Chateauneuf et al. 2005). This index captures the maximal relative drop of the slope of the function u along its domain and is given by

$$T(u) = \sup_{z_1 < z_2 \le z_3 < z_4} \left[\frac{u(z_2) - u(z_1)}{z_2 - z_1} \middle/ \frac{u(z_4) - u(z_3)}{z_4 - z_3} \right].$$

For functions defined on $[a, b]$ that are differentiable, strictly increasing and concave, the index of thriftiness becomes simply $T(u) = u'(a)/u'(b)$. In this case, this index measures how significative is the reduction in the slope of the utility function along its domain. It is plain that the same value of the thriftiness index is compatible with a plethora of local behaviors. For instance, the reduction in the slope can be uniformly distributed over the domain, as occurs with the CARA functions, or it can be concentrated on a very small interval. In the latter case, the utility function could exhibit a local ARA index that is zero at all points of its domain except on an arbitrarily small interval where the ARA index could become arbitrarily large. In fact, if we allow for non-differentiable functions, the drop of the slope can occur at a single point and, of course, all concave transformations of such a function will not be differentiable at that point. Notice that any increasing and strictly concave transformation of a given function u will exhibit an index of thriftiness larger than that of u. It should also be noticed that the index of thriftiness is a measure equivalent to the average value of the ARA index displayed by a twice continuously differentiable utility function u over its domain. Clearly, the average ARA of the function u is

$$\frac{1}{b-a} \int_a^b \frac{-u''(z)}{u'(z)} dz = \frac{1}{b-a} \left[-\ln\left(u'(b)\right) + \ln\left(u'(a)\right) \right]$$

$$= \frac{1}{b-a} \ln\left(\frac{u'(a)}{u'(b)}\right) = \frac{1}{b-a} \ln\left(T(u)\right).$$

Concerning the connection between the index of thriftiness and the SOSD ordering, only very partial results can be obtained since it can be established the existence of a lower bound t^* on the index of thriftiness so that stochastic dominance between two distributions holds for some utility function displaying an index of thriftiness larger than that lower bound. In fact, the threshold value of the index of thriftiness is given by $t^* = 1/\alpha^*$, where α^* is given by the expression (18) in the Appendix.

Finally, note that the three measures of global risk aversion we have considered allow us to induce three new ordering on the set of utility functions and, hence, we can say that u exhibits more risk aversion than v if the infimum (the supremum) of the ARA index (or the index of thriftiness) of u is larger than v. Notice that the traditional notion of more risk averse utility functions establishes that u exhibits more risk aversion than v if the ARA index of u is larger than the ARA index of v at every point of their common domain or, equivalently, if there exists a concave function g such that $u(\cdot) = g(v(\cdot))$. It is then clear that, if $u(\cdot) = g(v(\cdot))$ for some g concave, then u is more risk averse than v according to any of the three orderings that we have just introduced. However, the converse is not true.

6 Concluding remarks

The main result of our paper (Theorem 4.3) provides a sharp characterization of how controversial is the ordering of two distributions on the basis of the extreme most risk averse preferences when the global degree of concavity is measured by the infimum of the ARA index over the support. It establishes that s^* is indeed the lowest degree of global concavity for which we can obtain unanimity in the ranking of the two distributions. Below this threshold, we will always find preferences with the same degree of global concavity yielding transforms of the variables that cannot be ranked according to SOSD.

In contrast, the supremum of the ARA index does not provide such a sharp characterization. We can only find a lower bound on the degree of concavity below which there are no preferences such that all their concave transformation agree in ranking one distribution over the other. Finally, when we use the average ARA index as a global measure of concavity, we cannot find a threshold level of concavity above which we obtain unanimity.

