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Complete monotonicity, background risk, and risk aversion

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Abstract

This paper analyzes how the statistical properties of a risk affect the attitude of individuals towards accepting another independent risk. We conduct the analysis for the class of increasing utility functions having all their derivatives with alternating sign. Such utilities can be expressed as mixtures of negative exponential functions and they are fully described by distribution functions over the set of exponents. Our analysis exploits the relationship between the distribution functions characterizing utilities and the distribution functions characterizing risks. In particular, we find sufficient conditions for an additional background risk to either reduce or increase the index of absolute risk aversion. © 1997 Elsevier Science B.V.

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1. Introduction

In this paper we analyze how the introduction of an additional source of uncertainty affects the risk bearing attitude of individuals. We restrict our analysis to state-independent preferences over the space of random variables that are representable by increasing Bernouilli functions having all derivatives of alternating sign. Such utility functions are called mixed and they have been exhaustively characterized in Caballé and Pomansky (1996). The derivative of a mixed utility is a Laplace transform and, thus, it is fully characterized by a distribution function over the set of negative exponential functions. Another important property of this class of utilities is that it satisfies natural

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restrictions imposed on preferences defined on the space of random variables. In particular, mixed utilities display decreasing absolute risk aversion and decreasing prudence. The latter property means that, under additive time separability, the strength of precautionary saving when future income is uncertain is decreasing in wealth (see Kimball, 1990). On the other hand, the class of mixed utilities is very flexible and includes as particular cases all the DARA utility functions commonly found in the economics of uncertainty literature, like the HARA, the CARA, the logarithmic and the isoelastic utilities.

Our analysis exploits the relationship between the true distribution function of the risk being added to the initial background wealth and the artificial distribution function over exponents defining a mixed utility. This technique allows us to find sufficient conditions under which the introduction of a risk either increases or decreases the risk aversion attitude of individuals and, thus, their willingness to accept another independent risk. Moreover, by using such an approach, it is straightforward to construct examples of pairs of risk additions and utilities such that these risk additions are desirable for all levels of wealth while they uniformly increase the risk aversion of individuals. In other words, our analysis allows the construction of simple robust examples of individuals who become happier after the introduction of the additional risk while they become simultaneously more risk averse. Note that these examples are obtained without paying the cost of assuming weird preferences which violate reasonable restrictions.

The effects of changes in background wealth on risk aversion is a relevant issue since it may help to explain the equity premium puzzle posed by Mehra, Prescott (1985). According to these authors, the empirical difference between average returns to stocks and average returns to Treasury bills implies that investors are implausibly averse to risk. However, it can be argued that the interaction among different sources of uncertainty may dramatically increase the risk aversion of individuals. If some of the risks are disregarded, then the theoretical equilibrium risk premium could be underestimated. This may occur for instance in asset pricing models that disregard the risk associated with labor income (see Weil, 1992 and Constantinides, Duffie, 1995). Thus, our analysis finds conditions for additional sources of uncertainty to increase (or decrease) the equilibrium risk premium.

Our work is obviously related to the work of Kihlstrom et al. (1981) who study whether the conditions for more risk aversion are preserved under random background wealth. Another related paper is that of Eeckhoudt et al. (1996). These authors find necessary and sufficient conditions for a deterioration in background wealth to induce more risk aversion. Their analysis is restricted to shifts in the distribution of background wealth which can be ranked according to the ordering induced by a stochastic dominance relationship. Our analysis considers instead all possible additions of risk and identifies under which assumptions on the mixed utility such additions of risk either reduce or increase the value of the index of absolute risk aversion.

The paper is organized as follows. Section 2 introduces the class of mixed utility functions. Section 3 characterizes the possible changes in background wealth. The effects of such changes on the risk aversion attitude is analyzed in Section 4. The relationship between desirability of a risk and its effects on risk aversion is discussed in Section 5. Section 6 deals with the case in which the initial wealth is already random. Section 7 concludes the paper.

2. Assumptions on utility functions

Assume that agents have state-independent preferences defined over the space of random variables taking values on the interval $[a, \infty)$ with $a > -\infty$, and that such preferences are represented by a Bernouilli utility u. Thus, the random variable (or risk) $\tilde{x}_1$ is preferred to the random variable $\tilde{x}_2$ if and only if $E(u(\tilde{x}_1)) \geq E(u(\tilde{x}_2))$, where u is a real-valued Borel measurable function. Notice that, because of the state-independence assumption, the preference over the space of random variables induces a preference relationship over the space of distributions of random variables.

The family of Bernouilli utilities we will consider in this paper is the class of increasing utility functions having all derivatives of alternating sign. Such functions were initially considered by Pratt (1964) and, more recently, they were analyzed by Pratt, Zeckhauser (1987) and Caballé and Pomansky (1996). To introduce this class of utilities we need the following definitions and preliminary results concerning completely monotone functions (see section XIII.4 of Feller, 1971):

Definition 1. A real-valued function $\phi(w)$ defined on the interval (a, ∞) is completely monotone if it is nonnegative and has negative odd derivatives and positive even derivatives, that is,

$$(-1)^n \phi^n(w) \geq 0, \quad \text{for all } w \in (a, \infty) \quad \text{and} \quad n = 0, 1, 2, \ldots,$$

where $\phi^n(w)$ denotes the n-th order derivative of $\phi(w)$.

Definition 2. A distribution function $F(s)$ defined on $[c, \infty)$ is a real-valued, non-decreasing and right continuous map from $[c, \infty)$ into $[0, \infty)$.

Note that we do not exclude neither the possibility of $F(c)>0$ nor that of $\lim_{s\to\infty} F(s)=\infty$. We say that a distribution function is non-trivial if $F(s)>0$ for some $s\in[c, \infty)$. A distribution function F is constant if $F(s)=F(s')$ for all $s, s'\in[c, \infty)$. Obviously, if a distribution function is not constant then it is non-trivial.

There is a one-to-one correspondence between a distribution function on $[c, \infty)$ and a Lebesgue–Stieltjes measure (or distribution) on such an interval (see section 1.4 of Ash, 1972).[1] The distribution μ_F associated with the distribution function F on $[c, \infty)$ satisfies $\mu_F(a, b]=F(b)-F(a)$, $\mu_F\{c\}=F(c)$, and $\mu_F\{x\}=0$ if and only if x is a point of continuity of F on the open interval (c, ∞).

Definition 3. Consider the interval $[c, \infty]$ of extended real numbers. The essential infimum $\underline{s}$ and the essential supremum $\bar{s}$ of the distribution μ_F associated with the non-trivial distribution function $F(s)$ on $[c, \infty)$ are defined as follows:

$$\underline{s} = \inf_{[c,\infty]} \{s \in [c, \infty) | F(s) > 0\} .$$

[1] A Lebesgue–Stieltjes measure (or distribution) on $[c, \infty)$ is a measure on the Borel sets of $[c, \infty)$ such that it assigns finite values to bounded intervals.

$$\bar{s} = \begin{cases} c \text{ if } F \text{ is constant,} \\ \sup\limits_{[c,\infty]} \{s \in [c, \infty) | F(s+z) > F(s) \quad \text{for some } z > 0\} \text{ if } F \text{ is not constant .} \end{cases}$$

Let F be a non-trivial distribution. Since the support of the distribution μ_F denoted supp(μ_F) is the complement in $[c, \infty)$ of the largest open Borel subset of $[c, \infty)$ having μ_F-measure zero, it follows that

$$\underline{s} = \inf_{[c,\infty]} \{\text{supp}(\mu_F)\} \quad \text{and} \quad \bar{s} = \sup_{[c,\infty]} \{\text{supp}(\mu_F)\} .$$

On the other hand, if F is trivial, i.e., $F(s)=0$ for all $s \in [c, \infty)$, then supp$(\mu_F)=\emptyset$.

Definition 4. A real-valued function $\phi(w)$ defined on the open interval $(0, \infty)$ is a Laplace transform of a distribution function $F(s)$ on $[c, \infty)$ if it can be written as the following improper Stieltjes integral:

$$\phi(w) = \int_c^\infty e^{-sw} \, dF(s), \quad \text{for all } w \in (0, \infty) .$$

The following classical theorem, whose proof can be found for instance in section IV.12 of Widder (1941), provides an explicit representation of a completely monotone function:

Theorem 1. (*Bernstein's Theorem*). *The function, $\phi(w)$ defined on $(0, \infty)$ is completely monotone if and only if it is the Laplace transform of a distribution function $F(s)$ on $[0, \infty)$.*

The next definition introduces the class of utility functions which we are going to use throughout this paper:

Definition 5. A real-valued utility function $u(w)$ defined on the interval $[a, \infty)$ is mixed if it is continuous on $[a, \infty)$ and has a completely monotone first-order derivative on (a, ∞).

Note that by virtue of Theorem 1 the first-order derivative of a mixed utility on $[0, \infty)$ is a Laplace transform which is fully characterized by its associated distribution function $F(s)$. Conversely, a distribution function $F(s)$ is fully characterized by its associated Laplace transform $\phi(w)$. In fact, at every point of continuity of F, the following inversion formula applies:

$$F(s) = \lim_{a\to\infty} \sum_{n \le as} \frac{(-a)^n}{n!} \phi^n(w) .$$

Making the change of variable $x=w-a$, we can define the function $\hat{u}(x) \equiv u(x+a)$. Clearly $\hat{u}$ is mixed on $[0, \infty)$ if and only if u is mixed on $[a, \infty)$. Assuming that $\int_c^\infty 1/s \, dF(s)$ is finite for some $c>0$ (and thus for all $c>0$), we can integrate a completely monotone function $\phi(x)$ on $(0, \infty)$ to obtain a mixed function $\hat{u}$,

$$\hat{u}(x) = \int_0^x \phi(z)\,\mathrm{d}z + \hat{u}(0) = \int_0^x \left(\int_0^\infty \mathrm{e}^{-sz}\,\mathrm{d}F(s)\right)\mathrm{d}z + \hat{u}(0)$$
$$= \int_0^\infty \frac{1-\mathrm{e}^{sx}}{s}\,\mathrm{d}F(s) + \hat{u}(0)\,,$$

where the second equality comes from Theorem 1 and the third from exchanging the order of integration and performing the inner Riemann integral.[2] Therefore, taking into account the change of variable, we obtain

$$u(w) = \int_0^\infty \frac{1-\mathrm{e}^{s(w-a)}}{s}\,\mathrm{d}F(s) + u(a)\,.$$

Note also that the affine transformation of u,

$$u(w) = \int_0^\infty \frac{1-\mathrm{e}^{-s(w-a)}}{s}\,\mathrm{d}F(s)\,, \tag{1}$$

represents the same state-independent preferences over the space of random variables taking values on $[a, \infty)$. Such a transformation amounts to the normalization of $u(a)=\hat{u}(0)=0$. Therefore, we can conclude that a utility function is mixed on $[a, \infty)$ if and only if it has the representation given in (1) with $\int_c^\infty 1/s\,\mathrm{d}F(s)<\infty$ for some $c>0$. Equivalently, $u(w)$ is mixed on $[a, \infty)$ if and only if its first derivative can be written as

$$u'(w) = \int_0^\infty \mathrm{e}^{-s(w-a)}\,\mathrm{d}F(s)\,, \tag{2}$$

for some distribution function $F(s)$ on $[0, \infty)$. The set of mixed utilities is denoted by $\mathcal{M}$. We will also say that the utilities belonging to $\mathcal{M}$ display mixed risk aversion.

The importance of the class of mixed utilities relies on the fact that it includes the Bernouilli functions most commonly used in finance and other areas of applied economics involving uncertainty. For instance, if the distribution function F over exponents is gamma, then u becomes a HARA utility with decreasing risk aversion (see Feller, 1971, p. 430). The family of HARA utilities was considered by Cass, Stiglitz (1970) in order to analyze conditions under which two-fund monetary separation holds. Moreover, the set of HARA utilities is the maximal set which is closed under stochastic dynamic programming. That is to say, a stochastic dynamic programming problem exhibits a HARA value function if and only if the objective function is also HARA (see Hakansson, 1970).

Furthermore, we obtain the negative exponential or CARA utility when F is associated with a Dirac distribution, the power or isoelastic utility when F is also a power function, and the logarithmic utility when F is exponential. Note that the Dirac, the exponential and the power distributions are all limiting cases of the gamma distribution, which agrees with the fact that the CARA, the isoelastic and the logarithmic utilities are in turn all limiting cases of the HARA utility.

[2] The assumption that $\int_c^\infty 1/s\ \mathrm{d}F(s)$ is finite for some $c>0$ is necessary and sufficient for the convergence of the integral $\int_0^\infty (1-\mathrm{e}^{-sx}/s)\ \mathrm{d}F(s)$.

Another important feature of the class $\mathcal{M}$ is that it has an explicit functional representation that satisfies natural restrictions imposed on Bernouilli utilities. One of the strongest non-parametric restrictions is the one of standard risk aversion (Kimball, 1993). A utility displays standard risk aversion when an undesirable risk can never be made desirable by the presence of another independent risk which increases the expected marginal utility. As shown in Kimball (1990), standardness means that the precautionary saving motive is decreasing in wealth in a two-period context when second-period endowment is uncertain and there is additive time separability on preferences. It can also be shown that a utility $u(w)$ defined on $[a, \infty)$ is standard if and only if its index of prudence $-u'''(w)/u''(w)$ is nonincreasing over the interval (a, ∞). On the other hand, u is mixed if and only if $-u^{n+1}(w)/u^{n}(w)$ is nonincreasing for all $n = 1, 2, \ldots$, and, hence, mixed utilities are standard (see Proposition 3.2 in Caballé and Pomansky, 1996).

Since under decreasing risk aversion an undesirable risk increases the expected marginal utility, standard utilities constitute a strict subset of proper utilities, which are the ones for which an undesirable risk is never made desirable by the presence of another independent, undesirable risk (see Pratt, Zeckhauser, 1987). Moreover, since subfair risks (i.e., risks with nonpositive mean) are undesirable under risk aversion, properness implies risk vulnerability. The latter property means that an undesirable risk is never made desirable by the presence of another independent, subfair risk (see Gollier, Pratt, 1996). Finally, risk vulnerability implies decreasing absolute risk aversion, since a degenerate random variable taking only a nonpositive value is subfair under risk aversion.

Summing up, mixed utilities have appealing properties and possess a representation which has enough flexibility to handle several functional forms. These particular functional forms are obtained by controlling the distribution function $F(s)$ over exponents. In this line of research, Caballé and Pomansky (1996) analyze the relationship between the indexes of risk aversion of u and the properties of its associated distribution function F.