To conclude, we simply wish to establish the bridge between our results and the classical analysis of Hadar and Russell (1969) and Rothschild and Stiglitz (1970). In our paper, we have introduced a partition on the set of twice continuously differentiable utility functions according to the infimum of their ARA index over their common domain. The class corresponding to the value s of the infimum of the ARA index is $I(s)$. Notice that the set of increasing functions exhibiting a non-negative value of the infimum of the ARA index exactly corresponds to the set of increasing twice continuously differentiable concave functions. The aforementioned classical

analysis relating concavity of the Bernoulli utility functions and SOSD provides a limited answer to the question of whether it is possible to rank two risks by simply knowing that the utility function belongs to a particular set. The most celebrated result of that analysis says that, if we restrict to pairs of distributions satisfying the integral condition (2), then $F_{u(X)} \succsim_D F_{u(Y)}$ for all $u \in I(s)$ with $s \geq 0$. This corresponds to part (a) of our Theorem 4.3. In line with part (b) of the same theorem, when (2) is satisfied, it can be shown that there exist increasing functions $u \in I(s)$ such that neither $F_{u(X)} \succsim_D F_{u(Y)}$ nor $F_{u(Y)} \succsim_D F_{u(X)}$ for all $s < 0$. Furthermore, in line with Corollary 4.4, there exist increasing functions $u \in I(s)$ with $s < 0$ such that $F_{u(X)} \succsim_D F_{u(Y)}$. Therefore, we can find non-concave utility functions whose increasing and concave transformations would rank one distribution over the other. Finally, when the two distributions satisfy (2), it is also a known result that neither $F_{u(X)} \succsim_D F_{u(Y)}$ nor $F_{u(Y)} \succsim_D F_{u(X)}$ for all $u \in P(s)$ with $s < 0$, where $P(s)$ is the class of utility functions with a supremum of their ARA index equal to s.

In our analysis we show that, when two distributions cannot be ranked by SOSD, one can nevertheless obtain stochastic dominance, but restricted to a class of increasing and concave functions displaying sufficiently high global risk aversion, namely, the class $I(s)$ with s being greater than an appropriate value s^*. Thus, our results generalize the aforementioned classical results to any given arbitrary pair of distributions satisfying Assumption M. Finally, let us mention that in the theory of poverty measurement Le Breton (1994) and Tungodden (2005) have used a related approach to find the minimal degree of the index of relative risk aversion above which income distributions can be ranked for valuation functions displaying constant risk aversion. Our approach extends the analysis to general concave Bernoulli utilities by just focusing on the infimum of their ARA index.

Appendix

Proof of Proposition 3.1. Note first that the function $k(\cdot; \alpha, z_1)$ is continuous. From Assumption M and Definition 2.5, we know that the integral

$$\int_a^x [F_X(z) - F_Y(z)] \, dz \tag{8}$$

must change its sign at least once on (a, b) and be non-positive on an interval $[a, c]$, with $c \in (a, b)$, before becoming positive for the first time. We can thus define the interval $[a, c]$ with $c \in (a, b)$ satisfying

$$\int_a^x [F_X(z) - F_Y(z)] \, dz \leq 0 \text{ for all } x \in [a, c], \tag{9}$$

and

$$\int_a^x [F_X(z) - F_Y(z)] \, dz > 0 \text{ for all } x \in (c, c + h) \text{ and for some } h > 0. \tag{10}$$

We can also define the real number $z_1 \in (a, b)$ as

$$z_1 = \max \left\{ \underset{x \in [a,c]}{\arg \min} \int_a^x [F_X(z) - F_Y(z)] \, dz \right\}, \tag{11}$$

with $c \in (a, b)$, and the real number z_M as

$$z_M = \max \left\{ \underset{x \in [a,b]}{\arg \max} \int_a^x [F_X(z) - F_Y(z)] \, dz \right\}. \tag{12}$$

Therefore, z_1 is the largest value that minimizes the integral (8) on the interval $[a, c]$. Also notice that the function $F_X - F_Y$ must change sign at z_1. Moreover, z_M is the largest value that maximizes the integral (8) on the interval $[a, b]$.