3. Changes in background wealth

Let us now consider changes in the nonrandom background wealth w of individuals. Those changes take the form of adding some random variable $\tilde{\varepsilon}$ to w. The distribution function of the random variable $\tilde{\varepsilon}$ is $G_{\tilde{\varepsilon}}(z) \equiv \mathrm{Prob}(\tilde{\varepsilon} \leq z)$. Therefore, the distribution function of a random variable taking values on the interval $[c, \infty)$ is a distribution function defined on $[c, \infty)$ such that $\lim_{z \to \infty} G_{\tilde{\varepsilon}}(z) = 1$ (see Definition 2). Moreover, the distribution function of a random variable $\tilde{\varepsilon}$ taking values on $[c, \infty)$ defines a Lebesgue–Stieltjes measure on $[c, \infty)$ (see section 5.6 of Ash, 1972). Such a Lebesgue–Stieltjes measure will be called the distribution of the random variable $\tilde{\varepsilon}$.

We will assume that the random variable $w + \tilde{\varepsilon}$ takes values on the domain of u, $[a, \infty)$. Let $\underline{s}_{\tilde{\varepsilon}} > -\infty$ be the essential infimum of the distribution of the random variable $\tilde{\varepsilon}$ (see Definition 3), and define the function $v(w) \equiv E(u(w + \tilde{\varepsilon}))$ for all $w \geq a - \underline{s}_{\tilde{\varepsilon}}$. We are interested in finding conditions on both u and $\tilde{\varepsilon}$ such that, for all $w > a - \underline{s}_{\tilde{\varepsilon}}$, the function

$v(w)$ displays more absolute risk aversion than the function $u(w)$. That is, we want to know when

$$\frac{-E(u''(w+\tilde{\varepsilon}))}{E(u'(w+\tilde{\varepsilon}))} \geq \frac{-u''(w)}{u'(w)}, \quad \text{for all } w > a - \underline{s}_{\tilde{\varepsilon}}. \tag{3}$$

Expression (3) is equivalent to saying that the presence of the risk $\tilde{\varepsilon}$ makes the agents more risk averse, and therefore their willingness to accept another independent risk $\tilde{x}$ will decrease. It should be pointed that the assumption of statistical independence is crucial in this context since, under risk aversion, if $\tilde{\varepsilon}$ and $\tilde{x}$ were negatively correlated, the introduction of the risk $\tilde{\varepsilon}$ could in fact reduce the risk borne by the individual. Let $\pi(\tilde{x}, \tilde{y})$ be the risk premium required for the risk $\tilde{x}$ when the (independent) background wealth is $\tilde{y}$. On the other hand, let $c(\tilde{x}, \tilde{y})$ be the certainty equivalent of risk $\tilde{x}$ when the (independent) background wealth is $\tilde{y}$. Therefore,

$$E(u(\tilde{y}+\tilde{x})) = E(u(\tilde{y}+E(\tilde{x}) - \pi(\tilde{x}, \tilde{y}))) = E(u(\tilde{y}+c(\tilde{x}, \tilde{y}))).$$

Let $\underline{s}_{\tilde{x}} > -\infty$ be the essential infimum of the distribution of $\tilde{x}$. Then, according to Pratt (1964), condition (3) is equivalent to the condition $\pi(\tilde{x}, w+\tilde{\varepsilon}) \geq \pi(\tilde{x}, w)$ for all $w > a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{x}}$, which is in turn equivalent to the condition $c(\tilde{x}, w+\tilde{\varepsilon}) \leq c(\tilde{x}, w)$ for all $w > a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{x}}$. It is also well known that the quantity invested in a risky asset having a random rate of return independent of $\tilde{\varepsilon}$ will be lower when background wealth is $w+\tilde{\varepsilon}$ than when it is w if and only if (3) holds. Moreover, whenever (3) holds, the amount of insurance that individuals purchase against a risk will increase when the independent risk $\tilde{\varepsilon}$ is added to the initial wealth w.

From now on we will restrict our analysis to significative changes in background wealth, i.e., we will assume that the probability of $\tilde{\varepsilon}$ taking values different from zero is strictly positive. Let $\psi_{\tilde{\varepsilon}}(s)$ denote the Laplace transform of the distribution function $G_{\tilde{\varepsilon}}(z)$ of the significative random variable $\tilde{\varepsilon}$, that is, $\psi_{\tilde{\varepsilon}}(s) = \int_{\underline{s}_{\tilde{\varepsilon}}}^{\infty} \mathrm{e}^{-sz}\, \mathrm{d}G_{\tilde{\varepsilon}}(z)$. For the sake of brevity, we will also call $\psi_{\tilde{\varepsilon}}(s)$ the Laplace transform of $\tilde{\varepsilon}$. The following lemma, whose proof is straightforward, gives the properties of a Laplace transform of a significative random variable.

Lemma 1. *Let $\tilde{\varepsilon}$ be a significative random variable, i.e.,* $\mathrm{Prob}(\tilde{\varepsilon}=0)<1$, *then*
(*a*) $\psi_{\tilde{\varepsilon}}(0)=1$.
(*b*) $\psi'_{\tilde{\varepsilon}}(0)=-E(\tilde{\varepsilon})$.
(*c*) $\psi''_{\tilde{\varepsilon}}(s)>0$ *for all* $s\in(0, \infty)$.

Attending exclusively to their statistical properties, we divide the set of significative risks into the following three disjoint categories:

Definition 6. Consider the set of significative risks, then
(a) A risk $\tilde{\varepsilon}$ is subfair if $E(\tilde{\varepsilon})\leq 0$.
(b) A risk $\tilde{\varepsilon}$ is potentially undesirable if $E(\tilde{\varepsilon})>0$ and $\mathrm{Prob}(\tilde{\varepsilon}<0)>0$.
(c) A risk $\tilde{\varepsilon}$ is positive if $\mathrm{Prob}(\tilde{\varepsilon}<0)=0$.

The next lemma provides further obvious properties of the Laplace transform of a significative random variable depending on the category which it belongs to:

Lemma 2. *Let* $\tilde{\varepsilon}$ *be a significative risk.*

(*a*) *If* $\tilde{\varepsilon}$ *is subfair then* $\psi_{\tilde{\varepsilon}}(s)>1$ *for all* $s\in(0,\infty)$.

(*b*) *If* $\tilde{\varepsilon}$ *is potentially undesirable then there exists a real number* $s^*\in(0,\infty)$ *such that* $\psi_{\tilde{\varepsilon}}(s^*)=1$, $\psi_{\tilde{\varepsilon}}(s)<1$ *for all* $s\in(0,s^*)$, *and* $\psi_{\tilde{\varepsilon}}(s)>1$ *for all* $s\in(s^*,\infty)$.

(*c*) *If* $\tilde{\varepsilon}$ *is positive then* $\psi_{\tilde{\varepsilon}}(s)<1$ *for all* $s\in(0,\infty)$.

Proof. Parts (a) and (c) follow immediately from Lemma 1. Part (b) follows also from Lemma 1 and the fact that $\mathrm{Prob}(\tilde{\varepsilon}<0)>0$ implies that $\lim_{s\to\infty}\psi_{\tilde{\varepsilon}}(s)=\infty$. Q.E.D.

Note that s^* can be interpreted as the value of the index of absolute risk aversion which makes an individual having a negative exponential (or CARA) utility indifferent between accepting or not a potentially undesirable risk, i.e.,

$$E(-\mathrm{e}^{-s^*(w-a+\tilde{\varepsilon})})=E(-\mathrm{e}^{-s^*(w-a)}),$$

since $E(-e^{-s^*(w-a+\tilde{\varepsilon})})=\psi_{\tilde{\varepsilon}}(s^*)E(-e^{-s^*(w-a)})$ and $\psi_{\tilde{\varepsilon}}(s^*)=1$.

Lemma 3. *If the utility function* $u(w)$ *is mixed on* $[a,\infty)$ *then the function* $v(w)\equiv E(u(w+\tilde{\varepsilon}))$ *is mixed on* $[a-\underline{s}_{\tilde{\varepsilon}},\infty)$.

Proof. Observe that

$$\begin{aligned} v'(w)&=E(u'(w+\tilde{\varepsilon}))=E\left(\int_0^\infty \mathrm{e}^{-s(w-a+\tilde{\varepsilon})}\,\mathrm{d}F(s)\right)\\ &=\int_{\underline{s}_{\tilde{\varepsilon}}}^\infty\left(\int_0^\infty \mathrm{e}^{-s(w-a+z)}\,\mathrm{d}F(s)\right)\mathrm{d}G_{\tilde{\varepsilon}}(z)=\int_0^\infty \mathrm{e}^{-s(w-a)}\left(\int_{\underline{s}_{\tilde{\varepsilon}}}^\infty \mathrm{e}^{-sz}\,\mathrm{d}G_{\tilde{\varepsilon}}(z)\right)\mathrm{d}F(s)\\ &=\int_0^\infty \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\left(\int_{\underline{s}_{\tilde{\varepsilon}}}^\infty \mathrm{e}^{-s(z-\underline{s}_{\tilde{\varepsilon}})}\,\mathrm{d}G_{\tilde{\varepsilon}}(z)\right)\mathrm{d}F(s)\,, \end{aligned} \tag{4}$$

where the second equality comes from (2). Thus, making the change of variable $y=z-\underline{s}_{\tilde{\varepsilon}}$, we get

$$\int_{\underline{s}_{\tilde{\varepsilon}}}^\infty \mathrm{e}^{-s(z-\underline{s}_{\tilde{\varepsilon}})}\,\mathrm{d}G_{\tilde{\varepsilon}}(z)=\int_0^\infty \mathrm{e}^{-sy}\,\mathrm{d}G_{\tilde{\varepsilon}}(y+\underline{s}_{\tilde{\varepsilon}})=\int_0^\infty \mathrm{e}^{-sy}\,\mathrm{d}G_{\tilde{\varepsilon}-\underline{s}_{\tilde{\varepsilon}}}(y)=\psi_{\tilde{\varepsilon}-\underline{s}_{\tilde{\varepsilon}}}(s)\,.$$

Therefore, (4) becomes

$$v'(w)=\int_0^\infty \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\psi_{\tilde{\varepsilon}-\underline{s}_{\tilde{\varepsilon}}}(s)\,\mathrm{d}F(s)=\int_0^\infty \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\,\mathrm{d}H(s)\,, \tag{5}$$

where H is defined by the following Stieltjes integral:

$$H(s)=\int_0^s \psi_{\tilde{\varepsilon}-\underline{s}_{\tilde{\varepsilon}}}(z)\,\mathrm{d}F(z)\,.$$

The function H is a distribution function on $[0, \infty)$ since $\psi_{\tilde{\varepsilon}-\underline{s}_{\tilde{\varepsilon}}}(s)$ is positive for all $s \geq 0$. Therefore, the utility function v is mixed on $[a-\underline{s}_{\tilde{\varepsilon}}, \infty)$ as follows from (2) and (5). Q.E.D.

A more elegant but less constructive proof of Lemma 3 can be obtained by noticing that the distribution function $G_{\tilde{\varepsilon}}(z)$ of the random variable $\tilde{\varepsilon}$ can be approximated by a sequence of step distribution functions $\{G_{\tilde{\varepsilon}_n}(z)\}$ associated with the discrete random variables $\{\tilde{\varepsilon}_n\}$. Note that $E(u'(w+\tilde{\varepsilon}_n))$ is just a weighted sum of completely monotone functions with positive coefficients and, thus, it is completely monotone. Moreover, $\lim_{n\to\infty} E(u'(w+\tilde{\varepsilon}_n)) = E(u'(w+\tilde{\varepsilon}))$, and $E(u'(w+\tilde{\varepsilon}))$ is also completely monotone as a consequence of Theorem 2 in section XIII.1 of Feller (1971).

4. Comparing risk aversion attitudes

Since we are interested in knowing whether $v(w)$ displays more absolute risk aversion than $u(w)$ for all w in the relevant domain, we define the corresponding indexes of absolute risk aversion,

$$A_v(w) = \frac{-v''(w)}{v'(w)} = \frac{-E(u''(w+\tilde{\varepsilon}))}{E(u'(w+\tilde{\varepsilon}))} \quad \text{and} \quad A_u(w) = \frac{-u''(w)}{u'(w)}.$$

Let us consider the set of extended nonnegative real numbers $[0, \infty]$, and define $s_{\inf}$ as the extended nonnegative real number at which the Laplace transform $\psi_{\tilde{\varepsilon}}(s)$ of risk $\tilde{\varepsilon}$ reaches its unique infimum. It follows from Lemmas 1 and 2 that (a) if a risk $\tilde{\varepsilon}$ is subfair, then $s_{\inf}=0$, (b) if $\tilde{\varepsilon}$ is potentially undesirable, then $s_{\inf}\in(0, s^*)$, and (c) if $\tilde{\varepsilon}$ is positive, then $s_{\inf}=\infty$. On the other hand, let $\bar{s}$ and $\underline{s}$ be the essential supremum and the essential infimum, respectively, of the distribution μ_F on exponents associated with the distribution function F. Such a distribution function F characterizes in turn the mixed utility u. Our next result establishes sufficient conditions, relating the utility u and the change in background wealth $\tilde{\varepsilon}$, for which $v(w)\equiv E(u(w+\tilde{\varepsilon}))$ displays less or more risk aversion than $u(w)$ for all background wealth $w \in (a-\underline{s}_{\tilde{\varepsilon}},\infty)$, where $\underline{s}_{\tilde{\varepsilon}} > -\infty$ is the essential infimum of the distribution of the random variable $\tilde{\varepsilon}$ having the distribution function $G_{\tilde{\varepsilon}}(z)$.

Proposition 1. *Let u be a mixed utility defined on $[a, \infty)$. Assume that $A_u(w)$ is strictly decreasing and that $\tilde{\varepsilon}$ is a significative risk. Then*
(*a*) $A_v(w)<A_u(w)$, *for all* $w>a-\underline{s}_{\tilde{\varepsilon}}$ *whenever* $s_{\inf}\geq\bar{s}$.
(*b*) $A_v(w)>A_u(w)$, *for all* $w>a-\underline{s}_{\tilde{\varepsilon}}$ *whenever* $s_{\inf}\leq\underline{s}$.

In order to prove this proposition we need to state the following nice theorem whose proof can be found for instance in section V.1.6 of Pólya and Szegö (1976).

Theorem 2. (*Laguerre's Theorem*). *Let C denote the number of changes of sign of the* *function* $\phi(s)$ *in the interval* $(0, \infty)$, *and let Z denote the number of real zeros (counted according to their multiplicity) of the integral*

$$\Omega(x) = \int_0^\infty \phi(s)\, e^{-sx}\, ds\ . \tag{6}$$

Then $Z \le C$.

It is straightforward to see that the same result of Laguerre's theorem holds if we replace the Riemann integral in (6) with the Stieltjes integral of $\phi(s)$ with respect to a monotonic function on $[0, \infty)$. In particular, Laguerre's theorem holds when instead of (6) we have

$$\Omega(x) = \int_0^\infty \phi(s)\, e^{-sx}\, dF(s)\ ,$$

where $F(s)$ is a distribution function on $[0, \infty)$ and C would be now the number of changes of sign of $\phi(s)$ in the open interval $(\underline{s}, \bar{s})$ with $\underline{s}$ and $\bar{s}$ being the essential infimum and the essential supremum, respectively, of the distribution μ_F associated with the distribution function F. The open interval $(\underline{s}, \bar{s})$ constitutes in fact the interior of the convex hull of $\text{supp}(\mu_F)$.