Since $F_{k(X;\alpha,z_1)}(k) = F_X\left(k^{-1}(k; \alpha, z_1)\right)$ and $F_{k(Y;\alpha,z_1)}(k) = F_Y\left(k^{-1}(k; \alpha, z_1)\right)$, we have that the SOSD condition given in (2) for the transformed random variables,

$$\int_a^y \left[F_{k(X;\alpha,z_1)}(k) - F_{k(Y;\alpha,z_1)}(k) \right] \, dk \leq 0 \quad \text{for all } y \in [k(a; \alpha, z_1), k(b; \alpha, z_1)],$$

will be satisfied if and only if the following two inequalities hold:

$$\int_a^x [F_X(z) - F_Y(z)] \, dz \leq 0 \quad \text{for all } x \in [a, z_1], \tag{13}$$

and

$$\int_a^{z_1} [F_X(z) - F_Y(z)] \, dz + \int_{z_1}^y \left[F_X\left(\tfrac{k-(1-\alpha)z_1}{\alpha}\right) - F_Y\left(\tfrac{k-(1-\alpha)z_1}{\alpha}\right) \right] \, dk \leq 0,$$
$$\text{for all } y \in [z_1, \alpha b + (1-\alpha)z_1] \tag{14}$$

Making the change of variable, $z = [k - (1 - \alpha)z_1]/\alpha$, the integral condition (14) becomes

$$\int_a^{z_1} [F_X(z) - F_Y(z)] \, dz + \alpha \int_{z_1}^x [F_X(z) - F_Y(z)] \, dz \leq 0 \text{ for all } x \in [a, b]. \tag{15}$$

Note that condition (13) always holds, as dictated by the definition of z_1. Moreover, condition (15) holds if and only if

$$V(\alpha, z_1) \equiv \int_a^{z_1} [F_X(z) - F_Y(z)] \, dz + \alpha \int_{z_1}^{z_M} [F_X(z) - F_Y(z)] \, dz \leq 0 \tag{16}$$

since, from the definition of z_M and the fact that $z_1 < z_M$, we deduce that

$$\int_{z_1}^{z_M} [F_X(z) - F_Y(z)] \ dz \geq \int_{z_1}^{x} [F_X(z) - F_Y(z)] \ dz \quad \text{for all } x \in [a, b].$$

The function $V(\alpha, z_1,)$ defined in (16) is strictly increasing in α because

$$\frac{\partial V(\alpha, z_1, z_M)}{\partial \alpha} = \int_{z_1}^{z_M} [F_X(z) - F_Y(z)] \ dz > 0,$$

where the strict inequality comes from the definitions of z_1 and z_M. Moreover,

$$V(0, z_1) = \int_{a}^{z_1} [F_X(z) - F_Y(z)] \ dz < 0.$$

$$V(1, z_1) = \int_{a}^{z_M} [F_X(z) - F_Y(z)] \ dz > 0.$$

Therefore, we can choose the unique value $\alpha^* \in (0, 1)$ for which

$$V(\alpha^*, z_1) = 0. \tag{17}$$

The real number α^* is the largest value of α satisfying $F_{k(X;\alpha,z_1)} \succsim_{D} F_{k(Y;\alpha,z_1)}$, that is,

$$\alpha^* = \max \left\{ \alpha \in \mathbb{R} \text{ such that } \int_{a}^{y} \left[F_{k(X;\alpha,z_1)}(k) - F_{k(Y;\alpha,z_1)}(k) \right] \ dk \leq 0 \right.$$

$$\left. \text{for all } y \in [k(a; \alpha, z_1), k(b; \alpha, z_1)] \right\}.$$

Therefore, according to (16) and (17), α^* would be given by

$$\alpha^* = -\frac{\int_{a}^{z_1} [F_X(z) - F_Y(z)] \ dz}{\int_{z_1}^{z_M} [F_X(z) - F_Y(z)] \ dz}, \tag{18}$$

where z_1 and z_M are given in (11) and (12).