Proof of Proposition 1. (a) Let us first define the differentiable function $\Psi(w) = v'(w)/u'(w)$. Then, $A_u(w) > A_v(w)$ for all $w \in (a - \underline{s}_{\tilde{\varepsilon}}, \infty)$ if and only if $\Psi'(w) > 0$ for all $w \in (a - \underline{s}_{\tilde{\varepsilon}}, \infty)$. After normalizing $u(a - \underline{s}_{\tilde{\varepsilon}} = 0$ and using (5), we have the following explicit functional form for $\Psi(w)$:

$$\Psi(w) = \frac{\int_{\underline{s}_{\tilde{\varepsilon}}}^\infty \int_0^\infty e^{-s(w-a+\underline{s}_{\tilde{\varepsilon}}+z)}\, dF(s)\, dG_{\tilde{\varepsilon}}(z)}{\int_0^\infty e^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, dF(s)} = \frac{\int_{\underline{s}}^{\bar{s}} \psi_{\tilde{\varepsilon}}(s)\, e^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, dF(s)}{\int_{\underline{s}}^{\bar{s}} e^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, dF(s)}\ . \tag{7}$$

From Lemmas 1 and 2 we know that $\psi_{\tilde{\varepsilon}}(s)$ is strictly decreasing on $[0, s_{\text{inf}}]$. Since, by assumption, $(\underline{s}, \bar{s}) \subseteq (0, s_{\text{inf}})$, it follows that $\psi_{\tilde{\varepsilon}}(\bar{s}) < \psi_{\tilde{\varepsilon}}(s) < \psi_{\tilde{\varepsilon}}(\underline{s})$ for all $s \in (\underline{s}, \bar{s})$, which in turn implies that

$$\psi_{\tilde{\varepsilon}}(\bar{s}) < \Psi(w) < \psi_{\tilde{\varepsilon}}(\underline{s}), \quad \text{for all } w \in (a - \underline{s}_{\tilde{\varepsilon}}, \infty)\ . \tag{8}$$

The inequalities in (8) are strict because the assumption that $A_u(x)$ is strictly decreasing is equivalent to requiring that $\text{supp}(\mu_F)$ has at least two points.

Note that $\Psi(w)$ is monotonic (either decreasing or increasing) whenever the equation $\Psi(w) = \lambda$ has at most one solution for all $\lambda \in (0, \infty)$. From (7) we can rewrite $\Psi(w) = \lambda$ as

$$\Omega(x) \equiv \int_{\underline{s}}^{\bar{s}} (\psi_{\tilde{\varepsilon}}(s) - \lambda)\, e^{-sx}\, dF(s) = 0, \tag{9}$$

where $x = w - a + \underline{s}_{\tilde{\varepsilon}} \ge 0$. Eq. (9) has no solution for x (and, thus, it has no solution for

w) when $\lambda \not\in (\psi_{\tilde{\varepsilon}}(\bar{s}),\ \psi_{\tilde{\varepsilon}}(\underline{s}))$. On the other hand, when $\lambda \in (\psi_{\tilde{\varepsilon}}(\bar{s}),\ \psi_{\tilde{\varepsilon}}(\underline{s}))$. Theorem 2 implies that Eq. (9) has at most a single solution since the function $(\psi_{\varepsilon}(s) - \lambda)$ is monotonically decreasing on $(\underline{s}, \bar{s})$ and, hence, it has only one change of sign on such an interval. Therefore, $\lim_{w\to\infty} \Psi(w)$ exists and is finite as a consequence of the monotonicity and boundedness of $\Psi(w)$ on $(a - \underline{s}_{\tilde{\varepsilon}}, \infty)$.

Our next goal is to prove that $\lim_{w\to\infty} \Psi(w) = \psi_{\tilde{\varepsilon}}(\underline{s})$. To this end consider the function $\Theta(w) = v'(w) - \alpha u'(w)$ with $\alpha < \psi_{\tilde{\varepsilon}}(\underline{s})$. Therefore,

$$\Theta(w) = \int_{\underline{s}}^{\bar{s}} (\psi_{\tilde{\varepsilon}}(s) - \alpha)\, \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, \mathrm{d}F(s)\,.$$

Then, there exists a real number $c \in (\underline{s}, \bar{s})$ such that $\psi_{\tilde{\varepsilon}}(s) > \alpha$ for all $s \in (\underline{s}, c)$. Observe that

$$\int_{c}^{\bar{s}} (\psi_{\tilde{\varepsilon}}(s) - \alpha)\, \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, \mathrm{d}F(s) \le \int_{c}^{\bar{s}} |\psi_{\tilde{\varepsilon}}(s) - \alpha|\, \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, \mathrm{d}F(s)$$
$$\le \left(\int_{c}^{\bar{s}} |\psi_{\tilde{\varepsilon}}(s) - \alpha|\, \mathrm{d}F(s)\right) \mathrm{e}^{-c(w-a+\underline{s}_{\tilde{\varepsilon}})}\,, \qquad (10)$$

for all $w > a - \underline{s}_{\tilde{\varepsilon}}$, where $\int_c^{\bar{s}} |\psi_{\tilde{\varepsilon}}(s) - \alpha|\, \mathrm{d}F(s)$ is obviously finite. Moreover, since $\psi_{\tilde{\varepsilon}}(s) > \alpha$ for all $s \in (\underline{s}, c)$, there exists a real number $\hat{w} > a - \underline{s}_{\tilde{\varepsilon}}$ such that

$$\int_{\underline{s}}^{c} (\psi_{\tilde{\varepsilon}}(s) - \alpha)\, \mathrm{e}^{(c-s)(w-a+\underline{s}_{\tilde{\varepsilon}})}\, \mathrm{d}F(s) > \int_{c}^{\bar{s}} |\psi_{\tilde{\varepsilon}}(s) - \alpha|\, \mathrm{d}F(s), \quad \text{for all } w > \hat{w}\,. \qquad (11)$$

Therefore, multiplying both sides of (11) by $\mathrm{e}^{-c(w-a-\underline{s}_{\tilde{\varepsilon}})}$, we get

$$\int_{\underline{s}}^{c} (\psi_{\tilde{\varepsilon}}(s) - \alpha)\, \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, \mathrm{d}F(s)$$
$$> \left(\int_{c}^{\bar{s}} |\psi_{\tilde{\varepsilon}}(s) - \alpha|\, \mathrm{d}F(s)\right) \mathrm{e}^{-c(w-a+\underline{s}_{\tilde{\varepsilon}})}, \quad \text{for all } w > \hat{w}\,. \qquad (12)$$

Then,

$$\Theta(w) = \int_{\underline{s}}^{c} (\psi_{\tilde{\varepsilon}}(s) - \alpha)\, \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, \mathrm{d}F(s) + \int_{c}^{\bar{s}} (\psi_{\tilde{\varepsilon}}(s) - \alpha)\, \mathrm{e}^{-s(w-a+\underline{s}_{\tilde{\varepsilon}})}\, \mathrm{d}F(s) > 0\,,$$

for all $w > \hat{w}$, as follows from (10) and (12). Note that $\Theta(w) > 0$ if and only if $\Psi(w) > \alpha$ since $\Theta(w)/u'(w) = \Psi(w) - \alpha$. Taking an increasing sequence of values for α converging to $\psi_{\tilde{\varepsilon}}(\underline{s})$, we get that $\lim_{w\to\infty} \Psi(w) \ge \psi_{\tilde{\varepsilon}}(\underline{s})$. This, together with (8), implies that $\lim_{w\to\infty} \Psi(w) = \psi_{\tilde{\varepsilon}}(\underline{s})$.