As follows from (16), the inequality $V(\alpha, z_1) < 0$ holds for all $\alpha < \alpha^*$, and this implies that

$$\int_{a}^{y} [F_X(k) - F_Y(k)] \ dk < 0 \quad \text{for all } y \in [a, \alpha b - (1 - \alpha)z_1],$$

which in turn means that $F_{k(X;\alpha,z_1)} \succ_{D} F_{k(Y;\alpha,z_1)}$ for all $\alpha < \alpha^*$.

Finally, for all $\alpha > \alpha^*$ there exists a number $y \in (z_1, \alpha b - (1 - \alpha)z_1)$ such that

$$\int_a^y [F_X(k) - F_Y(k)] \ dk > 0,$$

while

$$\int_a^{z_1} [F_X(k) - F_Y(k)] \ dk < 0.$$

According to Proposition 2.2, the previous two inequalities mean that neither $F_{k(X;\alpha,z_1)} \underset{D}{\succsim} F_{k(Y;\alpha,z_1)}$ nor $F_{k(Y;\alpha,z_1)} \underset{D}{\succsim} F_{k(X;\alpha,z_1)}$ for all $\alpha > \alpha^*$. $\square$

Proof of Corollary 3.2. Note from (16) and (17) that the pairs $\{\hat{\alpha}, \hat{z}\}$ must satisfy

$$V(\hat{\alpha}, \hat{z}) \equiv \int_a^{\hat{z}} [F_X(z) - F_Y(z)] \ dz + \hat{\alpha} \int_{\hat{z}}^{z_M} [F_X(z) - F_Y(z)] \ dz = 0. \quad (19)$$

In order to preserve SOSD for the transformed random variables, $\hat{z}$ must belong to the interval $[a, c]$ satisfying conditions (9) and (10). Furthermore, both the definition of z_1 in (11) and the fact that $\alpha^* \in (0, 1)$ imply that $V(\alpha^*, \hat{z}) \geq V(\alpha^*, z_1) = 0$. Therefore, since the function $V(\alpha, z)$ is strictly decreasing in α, it follows from (19) that $\hat{\alpha} \leq \alpha^*$. $\square$

Proof of Corollary 3.3. We first prove that, if the distribution functions F_X and F_Y satisfy $E(F_X) = E(F_Y)$ and neither $F_X \underset{D}{\succsim} F_Y$ nor $F_Y \underset{D}{\succsim} F_X$. Then, the function $F_X - F_Y$ must change sign on (a, b) at least twice. To this end, Notice that, since neither $F_X \underset{D}{\succsim} F_Y$ nor $F_Y \underset{D}{\succsim} F_X$, it is well known that $F_X - F_Y$ must change sign at least once on (a, b).[6] Let us proceed by contradiction and assume that the right-continuous function $F_X - F_Y$ changes sign only once so that, without loss of generality, assume that $F_X(x) \leq F_Y(x)$ for all $x \in [a, x^*)$, and $F_X(x) > F_Y(x)$ for all $x \in (x^*, b)$. Therefore, letting $H(x) = \int_a^x [F_X(z) - F_Y(z)] dz$, we have that $H(x^*) \leq 0$. Clearly, $H(x)$ is increasing for $x \in [x^*, b]$. Moreover, $H(b) = 0$ since, by integrating by parts,

$$\int_a^b [F_X(z) - F_Y(z)] \ dz = - \int_{[a,b)} z \ dF_X(z) + \int_{[a,b]} z \ dF_Y(z)$$

$$= -E(F_X) + E(F_Y) = 0. \quad (20)$$

---

[6] See Hadar and Russell (1969).

Therefore $H(x) \le 0$ for all $x \in [a, b]$, which means that $F_X \underset{D}{\succsim} F_Y$, and this is the desired contradiction.

To finish the proof of the corollary just notice that, from (12) and the fact that $F_X \underset{D}{\sim} F_Y$ does not hold, we get

$$\int_a^{z_M} [F_X(z) - F_Y(z)] \, dz \ne 0.$$

Moreover, from (20), we have

$$\int_a^b [F_X(z) - F_Y(z)] \, dz = 0.$$

Therefore, $z_M < b$. Finally as z_M is interior, it is clear from (12) that $F_X - F_Y$ must change sign at z_M. □

Proof of Proposition 3.5. Construct a function $q(z; \varepsilon, \beta, z_1)$ having the functional form given in (4) with $\beta \in (\alpha^*, 1)$, where z_1 and α^* are defined in (11) and (18), respectively. The function $q(z; \varepsilon, \beta, z_1)$ can be obviously constructed so that neither $F_{q(X;\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(Y;\varepsilon,\beta,z_1)}$ nor $F_{q(Y;\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(X;\varepsilon,\beta,z_1)}$ for a sufficiently small real number $\varepsilon > 0$, by following the same steps of the proof of Proposition 3.4. □

Proof of Proposition 4.1. Consider first the class of continuously differentiable, increasing and concave utility functions with the following functional form:

$$v(z; s, z_0) = \begin{cases} -\dfrac{1}{s} e^{-s(z-z_0)} + z_0 + \dfrac{1}{s} & \text{for } x \in [a, z_0) \\ \\ z & \text{for } x \in [z_0, b], \end{cases} \tag{21}$$

where $s > 0$ and z_0 is the smallest value on $[a, b]$ at which the function $F_X(z) - F_Y(z)$ changes sign (see Definition 2.4). Let us find a value of s for which $F_{v(X;s,z_0)} \underset{D}{\succsim} F_{v(Y;s,z_0)}$, i.e.

$$\int_{v(a;s)}^y \left[F_{v(X;s,z_0)}(v) - F_{v(Y;s,z_0)}(v) \right] dv \le 0 \quad \text{for all } y \in [v(a; s, z_0), b].$$

By performing the corresponding change of variable, the previous inequality becomes:

$$\int_a^x [F_X(z) - F_Y(z)] \, v'(z; s, z_0) dz \le 0 \quad \text{for all } x \in [a, b],$$

which in turn can be decomposed into the following two inequalities:

$$\int_{a}^{x} [F_X(z) - F_Y(z)] \, v'(z; s, z_0) \, dz \le 0 \quad \text{for all } x \in [a, z_0], \tag{22}$$

$$\int_{a}^{z_0} [F_X(z) - F_Y(z)] \, v'(z; s, z_0) dz + \int_{z_0}^{x} [F_X(z) - F_Y(z)] \, dz \le 0$$
$$\text{for all } x \in [z_0, b]. \tag{23}$$

Note that inequality (22) always holds since the integrand is non-positive by the definition of z_0. Taking into account the definition of z_M in (12), we know that

$$\int_{z_M}^{x} [F_X(z) - F_Y(z)] \, dz \le 0, \quad \text{for all } x \in [z_M, b]$$

Define the real number z_N as

$$z_N = \max \left\{ \underset{x \in [a,b]}{\arg \min} \int_{a}^{x} [F_X(z) - F_Y(z)] \, dz \right\}. \tag{24}$$

Hence, we get

$$\int_{z_N}^{z_M} [F_X(z) - F_Y(z)] \, dz \ge \int_{z_0}^{x} [F_X(z) - F_Y(z)] \, dz \quad \text{for all } x \in [z_0, b].$$

Therefore, (23) holds whenever

$$\int_{a}^{z_0} [F_X(z) - F_Y(z)] \, v'(z; s) dz + \int_{z_N}^{z_M} [F_X(z) - F_Y(z)] \, dz = 0. \tag{25}$$

Let $K = \int_{z_N}^{z_M} [F_X(z) - F_Y(z)] \, dz$ and

$$H(x) = \int_{a}^{x} [F_Y(z) - F_X(z)] \, dz \quad \text{for } x \in [a, z_0].$$