Summing up, the function $\Psi(w)$ is monotonic and bounded above by $\psi_{\tilde{\varepsilon}}(\underline{s})$ for all $w > a - \underline{s}_{\tilde{\varepsilon}}$, and $\lim_{w\to\infty} \Psi(w) = \psi_{\tilde{\varepsilon}}(\underline{s})$. Hence, the function $\Psi(w)$ is strictly increasing on $(a - \underline{s}_{\tilde{\varepsilon}}, \infty)$, so that it increases monotonically towards its limit $\psi_{\tilde{\varepsilon}}(\underline{s})$. This proves that $\Psi'(w) \ge 0$. Note however that if there exists a value w^* such that $\Psi(w^*) = \lambda$, for some $\lambda \in (\psi_{\tilde{\varepsilon}}(\bar{s}), \psi_{\tilde{\varepsilon}}(\underline{s}))$, and $\Psi'(w^*) = 0$, then $x^* = w^* - a - \underline{s}_{\tilde{\varepsilon}}$ is a repeated root of Eq. (9). This means that Eq. (9) has at least two roots (counted according to their multiplicity) which contradicts Theorem 2. Therefore, $\Psi'(w)>0$ for all $w\in(a-\underline{s}_{\tilde{\varepsilon}},\infty)$, which is the desired result

(b) The proof of (b) follows exactly the same steps as the one of (a). We just have to reverse all the inequality signs and notice that $\psi_{\tilde{\varepsilon}}(s)$ is strictly increasing on $(\underline{s},\bar{s})$ since $(\underline{s},\bar{s})\subseteq(s_{\inf},\infty)$. Therefore, $\psi_{\tilde{\varepsilon}}(\underline{s})>\Psi(w)>\psi_{\tilde{\varepsilon}}(\underline{s})$ for all $w>a-\underline{s}_{\tilde{\varepsilon}}$ and $\Psi(w)$ is thus strictly decreasing on $(a-\underline{s}_{\tilde{\varepsilon}},\infty)$. Q.E.D.

The following Corollary establishes the unambiguous comparison when the significative risk $\tilde{\varepsilon}$ is either subfair or positive:

Corollary 1. *Assume that $A_u(w)$ is strictly decreasing. Then*

(*a*) $A_v(w)>A_v(w)$, *for all* $w>a-\underline{s}_{\tilde{\varepsilon}}$ *if* $\tilde{\varepsilon}$ *is subfair.*

(*b*) $A_v(w)<A_u(w)$, *for all* $w>a-\underline{s}_{\tilde{\varepsilon}}$ *if* $\tilde{\varepsilon}$ *is positive.*

Proof. It follows immediately from Proposition 1 since $s_{\inf}=0\le\underline{s}$ if $\tilde{\varepsilon}$ is subfair, whereas $s_{\inf}=\infty\ge\bar{s}$ if $\tilde{\varepsilon}$ is positive. Q.E.D.

Part (a) of the previous corollary has been already obtained under much weaker conditions than mixed risk aversion. Gollier, Pratt (1996) have in fact proved that (a) follows from the assumption of risk vulnerability, and we have already pointed out that the class of mixed utilities constitutes a strict subset of the class of risk vulnerable utilities.

We have assumed that $A_u(w)$ is strictly decreasing so as to exclude the trivial case of utility functions displaying constant absolute risk aversion (which are mixed, as we have already argued in Section 2). Indeed, it is obvious that for the CARA utilities the index of absolute risk aversion is unaffected by the presence of random background wealth.

5. Desirability and risk aversion

An interesting observation we should make is that $A_v(w)>A_u(w)$ may hold uniformly even if the significative risk $\tilde{\varepsilon}$ is desirable for all $w\ge a-\underline{s}_{\tilde{\varepsilon}}$. Therefore, bearing a risk may make individuals happier for all levels of initial wealth and, at the same time, to induce a more risk averse behavior. Of course, such a globally desirable risk cannot be subfair since all subfair risks are undesirable under risk aversion. On the other hand, the desirable risk $\tilde{\varepsilon}$ cannot be positive since then $A_v(w)<A_u(w)$, as follows from part (b) of Corollary 1. Therefore, a globally desirable risk which induces more risk aversion for all levels of background wealth will be potentially undesirable provided it exists. Moreover, the corresponding utility function must display strictly decreasing absolute risk aversion. The following proposition establishes the exact result:

Proposition 2. (*a*) *If $\tilde{\varepsilon}$ is a significative potentially undesirable risk, then there exists a utility function $u\in\mathcal{M}$ such that $A_v(w)>A_u(w)$ and $v(w)>u(w)$ for all $w>a-\underline{s}_{\tilde{\varepsilon}}$.*

(*b*) *If $u\in\mathcal{M}$, $\underline{s}>0$, and $\bar{s}$ is finite, then there exists a significative potentially undesirable risk $\tilde{\varepsilon}$ such that $A_v(w)>A_u(w)$ and $v(w)>u(w)$ for all $w>a-\underline{s}_{\tilde{\varepsilon}}$.*

Proof. (a) Our proof will consist of constructing the mixed utility with the desired properties. Consider the mixed utility $u(w)$ characterized by a distribution function $F(s)$ such that $\text{supp}(\mu_F)$ has just two points: $\underline{s}$ and $\bar{s}$, with $\bar{s}>\underline{s}>0$. Let $\alpha_1=F(\underline{s})>0$ and $\alpha_2=F(\bar{s})-F(\underline{s})>0$. Then, it follows from (1) that

$$u(w)=(1-e^{-\underline{s}(w-a)})\left(\frac{\alpha_1}{\underline{s}}\right)+(1-e^{-\bar{s}(w-a)})\left(\frac{\alpha_2}{\bar{s}}\right).$$

On the other hand,

$$\begin{aligned}v(w)&=\int_{\underline{s}_{\tilde{\varepsilon}}}^{\infty}\left[(1-e^{-\underline{s}(w-a+z)})\left(\frac{\alpha_1}{\underline{s}}\right)+(1-e^{-\bar{s}(w-a+z)})\left(\frac{\alpha_2}{\bar{s}}\right)\right]dG_{\tilde{\varepsilon}}(z)\\&=(1-e^{-\underline{s}(w-a)}\psi_{\tilde{\varepsilon}}(\underline{s}))\left(\frac{\alpha_1}{\underline{s}}\right)+(1-e^{-\bar{s}(w-a)}\psi_{\tilde{\varepsilon}}(\bar{s}))\left(\frac{\alpha_2}{\bar{s}}\right)1.\end{aligned}$$

Thus, the condition $v(w)>u(w)$ becomes

$$(1-\psi_{\tilde{\varepsilon}}(\underline{s}))\,e^{-\underline{s}(w-a)}\left(\frac{\alpha_1}{\underline{s}}\right)+(1-\psi_{\tilde{\varepsilon}}(\bar{s}))\,e^{-\bar{s}(w-a)}\left(\frac{\alpha_2}{\bar{s}}\right)>0. \tag{13}$$

The condition $A_v(w)>A_u(w)$ is equivalent to assuming that the function $\Psi(w)=v'(w)/u'(w)$ is strictly decreasing. From (7), we get

$$\Psi(w)=\frac{\alpha_1\psi_{\tilde{\varepsilon}}(\underline{s})\,e^{-\underline{s}(w-a)}+\alpha_2\psi_{\tilde{\varepsilon}}(\bar{s})\,e^{-\bar{s}(w-a)}}{\alpha_1\,e^{-\underline{s}(w-a)}+\alpha_2\,e^{-\bar{s}(w-a)}}.$$

Computing the derivative of $\Psi(w)$, we obtain after some tedious algebra,

$$\Psi'(w)=\frac{-\alpha_1\alpha_2(\bar{s}-\underline{s})[\psi_{\tilde{\varepsilon}}(\bar{s})-\psi_{\tilde{\varepsilon}}(\underline{s})]\,e^{-\bar{s}(w-a)-\underline{s}(w-a)}}{[\alpha_1\,e^{-\underline{s}(w-a)}+\alpha_2\,e^{-\bar{s}(w-a)}]^2},$$

and therefore $\Psi'(w)<0$ if and only if $\psi_{\tilde{\varepsilon}}(\bar{s})>\psi_{\tilde{\varepsilon}}(\underline{s})$. Notice that such a condition might be compatible with (13) since a sufficient condition for the latter is that $\psi_{\tilde{\varepsilon}}(\bar{s})$ and $\psi_{\tilde{\varepsilon}}(\underline{s})$ be both less than one. Summing up, the two conditions $A_v(w)>A_u(w)$ and $v(w)>u(w)$ hold simultaneously whenever

$$\psi_{\tilde{\varepsilon}}(\underline{s})<\psi_{\tilde{\varepsilon}}(\bar{s})\le 1. \tag{14}$$

Notice that (14) can be easily achieved if $\tilde{\varepsilon}$ is a potentially undesirable risk. To do so, recall Lemma 2 and select as a value of $\underline{s}$ a real number belonging to the interval $(0, s^*)$, which implies that $\psi_{\tilde{\varepsilon}}(\underline{s})<1$. If $\underline{s}<s_{\text{inf}}$ then, from the properties of the Laplace transform of a potentially undesirable risk given in Lemma 2 (b), there exists a real number $\hat{s}$ such that $\hat{s}>s_{\text{inf}}$ and $\psi_{\tilde{\varepsilon}}(\hat{s})=\psi_{\tilde{\varepsilon}}(\underline{s})$. Then, $\psi_{\tilde{\varepsilon}}(s)\in(\psi_{\tilde{\varepsilon}}(\underline{s}), 1]$ for all $s\in(\hat{s}, s^*]$. Hence, we finish the construction of a utility function satisfying (14) by selecting a value of $\bar{s}$ equal to any real number in the interval $(\hat{s}, s^*]$. If we had selected instead a value of $\underline{s}\in(s_{\text{inf}}, s^*)$, then we should choose a value of $\bar{s}$ belonging to the interval $(\underline{s}, s^*]$ since $\psi_{\tilde{\varepsilon}}(s)\in(\psi_{\tilde{\varepsilon}}(\underline{s}), 1]$ for all $s\in(\underline{s}, s^*]$.

(b) Consider a random variable $\tilde{\varepsilon}$ taking just two real values, $-L$ and H, with $L>0$ and $H>0$. Note that $\underline{s}_{\tilde{\varepsilon}}=-L$. The probabilities associated with these two values are p

and $1-p$, respectively. Clearly, the risk $\tilde{\varepsilon}$ is a significative potentially undesirable risk, and its Laplace transform is $\psi_{\tilde{\varepsilon}}(s)=p\,\mathrm{e}^{sL}+(1-p)\,\mathrm{e}^{-sH}$. Thus,

$$s_{\inf}=\frac{\ln\left(\dfrac{(1-p)H}{pL}\right)}{L+H}\,.$$

According to Proposition l, $A_v(w)>A_u(w)$ whenever $s_{\inf}\le\underline{s}$. Because of the convexity of $\psi_{\tilde{\varepsilon}}(s)$, the latter condition is achieved if

$$\psi'_{\tilde{\varepsilon}}(\underline{s})=pL\,\mathrm{e}^{\underline{s}L}+(1-p)H\,\mathrm{e}^{-\underline{s}H}\ge 0\,. \tag{15}$$

On the other hand, a sufficient condition for $v(w)>u(w)$ is that

$$\psi_{\tilde{\varepsilon}}(s)=p\,\mathrm{e}^{sL}+(1-p)\,\mathrm{e}^{-sH}\le 1,\quad \text{for all } s\in[\underline{s},\bar{s}]\,. \tag{16}$$

Conditions (15) and (16) may hold simultaneously by choosing a sufficiently high value for H and a sufficiently low value for p. This is so because $\lim_{H\to\infty}\psi'_{\tilde{\varepsilon}}(\underline{s})=pL\,\mathrm{e}^{\underline{s}L}>0$ and $\lim_{p\to 0}\psi_{\tilde{\varepsilon}}(\underline{s})=\mathrm{e}^{-\underline{s}H}<1$. Q.E.D.

The previous proposition has illustrated the procedure for obtaining examples of globally desirable risks which induce more risk aversion. Under the assumptions of the proposition, for a given mixed utility we can find a desirable random change on wealth which makes agents behave in a more risk averse fashion when they are faced with additional, independent sources of uncertainty. Conversely, given a potentially undesirable risk $\tilde{\varepsilon}$, we can find a utility function u which makes such a risk globally desirable, while $v(w)=E(u(w+\tilde{\varepsilon}))$ displays more risk aversion than $u(w)$ for all levels of w in the relevant domain.

It should also be pointed out that it is impossible to find examples of globally undesirable risks which uniformly decrease the index of absolute risk aversion. The reason is that, under mixed risk aversion, a risk is undesirable for all levels of wealth only if it is subfair. Thus, part (a) of Corollary 1 prevents any reduction in the index of absolute risk aversion in such a circumstance.

6. Random background wealth

Our analysis can be easily extended to a situation in which the initial background wealth is random. To this end, we just need to use the results of Kihlstrom et al. (1981). Thus, assume that background wealth is a random variable $\tilde{w}$, and let $\underline{s}_{\tilde{w}}>-\infty$ be the infimum of the distribution of the random variable $\tilde{w}$. Define the functions

$$U(x)=E(u(\tilde{w}+x))\,,$$

and

$$V(x)=E(v(\tilde{w}+x)),\quad \text{with } v(\tilde{w}+x)=E_{\tilde{\varepsilon}}(u(\tilde{w}+\tilde{\varepsilon}+x))\,, \tag{17}$$

where $x\in(a-\underline{s}_{\tilde{\varepsilon}}-\underline{s}_{\tilde{w}},\infty)$ and the subindex in the expectation operator denotes the random variable with respect to which the mathematical expectation is computed. Consider a random variable $\tilde{x}$ taking values in the interval $(a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}, \infty)$ and assume that the random variables $\tilde{w}$, $\tilde{\varepsilon}$ and $\tilde{x}$ are all mutually independent. Finally, define the indexes of absolute risk aversion

$$A_U(x) = \frac{-U''(x)}{U'(x)} \quad \text{and} \quad A_V(x) = \frac{-V''(x)}{V'(x)} .$$

Recalling our definition of risk premium in Section 3, it follows from Pratt (1964) that $\pi(\tilde{x}, \tilde{w} + \tilde{\varepsilon}) \geq \pi(\tilde{x}, \tilde{w})$ for every risk $\tilde{x}z$ taking values in $(a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}, \infty)$ if and only if $A_V(x) \geq A_U(x)$ for all $x > a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}$. Hence, to say that an individual increases its risk aversion when the risk $\tilde{\varepsilon}$ is added to the initial background wealth $\tilde{w}$ is equivalent to saying that $A_V(x) \geq A_U(x)$.