The mapping $H(x)$ is an increasing and right-continuous function which induces a Lebesgue–Stieltjes measure on $[a, z_0]$. Therefore, (25) can be written as

$$\int_{a}^{z_0} v'(z; s, z_0) dH(z) = K. \tag{26}$$

Moreover, by the definition of z_M,

$$
-\int_a^{z_0} dH(z) + K = \int_a^{z_0} [F_X(z) - F_Y(z)] \, dz + \int_{z_N}^{z_M} [F_X(z) - F_Y(z)] \, dz
$$

$$
\geq \int_a^{z_0} [F_X(z) - F_Y(z)] \, dz + \int_{z_0}^{z_M} [F_X(z) - F_Y(z)] \, dz
$$

$$
= \int_a^{z_M} [F_X(z) - F_Y(z)] \, dz > 0.
$$

Therefore, letting $C = \int_a^{z_0} dH(z)$, we can conclude that $K > C > 0$. Moreover, by noticing that $v'(z; s) = e^{-s(z-z_0)}$, Eq. (26) becomes

$$
\int_a^{z_0} e^{-s(z-z_0)} \, dH^*(z) = \frac{K}{C}, \tag{27}
$$

where $H^*(z) = H(z)/C$ is a distribution function on $[a, z_0]$ because $H^*(z_0) = 1$. Eq. (27) has a unique solution for s since $K/C > 1$, the LHS of (27) is strictly increasing in s, $\lim_{\to 0} \int_a^{z_0} e^{-s(z-z_0)} \, dH^*(z) = 1$ and $\lim_{s \to \infty} \int_a^{z_0} e^{-s(z-z_0)} \, dH^*(z) = \infty$. Let $\hat{s}$ be the unique solution of equation (27) . Clearly, the inequality in (23) becomes strict whenever $s > \hat{s}$.

Consider now the increasing and concave function

$$
w(z; \hat{s}, z_0) = \begin{cases} z & \text{for } z \in [a, z_0) \\ -\dfrac{1}{\hat{s}} e^{-\hat{s}(z-z_0)} + z_0 + \dfrac{1}{\hat{s}} & \text{for } z \in [z_0, b] . \end{cases}
$$

The increasing and concave transformation of $v(z; \hat{s}, z_0)$ given by $w(v(z; \hat{s}, z_0); \hat{s}, z_0)$ exhibits a constant ARA index since

$$
w(v(z; \hat{s}, z_0); \hat{s}, z_0) = -\frac{1}{\hat{s}} e^{-\hat{s}(z-z_0)} + z_0 + \frac{1}{\hat{s}} \quad \text{for } z \in [a, b].
$$

Obviously, $F_{w(v(X;s,z_0);s,z_0)} \underset{D}{\succsim} F_{w(v(Y;s,z_0);s,z_0)}$ for all $s > \hat{s}$. Hence, $F_{r(X;s)} \underset{D}{\succsim} F_{r(Y;s)}$ for all $s > \hat{s}$, where $r(\cdot; s)$ is a CARA utility function with an ARA index equal to s. Since, for s sufficiently close to zero, neither $F_{r(X;s)} \underset{D}{\succsim} F_{r(Y;s)}$ nor $F_{r(Y;s)} \underset{D}{\succsim} F_{r(X;s)}$, we can find by continuity the value $s^* \in (0, \hat{s})$ for which $F_{r(X;s^*)} \underset{D}{\succsim} F_{r(Y;s^*)}$ and $F_{r(X;s)} \underset{D}{\succsim} F_{r(Y;s)}$ for all $s > s^*$, whereas neither $F_{r(X;s)} \underset{D}{\succsim} F_{r(Y;s)}$ nor $F_{r(Y;s)} \underset{D}{\succsim} F_{r(X;s)}$ for all $s < s^*$. $\qquad \square$

Proof of Corollary 4.2. Obvious from the proof of Corollary 2.3 and from Pratt (1964) since u is an increasing and (strictly) concave transformation of the utility function $v(\cdot; \hat{s}, z_0)$, whose functional form is given in the expression (21) and that satisfies $F_{v(X;s^*,z_0)} \underset{D}{\succsim} F_{v(Y;s^*,z_0)}$. $\qquad \square$

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