Alternatively, consider the typical portfolio selection problem in which a risk averse individual with preferences characterized by the Bernouilli utility u has a random wealth $\tilde{y}$. Before observing the realization of $\tilde{y}$, the investor has to decide how much to invest in a risky asset with a random gross rate of return $\tilde{R}$ which is independent of $\tilde{y}$. Let $z(\tilde{y}, \tilde{R}; u)$ be the optimal investment in risky asset, that is,

$$z(\tilde{y}, \tilde{R}; u) = \underset{z \in \Delta}{\operatorname{argmax}} E(u(\tilde{y} + [\tilde{R} - 1]z)) ,$$

where Δ is the subset of real numbers such that, for all $z \in \Delta$, the random variable $\tilde{y} + [\tilde{R} - 1]z$ takes values in the domain of u. Then, from Arrow (1970) and Pratt (1964), we can conclude that the condition $A_V(x) \geq A_U(x)$ for all x in the relevant domain is necessary and sufficient for $z(\tilde{y}, \tilde{R}; V) \leq z(\tilde{y}, \tilde{R}; U)$, which is in turn equivalent to $z(\tilde{w} + \tilde{\varepsilon}, \tilde{R}; u) \leq z(\tilde{w}, \tilde{R}; u)$.

As follows from the Theorem in Kihlstrom et al. (1981), if

$$A_v(w) \underset{(\leq)}{\geq} A_u(w) \quad \text{for all } w \in (a - \underline{s}_{\tilde{\varepsilon}}, \infty) , \tag{18}$$

and either $A_v(w)$ or $A_u(w)$ are nonincreasing for all $w \in (a - \underline{s}_{\tilde{\varepsilon}}, \infty)$, then

$$A_V(x) \underset{(\leq)}{\geq} A_U(x) \quad \text{for all } x \in (a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}, \infty) . \tag{19}$$

Moreover, it can be easily seen that the inequalities in (19) become strict if the inequalities in (18) are also strict and either $A_v(w)$ or $A_u(w)$ are strictly decreasing for all $w \in (a - \underline{s}_{\tilde{\varepsilon}}, \infty)$.

If u is mixed, then it displays nonincreasing absolute risk aversion, and this allows one to conclude that the sufficient conditions for

$$A_v(x) \underset{(\leq)}{\geq} A_u(x)$$

will be also sufficient for

$$A_V(x) \underset{(\leq)}{\geq} A_U(x) .$$

A more direct approach, that uses the fact that the function v is also mixed (see Lemma 3), can be undertaken along the lines of Proposition 1. Note that if u is mixed on $[a - \underline{s}_{\tilde{\varepsilon}}, \infty)$, then

$$U'(x) = E(u'(\tilde{w} + x)) = \int_{\underline{s}_{\tilde{w}}}^{\infty} \left(\int_0^{\infty} \mathrm{e}^{-s(y+x-a+\underline{s}_{\tilde{\varepsilon}})} \, \mathrm{d}F(s) \right) \mathrm{d}G_{\tilde{w}}(y)$$

$$= \int_0^{\infty} \mathrm{e}^{-s(x-a+\underline{s}_{\tilde{\varepsilon}}+\underline{s}_{\tilde{w}})} \psi_{\tilde{w}-\underline{s}_{\tilde{w}}}(s) \, \mathrm{d}F(s) = \int_0^{\infty} \mathrm{e}^{-s(x-a+\underline{s}_{\tilde{\varepsilon}}+\underline{s}_{\tilde{w}})} \, \mathrm{d}\hat{F}(s) ,$$

where $\hat{F}(s)$ is the distribution function defined by the Stieltjes integral

$$\hat{F}(s) = \int_0^{s} \psi_{\tilde{w}-\underline{s}_{\tilde{w}}}(z) \, \mathrm{d}F(z) .$$

Therefore, the function $U(x)$ is mixed on $[a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}, \infty)$. Similarly,

$$V'(x) = \int_0^{\infty} \mathrm{e}^{-s(x-a+\underline{s}_{\tilde{\varepsilon}}+\underline{s}_{\tilde{w}})} \psi_{\tilde{\varepsilon}}(s) \, \mathrm{d}\hat{F}(s) .$$

We can define now the function $\hat{\Psi}(w) \equiv U'(x)/V'(x)$ and proceed as in the proof of Proposition 1 to obtain sufficient conditions for either $A_V(x) > A_U(x)$ or $A_V(x) < A_U(x)$. Note that both the essential infimum $\underline{s}$ and the essential supremum $\bar{s}$ of the distribution associated with $\hat{F}$ are the same as the ones of the distribution associated with F since the Laplace transform $\psi_{\tilde{w}-\underline{s}_{\tilde{w}}}(z)$ is strictly positive for all $z \geq 0$. Therefore, we can combine Proposition 1 and Corollary 1 with our previous discussion to obtain the following result:

Proposition 3. *Let u be a mixed utility function defined on $[a, \infty)$ such that $A_u(x)$ is strictly decreasing on (a, ∞). Let $\tilde{\varepsilon}$ be a significative risk taking values in $(\underline{s}_{\tilde{\varepsilon}}, \infty)$, whereas $\tilde{w}$ is a random variable taking values in $(\underline{s}_{\tilde{w}}, \infty)$. Assume that $\tilde{w}$ and $\tilde{\varepsilon}$ are independent.*

(a) If $s_{\inf} \geq \bar{s}$, then $A_V(x) < A_U(x)$ for all $x > a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}$ and $\pi(\tilde{x}, \tilde{w} + \tilde{\varepsilon}) < \pi(\tilde{x}, \tilde{w})$ for

every independent significative risk $\tilde{x}$ *taking values in* $(a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}, \infty)$. *In particular, the last two inequalities hold whenever* $\tilde{\varepsilon}$ *is positive.*

(*b*) If $s_{\inf} \leq \underline{s}$, *then* $A_V(x) > A_U(x)$ *for all* $x > a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}$ *and* $\pi(\tilde{x}, \tilde{w} + \tilde{\varepsilon}) > \pi(\tilde{x}, \tilde{w})$ *for every independent significative risk* $\tilde{x}$ *taking values in* $(a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}}, \infty)$. *In particular, the last two inequalities hold whenever* $\tilde{\varepsilon}$ *is subfair.*

7. A final remark

We have found conditions under which the introduction of an additional independent risk induces more (or less) risk aversion when preferences display mixed risk aversion. Our approach has used the relationship between the distribution function of background wealth and the (artificial) distribution function characterizing a mixed utility. Finally, our analysis has also allowed us to construct examples of globally desirable risks which increase the risk aversion of individuals for all levels of wealth.

The sufficient conditions obtained in Propositions 1 and 3 are indeed very strong and they could be relaxed if we just wanted to study the local behavior of risk aversion, that is, the willingness to accept another *small*, independent risk $\tilde{x}$. More precisely, regarding the analysis of Section 6, it can be asserted that for all $\eta > 0$ and $y \in (a - \underline{s}_{\tilde{\varepsilon}} - \underline{s}_{\tilde{w}} + \eta, \infty)$, there exists a real number $\delta > 0$ such that, for every mixed utility u characterized by a distribution function $F(s)$ satisfying $(F(\bar{s}) - F(s_{\inf})) < \delta$ (resp., $(F(s_{\inf}) - F(\underline{s})) < \delta$), it holds that

$$A_V(x) \underset{(>)}{<} A_U(x)$$

for all $x \in (y - \eta, y + \eta)$, where the functions U and V are defined in (17). This result is a consequence of the absolute continuity of a Laplace transform (and of all its derivatives) with respect to the measure associated with its distribution function. In other words, for the local analysis of the behavior of risk aversion, we just need to ensure that the distribution over the exponents of a mixed utility is sufficiently concentrated either below or above the point at which the Laplace transform of the risk $\tilde{\varepsilon}$ reaches its infimum.

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